NOTES ON GENERAL RELATIVITY (GR) AND GRAVITY

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Abstract. These are notes on General Relativity (GR) and Gravity.

As of March 23, 2015, I find that the Central Lectures given by Dr. Frederic P. Schuller for the WE Heraeus International Winter School to be, unequivocally, the best, most lucid, and well-constructed lecture series on General Relativity and Gravity. Instead of reinventing the wheel, I write these notes to build upon and supplement the video lectures and tutorials already created by them. This includes my corrections, comments, relations to other aspects of theoretical physics, and code implementing calculations in GR.

It should be noted that for symbolic computation, I heavily use the SageManifolds v.0.7 package for Sage Math. My goal in this area is this: we see a concept or idea from GR and we go from the equation on the blackboard or textbook and into (Python/Sage Math) code that immediately computes a calculation.

I keep these notes available online, openly accessible, and free for anyone, anytime (with your (financial) help and contribution at Tilt/Open, which is a subscription service). I want to keep these notes openly accessible because I want to encourage anyone to freely edit, copy, and make their own notes in the spirit of open-source software.

I continuously update these notes and post them here `ernestyalumni.wordpress.com`

The stated goal of the WE Heraeus International Winter School on Gravity and Light is to take the student from an introduction to the research frontier (cf. `http://www.gravity-and-light.org/lectures`). I want to get myself and other students or ambitious non-academic (maybe he or she is a working professional who had studied physics before in college, went to work in another field, maybe even, gasp, investment banking or mobile app developer, but still is curious and passionate about physics and want to contribute) equipped with all the tools available to do research, do calculations, to design experiments or collect data. Again, we're not here to reinvent the wheel. I'm not trying to make a General Relativity appreciation class, but this is a serious attempt towards training to do research.

Contents

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1991 *Mathematics Subject Classification.* General Relativity.
Key words and phrases. General Relativity, Gravity, Differential Geometry, Manifolds, Integration, MIT OCW, education, mathematics, physics.

I write notes, review papers, and code and make calculations for physics, math, and engineering to help with education and research. With your support, we can keep education and research material available online, openly accessible, and free for anyone, anytime. If you like what I'm trying to do for physics education research, please go to my Tilt/Open crowdfunding campaign, read the mission statement, share the page, and contribute financially if you can. ernestyalumni.tilt.com .

Part 1. WE Heraeus International Winter School on Gravity and Light

Introduction (from EY)

The International Winter School on Gravity and Light held *central lectures* given by Dr. Frederic P. Schuller. These lectures on General Relativity and Gravity are unequivocally and undeniably, the best and most lucid and well-constructed lecture series on General Relativity and Gravity. The mathematical foundation from topology and differential geometry from which General Relativity arises from is solid, well-selected in rigor. The lectures themselves are well-thought out and clearly explained.

Even more so, the International Winter School provided accompanying Tutorial Sessions for each of the lectures. I had given up hopes in seeing this component of the learning process ever be put online so that anyone and everyone in the world could learn through the Tutorial process as well. I was afraid that nobody would understand how the Tutorial or "Office Hours" session was important for students to digest and comprehend and work out-doing exercises-the material presented in the lectures. This International Winter School gets it and shows how online education has to be done, to do it in an excellent manner, moving forward.

For anyone who is serious about learning General Relativity and Gravity, I would simply point to these video lectures and tutorials.

What I want to do is to build upon the material presented in this International Winter School. Why it's important to me, and to the students and practicing researchers out there, is that the material presented takes the student from an introduction to the research frontier. That is the stated goal of the International Winter School. I want to dig into and help contribute to the cutting edge in research and this entire program with lectures and tutorials appears to be the most direct and sensible route directly to being able to do research in General Relativity and Gravity. -EY 20150323

1. Lecture 1: Topology

1.1. Lecture 1: Topological Spaces.

Definition 1. *Let M be a set.*
*A **topology** $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$, $\mathcal{P}(M)$ power set of M: set of all subsets of M. satisfying*

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$
(ii) $U \in \mathcal{O}, \quad V \in \mathcal{O} \Longrightarrow U \bigcap V \in \mathcal{O}$
(iii) $U_\alpha \in \mathcal{O}, \quad \alpha \in \mathcal{A} \quad \Longrightarrow \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) \in \mathcal{O}$

$\mathcal{O}\}$ utterly useless

Definition 2. $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$

EY : 20150524
I'll fill in the proof that $\mathcal{O}_{\text{standard}}$ is a topology.

Proof. $\emptyset \in \mathcal{O}_{\text{standard}}$
since $\forall\, p \in \emptyset$, $\exists\, r \in \mathbb{R}^+$: $\mathcal{B}_r(p) \subseteq \emptyset$ (i.e. satisfied "vacuously")

Suppose $U, V \in \mathcal{O}_{\text{standard}}$.

Let $p \in U \bigcap V$. Then $\exists\, r_1, r_2 \in \mathbb{R}^+$ s.t. $\begin{aligned} &\mathcal{B}_{r_1}(p) \subseteq U \\ &\mathcal{B}_{r_2}(p) \subseteq V \end{aligned}$

Let $r = \min\{r_1, r_2\}$.
Clearly $\mathcal{B}_r(p) \subseteq U$ and $\mathcal{B}_r(p) \subseteq V$. Then $\mathcal{B}_r(p) \subseteq U \bigcap V$. So $U \bigcap V \in \mathcal{O}_{\text{standard}}$.

Suppose, $U_\alpha \in \mathcal{O}_{\text{standard}}$, $\forall\, \alpha \in \mathcal{A}$.
Let $p \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Then $p \in U_\alpha$ for at least 1 $\alpha \in \mathcal{A}$.
$\exists\, r_\alpha \in \mathbb{R}^+$ s.t. $\mathcal{B}_{r_\alpha}(p) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. So $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{O}_{\text{standard}}$ □

1.2. 2. Continuous maps.

1.3. 3. Composition of continuous maps.

1.4. 4. Inheriting a topology.

EY : 20150524
I'll fill in the proof that given f continuous (cont.), then the restriction of f onto a subspace S is cont. If you want a reference, check out Klaus Jänich [2, pp. 13, Ch. 1 Fundamental Concepts, Sec. Continuous Maps]
If cont. $f : M \to N$, $S \subseteq M$, then $f|_S$ cont.

Proof. Let open $V \subseteq N$, i.e. $V \in \mathcal{O}_N$ i.e. V in the topology $\mathcal{O}_N$ of N.

$$f|_S^{-1}(V) = \{m \in M | \; f|_S(m) \in V\}$$

Now $f^{-1}(V) = \{m \in M | f(m) \in V\}$.
So $f^{-1}(V) \bigcap S = f|_S^{-1}(V)$

Now f cont. So $f^{-1}(V) \in \mathcal{O}_N$.
and recall $\mathcal{O}_S| := \{U \bigcap S | U \in \mathcal{O}_M\}$.
so $f^{-1}(V) \bigcap S = f|_S^{-1}(V) \in \mathcal{O}_S$ i.e. $f|_S^{-1}(V)$ open.
$\Longrightarrow f|_S$ cont. $\square$

TOPOLOGY TUTORIAL SHEET

filename : `main.pdf`
The WE-Heraeus International Winter School on Gravity and Light: Topology

EY : 20150524
What I won't do here is retype up the solutions presented in the Tutorial (cf. `https://youtu.be/_XkhZQ-hNLs`): the presenter did a very good job. If someone wants to type up the solutions and copy and paste it onto this LaTeX file, in the spirit of open-source collaboration, I would encourage this effort.

Instead, what I want to encourage is the use of as much CAS (Computer Algebra System) and symbolic and numerical computation because, first, we're in the 21st century, second, to set the stage for further applications in research. I use Python and Sage Math alot, mostly because they are open-source software (OSS) and fun to use. Also note that the structure of Sage Math modules matches closely to Category Theory.

In checking whether a set is a topology, I found it strange that there wasn't already a function in Sage Math to check each of the axioms. So I wrote my own; see my code snippet, which you can copy, paste, edit freely in the spirit of OSS here, titled `topology.sage`:
gist github ernestyalumni topology.sage
Download topology.sage
Loading `topology.sage`, after changing into (with the usual Linux terminal commands, cd, ls) by

```
sage: load(''topology.sage'')
```

Exercise 2: Topologies on a simple set.

Question Does $\mathcal{O}_1 := \ldots$ constitute a topology ...?.

Solution: Yes, since we check by typing in the following commands in Sage Math:

```
emptyset in O_1
Axiom2check(O_1) # True
Axiom3check(O_1) # True
```

Question What about $\mathcal{O}_2$ ...?.

Solution: No since the 3rd. axiom fails, as can be checked by typing in the following commands in Sage Math:

```
emptyset in O_2
Axiom2check(O_2) # True
Axiom3check(O_2) # False
```

2. LECTURE 2: TOPOLOGICAL MANIFOLDS

Lecture 2: Manifolds. Topological spaces: $\exists$ so man that mathematicians cannot even classify them.

For spacetime physics, we may focus on topological spaces $(M, \mathcal{O})$ that can be <u>charted</u>, analogously to how the surface of the earth is charted in an <u>atlas</u>.

2.1. Topological manifolds.

Definition 3. *A topological space $(M, \mathcal{O})$ is called a **d-dimensional topological method** if*
$\forall p \in M : \exists U \in \mathcal{O}, U \ni p : \exists x : U \subseteq M \to x(U) \subseteq \mathbb{R}^d$ $\qquad (M, \mathcal{O}), (\mathbb{R}^d, \mathcal{O}_{std})$

(i) x ***invertible*** *:*
$$x^{-1} : x(U) \to U$$
(ii) x ***continuous***
(iii) x^{-1} ***continuous***

2.2. Terminology.

2.3. **3. Chart transition maps.** Imagine 2 charts (U, x) and (V, y) with overlapping regions.

2.4. **4. Manifold philosophy.** Often it is desirable (or indeed the way) to define properties ("continuity") of real-world object ("$\mathbb{R} \xrightarrow{\gamma} M$") by judging suitable coordinates not on the "real-world" object itself, but on a chart-representation of that real world object.

EY's add-ons. This lecture gives me a good excuse to review Topology and Topological Manifolds from a mathematician's point of view. I find John M. Lee's **Introduction to Topological Manifolds** book good because it's elementary and thorough and it's fairly recent (2010) so it's up to date [**?**]. See my notes and solutions for the book; it's a file titled `LeeJM_IntroTopManifolds_sol.pdf` of which I'll try to keep the pdf and LaTeX file available for download on my ernestyalumni Google Drive (so try to search for it on Google).

TUTORIAL TOPOLOGICAL MANIFOLDS

filename: `Sheet_1.2.pdf`

Exercise 4: Before the invention of the wheel.
Another one-dimensional topological manifold. Another one?
Consider set $F^1 := \{(m, n) \in \mathbb{R}^2 | m^4 + n^4 = 1\}$, equipped with subset topology $\mathcal{O}_{\text{std}}|_{F^1}$

Question $x : F^1 \to \mathbb{R}$ is what?.

Solution . EY : 20150525 The tutorial video `https://youtu.be/ghfEQ3u_B6g` is really good and this solution is how I'd write it, but it's really the same (I needed the practice).

$$\boxed{\begin{aligned} x : F^1 \to \mathbb{R} \\ (m, n) \mapsto m \end{aligned}}$$

If $m = 0$, $n^4 = 1$ so $n = \pm 1$ so it's not injective.
Let the closed n-dim. upper half-space $\mathbb{H}^n \subseteq \mathbb{R}^1$. Then
$$\begin{aligned} \mathbb{H}^n &= \{(x_1 \ldots x_n) \in \mathbb{R}^n | x_n \geq 0\} \\ \text{int}\mathbb{H}^n &= \{(x_1 \ldots x_n) \in \mathbb{R}^n | x_n > 0\} \\ -\mathbb{H}^n &= \{(x_1 \ldots x_n) \in \mathbb{R}^n | x_n \leq 0\} \\ -\text{int}\mathbb{H}^n &= \{(x_1 \ldots x_n) \in \mathbb{R}^n | x_n < 0\} \end{aligned}$$

Question This map x may be made injective by restricting its domain to either of 2 maximal open subsets of F^1. Which ones?.

Solution .

Let

$$U_+ = F^1 \cap \text{int}\mathbb{H}^2$$
$$U_- = F^1 \cap -\text{int}\mathbb{H}^2$$

Look at

$$x^4 = 1 - n^4$$
$$\Longrightarrow x = \pm(1-n^4)^{1/4}$$

Then for

$$\begin{aligned} x_+^{-1} : (-1,1) \subseteq \mathbb{R} &\to U_+ \\ m &\mapsto (m, (1-m^4)^{1/4}) \\ x_-^{-1} : (-1,1) \subseteq \mathbb{R} &\to U_- \\ m &\mapsto (m, -(1-m^4)^{1/4}) \end{aligned}$$

x_+,x_- injective (since left inverse exists).

Question Construct injective y.

Solution .

Let

$$V_+ = F^1 \cap \text{int}\mathbb{H}^1$$
$$V_- = F^1 \cap -\text{int}\mathbb{H}^1$$

Then

$$\begin{aligned} y_+ : V_+ &\to (-1,1) \subseteq \mathbb{R} \\ (m,n) &\mapsto n \\ y_- : V_- &\to (-1,1) \subseteq \mathbb{R} \\ (m,n) &\mapsto n \end{aligned}$$

Question Construct inverse y^{-1}. Solution .

For

$$\begin{aligned} y_+^{-1} : (-1,1) \subseteq \mathbb{R} &\to V_+ \\ n &\mapsto ((1-n^4)^{1/4}, n) \\ y_-^{-1} : (-1,1) \subseteq \mathbb{R} &\to V_- \\ n &\mapsto (-(1-n^4)^{1/4}, n) \end{aligned}$$

y_+,y_- injective (since left inverse exists).

Note $(-1,0) \notin U_+, U_-$
$(1,0) \notin U_+, U_-$

and

$(0,1) \notin V_+, V_-$
$(0,-1) \notin V_+, V_-$

Question construct *transition map* $x \circ y^{-1}$.

Solution .

$$\begin{aligned} x_+ y_+^{-1} : (0,1) \subseteq \mathbb{R} &\to (0,1) \subseteq \mathbb{R} \\ n &\mapsto (1-n^4)^{1/4} \\ x_- y_+^{-1} : (-1,0) \subseteq \mathbb{R} &\to (0,1) \subseteq \mathbb{R} \\ n &\xrightarrow{y_+^{-1}} ((1-n^4)^{1/4}, n) \xrightarrow{x_-} (1-n^4)^{1/4} \\ x_+ y_-^{-1} : (0,1) \subseteq \mathbb{R} &\to (-1,0) \subseteq \mathbb{R} \\ n &\mapsto -(1-n^4)^{1/4} \\ x_- y_-^{-1} : (-1,0) \subseteq \mathbb{R} &\to (-1,0) \subseteq \mathbb{R} \\ n &\mapsto -(1-n^4)^{1/4} \end{aligned}$$

Question . . . Does the collection of these domains and maps form an atlas of F^1?.

Yes, with atlas

$$\mathcal{A} = \{ \begin{matrix} (U_+, x_+) \\ (U_-, x_-) \end{matrix}, \begin{matrix} (V_+, y_+) \\ (V_-, y_-) \end{matrix} \}$$

Clearly

$$\begin{aligned} U_+ \cup U_- \cup V_+ \cup V_- &= (F^1 \cap \text{int}\mathbb{H}^2) \cup (F^1 \cap -\text{int}\mathbb{H}^2) \cup (F^1 \cap \text{int}\mathbb{H}^1) \cup (F^1 \cap -\text{int}\mathbb{H}^1) = \\ &= F^1 \cap \mathbb{R}^2 \backslash \{(0,0)\} = F^1 \end{aligned}$$

and (the point is that) $x_\pm, y_\pm$ are homeomorphisms of open sets of F^1 onto open sets of 1 dim. $\mathbb{R}^1$ (namely $(-1,1) \subseteq \mathbb{R}^1$), and so $\mathcal{A}$ is an atlas of F^1.

3. Lecture 3: Multilinear Algebra

Lecture 3: Multilinear Algebra (International Winter School on Gravity and Light 2015)
We will **not** equip space(time) with a vector space structure. Do you know where

$$\begin{matrix} 5 \cdot \text{ Paris } =? \\ \text{lie ?} \end{matrix} \qquad \text{Paris } + \text{ Vienna } =?$$

Moreover, the tangent spaces T_pM (lecture 5) smooth manifolds (Lecture 4)
Beneficial to first study vector spaces abstractly for two reason

(i) for construction of T_pM one needs an intermediate vector space $C^\infty(M)$
(ii) tensor technique are most easily understood in an abstract setting.

3.1. **Vector spaces.**

Definition 4. *A vector space* $(V,+,-)$ *is*

(i) *a set* V
(ii) $+ : V \times V \to V$ *"addition"*
(iii) $\cdot : \mathbb{R} \times V \to V$ *"s-multiplication" EY : 20160317 s for "scalar"*

satisfying:

CANIADDU

$$\begin{aligned}
C^+ &: \quad v+w = w+v \\
A^+ &: \quad (u+v)+w = u+(v+w) \\
N^+ &: \quad \exists\, 0 \in V : \forall\, v \in V : v+0 = v \\
I^+ &: \quad \forall\, v \in V : \exists\, (-v) \in V : v+(-v) = 0 \\
A &: \quad \lambda \cdot (\mu + v) = (\lambda \cdot \mu) \cdot v \qquad (\forall\, \lambda, \mu \in \mathbb{R}) \\
D &: \quad (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v \\
D &: \quad \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v+w) \\
U &: \quad 1 \cdot v = v
\end{aligned}$$

Terminology. An element of a vector space is often referred to, informally as a vector.

Example. def. **set** of polynomials (fixed) degree $\mathcal{P} := \{p : (-1,+1) \to \mathbb{R} | p(x) = \sum_{n=0}^{N} p_n \cdot x^n\}$

Thought bubble: is $\square$ a vector?

$\square(x) = x^2$

No $\square \in \mathcal{P}$.

$$\begin{aligned} + : &\mathcal{P} \times \mathcal{P} \to \mathcal{P} \\ &(p,q) \mapsto p+q \end{aligned}$$

where $(p+q)(x) = p(x) +_{\mathbb{R}} q(x)$

$$\begin{aligned} \cdot : &\mathbb{R} \times \mathcal{P} \to \mathcal{P} \\ &(\lambda, p) \mapsto \lambda \cdot p \end{aligned}$$

where $(\lambda \cdot p)(x) := \lambda \cdot_{\mathbb{R}} p(x)$

Thought bubble: $\square$ a vector? Yes, but who cares?

$(\mathcal{P}, +, \cdot)$ is a vector space.

$\square \in \mathcal{P}$

3.2. **Linear maps.** These are the structure-respecting maps between vector spaces.

EY : 20160316 out of tradition, they're called "linear" maps

Definition 5. $(V, +_V, \cdot_V)$ *and* $(W, +_W, \cdot_W)$ *vector spaces*
Then a map

$$\varphi : V \to W$$

is called ***linear*** *if*

(i) $\varphi(v +_V \widetilde{v}) = \varphi(v) +_W \varphi(\widetilde{v})$
(ii) $\varphi(\lambda \cdot_V v) = \lambda \cdot_W \varphi(v)$

Example. : $\begin{aligned} \delta : &\mathcal{P} \to \mathcal{P} \\ &p \mapsto \delta(p) := p' \end{aligned}$

<u>linear</u>:

(i) $\delta(p+q) = (p +_{\mathcal{P}} q)' \underbrace{=}_{\text{sum rule}} p' +_{\mathcal{P}} q' = \delta(p) +_{\mathcal{P}} \delta(q)$
(ii) $\delta(\lambda p) = (\lambda p)' = \lambda \cdot p' = \lambda \cdot \delta(p)$

<u>Notation</u>: $\varphi : V \to W$ linear $\Longleftrightarrow: \varphi : V \xrightarrow{\sim} W$

$$\begin{tikzcd} V \arrow[r, "\psi", "\sim"'] \arrow[rr, bend right, "\sim", "\varphi \circ \psi"'] & W \arrow[r, "\varphi", "\sim"'] & U \end{tikzcd}$$

3.2.1. *Example*.* $\delta \circ \delta : \mathcal{P} \xrightarrow{\sim} \mathcal{P}$

3.3. **Vector space of Homomorphisms.** fun fact: $(V,+,\cdot)$ $(W,+,\cdot)$ vector spaces

def. $\mathrm{Hom}(V,W) := \{\varphi : V \xrightarrow{\sim} W\}$ set.

We can make this into a vector spaces.

$$\begin{aligned} \oplus : &\mathrm{Hom}(V,W) \times \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W) \\ &(\varphi, \psi) \mapsto \varphi \oplus \psi \end{aligned}$$

where $(\varphi \otimes \psi)(v) := \varphi(v) +_W \psi(v)$

$\otimes$: ... similarly.

$(\mathrm{Hom}(V,W), \oplus, \otimes)$ **is** a vector space.

3.3.1. *Example*.* $\mathrm{Hom}(\mathcal{P},\mathcal{P})$ is a vector space.

$\delta \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$

$\delta \circ \delta \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$

$\vdots$

$\underbrace{\delta \circ \cdots \circ \delta}_{M} \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$

$$\Longrightarrow 5 \circ \delta \oplus_{\mathrm{Hom}(\mathcal{P},\mathcal{P})} \delta \circ \delta \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$$

3.4. **Dual vector space.** heavily used special case:

$(V,+,\cdot)$ vector space:

Definition 6.

$$V^* := \{\varphi : V \xrightarrow{\sim} \mathbb{R}\} = Hom(V, \mathbb{R})$$

$\underbrace{(V^*, \oplus, \otimes)}_{\textit{dual vector space (to } V)}$ *is a vector space*

<u>Terminology</u>: $\varphi \in V^*$ is called, informally, a covector.

Example. $I : \mathcal{P} \xrightarrow{\sim} \mathbb{R}$

i.e. $I \in \mathcal{P}^*$

<u>def</u>. $I(p) := \int_0^1 dx p(x)$

linear: $\begin{aligned} I(p+q) &= \int_0^1 dx \underbrace{(p+q)(x)}_{p(x)+q(x)} \\ &= \cdots = I(q) + I(p) \end{aligned}$

$I(\lambda p) = \lambda \cdot I(p)$

i.e. $I = \int_0^1 dx$

3.5. **Tensors.**

Definition 7. *Let $(V,+,\cdot)$ be a vector space.*
An $\underline{(r,s)\text{-tensor } T \text{ over } V}$ $\quad r,s\in\mathbb{N}_0$
is a multi-linera map

$$T:\underbrace{V^*\times\dots\times V^*}_{r}\times\underbrace{V\times\dots\times V}_{s}\xrightarrow{\sim}\mathbb{R}$$

3.5.1. *Example.* : T $(1,1)$-tensor

$$T(\varphi+\psi,v)=T(\varphi,v)+T(\psi,v) \qquad T(\varphi,v+w)=T(\varphi,v)+T(\varphi,w)$$
$$T(\lambda\varphi,v)=\lambda\cdot T(\varphi,v) \qquad T(\varphi,\lambda\cdot v)=\lambda T(\varphi,v)$$

$$T(\varphi+\psi,v+w)=$$
$$=t(\varphi,v)+T(\varphi,w)+T(\psi,v)+t(\psi,w)$$

Excursion: Given $T:V^*\times V\xrightarrow{\sim}\mathbb{R}$
$\phi_T:\quad V\xrightarrow{\sim}(V*)^*\underbrace{=}_{\dim V<\infty}V$

Define $v\mapsto\underbrace{T(\cdot,v)}_{V^*\xrightarrow{\sim}\mathbb{R}}$

Given $\phi:V\xrightarrow{\sim}V$

Construct $T_\phi:V^*\times V\xrightarrow{\sim}\mathbb{R}$
$(\varphi,v)\mapsto\varphi(\phi(v))$
$\Longrightarrow$ given $T:T=T_{\varphi_T}$
given $\phi:\phi=\phi_{T_\phi}$

Example. $g:P\times P\xrightarrow{\sim}\mathbb{R}$
$(p,q)\mapsto\int_{-1}^{1}dxp(x)q(x)$
is a $(0,2)$-tensor over P.
Info: If $T\in\mathrm{Hom}(V,W)$

3.6. **Vectors and covectors as tensors.**

Theorem 1. *(including proof)*
"covector" $\varphi\in V^*\iff\varphi:V\xrightarrow{\sim}\mathbb{R}\iff\varphi(0,1)$-tensor.

Theorem 2. $v\in V\underbrace{=}_{\dim V<\infty}(V^*)^*\iff v:V^*\xrightarrow{\sim}\mathbb{R}\iff v$ *is $(1,0)$-tensor.*

3.7. **Bases.**

Definition 8. *$(V,+,\cdot)$ vector space.*
A subset $B\subset V$ is called
a basis if
Thought bubble: Hamel (L.A.) EY : 20160316 Hamel basis, Linear Algebra

$$\forall v\in V\quad\exists\ \underline{\text{finite}}\ \underbrace{F}_{\{f_1,\dots,f_n\}}\subset B:\exists!\underbrace{v^1,v^2,\dots,v^n}_{\in\mathbb{R}},\qquad v=v^1f_1+\dots+v^nf_n$$

Definition 9. *If $\exists$ basis $\mathcal{B}$ with finitely many elements, say d many, then we call $d=:\dim V$*
This is well-defined.

Remark: $(V,+,\cdot)$ be a finite-dim. vector space.
Having chosen a basis $e_1,\dots,e_n$ of $(V,+,\cdot)$ we may uniquely associate
(Thought bubble: this requires a chosen basis)

$$v\mapsto(v^1,\dots,v^n)\text{ called the components of }v\text{ w.r.t. chosen basis}$$

where: $v^1e_1+\dots+v^ne+n=v$

3.8. **Basis for the dual space.** choose Basis $e_1,\dots,e_n$ for V
can choose Basis $\epsilon^1,\dots,\epsilon^n$ for V^*
However, more economical to require
once $e_1,\dots,e_n$ on V has been chosen, that

$$\epsilon^a(e_b)=\delta^a_b$$

This uniquely determines choice of $\epsilon^1,\dots,\epsilon^n$ from choice of $e_1,\dots,e_n$

Definition 10. *If a basis $\epsilon^1,\dots,\epsilon^n$ of V^* satisfies this, it is called the* ***dual basis*** *(of the dual space)*

Example: P $(N=3)$

$$e_0,e_1,e_2,e_3\text{ basis if }\begin{matrix}e_0(x)=1\\ e_1(x)=x\\ e_2(x)=x^2\\ e_3(x)=x^3\end{matrix}\{e_a(x):=x^a$$

$\epsilon^0,\epsilon^1,\epsilon^2,\epsilon^3$ dual basis $\epsilon^a:=\frac{1}{a!}\,\partial^a|_{x=0}$

Proof. $\epsilon^a(e_b)=\delta^a_b$ ∎

3.9. **Components of tensors.** Let T be an (r,s)-tensor on a finite-dim. vs. V. Then define the $(r+s)^{\dim V}$ many real numbers.

$$\underbrace{T^{i_1\dots i_r}{}_{j_1\dots j_s}}_{\in\mathbb{R}}:=T(\epsilon^{i_1},\dots,\epsilon^{i_r},e_{j_1},e_{j_2},\dots,e_{j_s})$$

$i_1\dots i_r,j_1\dots j_s\in\{1,\dots,\dim V\}$
Thought bubble: $\underbrace{T^{i_1\dots i_r}{}_{j_1\dots j_s}}_{\in\mathbb{R}}$ are the components of the tensor w.r.t. chosen basis

Useful: Knowing components (and basis) one can reconstruct the entire tensor.

Example. T $(1,1)$- tensor

$$T^i{}_j:=T(\epsilon^i,e_j)$$

reconstruct

$$T(\varphi,v)=T\Big(\sum_{i=1}^{\dim V}\varphi_i\epsilon^i,\sum_{j=1}^{\dim V}v^je_j\Big)\qquad\varphi_i\in\mathbb{R},v^j\in\mathbb{R}$$
$$=\sum_{i=1}^{\dim V}\sum_{j=1}^{\dim V}\varphi_iv^j\underbrace{T(\epsilon^i,e_j)}_{T^i{}_j}$$
$$=:\varphi_iv^jT^i{}_j$$

4. Lecture 4: Differentiable Manifolds

so far: top. mfd. $\quad (M, \mathcal{O})$

$\dim M = d$

we wish to define a notion of differentiable

curves $\mathbb{R} \to M$

function $M \to \mathbb{R}$

maps $M \to N$

4.1. **1. Strategy.** choose a chart (U, x)

$\gamma : \mathbb{R} \to M$ portion of curve in chart domain

$$\begin{tikzcd} \gamma : \mathbb{R} \arrow[r] \arrow[dr, "x \circ \gamma"'] & U \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$

idea. try to "lift" the undergraduate notion of differentiability of a curve on $\mathbb{R}^d$ to a notion of differentiability of a curve on M

Problem Can this be well-defined under change of chart?

$$\begin{tikzcd} & y(U \cap V) \subseteq \mathbb{R}^d \\ \gamma : \mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r] \arrow[dr, "x \circ \gamma"'] & U \cap V \neq \emptyset \arrow[u, "y"] \arrow[d, "x"] \\ & x(U \cap V) \subseteq \mathbb{R}^d \arrow[uu, bend right, "y \circ x^{-1}"'] \end{tikzcd}$$

$x \circ \gamma$ undergraduate differentiable ("as a map $\mathbb{R} \to \mathbb{R}^d$")

$$\underbrace{y \circ \gamma}_{\text{maybe only continuous, but not undergraduate differentiable}} = \overbrace{\underbrace{(y \circ x^{-1})}_{\text{continuous}}}^{\mathbb{R}^d \to \mathbb{R}^d} \circ \overbrace{\underbrace{(x \circ \gamma)}_{\text{undergrad differentiable}}}^{\mathbb{R} \to \mathbb{R}^d} = y \circ (x^{-1} \circ x) \circ \gamma$$

At first sight, strategy does not work out.

4.2. **2. Compatible charts.** In section 1, we used any imaginable charts on the top. mfd. $(M, \mathcal{O})$.

To emphasize this, we may say that we took U and V from the *maximal atlas* $\mathcal{A}$ of $(M, \mathcal{O})$.

Definition 11. *Two charts (U, x) and (V, y) of a top. mfd. are called ❀-compatible if either*

(a) $U \cap V = \emptyset$ *or*

(b) $U \cap V \neq \emptyset$

chart transition maps have undergraduate ❀ property.

EY : 20151109 e.g. since $\mathbb{R}^d \to \mathbb{R}^d$, can use undergradate ❀ property such as continuity or differentiability.

$$y \circ x^{-1} : x(U \cap V) \subseteq \mathbb{R}^d \to y(U \cap V) \subseteq \mathbb{R}^d$$
$$x \circ y^{-1} : y(U \cap V) \subseteq \mathbb{R}^d \to x(U \cap V) \subseteq \mathbb{R}^d$$

[1] http://mathoverflow.net/questions/8789/can-every-manifold-be-given-an-analytic-structure

Philosophy:

Definition 12. *An atlas $\mathcal{A}_{❀}$ is a ❀-compatible atlas if any two charts in $\mathcal{A}_{❀}$ are ❀-compatible.*

Definition 13. *A* ***❀-manifold*** *is a triple* $(\underbrace{M, \mathcal{O}}_{top.\ mfd.}, \mathcal{A}_{❀}) \qquad \mathcal{A}_{❀} \subseteq \mathcal{A}_{maximal}$

❀	undergraduate ❀	
C^0	$C^0(\mathbb{R}^d \to \mathbb{R}^d) =$	continuous maps w.r.t. $\mathcal{O}$
C^1	$C^1(\mathbb{R}^d \to \mathbb{R}^d) =$	differentiable (once) and is continuous
C^k		k-times continuously differentiable
D^k		k-times differentiable
$\vdots$		
C^∞	$C^\infty(\mathbb{R}^d \to \mathbb{R}^d)$	
$\cup\!\!\mid$		
C^ω	$\exists$ multi-dim. Taylor exp.	
$\mathbb{C}^\infty$	satisfy Cauchy-Riemann equations, pair-wise	

EY : 20151109 Schuller says: C^k is easy to work with because you can judge k-times cont. differentiability from existence of all partial derivatives **and** their continuity. There are examples of maps that partial derivatives exist but are not D^k, k-times differentiable.

Theorem 3 (Whitney)**.** *Any $C^{k\geq 1}$-atlas, $\mathcal{A}_{C^{k\geq 1}}$ of a topological manifold contains a C^∞-atlas.*

Thus we may w.l.o.g. always consider C^∞-manifolds, "smooth manifolds", unless we wish to define Taylor expandibility/-complex differentiability ...

EY : 20151109 Hassler Whitney [1]

Definition 14. *A smooth manifold* $(\underbrace{M, \mathcal{O}}_{top.\ mfd.}, \underbrace{\mathcal{A}}_{C^\infty-atlas})$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & M \arrow[d, "x"] \\ & \mathbb{R}^d \end{tikzcd}$$

EY: 20151109 Schuller was explaining that the trajectory is real in M; the coordinate maps to obtain coordinates is $x \circ \gamma$

4.3. **4. Diffeomorphisms.** $M \xrightarrow{\phi} N$

If M, N are naked sets, the structure preserving maps are the bijections (invertible maps).

e.g. $\{1, 2, 3\} \to \{a, b\}$

Definition 15. *$M \cong_{set} N$ (set-theoretically) isomorphic if $\exists$ bijection $\phi : M \to N$*

Examples. $\mathbb{N} \cong_{\text{set}} \mathbb{Z}$

$\mathbb{N} \cong_{\text{set}} \mathbb{Q}$ (EY: 20151109 Schuller says from diagonal counting)

$\mathbb{N} \cancel{\cong_{\text{set}}} \mathbb{R}$

Now $(M, \mathcal{O}_M) \cong_{\text{top}} (N, \mathcal{O}_N)$ (topl.) isomorphic = "homeomorphic" $\exists$ bijection $\phi : M \to N$

ϕ, ϕ^{-1} are continuous.

$(V, +, \cdot) \cong_{\text{vec}} (W, +_w, \cdot_w)$ (EY: 20151109 vector space isomorphism) if $\exists$ bijection $\phi : V \to W$ linearly

finally

Definition 16. *Two C^∞-manifolds $(M,\mathcal{O}_M,\mathcal{A}_M)$ and $(N,\mathcal{O}_N,\mathcal{A}_N)$ are said to be **diffeomorphic** if $\exists$ bijection $\phi : M\to N$ s.t.*

$$\begin{aligned} &\phi : M \to N \\ &\phi^{-1} : N \to M \end{aligned}$$

are both C^∞-maps

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "\widetilde{y}\circ\phi\circ\widetilde{x}^{-1}"] & \mathbb{R}^e \\ M\supseteq U \arrow[u, "\widetilde{x}"] \arrow[r, "\phi"] \arrow[d, "x"'] & V\subseteq N \arrow[u, "\widetilde{y}"'] \arrow[d, "y"] \\ \mathbb{R}^d \arrow[r, "y\circ\phi\circ x^{-1}", "\text{undergraduate } C^\infty"'] \arrow[uu, bend left=60, "C^\infty"] & \mathbb{R}^e \arrow[uu, bend right=60] \end{tikzcd}$$

Theorem 4. *$\# =$ number of C^∞-manifolds one can make out of a given C^0-manifolds (if any) - up to diffeomorphisms.*

dimM	#	
1	*1*	*Morse-Radon theorems*
2	*1*	*Morse-Radon theorems*
3	*1*	*Morse-Radon theorems*
4	*uncountably infinitely many*	
5	*finite*	*surgery theory*
6	*finite*	*surgery theory*
$\vdots$	*finite*	*surgery theory*

EY : 20151109 cf. `http://math.stackexchange.com/questions/833766/closed-4-manifolds-with-uncountably-many-differentiable-structures`
The wild world of 4-manifolds

Tutorial 4 Differentiable Manifolds

EY : 20151109 The `gravity-and-light.org` website, where you can download the tutorial sheets *and* the full length videos for the tutorials and lectures, are no longer there. = (

Hopefully, the YouTube video will remain: `https://youtu.be/FXPdKxOq1KA?list=PLFeEvEPtX_0RQ1ys-7VIsKlBWz7RX-FaL`

Exercise 1: True or false?. *These basic questions are designed to spark discussion and as a self-test.*

Tick the correct statements, but not the incorrect ones!

(a) The function $f:\mathbb{R}\to\mathbb{R}$, ...
-
-
- ..., defined by $f(x) = |x^3|$, lies in $C^3(\mathbb{R}\to\mathbb{R})$.

EY : 20151109 **Solution 1a3.** For $f:\mathbb{R}\to\mathbb{R}$, $f(x) = |x^3| = \begin{cases} x^3 & \text{if } x\geq 0 \\ -x^3 & \text{if } x<0 \end{cases}$

$$f'(x) = \begin{cases} 3x^2 & \text{if } x\geq 0 \\ -3x^2 & \text{if } x<0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & \text{if } x\geq 0 \\ -6x & \text{if } x<0 \end{cases}$$

Thus,

$$\boxed{f(x) = |x^3| \in C^1(\mathbb{R}) \text{ but } f(x)\notin C^2(\mathbb{R})\subseteq C^3(\mathbb{R})}$$

-
-

(b)
(c)

Short Exercise 4: Undergraduate multi-dimensional analysis .

A good notation and basic results for partial differentiation.

For a map $f:\mathbb{R}^d\to\mathbb{R}$ we denote by the map $\partial_i f : \mathbb{R}^d\to\mathbb{R}$ the partial derivative with respect to the i-th entry.

Question :. Given a function

$$f:\mathbb{R}^3\to\mathbb{R};\ (\alpha,\beta,\delta)\mapsto f(\alpha,\beta,\delta) := \alpha^3\beta^2 + \beta^2\delta + \delta$$

calculate the values of the following derivatives:

Solution :.

- $(\partial_2 f)(x,y,z) =$
- $(\partial_1 f)(\square,\circ,*) =$
- $(\partial_1\partial_2 f)(a,b,c) =$
- $(\partial_3^2 f)(299, 1222, 0) =$

EY: 20151110

For $f(\alpha,\beta,\delta) := \alpha^3\beta^2+\beta^2\delta+\delta$, or $f(x,y,z) = x^3y^2+y^2z+z$,

$$\begin{aligned} (\partial_2 f) &= 2(x^3y+yz) \\ (\partial_1 f) &= 3x^2y^2 \\ (\partial_1\partial_2 f) &= 6x^2y \\ (\partial_3^2 f) &= 0 \end{aligned}$$

and so

- $(\partial_2 f)(x,y,z) = 2(x^3y+yz)$
- $(\partial_1 f)(\square,\circ,*) = 3\square^2\circ^2$
- $(\partial_1\partial_2 f)(a,b,c) = 6a^2b$
- $(\partial_3^2 f)(299,1222,0) = 0$

Exercise 5: Differentiability on a manifold.

How to deal with functions and curves in a chart

Let $(M,\mathcal{O},\mathcal{A})$ be a smooth d-dimensional manifold. Consider a chart (U,x) of the atlas $\mathcal{A}$ together with a smooth curve $\gamma:\mathbb{R}\to U$ and a smooth function $f:U\to\mathbb{R}$ on the domain U of the chart.

Question :. Draw a commutative diagram containing the chart domain, chart map, function, curveand the respective represen-

tatives of the function and the curve in the chart.
Solution :.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend left, "x \circ \gamma"] & U \arrow[r, shift left, "x"] \arrow[d, "f"'] & \mathbb{R}^d \arrow[l, shift left, "x^{-1}"] \arrow[dl, "(f \circ x^{-1})"] \\
 & \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau \in \mathbb{R} \arrow[r, mapsto, "\gamma"] \arrow[rr, mapsto, bend left, "x \circ \gamma"] & p \in U \arrow[r, mapsto, shift left, "x"] \arrow[d, mapsto, "f"'] & x(p) = (x \circ \gamma)(\tau) \in \mathbb{R}^d \arrow[l, mapsto, shift left, "x^{-1}"] \arrow[dl, mapsto, "(f \circ x^{-1})"] \\
 & f(p) \in \mathbb{R} &
\end{tikzcd}$$

Question :. Consider, for $d = 2$,

$$(x \circ \gamma)(\lambda) := (\cos(\lambda), \sin(\lambda)) \text{ and } (f \circ x^{-1})((x, y)) := x^2 + y^2$$

Using the chain rule, calculate

$$(f \circ \gamma)'(\lambda)$$

explicitly.
Solution :.
EY : 20151109 Indeed, the domains and codomains of this $f\gamma$ mapping makes sense, from $\mathbb{R} \to \mathbb{R}$ for

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend left, "x \circ \gamma"] \arrow[dr, "f \circ \gamma"] & U \arrow[r, shift left, "x"] \arrow[d, "f"'] & \mathbb{R}^d \arrow[l, shift left, "x^{-1}"] \arrow[dl, "(f \circ x^{-1})"] \\
 & \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau \in \mathbb{R} \arrow[r, mapsto, "\gamma"] \arrow[rr, mapsto, bend left, "x \circ \gamma"] \arrow[dr, mapsto, "f \circ \gamma"] & p \in U \arrow[r, mapsto, shift left, "x"] \arrow[d, mapsto, "f"'] & x(p) = (x \circ \gamma)(\tau) \in \mathbb{R}^d \arrow[l, mapsto, shift left, "x^{-1}"] \arrow[dl, mapsto, "(f \circ x^{-1})"] \\
 & f(p) \in \mathbb{R} &
\end{tikzcd}$$

$$(f \circ \gamma)'(\lambda) = (Df) \cdot \dot{\gamma}(\lambda) = \frac{\partial f}{\partial x^j} \dot{\gamma}^j(\lambda) = 2x(-\sin\lambda) + 2y\cos\lambda = 2(-\cos\lambda\sin\lambda + \sin\lambda\cos\lambda) = 0$$

5. Lecture 5: Tangent Spaces

lead question: "what is the velocity of a curve γ point p?

5.1. Velocities.

Definition 17. $(M, \mathcal{O}, \mathcal{A})$ *smooth mfd.*
curve $\gamma : \mathbb{R} \to M$ *at least* C^1.
Suppose $\gamma(\lambda_0) = p$
The ***velocity*** *of* γ p *is the linear map*

$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$$

$$C^\infty(M) := \{f : M \to \mathbb{R} | f \text{ smooth function } \} \text{ equipped with } \begin{aligned} (f \oplus g)(p) &:= f(p) + g(p) \\ (\lambda \otimes g)(p) &:= \lambda \cdot g(p) \end{aligned}$$

$\sim$ *denotes linear map on top of* $\to$.

$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & M \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

intuition
Schuller says: children run around the world. Temperature function as temperature contour lines. You feel the temperature. You observe the rate of change of temperature as you run around. f is temperature.
<u>past</u>: " $\underbrace{v^i}(\partial_i f) = (\underbrace{v^i \partial_i}_{\text{vector}}) f$

5.2. Tangent vector space.

Definition 18. *For each point* $p \in M$
def the ***set*** *"tangent space* $\neq_0$ M p *"*

$$T_pM := \{v_{\gamma,p} | \gamma \text{ smooth curves } \}$$

<u>picture</u>:
rather M than (embedded) p T_pM EY : 20151109 see https://youtu.be/pepU_7NJSGM?t=12m38s for the picture
<u>Observation</u>: T_pM can be made into a vector space.

$$\begin{aligned}
\oplus : & T_pM \times T_pM \to \\
& (v_{\gamma,p} \oplus v_{\delta,p})(\underbrace{f}_{\in C^\infty(M)}) := v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f) \\
\odot : & \mathbb{R} \times T_pM \to \mathrm{Hom}(C^\infty(\mathbb{R}), \mathbb{R}) \\
& (\alpha \odot v_{\gamma,p})(f) := \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f)
\end{aligned}$$

Remains to be shown that
(i) $\exists \sigma$ curve : $v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$
(ii) $\exists \tau$ curve : $\alpha \odot v_{\gamma,p} = v_{\tau,p}$

<u>Claim</u>: $\begin{aligned} \tau : & \mathbb{R} \to M \\ & \mapsto \tau(\lambda) := \gamma(\alpha\lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \end{aligned}$ where $\begin{aligned} \mu_\alpha : & \mathbb{R} \to \mathbb{R} \\ & r \mapsto \alpha \cdot r + \lambda_0 \end{aligned}$, does the trick.
$\tau(0) = \gamma(\lambda_0) = p$

$$\begin{aligned}
v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) \\
&= (f \circ \gamma)'(\lambda_0) \cdot \alpha = \\
&= \alpha \cdot v_{\gamma,p}
\end{aligned}$$

Now for the sum:
$v_{\gamma,p} \oplus v_{\delta,p} \overset{?}{=} v_{\sigma,p}$
make a <u>choice</u> of chart $(\underbrace{U}_{\ni p}, x)$ In cloud: ill definition alarm bells.
and define:

Claim:

$$\sigma : \mathbb{R} \to M$$
$$\sigma(\lambda) := x^{-1}(\underbrace{(x\circ\gamma)(\lambda_0+\lambda)}_{\mathbb{R}\to\mathbb{R}^d} + (x\circ\delta)(\lambda_1+\lambda) - (x\circ\gamma)(\lambda_0))$$

does the trick.

Proof. Since:

$$\begin{aligned} \sigma_x(0) &= x^{-1}((x\circ\gamma)(\lambda_0) + (x\circ\delta)(\lambda_1) - (x\circ\gamma)(\lambda_0)) \\ &= \delta(\lambda_1) = p \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma_x,p}(f) &:= (f\circ\sigma_x)'(0) = \\ &= (\underbrace{(f\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}} \circ \underbrace{(x\circ\sigma_x)}_{\mathbb{R}\to\mathbb{R}^d})'(\gamma) = \underbrace{(x\circ\sigma_x)'(0)}_{(x\circ\gamma)'(\lambda_0)+(x\circ\delta)'(\lambda_1)} \cdot \left(\partial_i(f\circ x^{-1})\right)(x(\underbrace{\sigma(0)}_{p})) = \\ &= (x\circ\gamma)'(\lambda_0)(\partial_i(f\circ x^{-1}))(x(p)) + (x\circ\delta)(\lambda_1)(\partial_i(f\circ x^{-1}))(x(p)) \\ &= (f\circ\gamma)'(\lambda_0) + (f\circ\delta)'(\lambda_1) = \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall\, f \in C^\infty(M) \end{aligned}$$

$$\boxed{v_{\gamma,p}\oplus v_{\delta,p} = v_{\sigma,p}}$$

$\square$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\sigma"] \arrow[dr, "x\circ\sigma"'] & M \arrow[r, "f"] \arrow[d, "x"] & \mathbb{R} \\ & \mathbb{R}^d \arrow[ur, "f\circ x^{-1}"'] & \end{tikzcd}$$

picture: (cf. https://youtu.be/pepU_7NJSGM?t=39m5s)

$$\begin{aligned} \gamma &: \mathbb{R} \to M \\ \delta &: \mathbb{R} \to M \end{aligned}$$

$(\gamma\oplus)(\lambda) := \gamma(\lambda) + \delta(\lambda)$

EY : 20151109 Schuller says adding trajectories is chart dependent, bad. Adding velocities is good.

5.3. Components of a vector wrt a chart.

Definition 19. *Let* $(U,x)\in\mathcal{A}_{smooth}$.

Let $\begin{aligned} \gamma &: \mathbb{R}\to U \\ \gamma(0) &= p \end{aligned}$.

Calculate

$$\begin{aligned} v_{\gamma,p}(f) &:= (f\circ\gamma)'(0) = (\underbrace{(f\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}} \circ \underbrace{(x\circ\gamma)}_{\mathbb{R}\to\mathbb{R}^d})'(0) \\ &= \underbrace{(x\circ\gamma)^{i'}(0)}_{\dot{\gamma}^i_x(0)} \cdot \underbrace{(\partial_i(f\circ x^{-1}))(x(p))}_{=:\left(\frac{\partial f}{\partial x^i}\right)_p} \end{aligned}$$

think cloud $f : M \to \mathbb{R}$

$$= \boxed{\dot{\gamma}^i_x(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p} f \quad \forall\, f\in C^\infty(M)$$

$\therefore$ *as a map.*

$$v_{\gamma,p} \underbrace{=}_{\text{use of chart}} \underbrace{\gamma^i_x(0)}_{\text{"components of the velocity } v_{\gamma,p}\text{"}} \underbrace{\left(\frac{\partial}{\partial x^i}\right)}_{\substack{\text{basis elements of the } T_pM \text{ wrt which the components need to be understood.} \\ \text{"chart induced basis of } T_pM\text{"}}}$$

Picture: https://youtu.be/pepU_7NJSGM?t=1h16s

5.4. 4. Chart-induced basis.

Definition 20. $(U,x)\in\mathcal{A}_{smooth}$
the $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_pU \subseteq T_pM$
constitute a ***basis*** *of* T_pU

Proof. remains: linearly independent

$$\lambda^i\left(\frac{\partial}{\partial x^i}\right)_p \overset{!}{=} 0$$

$$\implies \lambda^i\left(\frac{\partial}{\partial x^i}\right)_p (x^j) = \lambda^i\partial_i(\underbrace{x^j\circ x^{-1}}_{})(x(p)) = \qquad \begin{aligned} x^j\circ x^{-1} &: \mathbb{R}^d\to\mathbb{R} \\ (\alpha^1,\dots,\alpha^d) &\mapsto \alpha^j \end{aligned}$$

$$= \lambda^i\delta_i^{\ j} = \lambda^j \qquad j = 1,\dots,d$$

in cloud: $x^j : U\to\mathbb{R}$ differentiable

$\square$

Corollary 1. $dimT_pM = d = dimM$

Terminology: $X\in T_pM \to \exists\,\gamma : \mathbb{R}\to M : X = v_{\gamma,p}$ and
$\exists\, \underbrace{X_1^1,\dots,X^d}_{\in\mathbb{R}} : X = X^i\left(\frac{\partial}{\partial x^i}\right)_p$

5.5. 5. Change of vector ***components*** under a change of chart. ✘ vector does **not** change under change of chart.

Let (U,x) and (V,y) be overlapping charts and $p\in U\cap V$.
Let $X\in T_pM$

$$X^i_{(y)}\cdot\left(\frac{\partial}{\partial y^i}\right)_p \underbrace{=}_{(V,y)} X \underbrace{=}_{(U,x)} X^i_x\left(\frac{\partial}{\partial x^i}\right)_p$$

to study change of components formula:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i(f\circ x^{-1})(x(p)) = \\ &= \partial_i(\underbrace{(f\circ y^{-1})}_{\mathbb{R}^d\to\mathbb{R}} \circ \underbrace{(y\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}^d})(x(p)) \\ &= (\partial_i(y^i\circ x^{-1}))(x(p))\cdot(\partial_j(f\circ y^{-1}))(y(p)) = \\ &= \boxed{\left(\frac{\partial y^p}{\partial x^i}\right)_p \cdot \left(\frac{\partial f}{\partial y^j}\right)_p} f \end{aligned}$$

$$\Longrightarrow X^i_{(x)} \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_p = X^j_{(y)} \left(\frac{\partial}{\partial y^j}\right)_p$$

$$\Longrightarrow \boxed{X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)}}$$

5.6. **6. Cotangent spaces.** $T_pM = V$
trivial $(T_pM)^* := \{\varphi : T_pM \xrightarrow{\sim} \mathbb{R}\}$
Example: $f \in C^\infty(M)$

$$\begin{aligned} (df)_p : & T_pM \xrightarrow{\sim} \mathbb{R} \\ & X \mapsto (df)_p(X) \end{aligned}$$

i.e. $\boxed{(df)_p \in T_pM^*}$
$(df)_p$ called the gradient of f $p \in M$.
Calculate components of gradient w.r.t. chart-induced basis (U, x)

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left(\left(\frac{\partial}{\partial x^j}\right)_p\right) \\ &= \left(\frac{\partial f}{\partial x^j}\right)_p = \partial_j (f \circ x^{-1})(x(p)) \end{aligned}$$

Theorem 5. *Consider chart* $(U, x) \Longrightarrow x^i : U \to \mathbb{R}$
Claim: $(dx^1)_p, (dx^2)_p, \ldots, (dx^d)_p$ *basis of* T_p^*M
$\Longrightarrow$ *In fact: dual basis:*

$$(dx^a)_p \left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\frac{\partial x^a}{\partial x^b}\right)_p = \cdots = \delta^a_b$$

5.7. **7. Change of *components* of a covector under a change of chart:**

$$\underbrace{T_p^*M}_{\ni \omega} \text{ with } \omega_{(y)}(dy^j)_p = \omega = \omega_{(x)i}(dx^i)_p$$

$$\Longrightarrow \boxed{\omega_{(y)i} = \frac{\partial x^j}{\partial y^i} \omega_{(x)j}}$$

6. Lecture 6: Fields

So far:

$$\begin{array}{c} T_pM \\ \vdots \downarrow \\ T_p^*M \\ \vdots \downarrow \\ \vdots \end{array},$$

now
in Thought Cloud: theory of bundles

6.1. Bundles.

Definition 21. *A **bundle** is a triple*

$$E \xrightarrow{\pi} M$$

E smooth manifold *"**total space**"*
π smooth map (surjective) "projection map"
M smooth manifold "base space"

Example E = cylinder M = circle

Definition 22. *define **fibre over** p*
$:= preim_\pi(\{p\})$

Definition 23. *A **section** σ of a bundle*

$$\begin{tikzcd} E \arrow[d, "\phi\, d"'] \\ M \arrow[u, bend right, "*"'] \end{tikzcd}$$

require $\pi \circ \sigma = id_M$
Schuller says: in quantum mechanics, Aside: $\psi : M \to \mathbb{C}$

6.2. Tangent bundle of smooth manifold.
$(M, \mathcal{O}, \mathcal{A})$ smooth manifold

(a) as a **set** $TM := \dot{\bigcup}_{p \in M} T_pM$

(b) surjective $\pi : TM \to M$ the *unique* point $p \in M$, $X \in T_pM$
$X \mapsto p$
situation: $\underbrace{TM}_{\text{set}} \quad \underbrace{\xrightarrow{\pi}}_{\text{surjective map}} \quad \underbrace{M}_{\text{smooth manifold}}$

(c) Construct topology on TM that is the coarsest topology such that π (just) continuous. ("initial topology with respect to π").

$$\mathcal{O}_{TM} := \{\text{preim}_\pi(U) | \mathcal{U} \in \mathcal{O}\}$$

Show: Tutorial $\mathcal{O}_{TM}$ Schuller says this is shown in the tutorial
$(TM, \mathcal{O}_{TM})$

Construction of a C^∞-atlas on TM from the C^∞-atlas $\mathcal{A}$ on M.

$$\mathcal{A}_{TM} := \{(T\mathcal{U}, \xi_x) | (U, x) \in \mathcal{A}\}$$

where

$$\begin{aligned} \xi_x : & T\mathcal{U} \to \mathbb{R}^{2 \cdot \dim M} \\ & X \mapsto (\underbrace{(x^1 \circ \pi)(X), \ldots, (x^d \circ \pi)(X)}_{(U,x)-\text{ coords of } \pi(X) \ (d \text{ many })}, (dx^1)_{\pi(X)}(X), \ldots, (dx^d)_{\pi(X)}(X)) \end{aligned}$$

where $X \in T_{\pi(X)}M$
$X = X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)}$

$$\begin{aligned} (dx^j)_{\pi(X)}(X) = (dx^j)_{\pi(X)} \left(X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)}\right) = \\ = X^i_{(x)} \delta^j_i = X^j_{(x)} \end{aligned}$$

Write $\xi_x^{-1}:\xi_x(TU)\subseteq\mathbb{R}^{2\dim M}\to TU$

$$(\alpha^1,\dots,\alpha^d,\beta^1,\dots,\beta^d):=\beta^i\left(\frac{\partial}{\partial x^i}\right)_{\underbrace{x^{-1}(\alpha^1,\dots,\alpha^d)}_{\pi(X)}}$$

Check:

$$(\xi_y\circ\xi_x^{-1})(\alpha^1,\dots,\alpha^d,\beta^1,\dots,\beta^d)=$$
$$=\xi_y\left(\beta^i\left(\frac{\partial}{\partial x^i}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)}\right)$$
$$=\left(\dots,(y^i\circ\pi)(\beta^m\cdot\left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1\dots\alpha^d)}),\dots,\dots(dy^i)_{x^{-1}(\alpha^1,\dots\alpha^d)}\left(\beta^m\left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1\dots\alpha^d)}\right),\dots\right)=$$
$$=(\dots,(y^i\circ x^{-1})(\alpha^1,\dots,\alpha^d),\dots,\dots,\underbrace{\beta^m(dy^i)_{x^{-1}(\alpha^1,\dots,\alpha^d)}\left(\left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1\dots\alpha^d)}\right)}_{\beta^m\left(\frac{\partial y}{\partial x^m}\right)_{x^{-1}(\alpha^1\dots\alpha^d)}})$$

$\left(\frac{\partial y}{\partial x^m}\right)_{x^{-1}(\alpha^1\dots\alpha^d)}=\partial_m(y^i\circ x^{-1})(x\circ(x^{-1}(\alpha^1\dots\alpha^d)))=\partial_m(y^i\circ x^{-1})(\alpha^1\dots\alpha^d)$ smooth.

upshot

$$\underbrace{TM}_{\text{smooth manifold}}\ \underbrace{\xrightarrow{\pi}}_{\text{smooth map}}\ \underbrace{M}_{\text{smooth manifold}}$$

bundle, called the tangent bundle.

6.3. Vector fields.

Definition 24. *A **smooth vector field** χ is a* smooth *map*

6.4. The $C^\infty(M)$-module $\Gamma(TM)$. set $\Gamma(TM)=\{\chi\quad M\to TM|$ smooth section $\}$

$(\chi\oplus\widetilde{\chi})(f):=(\chi f)+\underbrace{\widetilde{\chi}}_{C^\infty(M)}(f)$

$(g\odot\xi)(f):=g\cdot\underbrace{\chi}_{C^\infty(M)}(f)$

upshot: set of all smooth vector fields can be made into a $C^\infty(M)$-module.
Fact:

1. ZFC $\Longrightarrow$ every vector space has a basis.
2. no such result exists for modules.

This is a shame, because otherwise, we could have chosen (for any manifolds) vector fields,

$$\Xi_{(1)},\dots,\Xi_{(d)}\in\Gamma(TM)$$

and would be able to write every vector field Ξ

$$\Xi=\underbrace{f^i}_{\text{component functions}}\cdot\Xi_{(i)}$$

Simple counterexample
Schuller says: Take a sphere, Morse Theorem, every smooth vector field must vanish at 2 pts. "mustn't choose a global basis"

However: $\frac{\partial}{\partial x^i}:U\xrightarrow{\text{smooth}}TU$
$p\mapsto\left(\frac{\partial}{\partial x^i}\right)_p$

6.5. Tensor fields. so far

$\Gamma(M)=$"set of vector fields" $C^\infty(M)$-module
$\Gamma(T^*M)=$ "covector fields" $C^\infty(M)$-module

Definition 25. *An (r,s)-tensor field T is a multi-linear map*

$$T:\underbrace{\Gamma(T^*M)\times\dots\times\Gamma(T^*M)}_{r}\times\Gamma(TM)\times\dots\times\Gamma(TM)\xrightarrow{\sim}C^\infty(M)$$

Example: $f\in C^\infty(M)$

$$df:\Gamma(TM)\xrightarrow{\sim}C^\infty(M)$$
$$\Xi\mapsto df(\Xi):=\Xi[f]$$

df $(0,1)$-T.F. (tensor field)
where $(\Xi f)(\underbrace{p}_{\in M}):=\underbrace{\Xi(p)}_{\in T_pM}f$
can check: df is $C^\infty-$linear

7. Lecture 7: Connections

$$\nabla_Xf=Xf=(df)(X)\text{ but (not quite)}$$
$$X:C^\infty(M)\to C^\infty(M)$$
$$df:\Gamma(TM)\to C^\infty(M)$$
$$\nabla_X:C^\infty(M)\to C^\infty(M)$$

$$\begin{array}{ccc}\nabla_X:C^\infty(M) & \longrightarrow & C^\infty(M)\\ \vdots\downarrow & & \vdots\downarrow\\ \nabla_X:\ \begin{array}{l}TM^p\otimes T^*M^q\text{ i.e.}\\ \binom{p}{q}\text{ tensor field}\end{array} & \longrightarrow & \begin{array}{l}TM^p\otimes T^*M^q\text{ i.e.}\\ \binom{p}{q}\text{ tensor field}\end{array}\end{array}$$

7.1. Directional derivatives of tensor fields. manifold with connection is quadruple $(M,\mathcal{O},\mathcal{A},\nabla)$

topology $\mathcal{O}$
atlas $\mathcal{A}$
Consider chart $(U,x)\in\mathcal{A}$

Definition 26. $\forall$ *pair $(X,(p,q)-$ tensor field$)\equiv(X,(p,q)-TF)$,
connection ∇ on smooth manifold $(M,\mathcal{O},\mathcal{A})$
$\nabla:(X,(p,q)-TF)\to(p,q)-TF$ s.t.*

(i) $\nabla_Xf=Xf$
(ii) $\nabla_X(T+S)=\nabla_XT+\nabla_XS$

(iii)

$$\nabla_X(T(\omega,Y)) = (\nabla_X T)(\omega,T) + T(\nabla_X\omega,Y) + T(\omega,\nabla_X Y)$$

"Leibnitz" rule.
As

$$T\otimes S(\omega_{(1)}\dots\omega_{(p+r)},Y_{(1)}\dots Y_{(q+s)}) = T(\omega_{(1)}\dots\omega_{(p)},Y_{(1)}\dots Y_{(q)})\cdot S(\omega_{(p+1)}\dots\omega_{(p+r)},Y_{(q+1)}\dots Y_{(q+s)})$$

so

$$\nabla_X(T\otimes S) = (\nabla_X T)\otimes S + T\otimes\nabla_X S$$

(iv) $\nabla_{fX+Z}T = f\nabla_X T + \nabla_Z T$ *C^∞-linear*

7.2. **New structure on $(M,\mathcal{O},\mathcal{A})$ required to fix ∇.** There are $(\dim M)^3$ many ${\Gamma^i}_{jk}$

$$\begin{aligned} &{\Gamma^i}_{jk} : U\to\mathbb{R} \\ &p\mapsto\left(dx^i(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x^j})\right)(p)\end{aligned}$$

Now $\nabla_{\frac{\partial}{\partial x^m}}(dx^i) = ?$

$$\underbrace{\nabla_{\frac{\partial}{\partial x^m}}(\underbrace{dx^i\left(\frac{\partial}{\partial x^j}\right)}_{{\delta^i}_j})}_{\overset{(\text{iii})}{\|}} = \frac{\partial}{\partial x^m}({\delta^i}_j) = 0$$

$$= \left(\nabla_{\frac{\partial}{\partial x^m}}dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + dx^i(\underbrace{\nabla_{\frac{\partial}{\partial x^m}}\frac{\partial}{\partial x^j}}_{\Gamma^q_{jm}\frac{\partial}{\partial x^q}}) = 0$$

$$\Longrightarrow\left(\nabla_{\frac{\partial}{\partial x^m}}dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = -\Gamma^i_{jm}$$

$$\nabla_{\frac{\partial}{\partial x^m}}dx^i = -\Gamma^i_{jm}dx^j$$

Hence

$$(\nabla_X Y)^i = X(Y^i) + \Gamma^i_j\underbrace{m}_{\text{last entry goes in direction of }X}Y^jX^m$$

$$(\nabla_X\omega)_i = X(\omega_i) + -\Gamma^j_{im}\omega_jX^m$$

Note that for the immediately above expression for $(\nabla_X Y)^i$, in the second term on the right hand side, Γ^i_{jm} has the last entry at the bottom, m going in the direction of X, so that it matches up with X^m. This is a good mnemonic to memorize the index positions of Γ.

summary so far:

$$\begin{aligned}(\nabla_X Y)^i &= X(Y^i) + \Gamma^i_{jm}Y^jX^m \\ (\nabla_X\omega)_i &= X(\omega_i) + -\Gamma^j_{im}\omega_jX^m\end{aligned}$$

similarly, by further application of Leibnitz
T a $(1,2)$-TF (tensor field)

$$(\nabla_X T)^i{}_{jk} = X({T^i}_{jk}) + {\Gamma^i}_{sm}{T^s}_{jk}X^m - {\Gamma^s}_{jm}{T^i}_{sk}X^m - {\Gamma^s}_{km}{T^i}_{js}X^m$$

What is a Euclidean space:
$(M=\mathbb{R}^n,\mathcal{O}_{\text{st}},\mathcal{A})$ smooth manifold.
Assume $(\mathbb{R}^n,\text{id}_{\mathbb{R}^n})\in\mathcal{A}$ and

$$(\Gamma^i_{(x)})_{jk} = dx^i\left((\nabla_{\underline{\mathrm{E}}})_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right)\overset{!}{=}0$$

7.3. **Change of Γ's under change of chart.** (U,x), $(V,y)\in\mathcal{A}$ and $U\cap V\neq\emptyset$

$$\Gamma^i_{jk}(y) := dy^i\left(\nabla_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j}\right) = \frac{\partial y^i}{\partial x^q}dx^q\left(\nabla_{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\frac{\partial}{\partial x^s}\right)$$

Note ∇_{fX} is C^∞-linear for fX
covector dy^i is C^∞-linear in its argument

$$\begin{aligned}\Longrightarrow\Gamma^i_{jk}(y) &= \frac{\partial y^i}{\partial x^q}dx^q\left(\frac{\partial x^p}{\partial y^k}\left[\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j}\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial}{\partial x^s}\right)\right]\right) = \\ &= \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}\frac{\partial x^s}{\partial y^j}{\delta^q}_s + \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}{\Gamma^q}_{sp}(x)\end{aligned}$$

$$\Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j\partial y^k} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{sp}(x) \quad (1)$$

Eq. (1) is the change of connection coefficient function under the change of chart $(U\cap V,x)\to(U\cap V,y)$

7.4. **Normal Coordinates.**

Tutorial 7 Connections. Exercise 1. : True or false?

(a)
- $\nabla_{fX}Y = f\nabla_X Y$ by definition so $\nabla_{fX} = f\nabla_X$ i.e. ∇_X is $C^\infty(M)$-linear in X
- $f\in C^\infty(M)$ is a $(0,0)$-tensor field. $\nabla_X f = Xf\equiv X(f)$ by definition.
- If the manifold is flat, I'm assuming that means that the manifold is globally a Euclidean space, and by definition, $\Gamma=0$.
$$\nabla_X Y = X^j\frac{\partial}{\partial x^j}(Y^i)\frac{\partial}{\partial x^i} + \Gamma^i_{jk}Y^kX^k\frac{\partial}{\partial x^i} = X^j\frac{\partial Y^i}{\partial x^j}\frac{\partial}{\partial x^i} + 0$$
and similarly for any (p,q)-tensor field, i.e.
$$\nabla_X T = X^j\frac{\partial T^{i_1\dots i_p}_{j_1\dots j_q}}{\partial x^j}$$
-
$$\nabla_X f = X^j\frac{\partial f}{\partial x^j} = X\cdot\text{grad}(f)$$
- $\forall\,(U,x)\in\mathcal{A}$, locally (after working out the first few cases, and doing induction, one can look up the expression for the local form; I found it in Nakahara's **Geometry, Topology and Physics**, Eq. 7.26, and it needs to be modified for the convention of order of bottom indices for Γ:
$$\nabla_\nu t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_q} = \partial_\nu t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_q} + \Gamma^{\lambda_1}_{\kappa\nu}t^{\kappa\lambda_2\dots\lambda_p}_{\mu_1\dots\mu_q} + \dots + \Gamma^{\lambda_p}_{\kappa\nu}t^{\lambda_1\dots\lambda_{p-1}\kappa}_{\mu_1\dots\mu_q} - \Gamma^\kappa_{\mu_1\nu}t^{\lambda_1\dots\lambda_p}_{\kappa\mu_2\dots\mu_q} - \dots - \Gamma^\kappa_{\mu_q\nu}t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_{q-1}\kappa}$$
Clearly, ∇_X is uniquely fixed $\forall\,p\in M$ by choosing each of the $(\dim M)^3$ many connection coefficient functions Γ.

(b)
- $\nabla:\mathfrak{X}(M)\to\mathfrak{X}(M)$
$\nabla:(p,q)$-tensor field $\mapsto(p,q)$-tensor field
- By definition, ∇ satisfies the Leibniz rule.
-
-
-

Exercise 2. : Practical rules for how ∇ acts Torsion-free covariant derivative boils down to a connection coefficient function Γ that is symmetric in the bottom indices.

- $$\nabla_X f = X(f) = X^i \frac{\partial f}{\partial x^i}$$
- $$(\nabla_X Y)^a = X^i \frac{\partial Y^a}{\partial x^i} + \Gamma^a_{jk} Y^j X^k$$
- $$(\nabla_X \omega)_a = X^i \frac{\partial \omega_a}{\partial x^j} - \Gamma^i_{ak}\omega_i X^k$$
- $$(\nabla_m T)^a{}_{bc} = \frac{\partial}{\partial x^m}(T^a{}_{bc}) + \Gamma^a{}_{im}T^i_{bc} - \Gamma^i_{bm}T^a_{ic} - \Gamma^j_{cm}T^a_{bj}$$
- $$(\nabla_{[m} A)_{n]} = (\nabla_m A)_n - (\nabla_n A)_m = \frac{\partial A_n}{\partial x^m} - \Gamma^i{}_{nm}A_i - \left(\frac{\partial A_m}{\partial x^n} - \Gamma^i{}_{mn}A_i \right) = \frac{\partial A_m}{\partial x^m} - \frac{\partial A_m}{\partial x^n}$$
- $$(\nabla_m \omega)_{nr} = \frac{\partial \omega_{nr}}{\partial x^m} - \Gamma^i{}_{nm}\omega_{ir} - \Gamma^i{}_{rm}\omega_{ni}$$

Exercise 3. : Connection coefficients

Question .

The connection coefficient functions Γ in chart $(U\cap V, y)$ is given, in terms of chart $(U \cap V, x)$ as follows:
Recall Eq. (1)

$$\Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{sp}(x)$$

8. Lecture 8: Parallel Transport & Curvature (International Winter School on Gravity and Light 2015)

8.1. Parallelity of vector fields.

Definition 27. (1) ***parallely transported*** *along smooth curve* $\gamma : \mathbb{R} \to M$ *if*

(2)
$$\boxed{\nabla_{v_\gamma} X = 0}$$

(2) *A slightly weaker condition is "parallel"*

$$(\nabla_{v_{\gamma,\gamma(\lambda)}} X)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}$$

8.2. Autoparallely transported curves.

Definition 28. *curve* $\gamma : \mathbb{R}\to M$ *is called* ***autoparallely transported*** *if*

(3)
$$\nabla_{v_\gamma} v_\gamma \stackrel{!}{=} 0$$

8.3. Autoparallel equation.

$$\nabla_{v_\gamma} v_\gamma = 0$$

in summary:

(4)
$$\ddot{\gamma}^m_{(x)}(\lambda) + (\Gamma^m_{(x)})_{ab}(\gamma(\lambda))\dot{\gamma}^a_{(x)}(\lambda)\dot{\gamma}^b_{(x)}(\lambda) = 0$$

8.4. Torsion.

Definition 29. ***torsion*** *of a connection* ∇ *is the* $(1,2)$-***tensor field***

(5)
$$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X,Y])$$

(Inside a cloud)
$[X,Y]$ vector field defined by

$$[X,Y]f := X(Yf) - Y(Xf)$$

Proof. check T is C^∞-linear in each entry

$$T(\omega, fX, Y) = \omega(\nabla_{fX}Y - \nabla_Y(fX) - [fX, Y])$$

□

Definition 30. *A* $(M, \mathcal{O}, \mathcal{A}, \nabla)$ *is called torsion-free if* $T = 0$

In a chart

$$\begin{aligned} T^i_{ab} := T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= dx^i(\dots) \\ &= \Gamma^i{}_{ab} - \Gamma^i{}_{ba} = 2\Gamma^i{}_{[ab]} \end{aligned}$$

From now on, in these lectures, we only use torsion-free connections.

8.5. 4. Curvature.

Definition 31. ***Riemann curvature*** *of a connection* ∇ *is the* $(1,3)$*-tensor field*

(6)
$$Riem(\omega, Z, X, Y) := \omega(\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z)$$

Proof. do it: C^∞-linear in each slot. □

<u>Tutorials</u> $\text{Riem}^i{}_{jab} = \dots$

Tutorial 8 Parallel transport & Curvature

Exercise 1.

Exercise 2. : Where connection coefficients appear

It was suggested in the tutorial sheets and hinted in the lecture that the following should be committed to memory.

Question : Recall the autoparallel equation for a curve γ.

(a)
$$\nabla_{v_\gamma} v_\gamma = 0$$

(b)
$$\nabla_{v_\gamma} v_\gamma = \nabla_{\dot{\gamma}\frac{\partial}{\partial x^\mu}} v_\gamma = \dot{\gamma}^\nu \nabla_{\partial_\nu} v_\gamma = \dot{\gamma}^\nu \left[\frac{\partial v^\mu_\gamma}{\partial x^\nu} + \Gamma^\rho_{\mu\nu} v^\mu_\gamma \right] \frac{\partial}{\partial x^\rho} = \dot{\gamma}^\nu \left[\frac{\partial \dot{\gamma}^\rho}{\partial x^\nu} + \Gamma^\rho_{\mu\nu}\dot{\gamma}^\mu \right]\frac{\partial}{\partial x^\rho} = 0$$

$$\implies \boxed{\ddot{\gamma}^\rho + \Gamma^\rho_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu}$$

as, for example, for $F(x(t))$,

$$\frac{dF(x(t))}{dt} = \dot{x}\frac{\partial F}{\partial x} = \frac{d}{dt}F$$

so that

$$\dot{\gamma}^\nu \frac{\partial v^\mu_\gamma}{\partial x^\nu} = \frac{d}{d\lambda}v^\mu_\gamma = \frac{d^2}{d\lambda^2}\gamma^\mu$$

Question : Determine the coefficients of the Riemann tensor with respect to a chart (U,x).

Recall this manifestly covariant definition

$$\text{Riem}(\omega, Z, X, Y) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$

We want $R^i_{\ jab}$.
now

$$\nabla_X \nabla_Y Z = \nabla_X ((Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma^\rho_{\mu\nu} Z^\mu Y^\nu) \frac{\partial}{\partial x^\rho}) = (X^\alpha \frac{\partial}{\partial x^\alpha}(Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma^\rho_{\mu\nu} Z^\mu Y^\nu) + \Gamma^\rho_{\alpha\beta}(Y^\mu \frac{\partial}{\partial x^\mu} Z^\alpha + \Gamma^\alpha_{\mu\nu} Z^\mu Y^\nu) X^\beta) \frac{\partial}{\partial x^\rho}$$

For $X = \partial_a$, $Y = \partial_b$, $Z = \partial_j$, then the partial derivatives of the coefficients of the input vectors become zero.

$$\implies \nabla_{\partial_a} \nabla_{\partial_b} \partial_j = \frac{\partial}{\partial x^a}(\Gamma^i_{jb}) + \Gamma^i_{\alpha a} \Gamma^\alpha_{jb}$$

Now

$$[X,Y]^i = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial X^i}{\partial x^j}$$

For coordinate vectors, $[\partial_i, \partial_j] = 0 \ \forall i, j = 0, 1 \dots d$.
Thus

$$\boxed{R^i_{\ jab} = \frac{\partial}{\partial x^a} \Gamma^i_{jb} - \frac{\partial}{\partial x^b} \Gamma^i_{ja} + \Gamma^i_{\alpha a} \Gamma^\alpha_{jb} - \Gamma^i_{\alpha b} \Gamma^\alpha_{ja}}$$

Question :Ric$(X,Y) :=$ Riem$^m_{\ amb} X^a Y^b$ define $(0,2)$-tensor?.

Yes, transforms as such:

EY developments. I roughly follow the spirit in Theodore Frankel's **The Geometry of Physics: An Introduction** Second Ed. 2003, Chapter 9 Covariant Differentiation and Curvature, Section 9.3b. The Covariant Differential of a Vector Field. P.S. EY : 20150320 I would like a copy of the Third Edition but I don't have the funds right now to purchase the third edition: go to my tilt crowdfunding campaign, http://ernestyalumni.tilt.com, and help with your financial support if you can or send me a message on my various channels and ernestyalumni gmail email address if you could help me get a hold of a digital or hard copy as a pro bono gift from the publisher or author.

The spirit of the development is the following:

"How can we express connections and curvatures in terms of forms?" -Theodore Frankel.

From Lecture 7, connection ∇ on vector field Y, in the "direction" X,

$$\nabla_{\frac{\partial}{\partial x^k}} Y = \left(\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j \right) \frac{\partial}{\partial x^i}$$

Make the ansatz (approche, impostazione) that the connection ∇ acts on Y, the vector field, first:

$$\nabla Y(X) = \left(X^k \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i} = X^k \left(\nabla_{\frac{\partial}{\partial x^k}} Y \right)^i \frac{\partial}{\partial x^i} = (\nabla_X Y)^i \frac{\partial}{\partial x^i} = \nabla_X Y$$

Now from Lecture 7, Definition for Γ,

$$dx^i \left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) = \Gamma^i_{jk}$$

Make this ansatz (approche, impostazine)

$$\nabla \frac{\partial}{\partial x^j} = (\Gamma^i_{jk} dx^k) \otimes \frac{\partial}{\partial x^i} \in \Omega^1(M, TM) = T^*M \otimes TM$$

where $\Omega^1(M,TM) = T^*M \otimes TM$ is the set of all TM or vector-valued 1-forms on M, with the 1-form being the following:

$$\Gamma^i_{jk} dx^k = \Gamma^i_{\ j} \in \Omega^1(M) \qquad \begin{aligned} i &= 1 \dots \dim(M) \\ j &= 1 \dots \dim(M) \end{aligned}$$

So $\Gamma^i_{\ j}$ is a $\dim M \times \dim M$ matrix of 1-forms (EY !!!).
Thus

$$\nabla Y = (d(Y^i) + \Gamma^i_j Y^j) \otimes \frac{\partial}{\partial x^i}$$

So the connection is a (smooth) map from TM to the set of all vector-valued 1-forms on M, $\Omega^1(M,TM)$, and then, after "eating" a vector Y, yields the "covariant derivative":

$$\begin{aligned} &\nabla : TM \to \Omega^1(M,TM) = T^*M \otimes TM \\ &\nabla : Y \mapsto \nabla Y \\ &\nabla Y : TM \to TM \\ &\nabla Y(X) \mapsto \nabla Y(X) = \nabla_X(Y) \end{aligned}$$

Now

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] f = \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^i} \right) = 0$$

(this is okay as on $p \in (U,x)$; x-coordinates on same chart (U,x))
EY : 20150320 My question is when is this nontrivial or nonvanishing (i.e. not equal to 0).

$$[e_a, e_b] = ?$$

for a frame (e_c) and would this be the difference between a tangent bundle TM vs. a (general) vector bundle?
Wikipedia helps here. cf. wikipedia, "Connection (vector bundle)"

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M,E)$$
$$\nabla e_a = \omega^c_{ab} f^b \otimes e_c$$
$$f^b \in T^*M \text{ (this is the dual basis for } TM \text{ and, note, this is for the manifold, } M$$
$$\nabla_{f_b} e_a = \omega^c_{ab} e_c \in E$$

$$\omega^c_a = \omega^c_{ab} f^b \in \Omega^1(M)$$

is the connection 1-form, with $a, c = 1 \dots \dim V$. EY : 20150320 This V is a vector space living on each of the fibers of E. I know that $\Gamma(T^*M \otimes E)$ looks like it should take values in E, but it's meaning that it takes vector values of V. Correct me if I'm wrong: ernestyalumni at gmail and various social media.
Let $\sigma \in \Gamma(E)$, $\sigma = \sigma^a e_a$

$$\nabla \sigma = (d\sigma^c + \omega^c_{ab} \sigma^a f^b) \otimes e_c \text{ with}$$
$$d\sigma^c = \frac{\partial \sigma^c}{\partial x^b} f^b$$

$$\implies \nabla_X \sigma = \left(X^b \frac{\partial \sigma^c}{\partial x^b} + \omega^c_{ab} \sigma^a X^b \right) e_c = X^b \left(\frac{\partial \sigma^c}{\partial x^b} + \omega^c_{ab} \sigma^a \right) e_c$$

9. Lecture 9: Newtonian spacetime is curved!

Axiom 1 (Newton I:)**.** *A body on which no force acts moves uniformly along a straight line*

Axiom 2 (Newton II:)**.** *Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.*

Remark:

(1) 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space ...
(2) Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

9.1. Laplace's questions.

Laplace $*1749$
$\dagger 1827$

Q: "Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to more along straight lines in this curved space?"

Answer: No!

Proof. gravity is a force point of view

$$m\ddot{x}^\alpha(t) = F^\alpha(x(t))$$
$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t))$$

$-\partial_\alpha f^\alpha = 4\pi G\rho$ (Poisson)
ρ mass density of matter
(EY : 20150330) You know this, $F = Gm_1m_2/r^2$

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

Laplace asks: Is this $(\ddot{x}(t))$ of the form

$$\ddot{x}^\alpha(t) + \Gamma^\alpha_{\ \beta\gamma}(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

Conclusion: One cannot find Γ s such that Newton's equation takes the form of an autoparallel.

□

9.2. The full wisdom of Newton I.

use also the information from Newton's first law that particles (no force) move uniformly
introduce the appropriate setting to talk about the difference easily
insight: in spacetime $\boxed{\text{uniform \& straight motion}}$ is simply straight motion
So let's try in spacetime:
let $x : \mathbb{R} \to \mathbb{R}^3$

be a particle's trajectory in space $\longleftrightarrow$ worldline (history) of the particle
$$\begin{aligned} X &: \mathbb{R} \to \mathbb{R}^4 \\ t &\mapsto (t, x^1(t), x^2(t), x^3(t)) := \\ &:= (X^0(t), X^1(t), X^2(t), X^3(t)) \end{aligned}$$

That's all it takes:
Trivial rewritings:

$$\dot{X}^0 = 1$$

$$\Longrightarrow \boxed{\begin{aligned} \ddot{X}^0 &= 0 \\ \ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 &= 0 \end{aligned}} \quad (\alpha = 1,2,3) \Longrightarrow \begin{gathered} a = 0,1,2,3 \\ \boxed{\ddot{X}^a + \Gamma^a_{\ bc}\dot{X}^b\dot{X}^c = 0} \end{gathered}$$

antoparallel eqn in spacetime

Yes, choosing $\Gamma^0_{\ ab} = 0$
$$\Gamma^\alpha_{\ \beta\gamma} = 0 = \Gamma^\alpha_{\ 0\beta} = \Gamma^\alpha_{\ \beta 0}$$
only: $\boxed{\Gamma^\alpha_{\ 00} \overset{!}{=} -f^\alpha}$
Question: Is this a coordinate-choice artifact?
No, since $R^\alpha_{\ 0\beta 0} = -\frac{\partial}{\partial x^\beta} f^\alpha$ (only non-vanishing components) (tidal force tensor, $-$ the Hessian of the force component)
Ricci tensor $\Longrightarrow R_{00} = R^m_{\ 0m0} = -\partial_\alpha f^\alpha = 4\pi G\rho$
Poisson: $-\partial_\alpha f^\alpha = 4\pi G \cdot \rho$
writing: $T_{00} = \frac{1}{2}s$

$$\Longrightarrow \boxed{R_{00} = 8\pi G T_{00}}$$

Einstein in 1912 $\boxed{\cancel{R_{ab} = 8\pi G T_{ab}}}$
Conclusion: Laplace's idea works in spacetime
Remark

$$\Gamma^\alpha_{\ 00} = -f^\alpha$$
$$R^\alpha_{\ \beta\gamma\delta} = 0 \qquad \alpha,\beta,\gamma,\delta = 1,2,3$$
$$\boxed{R_{00} = 4\pi G\rho}$$

Q: What about transformation behavior of LHS of

$$\underbrace{\underbrace{\ddot{x}^a + \Gamma^a_{\ bc}\dot{X}^b\dot{X}^c}_{(\nabla_{v_X} v_X)^a}}_{:=a^a \text{ "acceleration vector"}} = 0$$

9.3. The foundations of the geometric formulation of Newton's axiom.

new start

Definition 32. *A **Newtonian spacetime** is a quintuple*

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where $(M, \mathcal{O}, \mathcal{A})$ 4-dim. smooth manifold

$$t : M \to \mathbb{R} \text{ smooth function}$$

(i) *"There is an absolute space"*
$$(dt)_p \neq 0 \qquad \forall p \in M$$
(ii) *"absolute time flows uniformly"*
$$\nabla dt \underbrace{=}_{\text{space of } (0,2)\text{-tensor fields}} 0 \qquad \text{everywhere}$$
∇dt is a $(0,2)$-tensor field
(iii) *add to axioms of Newtonian spacetime $\nabla = 0$ torsion free*

Definition 33. *absolute space at time τ*

$$S_\tau := \{p \in M | t(p) = \tau\}$$
$$\xrightarrow{dt \neq 0} M = \coprod S_\tau$$

Definition 34. *A vector $X \in T_pM$ is called*

(a) *future-directed if*
$$dt(X) > 0$$
(b) *spatial if*
$$dt(X) = 0$$
(c) *past-directed if*
$$dt(X) < 0$$

picture

Newton I: The worldline of a particle under the influence of no force (gravity isn't one, anyway) is a future-directed autoparallel i.e.

$$\nabla_{v_X} v_X = 0$$
$$dt(v_X) > 0$$

Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m} \Longleftrightarrow m \cdot a = F$$

where F is a spatial vector field:

$$dt(F) = 0$$

Convention: restrict attention to atlases $\mathcal{A}_{\text{stratefied}}$ whose charts $(\mathcal{U}, x)$ have the property

$$\begin{aligned} & x^0 : \mathcal{U} \to \mathbb{R} \\ & x^1 : \mathcal{U} \to \mathbb{R} \\ & \vdots \quad \vdots \\ & x^3 \end{aligned} \qquad x^0 = t|_{\mathcal{U}} \qquad \Longrightarrow \begin{aligned} & 0 \overset{\text{“absolute time flows uniformly”}}{=} \nabla dt \\ & 0 = \nabla_{\frac{\partial}{\partial x^a}} dx^0 = -\Gamma^0{}_{ba} \qquad a = 0, 1, 2, 3 \end{aligned}$$

Let's evaluate in a chart $(\mathcal{U}, x)$ of a stratified atlas $\mathcal{A}_{\text{sheet}}$: Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m}$$

in a chart.

$$(X^0)'' + \cancel{\Gamma^0{}_{cd}(X^a)'(X^b)'}^{\text{stratified atlas}} = 0$$
$$(X^\alpha)'' + \Gamma^\alpha_{\gamma\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha_{00} X^{0'} X^{0'} + 2\Gamma^\alpha_{\gamma 0} X^{\gamma'} X^{0'} = \frac{F^\alpha}{m} \qquad \alpha = 1, 2, 3$$
$$\Longrightarrow (X^0)''(\lambda) = 0 \Longrightarrow X^0(\lambda) = a\lambda + b \qquad \text{constants } a, b \text{ with}$$
$$X^0(\lambda) = (x^0 \circ X)(\lambda) \overset{\text{stratified}}{=} (t \circ X)(\lambda)$$

convention parametrize worldline by absolute time

$$\frac{d}{d\lambda} = a\frac{d}{dt}$$
$$a^2 \ddot{X}^\alpha + a^2 \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + a^2 \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0 = \frac{F^\alpha}{m}$$
$$\Longrightarrow \underbrace{\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0}_{a^\alpha} = \frac{1}{a^2} \frac{F^\alpha}{m}$$

10. Lecture 10: Metric Manifolds

cf. Lecture 10: Metric Manifolds (International Winter School on Gravity and Light 2015)

We establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space).

From this structure, one can then define a notion of length of a curve.
Then we can look at shortest curves.

Requiring then that the shortest curves coincide with the straightest curves (wrt ∇) will result in ∇ being determined by the metric structure.

$$g \overset{\text{straight=short}}{\underset{T=0}{\rightsquigarrow}} \nabla \rightsquigarrow \text{Riem}$$

10.1. Metrics.

Definition 35. *A metric g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying*

(i) *symmetry $g(X,Y) = g(Y,X) \quad \forall X, Y$ vector fields*
(ii) *non-degeneracy: the musical map*
$$\text{“flat”} \flat : \Gamma(TM) \to \Gamma(T^*M)$$
$$X \mapsto \flat(X)$$

where $\quad \flat(X)(Y) := g(X,Y)$
$\qquad \flat(X) \in \Gamma(T^*M)$

In thought bubble: $\flat(X) = g(X, \cdot)$

…is a C^∞-isomorphism in other words, it is invertible.

Remark: $(\flat(X))_a$ or
X_a
$(\flat(X))_a := g_{am} X^m$

Thought bubble: $\flat^{-1} = \sharp$

$\flat^{-1}(\omega)^a := g^{am}\omega_m$
$\flat^{-1}(\omega)^a := (g^{\text{“}-1\text{''}})^{am}\omega_m \Longrightarrow$ not needed. (all of this is not needed)

Definition 36. *The $(2,0)$-tensor field $g^{\text{“}-1\text{''}}$ with respect to a metric g is the symmetric*

$$g^{\text{“}-1\text{''}} : \Gamma(T^*M) \times \Gamma(T^*M) \to C^\infty(M)$$
$$(\omega, \sigma) \mapsto \omega(\flat^{-1}(\sigma)) \qquad \flat^{-1}(\sigma) \in \Gamma(TM))$$

chart: $g_{ab} = g_{ba}$
$\qquad (g^{-1})^{am} g_{mb} = \delta^a_b$

Example: $(S^2, \mathcal{O}, \mathcal{A})$
chart $(\mathcal{U}, x)$
$\varphi \in (0, 2\pi)$
$\theta \in (0, \pi)$
define the metric

$$g_{ij}(x^{-1}(\theta, \varphi)) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{bmatrix}_{ij}$$

$R \in \mathbb{R}^+$

“the metric of the round sphere of radius R”

$$A^a{}_m v^m = \lambda v^a \qquad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

10.2. **Signature.** Linear algebra:

$$g_{am} v^m = \lambda \cdot v^a ? \rightsquigarrow \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

$(1,1)$ tensor has eigenvalues
$(0,2)$ has signature (p,q) (well-defined)

$$\left.\begin{matrix} (+++) \\ (++-) \\ (+--) \\ (---) \end{matrix}\right\} d+1 \text{ if } p+q = \dim V$$

Definition 37. *A metric is called*
Riemannian *if its signature is* $(++\cdots+)$
Lorentzian *if* $(+-\cdots-)$

10.3. **Length of a curve.** Let γ be a smooth curve.
Then we know its veloctiy $v_{\gamma,\gamma(\lambda)}$ at each $\gamma(\lambda) \in M$.

Definition 38. *On a Riemannian metric manifold* $M, \mathcal{O}, \mathcal{A}, g)$*, the* ***speed*** *of a curve at* $\gamma(\lambda)$ *is the number*

$$(\sqrt{g(v_\gamma, v_\gamma)})_{\gamma(\lambda)} = s(\lambda)$$

F. Schuller: "I feel the need for speed." -Top Gun.
(I feel the need for speed, then I feel the need for a metric)
Aside: $[v^a] = \frac{1}{T}$
$[g_{ab}] = L^2$
$[\sqrt{g_{ab} v^a v^b}] = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$

Definition 39. *Let* $\gamma : (0,1) \to M$ *a smooth curve.*
Then the ***length of*** γ *is the number*

$$\mathbb{R} \ni L[\gamma] := \int_0^1 d\lambda s(\lambda) = \int_0^1 d\lambda \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}$$

F. Schuller: "velocity is more fundamental than speed, speed is more fundamental than length"
Example: reconsider the round sphere of radius R
Consider its equator:

$$\theta(\lambda) := (x^1 \circ \gamma)(\lambda) = \frac{\pi}{2}$$
$$\varphi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3$$
$$\theta'(\lambda) = 0$$
$$\varphi'(\lambda) = 6\pi\lambda^2$$

on the same chart $g_{ij} = \begin{bmatrix} R^2 & \\ & R^2 \sin^2\theta \end{bmatrix}$
F.Schuller: do everything in this chart

$$L[\gamma] = \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} = \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda)) 36\pi^2\lambda^4} =$$
$$= 6\pi R \int_0^1 d\lambda \lambda^2 = 6\pi R [\frac{1}{3}\lambda^3]_0^1 = 2\pi R$$

Theorem 6. $\gamma : (0,1) \to M$ *and*
$\sigma : (0,1) \to (0,1)$ *smooth bijective and increasing "reparametrization"*

$$L[\gamma] = L[\gamma \circ \sigma]$$

Proof. $\Longrightarrow$ Tutorials ☐

10.4. **Geodesics.**

Definition 40. *A curve* $\gamma : (0,1) \to M$ *is called a* ***geodesic*** *on a Riemannian manifold* $(M, \mathcal{O}, \mathcal{A}, g)$ *if its a* stationary *curve with respect to a length functional* L*.*

Thought bubble: in classical mechanics, deform the curve a little, ϵ times this deformation, to first order, it agrees with $L[\gamma]$

Theorem 7. γ *geodesic iff it satisfies the Euler-Lagrange equations for the Lagrangian*

$$\begin{aligned} \mathcal{L} :& TM \to \mathbb{R} \\ & X \mapsto \sqrt{g(X,X)} \end{aligned}$$

In a chart, the Euler Lagrange equations take the form:

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m} \right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial x^m} = 0$$

F.Schuller: this is a chart dependent formulation
here:

$$\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda)}$$

Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} = \frac{1}{\sqrt{\cdots}} g_{mj}(\gamma(\lambda))\dot{\gamma}^j(\lambda)$$
$$\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right)^{\cdot} = \left(\frac{1}{\sqrt{\cdots}} \right)^{\cdot} g_{mj}(\gamma(\lambda)) \cdot \dot{\gamma}^j(\lambda) + \frac{1}{\sqrt{\cdots}} \left(g_{mj}(\gamma(\lambda))\ddot{\gamma}^j(\lambda) + \dot{\gamma}^s(\partial_s g_{mj})\dot{\gamma}^j(\lambda) \right)$$

Thought bubble: reparametrize $g(\dot{\gamma}, \dot{\gamma}) = 1$ (it's a condition on my reparametrization)
By a clever choice of reparametrization $(\frac{1}{\sqrt{\cdots}})^{\cdot} = 0$

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{\cdots}} \partial_m g_{ij}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda)$$

putting this together as Euler-Lagrange equations:

$$g_{mj}\ddot{\gamma}^j + \partial_s g_{mj}\dot{\gamma}^s\dot{\gamma}^j - \frac{1}{2}\partial_m g_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$$

Multiply on both sides $(g^{-1})^{qm}$

$$\ddot{\gamma}^q + (g^{-1})^{qm}(\partial_i g_{mj} - \frac{1}{2}\partial_m g_{ij})\dot{\gamma}^i\dot{\gamma}^j = 0$$

$$\boxed{\ddot{\gamma}^q + (g^{-1})^{qm}\frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})\dot{\gamma}^i\dot{\gamma}^j = 0}$$

geodesic equation for γ in a chart.

$$\boxed{(g^{-1})^{qm}\frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) =: \Gamma^q{}_{ij}(\gamma(\lambda))}$$

Thought bubble: $\left(\frac{\partial \mathcal{L}}{\partial \xi_x^{a+\dim M}}\right)^{\cdot}_{\sigma(x)} - \left(\frac{\partial \mathcal{L}}{\partial x i_x^a}\right)_{\sigma(x)} = 0$

Definition 41. *"Christoffel symbol"* ${}^{L.C.}\Gamma$ *are the connection coefficient functions of the so-called Levi-Civita connection* ${}^{L.C.}\nabla$

We usually make this choice of ∇ if g is given.
$(M, \mathcal{O}, \mathcal{A}, g) \to (M, \mathcal{O}, \mathcal{A}, g, {}^{\text{L.C.}}\nabla)$
abstract way: $\nabla g = 0$ and $T = 0$ (torsion)
$\implies \nabla = {}^{\text{L.C.}}\nabla$

Definition 42. (a) *The Riemann-Christoffel curvature is defined by*

$$R_{abcd} := g_{am}R^m{}_{bcd}$$

(b) *Ricci:* $R_{ab} = R^m{}_{amb}$
Thought bubble: with a metric, ${}^{L.C.}\nabla$
(c) *(Ricci) scalar curvature:*

$$R = g^{ab}R_{ab}$$

Thought bubble: ${}^{L.C.}\nabla$

Definition 43. *Einstein curvature* $(M, \mathcal{O}, \mathcal{A}, g)$

$$G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R$$

Convention: $g^{ab} := (g^{\text{“}-1''})^{ab}$
F. Schuller: these indices are not being pulled up, because what would you pull them up with
(student) Question: Does the Einstein curvature yield new information?
Answer:

$$g^{ab}G_{ab} = R_{ab}g^{ab} - \frac{1}{2}g_{ab}g^{ab}R = R - \delta^a_a R = R - \frac{1}{2}\dim M\, R = (1 - \frac{d}{2})R$$

Tutorial 9: Metric manifolds. Exercise 3: Levi-Civita Connection. Suppose torsion-free $T = 0$ and metric-compatible connection $\nabla g = 0$

Question Recall $T = 0$ on a chart.

$$\boxed{\Gamma^c_{ba} = \frac{1}{2}(g^{-1})^{cm}\left(\frac{\partial g_{bm}}{\partial x^a} + \frac{\partial g_{ma}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^m}\right)}$$

or

$$\Gamma^a_{bc} = \frac{1}{2}(g^{-1})^{am}\left(\frac{\partial g_{bm}}{\partial x^c} + \frac{\partial g_{mc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^m}\right)$$

11. Symmetry

EY : 20150321 This lecture tremendously and lucidly clarified, for me at least, what a symmetry of the Lie algebra is, and in comparing structures $(M, \mathcal{O}, \mathcal{A})$ vs. $(M, \mathcal{O}, \mathcal{A}, \nabla)$, clarified differences, and asking about differences is a good way to learn, the difference between $\mathcal{L}$ and ∇, respectively.

Feeling that the round sphere

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$$

has rotational symmetry, while
the potato

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$$

does not.

11.1.

11.2. Important

11.3. **Flow of a complete vector field.** Let $(M, \mathcal{O}, \mathcal{A})$ smooth X vector field on M

Definition 44. *A curve* $\gamma : I \subseteq \mathbb{R} \to M$ *is called an integral curve of X if*

$$v_{\gamma,\gamma(\lambda)} = X_{\gamma(\lambda)}$$

Definition 45. *A vector filed X is* **complete** *if all integral curves have* $I = \mathbb{R}$ *EY: 20150321 (i.e. domain is all of* $\mathbb{R}$*)*

Ex. minute 48:30 EY : reall good explanation by F.P.Schuller; take a pt. out for an incomplete vector field.

Theorem 8. *compactly supported smooth vector field is complete.*

Definition 46. *The flow of a complete vector field X is a 1-parameter family*

$$h^X = \mathbb{R} \times M \to M$$

where $\gamma_p : \mathbb{R} \to M$ *is the integral curve of X with*
$\gamma(0) = p$
Then for fixed $\lambda \in \mathbb{R}$

$$h^X_\lambda : M \to M \textit{ smooth}$$

picture $h^X_{\underline{\lambda}}(S) \neq S($ *if* $X \neq 0)$

11.4. **Lie subalgebras of the Lie algebra** $(\Gamma(TM), [\cdot,\cdot])$ **of vector fields.**

(a) $\Gamma(TM) = \{$ set of all vector fields $\}$ $\quad C^\infty(M)$-module $= \mathbb{R}$-vector space

$$\implies [X, Y] \in \Gamma(TM) \qquad [X, Y]f := X(Yf) - Y(Xf)$$

(i) $[X, Y] = -[Y, X]$
(ii) $[\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]$
(iii) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$
$(\Gamma(TM), [\cdot,\cdot])$ Lie algebra

(b) Let $X_1 \ldots X_s$ for s (many) vector fields on M, such that

Tutorial 11 Symmetry. Exercise 1. : True or false?

(a)
-
- $\phi^* : T^*N \to T*M$ i.e. $\phi^*\nu(X) = \nu(\phi_* X)$ for smooth $\phi : M \to N$, so the pullback of a covector $\nu \in T^*N$ maps to a covector in $T*M$.
-
-
-
-

(b)
(c)

Exercise 2. : Pull-back and push-forward

Question . Let's check this locally

$$\phi^*(df)(X) = (df)(\phi_* X) = (df)(X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j} \text{ where } \quad \phi_* X = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$
$$d(\phi^* f)(X) = d(f(\phi))(X) = \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i} dx^i(X) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}$$

So

$$\boxed{\phi^*(df) = d(\phi^* f)} \qquad \forall\, p \in M,\ \forall\, X \in \mathfrak{X}(M)$$

The big idea is that this is a showing of the **naturality** of the pullback ϕ^* with d, i.e. that this commutes:

$$\begin{tikzcd} \Omega^1(M) & \Omega^1(N) \arrow[l, "\phi^*"'] \\ C^\infty(M) \arrow[u, "d"] & C^\infty(N) \arrow[l, "\phi^*"] \arrow[u, "d"'] \end{tikzcd}$$

Question .

$$(\phi_*)^a_{\ b} := (dy^a)(\phi_*(\frac{\partial}{\partial x^b}))$$
Let $g \in C^\infty(N)$
$$\phi_*\left(\frac{\partial}{\partial x^b} \right) g = \frac{\partial x^b}{g}\phi(p) = \frac{\partial}{\partial x^b} g\phi x^{-1} x(p) = \frac{\partial}{\partial x^b}(gyy^{-1}\phi x^{-1})(x) =$$
$$= \frac{\partial}{\partial x^b}(gy^{-1}(y\phi x^{-1}(x(p)))) = \left.\frac{\partial g}{\partial y}^b\right|_y \left.\frac{\partial y^a}{\partial x^b}\right|_x = \frac{\partial y^a}{\partial x^b}\frac{\partial g}{\partial y^a}$$

Then

$$\phi_*\left(\frac{\partial}{\partial x^b} \right) = \frac{\partial y^a}{\partial x^b}\frac{\partial}{\partial y^a}$$

and so

$$(\phi_*)^a_{\ b} = \frac{\partial y^a}{\partial x^b}$$

Question .

Exercise 3. :Lie derivative-the pedestrian way

Question . While it is true that $\forall\, p \in S^2$, for $x(p) = (\theta,\varphi)$, and $(yix^{-1})(\theta,\varphi) = (y^1,y^2,y^3) \in \mathbb{R}^3$ and that, at this point

p, $(y^1)^2/a^2 + (y^2)^2/b^2 + (y^3)^2/c^3 = 1$, this doesn't imply (EY: 20150321 I think) that, globally, it's an ellipsoid (yet). In the familiar charts given,
spherical chart $(U,x) \in \mathcal{A}$ and
$(\mathbb{R}^3, y = \mathrm{id}_{\mathbb{R}^3}) \in \mathcal{B}$
it looks like an ellipsoid, but change to another choice of charts, and it could look something very different.

Question .

Equip $(\mathbb{R}^3, \mathcal{O}_{\mathrm{st}}, \mathcal{B})$ with the Euclidean metric g, and pullback g.
Note that the pullback of the inclusion from $\mathbb{R}^3$ onto S^2 for the Euclidean metric is the following:

$$i^* g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j} \right) = g\left(i_*\frac{\partial}{\partial\theta^i}, i_*\frac{\partial}{\partial\theta^j} \right) = g\left(\frac{\partial x^a}{\partial\theta^i}\frac{\partial}{\partial x^a}, \frac{\partial x^b}{\partial\theta^j}\frac{\partial}{\partial x^b} \right) = g_{ab}\frac{\partial x^a}{\partial\theta^i}\frac{\partial x^b}{\partial\theta^j}$$

With $g_{ab} = \delta_{ab}$, the usual Euclidean metric, this becomes the following:

$$g_{ij}^{\text{ellipsoid}} = \frac{\partial x^a}{\partial\theta^i}\frac{\partial x^a}{\partial\theta^j}$$

At this point, one should get smart (we are in the 21st century) and use some sort of CAS (Computer Algebra System). I like Sage Math (version 6.4 as of 20150322). I also like the Sage Manifolds package for Sage Math.

I like Sage Math for the following reasons:

- Open source, so it's open and freely available to anyone, which fits into my principle of making online education open and freely available to anyone, anytime
- Sage Math structures everything in terms of Category Theory and Categories and Morphisms naturally correspond to Classes and Class methods or functions in Object-Oriented Programming in Python and they've written it that way

and I like Sage Manifolds for roughly the same reasons, as manifolds are fit into a category theory framework that's written into the Python code. e.g.

```
sage: S2 = Manifold(2, 'S^2', r'\mathbb{S}^2', start_index=1) ; print S2
sage: print S2
2-dimensional manifold 'S^2'
sage: type(S2)
<class 'sage.geometry.manifolds.manifold.Manifold_with_category'>
```

With code (I've provided for convenience; you can make your own as I wrote it based upon to example of S^2 on the sage-manifolds documentation website page), load it and do the following:

cf. https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/S2.sage
http://sagemanifolds.obspm.fr/examples.html

```
sage: load("S2.sage")
sage: U_ep = S2.open_subset('U_{ep}')
sage: eps.<the,phi> = U_ep.chart()
sage: a = var(\a")
sage: b = var(\b")
sage: c = var("c")
sage: inclus = S2.diff_mapping(R3, {(eps, cart): [ a*cos(phi)*sin(the), b*sin(phi)*sin(the),c*cos(the) ]} , name="inc",latex_name=r'\mathcal{i}')
sage: inclus.pullback(h).display()
inc_*(h) = (c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2) dthe*dthe - (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dthe*dphi
- (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dphi*dthe + (b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2 dphi*dphi
sage: inclus.pullback(h)[2,2].expr()
(b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2
```

A new open subset U_{ep} was declared in S^2, a new chart $(U_{\text{ep}}, (\theta, \phi))$ was declared, the constants, a, b, c, were declared, and the inclusion map given in the problem

$$y \circ \mathrm{i} \circ x^{-1} : (\theta, \phi) \mapsto (a \cos\phi \sin\theta, b \sin\phi \sin\theta, c \cos\theta)$$

Then the pullback of the inclusion map $\rangle$ was done on the Euclidean metric h, defined earlier in the file

`S2.sage`

. Then one can access the components of this metric and do, for example,

`simplify_full(),full_simplify(), reduce_trig()`

on the expression.

In Python, I could easily do this, and give an answer quick in LaTeX:

```
sage: for i in range(1,3):
....:     for j in range(1,3):
....:         print inclus.pullback(h)[i,j].expr()
....:         latex(inclus.pullback(h)[i,j].expr() )
....:
c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2
```

(EY: I'll suppress the LaTeX output but this sage math function gives you LaTeX code)

and so

$$\boxed{\begin{gathered} i^* g = c^2 \sin(the)^2 + \left(a^2 \cos(\phi)^2 + b^2 \sin(\phi)^2\right) \cos(the)^2 \, d\theta \otimes d\theta + \\ -2\left(a^2 - b^2\right) \cos(\phi) \cos(the) \sin(\phi) \sin(the) \, d\theta \otimes d\phi + \\ + \left(b^2 \cos(\phi)^2 + a^2 \sin(\phi)^2\right) \sin(the)^2 \, d\phi \otimes d\phi \end{gathered}}$$

Question .

```
sage: polar_vees = eps.frame()
sage: X_1 = - sin(phi) * polar_vees[1] - cot( the ) * cos(phi) * polar_vees[2]
sage: X_2 = cos( phi ) * polar_vees[1] - cot( the ) * sin( phi) * polar_vees[2]
sage: X_3 = polar_vees[2]
sage: X_2.lie_der(X_1).display()
(cos(the)^2 - 1)/sin(the)^2 d/dphi
sage: X_3.lie_der(X_1).display()
cos(phi) d/dthe - cos(the)*sin(phi)/sin(the) d/dphi
sage: X_3.lie_der(X_2).display()
sin(phi) d/dthe + cos(phi)*cos(the)/sin(the) d/dphi
```

Indeed, one can check on a scalar field $f_{\text{eps}} \in C^\infty(S^2)$:

```
sage: f_eps = S2.scalar_field({eps: function('f', the, phi ) }, name='f' )
sage: (X_1( X_2(f_eps)) - X_2(X_1(f_eps) ) ).display()
U_{ep} --> R
(the, phi) |--> -D[1](f)(the, phi)
sage: X_2.lie_der(X_1) == -X_3
True
sage: X_3.lie_der(X_1) == X_2
True
sage: X_3.lie_der(X_2) == -X_1
True
```

$$\Longrightarrow \boxed{[X_i, X_j] = -\epsilon_{ijk} X_k}$$

So $\text{span}_{\mathbb{R}}\{X_1, X_2, X_3\}$ equipped with $[\,,\,]$ constitute a Lie subalgebra on S^2 (It's closed under $[\,,\,]$

12. Integration

12.1.

12.2.

12.3. Volume forms.

Definition 47. *On a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ a $(0, dimM)$-tensor field Ω is called a <u>volume form</u> if*

(a) *Ω vanishes nowhere (i.e. $\Omega \neq 0 \; \forall \, p \in M$)*
(b) *totally antisymmetric*

$$\Omega(\dots, \underbrace{X}_{ith}, \dots, \underbrace{Y}_{jth} \dots) = -\Omega(\dots, \underbrace{Y}_{ith}, \dots, \underbrace{X}_{jth} \dots)$$

In a chart:

$$\Omega_{i_1 \dots i_d} = \Omega_{[i_1 \dots i_d]}$$

<u>Example</u> $(M, \mathcal{O}, \mathcal{A}, g)$ metric manifold
construct volume form Ω from g
In <u>any</u> chart: (U, x)

$$\Omega_{i_1 \dots i_d} := \sqrt{\det(g_{ij}(x))} \epsilon_{i_1 \dots i_d}$$

where **Levi-Civita symbol** $\epsilon_{i_1 \dots i_d}$ is <u>defined</u> as $\epsilon_{123\dots d} = +1$
$\epsilon_{1 \dots d} = \epsilon_{[i_1 \dots i_d]}$

Proof. (well-defined) Check: What happens under a change of charts

$$\begin{aligned} \Omega(y)_{i_1 \dots i_d} &= \sqrt{\det(g(y)_{ij})} \epsilon_{i_1 \dots i_d} = \\ &= \sqrt{\det(g_{mn}(x) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j})} \frac{\partial y^{m_1}}{\partial x^{i_1}} \dots \frac{\partial y^{m_d}}{\partial x^{i_d}} \epsilon_{[m_1 \dots m_d]} = \\ &= \sqrt{|\det g_{ij}(x)|} \left| \det \left(\frac{\partial x}{\partial y} \right) \right| \det \left(\frac{\partial y}{\partial x} \right) \epsilon_{i_1 \dots i_d} = \sqrt{\det g_{ij}(x)} \epsilon_{i_1 \dots i_d} \text{sgn} \left(\det \left(\frac{\partial x}{\partial y} \right) \right) \end{aligned}$$

$\square$

EY : 20150323

Consider the following:

$$\begin{aligned}
\Omega(y)(Y_{(1)}\dots Y_{(d)}) &= \Omega(y)_{i_1\dots i_d}Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} = \\
&= \sqrt{\det(g_{ij}(y))}\epsilon_{i_1\dots i_d}Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} = \\
&= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\epsilon_{i_1\dots i_d}\frac{\partial y^{i_1}}{\partial x^{m_1}}\dots\frac{\partial y^{i_d}}{\partial x^{m_d}}X^{m_1}\dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d}X^{m_1}\dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))}\left|\det\left(\frac{\partial x}{\partial y}\right)\right|\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d}X^{m_1}\dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))}\epsilon_{m_1\dots m_d}\mathrm{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right)X^{m_1}\dots X^{m_d} = \mathrm{sgn}(\det\left(\frac{\partial x}{\partial y}\right))\Omega_{m_1\dots m_d}(x)X^{m_1}\dots X^{m_d}
\end{aligned}$$

If $\det\left(\frac{\partial y}{\partial x}\right) > 0$,

$$\Omega(y)(Y_{(1)}\dots Y_{(d)}) = \Omega(x)(X_{(1)}\dots X_{(d)})$$

This works also if Levi-Civita symbol $\epsilon_{i_1\dots i_d}$ doesn't change at all under a change of charts. (around 42:43 https://youtu.be/2XpnbvPy-Zg)

Alright, let's require,
restrict the smooth atlas $\mathcal{A}$
to a subatlas ($\mathcal{A}^\uparrow$ still an atlas)

$$\mathcal{A}^\uparrow \subseteq \mathcal{A}$$

s.t. $\forall (U,x),(V,y)$ have chart transition maps $y\circ x^{-1}$
$x\circ y^{-1}$

s.t. $\det\left(\frac{\partial y}{\partial x}\right) > 0$
such $\mathcal{A}^\uparrow$ called an **oriented** atlas

$$(M,\mathcal{O},\mathcal{A},g) \Longrightarrow (M,\mathcal{O},\mathcal{A}^\uparrow,g)$$

Note: associated bundles.
Note also: $\det\left(\frac{\partial y^b}{\partial x^a}\right) = \det(\partial_a(y^b x^{-1}))$ $\frac{\partial y^b}{\partial x^a}$ is an endomorphism on vector space V.

$\varphi : V \to V$
$\det\varphi$ independent of choice of basis
g is a $(0,2)$ tensor field, not endomorphism (not independent of choice of basis) $\sqrt{|\det(g_{ij}(y))|}$

Definition 48. *Ω be a volume form on $(M,\mathcal{O},\mathcal{A}^\uparrow)$ and consider chart (U,x)*

Definition 49. $\omega_{(X)} := \Omega_{i_1\dots i_d}\epsilon^{i_1\dots i_d}$ *same way* $\epsilon^{12\dots d} = +1$
$\epsilon^{[\dots]}$

one can show

$$\boxed{\omega_{(y)} = \det\left(\frac{\partial x}{\partial y}\right)\omega_{(x)}} \qquad \textit{scalar density}$$

12.4. **Integration on one chart domain U.**

Definition 50.

$$\boxed{\int_U f :\overset{(U,y)}{=} \int_{y(U)} d^d\beta\,\omega_{(y)}(y^{-1}(\beta))f_{(y)}(\beta)} \tag{7}$$

Proof. : Check that it's (well-defined), how it changes under change of charts

$$\begin{aligned}
\int_U f :\overset{(U,y)}{=} \int_{y(U)} d^d\beta\,\omega_{(y)}(y^{-1}(\beta))f_{(y)}(\beta) &\underset{(U,y)}{=} \int_{x(U)}\int d^d\alpha\left|\det\left(\frac{\partial y}{\partial x}\right)\right| f_{(x)}(\alpha)\omega_{(x)}(x^{-1}(\alpha)\det\left(\frac{\partial x}{\partial y}\right) = \\
&= \int_{x(U)} d^d\alpha\,\omega_{(x)}(x^{-1}(x))f_{(x)}(\alpha)
\end{aligned}$$

□

On an oriented metric manifold $(M,\mathcal{O},\mathcal{A}^\uparrow,g)$

$$\int_U f := \int_{x(U)} d^d\alpha\,\underbrace{\sqrt{\det(g_{ij}(x))(x^{-1}(\alpha))}}_{\sqrt{g}}f_{(x)}(\alpha)$$

12.5. **Integration on the entire manifold.**

13. Lecture 13: Relativistic spacetime

Recall, from Lecture 9, the definition of Newtonian spacetime

$$(M,\mathcal{O},\mathcal{A},\nabla,t) \quad \begin{aligned} &\nabla \text{ torsion free} \\ &t\in C^\infty(M) \\ &dt\neq 0 \\ &\nabla dt = 0 \quad \text{(uniform time)}\end{aligned}$$

and the definition of relativistic spacetime (before Lecture)

$$(M,\mathcal{O},\mathcal{A}^\uparrow,\nabla,g,T) \quad \begin{aligned} &\nabla \text{ torsion-free} \\ &g \text{ Lorentzian metric}(+---) \\ &T \text{ time-orientation}\end{aligned}$$

13.1. **Time orientation.**

Definition 51. *$(M,\mathcal{O},\mathcal{A}^\uparrow,g)$ a Lorentzian manifold. Then a time-orientation is given by a vector field T that*
(i) *does **not** vanish anywhere*
(ii) *$g(T,T) > 0$*

Newtonian vs. relativistic
Newtonian
X was called future-directed if

$$dt(X) > 0$$

$\forall p\in M$, take half plane, half space of T_pM
also stratified atlas so make planes of constant t straight
relativistic

half cone $\forall p, q \in M$, half-cone $\subseteq T_pM$

This definition of spacetime
Question
I see how the cone structure arises from the new metric. I don't understand however, how the T, the time orientation, comes in

Answer
$(M, \mathcal{O}, \mathcal{A}, g)$ $g \overset{(}{\leftarrow} + - - -)$
requiring $g(X, X) > 0$, select cones
T chooses which cone

This definition of spacetime has been made to enable the following physical postulates:

(P1) The worldline γ of a massive particle satisfies
  (i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$
  (ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$

(P2) Worldlines of massless particles satisfy
  (i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0$
  (ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$
  picture: spacetime:

Answer (to a question) T is a smooth vector field, T determines future vs. past, "general relativity: we have such a time orientation; smoothness makes it less arbitrary than it seems" -FSchuller,
Claim: 9/10 of a metric are determined by the cone
spacetime determined by distribution, only one-tenth error

13.2. **Observers.** $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$

Definition 52. *An observer is a worldline γ with*

$$g(v_\gamma, v_\gamma) > 0$$
$$g(T, v_\gamma) > 0$$

together with a choice of basis

$$v_{\gamma,\gamma(\lambda)} \equiv e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$

of each $T_{\gamma(\lambda)}M$ where the observer worldline passes, if $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{ab}$

precise: observer = smooth curve in the frame bundle LM over M

13.2.1. *Two physical postulates.*

(P3) A **clock** carried by a specific observer (γ, e) will measure a **time**

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

between the two "events"

$\gamma(\lambda_0)$ "start the clock"

and

$\gamma(\lambda_1)$ "stop the clock"

Compare with Newtonian spacetime:

$$t(p) = 7$$

Thought bubble: proper time/eigentime τ

Application/Example.
$$M = \mathbb{R}^4$$
$$\mathcal{O} = \mathcal{O}_{\text{st}}$$
$$\mathcal{A} \ni (\mathbb{R}^4, \text{id}_{\mathbb{R}^4})$$
$$g : g_{(x)ij} = \eta_{ij} \quad ; \quad T^i_{(x)} = (1,0,0,0)^i$$
$$\Longrightarrow \Gamma^i_{(x)\ jk} = 0 \text{ everywhere}$$

$\Longrightarrow (M, \mathcal{O}, \mathcal{A}^{\uparrow}, g, T, \nabla)$ Riemm $= 0$
$\Longrightarrow$ spacetime is flat
This situation is called special relativity.
Consider two observers:

$$\gamma : (0,1) \to M$$
$$\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i$$
$$\delta : (0,1) \to M$$
$$\alpha \in (0,1) : \delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases}$$

let's calculate:

$$\tau_\gamma := \int_0^1 \sqrt{g_{(x)ij}\dot{\gamma}^i_{(x)}\dot{\gamma}^j_{(x)}} = \int_0^1 d\lambda 1 = 1$$
$$\tau_\delta := \int_0^{1/2} d\lambda \sqrt{1-\alpha^2} + \int_{1/2}^1 \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1-\alpha^2} = \sqrt{1-\alpha^2}$$

Note: piecewise integration
Taking the clock postulate (P3) seriously, one better come up with a realistic clock design that supports the postulate.
idea.
2 little mirrors

(P4) Postulate
Let (γ, e) be an observer, and
δ be a *massive* particle worldline that is parametrized s.t. $g(v_\gamma, v_\gamma) = 1$ (for parametrization/normalization convenience)
Suppose the observer and the particle *meet* somewhere (in spacetime)

$$\delta(\tau_2) = p = \gamma(\tau_1)$$

This observer measures the 3-velocity (spatial velocity) of this particle as

$$v_\delta : \epsilon^\alpha(v_{\delta,\delta(\tau_2)})e_\alpha \qquad \alpha = 1,2,3 \tag{8}$$

where $\epsilon^0, \boxed{\epsilon^1, \epsilon^2, \epsilon^3}$ is the unique dual basis of $e_0, \boxed{e_1, e_2, e_3}$

EY:20150407
There might be a major correction to Eq. (8) from the Tutorial 14 : Relativistic spacetime, matter, and Gravitation, see the second exercise, Exercise 2, third question:

$$v := \frac{\epsilon^\alpha(v_\delta)}{\epsilon^0(v_\delta)} e_\alpha \tag{9}$$

Consequence: An observer (γ, e) will extract quantities measurable in his laboratory from objective spacetime quantities always like that.
Ex: F Faraday $(0,2)$-tensor of electromagnetism:

$$F(e_a, e_b) = F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

observer frame e_a, e_b
$E_\alpha := F(e_0, e_\alpha)$
$B^\gamma := F(e_\alpha, e_\rho)\epsilon^{\alpha\beta\gamma}$ where $\epsilon^{123} = +1$ totally antisymmetric

13.3. **Role of the Lorentz transformations.** Lorentz transformations emerge as follows:
Let (γ, e) and $(\widetilde{\gamma}, \widetilde{e})$ be observers with $\gamma(\tau_1) = \widetilde{\gamma}(\tau_2)$
(for simplicity $\gamma(0) = \widetilde{\gamma}(0)$)
Now

$$\begin{aligned} & e_0, \dots, e_1 && \text{at } \tau = 0 \\ \text{and } & \widetilde{e}_0, \dots, \widetilde{e}_1 && \text{at } \tau = 0 \end{aligned}$$

both bases for the same $T_{\gamma(0)}M$
Thus: $\widetilde{e}_a = \Lambda^b{}_a e_b \qquad \Lambda \in GL(4)$
Now:

$$\begin{aligned} \eta_{ab} = g(\widetilde{e}_a, \widetilde{e}_b) &= g(\Lambda^m{}_a e_m, \Lambda^n{}_b e_n) = \\ &= \Lambda^m{}_a \Lambda^n{}_b \underbrace{g(e_m, e_n)}_{\eta_{mn}} \end{aligned}$$

i.e. $\Lambda \in O(1,3)$
Result: Lorentz transformations relate the *frames* of *any two observers* at the same point.
"$\widetilde{x}^\mu - \Lambda^\mu{}_\nu x^\nu$" is utter nonsense

Tutorial. I didn't see a tutorial video for this lecture, but I saw that the Tutorial sheet number 14 had the relevant topics. Go there.

14. Lecture 14: Matter

two types of matter
point matter
field matter
point matter
massive point particle
more of a phenomenological importance
field matter
electromagnetic field
more fundamental from the GR point of view
both classical matter types

14.1. **Point matter.** Our postulates (P1) and (P2) already constrain the possible particle worldlines.
But what is their precise law of motion, possibly in the presence of "forces",

(a) without external forces

$$S_{\text{massive}}[\gamma] := m \int d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

with:

$$g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$$

dynamical law Euler-Lagrange equation

similarly

$$\begin{aligned} & S_{\text{massless}}[\gamma, \mu] = \int d\lambda \mu g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) \\ & \delta_\mu \qquad g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0 \\ & \delta_\gamma \qquad \text{e.o.m.} \end{aligned}$$

Reason for describing equations of motion by actions is that composite systems have an action that is the sum of the actions of the parts of that system, possibly including "interaction terms."
Example.

$$S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$$

(b) presence of external forces
or rather presence of fields to which a particle "couples"
Example

$$S[\gamma; A] = \int d\lambda m \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})} + qA(v_{\gamma,\gamma(\lambda)})$$

where A is a **covector field** on M. A fixed (e.g. the electromagnetic potential)
Consider Euler-Lagrange eqns. $L_{\text{int}} = qA_{(x)}\dot{\gamma}^m_{(x)}$

$$m(\nabla_{v_\gamma} v_\gamma)_a + \underbrace{\dot{\left(\frac{\partial L_{\text{int}}}{\partial \cdot^{m}_{(x)}}\right)}\gamma - \frac{\partial L_{\text{int}}}{\partial \gamma^m_{(x)}}}_{*} = 0 \Longrightarrow \boxed{m(\nabla_{v_\gamma} v_\gamma)^a = \underbrace{-qF^a{}_m \dot{\gamma}^m}_{\text{Lorentz force on a charged particle in an electromagnetic field}}}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{\gamma}^a} = qA_{(x)a}, \qquad \dot{\left(\frac{\partial L}{\partial \cdot^m}\right)}\gamma &= q \cdot \frac{\partial}{\partial x^m}(A_{(x)m}) \cdot \dot{\gamma}^m_{(x)} \\ \frac{\partial L}{\partial \gamma^a} &= q \cdot \frac{\partial}{\partial x^a}(A_{(x)m})\dot{\gamma}^m \\ * = q\left(\frac{\partial A_a}{\partial x^m} - \frac{\partial A_m}{\partial x^a}\right)\dot{\gamma}^m_{(x)} &\quad = q \cdot F_{(x)am}\dot{\gamma}^m_{(x)} \end{aligned}$$

$F \leftarrow$ Faraday

$$S[\gamma] = \int (m\sqrt{g(v_\gamma, v_\gamma)} + qA(v_\gamma))d\lambda$$

14.2. **Field matter.**

Definition 53. *Classical (non-quantum) field matter is any tensor field on spacetime where equations of motion derive from an action.*

Example:

$$S_{\text{Maxwell}}[A] = \frac{1}{4}\int_M d^4x \sqrt{-g} F_{ab} F_{cd} g^{ac} g^{bd}$$

A $(0,1)$-tensor field
$=$ thought cloud: for simplicity one chart covers all of M
$-$ for $\sqrt{-g}$ $(+---)$

$F_{ab} := 2\partial_{[a}A_{b]} = 2(\nabla_{[a}A)_{b]}$
Euler-Lagrange equations for fields

$$0 = \frac{\partial \mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s}\left(\frac{\partial \mathcal{L}}{\partial \partial_s A_m}\right) + \frac{\partial}{\partial x^s}\frac{\partial}{\partial x^t}\frac{\partial^2 \mathcal{L}}{\partial \partial_t \partial_s A_m}$$

Example ...

$$(\nabla_{\frac{\partial}{\partial x^m}} F)^{ma} = j^a$$

inhomogeneous Maxwell
thought bubble $j = qv_\gamma$

$$\partial_{[a} F_{b]} - ()$$

homogeneous Maxwell
Other example well-liked by textbooks

$$S_{\text{Klein-Gordon}}[\phi] := \int_M d^4x \sqrt{-g}[g^{ab}(\partial_a \phi)(\partial_b \phi) - m^2\phi^2]$$

ϕ $(0,0)$-tensor field

14.3. **Energy-momentum tensor of matter fields.** At some point, we want to write down an action for the metric tensor field itself.

But then, this action $S_{\text{grav}}[g]$ will be added to any $S_{\text{matter}}[A, \phi, \dots]$ in order to describe the total system.

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]$$

$$\delta A \quad :\Longrightarrow \text{ Maxwell's equations}$$

$$\delta g_{ab} \quad : \boxed{\frac{1}{16\pi G} G^{ab}} + (-2T^{ab}) = 0$$

G Newton's constant

$$G^{ab} = 8\pi G_N T^{ab}$$

Definition 54. *$S_{matter}[\Phi, g]$ is a matter action, the* ***so-called energy-momentum tensor*** *is*

$$T^{ab} := \frac{-2}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_{matter}}{\partial g_{ab}} - \partial_s \frac{\partial \mathcal{L}_{matter}}{\partial \partial_s g_{ab}} + \dots \right)$$

$-$ of $\frac{-2}{\sqrt{g}}$ is Schrödinger minus (EY : 20150408 F.Schuller's joke? but wise)
choose all sign conventions s.t.

$$T(\epsilon^0, \epsilon^0) > 0$$

Example: For S_{Maxwell}:

$$T_{ab} = F_{am} F_{bn} g^{mn} - \frac{1}{4} F_{mn} F^{mn} g_{ab}$$

$T_{ab} \equiv T_{\text{Maxwell}ab}$

$$T(e_0, e_0) = \underline{E}^2 + \underline{B}^2$$
$$T(e_0, e_\alpha) = (E \times B)_\alpha$$

Fact: One often does not specify the fundamental action for some matter, but one is rather satisfied to assume certain properties / forms of

$$T_{ab}$$

Example Cosmology: (homogeneous & isotropic)
perfect fluid

of pressure p and density ρ modelled by

$$T^{ab} = (\rho + p) u^a u^b - p g^{ab}$$

radiative fluid

What is a fluid of photons:

$$T^{ab}_{\text{Maxwell}} g_{ab} = 0$$

observe: $T^{ab}_{\text{p.f.}} g_{ab} \overset{!}{=} 0$

$$= (\rho + p) u^a u^b g_{ab} - p \underbrace{g^{ab} g_{ab}}_{4}$$

$$\leftrightarrow \rho_p 04p = 0$$
$$\rho = 3p$$

$p = \frac{1}{3}\rho$

Reconvene at 3 pm? (EY : 20150409 I sent a Facebook (FB) message to the International Winter School on Gravity and Light: there was no missing video; it continues on Lecture 15 immediately)

Tutorial 14: Relativistic Spacetime, Matter and Gravitation. Exercise 2: Lorentz force law.

Question electromagnetic potential.

15. Lecture 15: Einstein gravity

Recall that in Newtonian spacetime, we were able to reformulate the Poisson law $\Delta\phi = 4\pi G_N \rho$ in terms of the Newtonian spacetime curvature as

$$R_{00} = 4\pi G_N \rho$$

R_{00} with respect to ∇_{Newton}
G_N = Newtonian gravitational constant
This prompted Einstein to postulate < 1915 that the relativistic field equations for the Lorentzian metric g of (relativistic) spacetime

$$R_{ab} = 8\pi G_N T_{ab}$$

However, this equation suffers from a problem
LHS: $(\nabla_a R)^{ab} \neq 0$
generically
RHS:

$$(\nabla_a T)^{ab} = 0$$

thought bubble: = formulated from an action
Einstein tried to argue this problem away.
Nevertheless, the equations cannot be upheld.

15.1. **Hilbert.** Hilbert was a specialist for variational principles.

To find the appropriate left hand side of the gravitational field equations, Hibert suggested to start from an action

$$S_{\text{Hilbert}}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}$$

thought bubble = "simplest action"
aim: varying this w.r.t. metric g_{ab} will result in some tensor

$$G^{ab} = 0$$

15.2. **Variation of S_{Hilbert}.**

$$0 \overset{!}{=} \underbrace{\delta}_{g_i} S_{\text{Hilbert}}[g] = \int_M [\underbrace{\delta\sqrt{-g}g^{ab}R_{ab}}_{1} + \underbrace{\sqrt{-g}\delta g^{ab}R_{ab}}_{2} + \underbrace{\sqrt{-g}g^{ab}\delta R_{ab}}_{3}]$$

$$\text{and } 1: \delta\sqrt{-g} = \frac{-(\det g) g^{mn}\delta g_{mn}}{2\sqrt{-g}} = \frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}$$

thought bubble

$$\delta \det(g) = \det(g) g^{mn}\delta g_{mn}$$

e.g. from

$$\det(g) = \exp \operatorname{tr} \ln g$$

ad 2: $g^{ab}g_{bc} = \delta^a_c$

$$\Longrightarrow (\delta g^{ab})g_{bc} + g^{ab}(\delta g_{bc}) = 0$$
$$\Longrightarrow \delta g^{ab} = -g^{am}g^{bn}\delta g_{mn}$$

ad 3:

$$\Delta R_{ab} \underbrace{=}_{\text{normal coords at point}} \delta\partial_b \Gamma^m_{\ am} - \delta\partial_m\Gamma^m_{\ ab} + \Gamma\Gamma - \Gamma\Gamma =$$
$$= \partial_b\delta\Gamma^m_{\ am} - \partial_m\delta\Gamma^m_{\ ab} =$$
$$= \nabla_b(\delta\Gamma)^m_{\ am} - \nabla_m(\delta\Gamma)^m_{\ ab}$$
$$\Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab} = \sqrt{-g}$$

"if you formulate the variation properly, you'll see the variation δ commute with ∂_b" EY : 20150408 I think one uses the integration at the bounds, integration by parts trick

$\Gamma^i_{(x)\ jk} - \widetilde{\Gamma}^i_{(x)\ jk}$ are the components of a $(1,2)$-tensor.

Notation: $(\nabla_b A)^i_{\ g} =: A^i_{\ j;b}$

$$\Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab}$$

$$\underbrace{=}_{\nabla g = 0} \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ am})_{;b} - \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ ab})_{;m} = \sqrt{-g}A^b_{\ ;b} - \sqrt{-g}B^m_{\ ,m}$$

Question: Why is the difference of coefficients a tensor?

Answer:

$$\Gamma^i_{(y)\ jk} = \frac{\partial y^i}{\partial x^m}\frac{\partial x^m}{\partial y^j}\frac{\partial x^q}{\partial y^k}\Gamma^m_{(x)\ \ ,nq} + \frac{\partial y^i}{\partial x^m}\frac{\partial^2 x^m}{\partial y^j\partial y^k}$$

Collecting terms, one obtains

$$0 \overset{!}{=} \delta S_{\text{Hilbert}} = \int_M [\frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}g^{ab}R_{ab} - \sqrt{-g}g^{am}g^{bn}\delta g_{mn}R_{ab} + \underbrace{(\sqrt{-g}A^a)_{,a}}_{\text{surface}} - \underbrace{(\sqrt{-g}B^b)_{,b}}_{\text{surface term}}]$$

$$= \int_M \sqrt{-g}\delta \underbrace{g_{mn}}_{\text{arbitrary variation}} [\frac{1}{2}g^{mn}R - R^{mn}] \Longrightarrow G^{mn} = R^{mn} - \frac{1}{2}g^{mn}R$$

Hence Hilbert, from this "mathematical" argument, concluded that one may take

$$\boxed{R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab}}$$

Einstein equations

$$S_{E-H}[g] = \int_M \sqrt{-g}R$$

15.3. **3. Solution of the $\nabla_a T^{ab} = 0$ issue.** One can show ($\to$ Tutorials) that the Einstein curvature

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$

satisfy the so-called contracted differential Bianchi identity

$$(\nabla_a G)^{ab} = 0$$

15.4. **Variants of the field equations.**

(a) a simple rewriting:

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab} = T_{ab}$$

$G_N = \frac{1}{8\pi}$

Contract on both sides g^{ab}

$$R_{ab} - \frac{1}{2}g_{ab}R = T_{ab} || g^{ab}$$
$$R - 2R = T := T_{ab}g^{ab}$$
$$\Longrightarrow R = -T$$
$$\Longrightarrow R_{ab} + \frac{1}{2}g_{ab}T = T_{ab}$$
$$\Longleftrightarrow R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}) =: \widehat{T}_{ab}$$
$$\boxed{R_{ab} = \widehat{T}_{ab}}$$

(b)

$$S_{E-H}[g] := \int_M \sqrt{-g}(R + 2\Lambda)$$

thought bubble: Λ cosmological constant

History:

1915: $\Lambda < 0$ (Einstein) in order to get a non-expanding universe

>1915: $\Lambda = 0$ Hubble

today $\Lambda > 0$ to account for an accelerated expansion

$\Lambda \neq 0$ can be interpreted as a contribution

$-\frac{1}{2}\Lambda g$ to the energy-momentum

"dark energy"

Question: surface terms scalar?

Answer: for a careful treatment of the surface terms which we discarded, see, e.g. E. Poisson, "A relativist's toolkit" C.U.P. "excellent book"

Question: What is a constant on a manifold?

Answer: $\int \sqrt{-g}\Lambda = \Lambda \int \sqrt{-g}1$

[back to dark energy]

[Weinberg, QCD, calculated]

idea: 1 could arise as the vacuum energy of the standard model fields

$\Lambda_{\text{calculated}} = 10^{120} \times \Lambda_{\text{obs}}$

"worst prediction of physics"

Tutorials: check that

- Schwarzscheld metric (1916)
- FRW metric
- pp-wave metric
- Reisner-Nordstrom

$\implies$ are solutions to Einstein's equations

in high school
$m\ddot{x} + m\omega^2 x^2 = 0$
$x(t) = \cos(\omega t)$
ET: [elementary tutorials]
study motion of particles & observers in Schwarzscheld S.T.
Satellite: Marcus C. Werner
Gravitational lensing
odd number of pictures Morse theory (EY:20150408 Morse Theory !!!)
Domenico Giulini
Hamiltonian form Canonical Formulations
Key to Quantum Gravity

Tutorial 13 Schwarzschild Spacetime

EY : 20150408 I'm not sure which tutorial follows which lecture at this point.

The tutorial video is excellent itself. Here, I want to encourage the use of CAS to do calculations. There are many out there. Again, I'm partial to the Sage Manifolds package for Sage Math which are both open-source and based on Python. I'll use that here.

Exercise 1. Geodesics in a Schwarzschild spacetime

Question Write down the Lagrangian.

Load "Schwarzschild.sage" in Sage Math, which will always be available freely here https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/Schwarzschild.sage:

```
sage: load("Schwarzschild.sage")
4-dimensional manifold 'M'
open subset 'U_sph' of the 4-dimensional manifold 'M'
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
```

and so on.

Look at the code and I had defined the Lagrangian to be

```
L
```

. To get out the coefficients of L of the components of the tangent vectors to the curve, i.e. t', r', θ', ϕ', denoted

```
tp,rp,thp,php
```

in my .sage file, do the following:

```
sage: L.expr().coefficients(tp)[1][0].factor().full_simplify()
(2*G_N*M_0 - r)/r
sage: L.expr().coefficients(rp)[1][0].factor().full_simplify()
-r/(2*G_N*M_0 - r)
sage: L.expr().coefficients(php)[1][0].factor().full_simplify()
r^2
sage: L.expr().coefficients(thp)[1][0].factor().full_simplify()
r^2*sin(th)^2
```

Question There are 4 Euler-Lagrange equations for this Lagrangian. Derive the one with respect to the function $t(\lambda)$!.

```
sage: L.expr().diff(t)
0
```


This confirms that $\frac{\partial L}{\partial t} = 0$

For $\frac{d}{d\lambda}\frac{\partial L}{\partial t'}$, then one needs to consider this particular workaround for Sage Math (computer technicality). One takes derivatives with respect to declared variables (declared with var) and then substitute in functions that are dependent upon λ, and then take the derivative with respect to the parameter λ. This does that:

```
sage: L.expr().diff( thp ).factor().subs( r == gamma1 ).subs( thp == gamma3.diff( tau ) ).subs( th == gamma3 ).diff(tau)\
....: .factor()
2*(2*cos(gamma3(tau))*gamma1(tau)*D[0](gamma3)(tau)^2 + 2*sin(gamma3(tau))*D[0](gamma1)(tau)*D[0](gamma3)(tau)
+ gamma1(tau)*sin(gamma3(tau))*D[0, 0](gamma3)(tau))*gamma1(tau)*sin(gamma3(tau))
```

Question Show that the Lie derivative of g with respect to the vector fields $K_t := \frac{\partial}{\partial t}$.

The first line defines the vector field by accessing the frame defined on a chart with spherical coordinates and getting the time vector. The second line is the Lie derivative of g with respect to this vector field.

```
sage: K_t = espher[0]
sage: g.lie_der(K_t).display() # 0, as desired
0
```

EY : 20150410 My question is this: $\forall X \in \Gamma(TM)$ i.e. X is a vector field on M, or, specifically, a section of the tangent bundle, then does

$$\mathcal{L}_X g = 0$$

instantly mean that X is a symmetry for (M, g)? $\mathcal{L}_X g$ is interpreted geometrically as how g changes along the flow generated by X, and if it equals 0, then g doesn't change.

16.

17.

18. Canonical Formulation of GR I

Dynamical and Hailtonian formulation of General Relativity.
Purpose

(1) formulate and solve initial-value problems
(2) integrate Einstein's Equations by numerical codes
(3) characterize degrees of freedom
(4) characterize isolated systems, associated symmetry groups and conserved quantities, like Energy/Mass, Momenta (linear and angular), Poincaré charges
(5) starting point for "canonical quantization" program.

How. We will rewrite Einstein's Eq. in form of a *constrained Hamiltonian system.*

$$\underbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}_{G_{\mu\nu}} + \underbrace{\Lambda}_{\text{kosm.}const.} g_{\mu\nu} = \underbrace{k}_{\frac{8\pi G}{c^4}} T_{\mu\nu}$$

$(-+++)$

$$T^{\mu\nu} = \begin{pmatrix} W & \frac{1}{c}S^m \\ cg^m & \mathbf{t}^{mn} \end{pmatrix}$$

W = Energy density (1 component)
g^m = Momentum density, (3 components)
S^m = Energy current-density (3 components)
$\mathbf{t}^{mn}$ = Momentum current-density (6 components)

$$T^{\mu\nu} = T^{\nu\mu} \Longrightarrow S^m = c^2 g^m$$

10 independent komp. (components)

Phys. dim. $[T^{\mu\nu}] = \frac{J}{m^3}$

$[G^{\mu\nu}] = \frac{1}{m^2}$

$$k = \frac{\text{curvature}}{\text{Energy} \cdot \text{density}}$$

$$[k] = \frac{1}{m^2} / \frac{J}{m^3}, \quad {}^2k \quad \frac{\text{Curvature}}{\text{mass density}} = \left(\frac{1}{1.5\,\text{AU}} \right)^2 / \text{ Density of water}$$

$$= \left(\frac{1}{10\,km} \right)^2 / \text{ Nuclear density in core of neutron star } \simeq 5 \cdot 10^{17}\, kg/m^3$$

If "Ein" for Einstein Tensor, $G_{\mu\nu} = \text{Ein}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right)$

$$\text{Ein}(v,w) = \frac{1}{4}[\text{Ein}(v+w,v+w) - \text{Ein}(v-w,v-w)$$

$$\text{Ein}(w,w) = -g(w,w) \sum_{\perp w} \text{Sec}$$

where $\perp w$ is take the sum over any triple of mutually perp. 2-planes in $\perp w$

$\text{Sec}(\text{Span}\{v,w\}) = \frac{\text{Riem}(v,w,v,w)}{[g(v,w)]^2 - g(v,v)g(w,w)}$

"sectional curvature"

Identity: $\nabla_\mu G^{\mu\nu} = 0$ (follows from twice-contracted II. Bianchi Identity $\sum_{\lambda\mu\nu \text{ cycl}} \nabla_\lambda R_{\alpha\beta\mu\nu} = 0$)

$$\underbrace{\partial_0 G^{0\nu}}_{\text{contains at most 1st time der.}} + \underbrace{\partial_k G^{k\nu} + \Gamma G + \Gamma G \equiv 0}_{\text{contains at most 2nd. time derivatives}}$$

$\Longrightarrow$ 4 out of 10 Einstein Eq. do not evolve the fields but rather constrain the initial data. The space-space components (6 Eqns.) are the evolution Eqns.

10 Einstein Eq. - 4 constraints (underdetermined elliptic type)

\ - 6 evolution equations (undetermined hyperbolic type)

19.

20.

21.

22. Lecture 22: Black Holes

Only depends on Lectures 1-15, so does lecture on "Wednesday"

Schwarzschild solution also vacuum solution (from tutorial EY : oh no, must do tutorial)

Study the Schwarzschild as a vacuum solution of the Einstein equation:

$m = G_N M$ where M is the "mass"

$$g = \left(1 - \frac{2m}{r} \right) dt \otimes dt - \frac{1}{1 - \frac{2m}{r}} dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi$$

in the so-called Schwarzschild coordinates

t	r	θ	φ
$(-\infty, \infty)$	$(0, \infty)$	$(0, \pi)$	$(0, 2\pi)$

What staring at this metric for a while, two questions naturally pose themselves:

(i) What exactly happens $r = 2m$?

t	r	θ	φ
$(-\infty, \infty)$	$(0, 2m) \cup (2m, \infty)$	$(0, \pi)$	$(0, 2\pi)$

(ii) Is there anything (in the real world) beyond $t \to -\infty$?
$t \to +\infty$

idea: Map of Linz, blown up

Insight into these two issues is afforded by stopping to stare.

Look at *geodesic* of g, instead.

22.1. Radial null geodesics.

null - $g(v_\gamma, v_\gamma) = 0$

Consider null geodesic in "Schd"

$$S[\gamma] = \int d\lambda \left[\left(1 - \frac{2m}{r} \right) \dot{t}^2 - \left(1 - \frac{2m}{r} \right)^{-1} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) \right]$$

with $[\dots] = 0$

and one has, in particular, the t-eqn. of motion:

$$\left(\left(1 - \frac{2m}{r} \right) \dot{t} \right)^{\cdot} = 0$$

$\Longrightarrow$

$$\boxed{\left(1 - \frac{2m}{r} \right) \dot{t} = k} = \text{ const.}$$

Consider radial null geodesics

$\theta \overset{!}{=}$ const. $\qquad \varphi =$ const.

From $\square$ and $\square$

$$\Longrightarrow \dot{r}^2 = k^2 \leftrightarrow \dot{r} = \pm k$$

$$\Longrightarrow r(\lambda) = \pm k \cdot \lambda$$

Hence, we may consider

$$\widetilde{t}(r) := t(\pm k \lambda)$$

Case A: $\oplus$

$$\frac{d\widetilde{t}}{dr} = \frac{\dot{\widetilde{t}}}{\dot{r}} = \frac{k}{\left(1 - \frac{2m}{r} \right) k} = \frac{r}{r - 2m}$$

$$\Longrightarrow \widetilde{t}_+(r) = r + 2m \ln |r - 2m|$$

(**outgoing** null geodesics)

Case b. $\pm$ (Circle around $-$, consider $-$):

$$\widetilde{t}_-(r) = -r - 2m \ln |r - 2m|$$

(**ingoing** null geodesics)

Picture

22.2. **Eddington-Finkelstein.** Brilliantly simple idea:

change (on the domain of the Schwarzschild coordinates) to different coordinates, s.t. in those new coordinates,
ingoing null geodesics appear as straight lines, of slope -1

This is achieved by

$$\bar{t}(t,r,\theta,\varphi) := t + 2m \ln |r-2m|$$

<u>Recall</u>: ingoing null geodesic has

$$\widetilde{t}(r) = -(r + 2m\ln|r-2m|) \qquad (Schdcoords)$$

$$\Longleftrightarrow \bar{t} - 2m\ln|r-2m| = -r - 2m\ln|r-2m| + \text{ const.}$$

$$\therefore \bar{t} = -r + \text{ const.}$$

(Picture)

outgoing null geodesics

$$\bar{t} = r + 4m\ln|r-2m| + \text{ const.}$$

Consider the new chart (V,g) while (U,x) was the Schd chart.

$$\underbrace{U}_{\text{Schd}} \bigcup \{ \text{ horizon } \} = V$$

"chart image of the horizon"

Now calculate the *Schd metric* g w.r.t. Eddington-Finkelstein coords.

$$\begin{aligned} \bar{t}(t,r,\theta,\varphi) &= t + 2m\ln|r-2m| \\ \bar{r}(t,r,\theta,\varphi) &= r \\ \bar{\theta}(t,r,\theta,\varphi) &= \theta \\ \bar{\varphi}(t,r,\theta,\varphi) &= \varphi \end{aligned}$$

EY : 20150422 I would suggest that after seeing this, one would calculate the metric by your favorite CAS. I like the Sage Manifolds package for Sage Math.

Schwarzschild_BH.sage on github

Schwarzschild_BH.sage on Patreon

Schwarzschild_BH.sage on Google Drive

```
sage: load(‘‘Schwarzschild_BH.sage’’)
4-dimensional manifold ’M’
  expr = expr.simplify_radical()
Levi-Civita connection ’nabla_g’ associated with the Lorentzian metric ’g’ on the 4-dimensional manifold ’M’
Launched png viewer for Graphics object consisting of 4 graphics primitives
```

Then calculate the Schwarzschild metric g but in Eddington-Finkelstein coordinates. Keep in mind to calculate the set of coordinates that uses $\bar{t}$, not $\widetilde{t}$:

```
sage: gI.display()
gI = (2*m - r)/r dt*dt - r/(2*m - r) dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: gI.display( X_EF_I_null.frame())
gI = (2*m - r)/r dtbar*dtbar + 2*m/r dtbar*dr + 2*m/r dr*dtbar + (2*m + r)/r dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
```


Part 2. Special Relativity

Physics Stackexchange question (from Wesley) and answer (from David Bar Moshe had an excellent question and explanation for special relativity.

http://physics.stackexchange.com/questions/12221/what-does-a-frame-of-reference-mean-in-terms-of-manifolds

Recall that for charts $\begin{aligned} (U,x) &\in \mathcal{A}_M \\ (U',x') &\in \mathcal{A}_M \end{aligned}$, of *smooth* atlas $\mathcal{A}_M$ of smooth M. Now $\begin{aligned} x &: U \subset M \to \mathbb{R}^n \\ x' &: U' \subset M \to \mathbb{R}^n \end{aligned}$

If $U \cap U' \neq \emptyset$, by def. of smooth atlas $\mathcal{A}_M$, $\begin{aligned} x' \circ x &: \mathbb{R}^n \to \mathbb{R}^n \\ x \circ x' &: \mathbb{R}^n \to \mathbb{R}^n \end{aligned}$ are diffeomorphisms.

For notation,

$A, B \in \Omega^1(M)$

$A \wedge B \in \Omega^2(M)$

$A \wedge B = A_i dx^i \wedge B_j dx^j = A_i B_j dx^i \wedge dx^j$

$A, B \in T_pM$ or TM

$A \wedge B \in \wedge^2(TM)$, $A \wedge B = A^i e_i \wedge B^j e_j = A^i B^j e_i \wedge e_j$

$(A \times B)_i = \epsilon_{ijk} A^j B^k$ or $A \times B = A^j B^k \epsilon_{ijk} e_i$

if $\dim M = 3$, $*(A \wedge B) = \frac{\sqrt{g}}{(3-2)!} A_i B_j g^{il} g^{jm} \epsilon_{lmn} dx^n = \sqrt{g} g^{il} A_i g^{jm} B_j \epsilon_{lmn} dx^n$

Part 3. Euclidean space

23. $\mathbb{R}^2$

Consider $\mathbb{R}^2$ as a smooth manifold.

Clearly, $\mathbb{R}^2$ is covered by 1 chart, with so-called *Cartesian* coordinates (that also happen to be global coordinates, in this particular case): $(\mathbb{R}^2, \mathcal{O}_{\text{std}}, \mathcal{A} = \{= \{(\mathbb{R}^2, \varphi_{\text{cart}})\})$, with the chart (which is a homeomorphism, by definition)

$$\begin{aligned} \varphi_{\text{cart}} &: \mathbb{R}^2 \to \mathbb{R}^2 \\ \varphi_{\text{cart}} &: p \equiv (x,y) \mapsto (x,y) \end{aligned}$$

Clearly, $\varphi_{\text{cart}} = 1_{\mathbb{R}^2}$ is a homeomorphism; in fact, it's a smooth diffeomorphism.

$\mathcal{O}_{\text{std}}$ denotes the so-called "standard" topology, that open sets consists of Euclidean d-balls. I will assume this topology for this discussion of $\mathbb{R}^d$'s.

Clearly, the above discussion generalizes to $\mathbb{R}^d$.

Consider the so-called *polar* coordinates

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned} \tag{10}$$

where $r \geq 0$ and $\varphi \in \mathbb{R}$. We want to construction a homeomorphism $\varphi_{\text{pol}}^{-1}$ from an open subset $V \subseteq \mathbb{R}^2$ to $\mathbb{R}^2$ according to these polar coordinates:

$$\begin{aligned} \varphi_{\text{pol}}^{-1} &: V \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \\ \varphi_{\text{pol}}^{-1} &: (r,\varphi) \mapsto (x,y) \end{aligned}$$

Immediately, from this choice of polar coordinates, we see problems with defining a homeomorphism, which is a bijection by definition. For instance, let $(\varphi_{\text{pol}}^{-1})^{-1} \equiv \varphi_{\text{pol}}$ (defining this highly suggestive notation), and so

$$\begin{aligned} \varphi_{\text{pol}} &: \mathbb{R}^2 \to \mathbb{R}^2 \\ \varphi_{\text{pol}} &: (x,y) \mapsto (r,\varphi) \end{aligned}$$

The single point, the origin of $\mathbb{R}^2$, gets mapped to many possible values! In other words,

$$\varphi_{\text{pol}}(0,0) = \{(r,\varphi) | r = 0 \text{ or } \varphi \in \mathbb{R}\}$$

Clearly this must be the case for any kind of coordinates involving cos's, sin's, etc. (i.e. transcendental functions). So the choice of these kinds of coordinates necessitate giving up on the possibility of a single chart covering the entire manifold of the Euclidean space.

Consider

$$\begin{aligned} V &:= \{(r,\varphi)|r>0, \varphi \in (0,2\pi)\} \subset \mathbb{R}^2 \\ [0,\infty)\times[0,2\pi) - V &= \{(r,\varphi)|r=0 \text{ or } \varphi = 0\} \end{aligned}$$

Now $\varphi_{\text{pol}}^{-1}([0,\infty)\times[0,2\pi)) = \mathbb{R}^2$, i.e. $\varphi^{-1}_{\text{pol}}$ maps $[0,\infty)\times[0,2\pi)$ *onto* $\mathbb{R}^2$ "once."

Now consider if $\varphi = 0$. Then $\begin{aligned} x &= r \\ y &= 0 \end{aligned}$ for $r > 0$. So $x > 0$ and $y = 0$. If $r = 0$, then $(x,y) = (0,0)$. Thus

$$\varphi^{-1}_{\text{pol}}([0,\infty)\times[0,2\pi) - V) = \{(x,y)|x \geq 0 \text{ and } y = 0\}$$

(If you have trouble seeing this, think of the union of the cases when $r = 0$ or when $\varphi = 0$). Thus

$$\varphi^{-1}_{\text{pol}}(V) = \{(x,y)|x<0 \text{ or } y \neq 0\}$$

Thus, in summary, we can include this chart into atlas $\mathcal{A}$:

$$\begin{gathered} (U,\varphi_{\text{pol}}) \in \mathcal{A} \\ \begin{aligned} U &= \{(x,y)|x<0 \text{ or } y\neq 0\} \\ \varphi_{\text{pol}}(U) &= \{(r,\varphi)|r>0, \varphi\in(0,2\pi)\} \end{aligned} \quad \begin{aligned} &\varphi_{\text{pol}} : U \to \mathbb{R}^2 \\ &\varphi_{\text{pol}}(x,y) = (\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right) \\ &\varphi^{-1}_{\text{pol}} : \varphi_{\text{pol}}(U) \to \mathbb{R}^2 \\ &\varphi^{-1}_{\text{pol}}(r,\varphi) = (r\cos\varphi, r\sin\varphi) = (x,y) \end{aligned} \end{gathered} \tag{11}$$

If someone really wanted to, one can try to cover $\mathbb{R}^2$ with another polar coordinate chart, but I would argue that we already have the cartesian coordinate chart that singlely covers $\mathbb{R}^2$. Also note that the transition map between polar coordinate charts and the cartesian coordinate chart is easy, since the cartesian chart is the identity.

24. $\mathbb{R}^3$

Consider the spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{12}$$

We seek to construct a chart and open subset/domain for these spherical coordinates. So consider, in parallel, following closely the discussion above for $\mathbb{R}^2$,

$$\begin{aligned} &\varphi^{-1}_{\text{sph}} : V \subset \mathbb{R}^3 \to \mathbb{R}^3 \\ &\varphi^{-1}_{\text{sph}} : (r,\theta,\varphi) \mapsto (x,y,z) \\ &V := \{(r,\theta,\varphi) \in \mathbb{R}^3 | r>0, \theta\in(0,\pi), \varphi \in (0,2\pi)\} \\ &\varphi^{-1}_{\text{sph}}([0,\infty)\times[0,\pi)\times[0,2\pi)) = \mathbb{R}^3 (\text{ once over }) \\ [0,\infty)\times[0,\pi)\times[0,2\pi) - V &= \{(r,\theta,\varphi)|r=0 \text{ or } \theta = 0 \text{ or } \varphi = 0\} \end{aligned}$$

Now if $\varphi = 0$, $\begin{aligned} x &= r\sin\theta \\ y &= 0 \\ z &= r\cos\theta \end{aligned}$, and if $\theta = 0$, $\begin{aligned} x &= 0 \\ y &= 0 \\ z &= r \end{aligned}$, for $r > 0$ and $\theta \in (0,\pi)$.

Thus,

$$\begin{aligned} \varphi^{-1}_{\text{sph}}([0,\infty)\times[0,\pi)\times[0,2\pi) - V) &= \{(x,y,z)|y=0 \text{ and } x \geq 0\} \\ \varphi^{-1}_{\text{sph}}(V) &= \{(x,y,z)|y\neq 0 \text{ or } x<0\} \end{aligned}$$

So, in summary,

$$\begin{gathered} (U,\varphi_{\text{sph}}) \in \mathcal{A} \text{ with} \\ \begin{aligned} U &= \{(x,y,z)|x<0 \text{ or } y\neq 0\} \\ \varphi_{\text{sph}}(U) &= \{(r,\theta,\varphi)|r>0, \theta\in(0,\pi), \varphi\in(0,2\pi)\} \end{aligned} \quad \begin{aligned} &\varphi_{\text{sph}} : U \to \mathbb{R}^3 \\ &\varphi_{\text{sph}}(x,y,z) = (\sqrt{x^2+y^2+z^2}, \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right), \arctan\left(\frac{y}{x}\right)) \\ &\varphi^{-1}_{\text{sph}} : \varphi_{\text{sph}}(U) \to \mathbb{R}^3 \\ &\varphi^{-1}_{\text{sph}}(r,\theta,\varphi) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \end{aligned} \end{gathered} \tag{13}$$

Consider another chart for *cylindrical* coordinates for $\mathbb{R}^3$:

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned} \tag{14}$$

Following a similar procedure as above, let

$$\begin{aligned} &\varphi^{-1}_{\text{cyl}} : V \subseteq \mathbb{R}^3 \to \mathbb{R}^3 \\ &\varphi^{-1}_{\text{cyl}} : (r,\varphi,z) \mapsto (x,y,z) = (r\cos\varphi, r\sin\varphi, z) \\ &\varphi^{-1}_{\text{cyl}}([0,\infty)\times[0,2\pi)\times\mathbb{R}) = \mathbb{R}^3 \qquad (\text{once over}) \\ &V := \{(r,\varphi,z)\in\mathbb{R}^3 | r>0,\ \varphi\in(0,2\pi),\ z\in\mathbb{R}\} \end{aligned}$$

and so

$$[0,\infty)\times[0,\pi)\times\mathbb{R} - V = \{(r,\varphi,z)|r=0 \text{ or } \varphi = 0\}$$

if $\varphi = 0$, $\begin{aligned} x &= r \\ y &= 0 \\ z &= z \end{aligned}$, for $r > 0$.

Thus

$$\varphi^{-1}_{\text{cyl}}([0,\infty)\times[0,2\pi)\times\mathbb{R} - V) = \{(x,y,z)|y=0 \text{ and } x\geq 0\}$$

and so

$$\begin{gathered} (U,\varphi_{\text{cyl}}) \in \mathcal{A} \text{ with} \\ U = \{(x,y,z)|y\neq 0 \text{ or } x<0\} \\ \varphi_{\text{cyl}}(x,y,z) = (\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right), z) \end{gathered} \tag{15}$$

25. $\mathbb{R}^n$

For "spherical coordinates" of $\mathbb{R}^n$,

$$
\begin{aligned}
x_1 &= r\sin(\theta_1)\dots\sin(\theta_{n-2})\cos(\phi) \\
x_2 &= r\sin(\theta_1)\dots\sin(\theta_{n-2})\sin(\phi) \\
x_3 &= r\sin(\theta_1)\dots\sin(\theta_{n-3})\cos(\theta_{n-2}) \\
x_4 &= r\sin(\theta_1)\dots\sin(\theta_{n-4})\cos(\theta_{n-3}) \\
&\vdots \\
x_{n-1} &= r\sin(\theta_1)\cos(\theta_2) \\
x_n &= r\cos(\theta_1)
\end{aligned} \tag{16}
$$

Consider $V = \{(r,\theta_1\dots\theta_{n-2},\phi) | r>0, \theta_1,\theta_2,\dots\theta_{n-2}\in(0,\pi), \phi\in(0,2\pi)\}$.

Now for $\theta_i = 0$, $i\in 1\dots n-2$, $x_1 = x_2 = \dots = x_{n-i} = 0$.

for $\phi = 0$, $x_2 = 0$, $x_1 > 0$, since $\theta_i\in(0,\pi)\quad\forall i\in 1\dots n-2$.

So let

$$V = \{(x_1\dots x_n) | x_2 = 0 \text{ and } x_1 \geq 0\}$$

and so

$$U = \{(x_1\dots x_n) | x_2 \neq 0 \text{ or } x_1 < 0\} \tag{17}$$

and we have chart (homeomorphism)

$$\varphi_{\text{sph}} : U \to \mathbb{R}^n$$

$$\varphi_{\text{sph}}(x_1\dots x_n) = (\sqrt{x_1^2+\dots+x_n^2},$$
$$\arctan\left(\frac{\sqrt{x_1^2+\dots+x_{n-1}^2}}{x_n}\right), \arctan\left(\frac{\sqrt{x_1^2+\dots+x_{n-2}^2}}{x_{n-1}}\right), \dots \arctan\left(\frac{\sqrt{x_1^2+x_2^2}}{x_3}\right), \arctan\left(\frac{x_2}{x_1}\right))$$

For "cylindrical coordinates" of $\mathbb{R}^n$, with coordinate x_n being a parameter of each "sphere",

$$
\begin{aligned}
x_1 &= r\sin(\theta_1)\dots\sin(\theta_{n-3})\cos(\phi) \\
x_2 &= r\sin(\theta_1)\dots\sin(\theta_{n-3})\sin(\phi) \\
x_3 &= r\sin(\theta_1)\dots\sin(\theta_{n-4})\cos(\theta_{n-3}) \\
x_4 &= r\sin(\theta_1)\dots\sin(\theta_{n-5})\cos(\theta_{n-4}) \\
&\vdots \\
x_{n-2} &= r\sin(\theta_1)\cos(\theta_2) \\
x_{n-1} &= r\cos(\theta_1) \\
x_n &= x_n
\end{aligned} \tag{18}
$$

If $\theta_i = 0$, $\quad i\in 1\dots n-3$, $x_1 = x_2 = \dots = x_{n-1-i} = 0$.

If $\phi = 0$, $x_2 = 0$.

For $V = \{(r,\theta_1\dots\theta_{n-3},\phi,z) | r\in(0,\infty),\ \theta_i\in(0,\pi)\,\forall i\in 1,\dots n-3,\ \phi\in(0,2\pi),\ z\in\mathbb{R}\}$,

$$\varphi_{\text{cyl}}^{-1}([0,\infty)\times[0,\pi)\dots\times[0,\pi)\times[0,2\pi)\times\mathbb{R} - V) = \{(x_1\dots x_n) | x_2 = 0 \text{ and } x_1\geq 0\}$$

For if $\phi = 0$, $x_2 = 0$, and $x_1(\varphi = 0) = r\sin(\theta_1)\dots\sin(\theta_{n-3}) > 0$ for $\theta_1\dots\theta_{n-3}\in(0,\pi)$.

Thus,

$$\varphi_{\text{cyl}}^{-1}(V) = \{(x_1\dots x_n) | x_2\neq 0 \text{ or } x_1 < 0\} =: U_{\text{cyl}} \tag{19}$$

and the chart is

$$\varphi_{\text{cyl}} : U \to \mathbb{R}^n$$

$$\varphi_{\text{cyl}}(x_1\dots x_n) = (\sqrt{x_1^2+\dots+x_{n-1}^2},$$
$$\arctan\left(\frac{\sqrt{x_1^2+\dots+x_{n-2}^2}}{x_{n-1}}\right), \arctan\left(\frac{\sqrt{x_1^2+\dots+x_{n-3}^2}}{x_{n-2}}\right), \dots \arctan\left(\frac{\sqrt{x_1^2+x_2^2}}{x_3}\right), \arctan\left(\frac{x_2}{x_1}\right), x_n)$$

25.0.1. *Sage Math, sagemanifolds, implementation.* As described above, with our chart constructions for spherical (or polar for 2-dimensions) and cylindrical coordinates, the Euclidean spaces $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^n$, in general, (for instance, say $n = 4$), can be implemented in Sage Math, using the `sagemanifolds` package. This implementation is given in Rn.sage of the github repository Gravite.

What `Rn.sage` provides are Python classes so that one can create automatically or *instantiate* Euclidean spaces as a manifold, and the "usual" atlas with charts for spherical and/or cylindrical coordinates. It was coded (you should be welcomed to take a look at the code directly) to follow precisely the logic of the above developments, e.g. the open subset U in Equations 11, 13, 15, 17, 19 is written into the code for the classes that represent the Euclidean spaces as such:

```
class R2(object):
      def __init__(self):
        self.M = Manifold(2,'R2',r'\mathbb{R}^2',start_index=1)
        self.cart_ch = self.M.chart('x y')
        self.U       = self.M.open_subset('U',coord_def={self.cart_ch: (self.cart_ch[1]<0, self.cart_ch[2]!=0)})

class R3(object):
      def __init__(self):
        self.M = Manifold(3,'R3',r'\mathbb{R}^3',start_index=1)
        self.cart_ch = self.M.chart('x y z')
        self.U = self.M.open_subset('U',coord_def={self.cart_ch: (self.cart_ch[1]<0, self.cart_ch[2]!=0)})

class Rn(object):
      def __init__(self,n):
        assert n>0
        if n == 2:
          print ``Use the class R2''
        elif n == 3:
          print ``Use the class R3''
        else:
          self.M = Manifold(n,'R'+str(n),r'\mathbb{R}^'+str(n),start_index=1)
          self.cart_ch = self.M.chart(r'' ``.join([r''x''+str(i) for i in range(1,n+1)]))
          xis = [self.cart_ch[i[0]] for i in self.M.index_generator(1)]
          self.U = self.M.open_subset('U',coord_def={self.cart_ch:(xis[0]<0,xis[1]!=0)})
```

so that the "spirit" in coding up `Rn.sage` is to follow the (abstract) math developed above closely and exactly, if possible.

Now, for instance, to implement

$$
\begin{aligned}
&(\mathbb{R}^2, \mathcal{A}_{\mathbb{R}^2}) \qquad & \mathcal{A}_{\mathbb{R}^2} &= \{(\mathbb{R}^2, \varphi_{\text{cart}}\equiv x^i), (U, \varphi_{\text{pol}}\equiv\varphi_{\text{sph}})\} \\
&(\mathbb{R}^3, \mathcal{A}_{\mathbb{R}^3}) & \mathcal{A}_{\mathbb{R}^3} &= \{(\mathbb{R}^3, x^i), (U, \varphi_{\text{sph}}), (U, \varphi_{\text{cyl}})\} \\
&(\mathbb{R}^n, \mathcal{A}_{\mathbb{R}^n}) & \mathcal{A}_{\mathbb{R}^n} &= \{(\mathbb{R}^n, x^i), (U, \varphi_{\text{sph}}), (U, \varphi_{\text{cyl}})\}
\end{aligned}
$$

one does, after loading the file, which is assumed to be in the working directory for Sage (simply `cd` or change directory into it) into Sage Math

```
sage: load(``Rn.sage'')
sage: R2eg = R2()
sage: R3eg = R3()
sage: R4 = Rn(4)
```

Indeed, the coordinate transformations in Equations 10, 12, 14, 16, 18 are implemented with this file `Rn.sage` as follows:

```
sage: R2eg.transit_sph_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
sage: R3eg.transit_sph_to_cart.display()
x = rh*cos(ph)*sin(th)
y = rh*sin(ph)*sin(th)
z = rh*cos(th)
sage: R3eg.transit_cyl_to_cart.display()
x = r*cos(phi)
y = r*sin(phi)
z = zc
sage: R4.transit_sph_to_cart.display()
x1 = rh*cos(ph)*sin(th1)*sin(th2)
x2 = rh*sin(ph)*sin(th1)*sin(th2)
x3 = rh*cos(th2)*sin(th1)
x4 = rh*cos(th1)
sage: R4.transit_cyl_to_cart.display()
x1 = r*cos(phi)*sin(the1)
x2 = r*sin(phi)*sin(the1)
x3 = r*cos(the1)
x4 = z
```

I would welcome readers at any point to open up `Rn.sage` and play with the code and to run it in Sage Math and to explore the methods associated with the classes; I use command `dir()`. I want to point out that the *Jacobian* for the coordinate changes can be obtained easily by the following:

```
sage: to_orthonormal2, e2, Jacobians2 = R2eg.make_orthon_frames(R2eg.sph_ch)
sage: Jacobians2[0].inverse()[:,R2eg.sph_ch]
[   cos(ph) -r*sin(ph)]
[   sin(ph)  r*cos(ph)]
sage: to_orthonormal3sph, e3sph, Jacobians3sph = R3eg.make_orthon_frames(R3eg.sph_ch)
sage: to_orthonormal3cyl, e3cyl, Jacobians3cyl = R3eg.make_orthon_frames(R3eg.cyl_ch)
sage: Jacobians3sph[0].inverse()[:,R3eg.sph_ch]
[    cos(ph)*sin(th)  rh*cos(ph)*cos(th) -rh*sin(ph)*sin(th)]
[    sin(ph)*sin(th)  rh*cos(th)*sin(ph)  rh*cos(ph)*sin(th)]
[            cos(th)         -rh*sin(th)                   0]
sage: Jacobians3cyl[0].inverse()[:,R3eg.cyl_ch]
[   cos(phi) -r*sin(phi)           0]
[   sin(phi)  r*cos(phi)           0]
[          0           0           1]
sage: to_orthonormal4sph, e4sph, Jacobians4sph = R4.make_orthon_frames(R4.sph_ch)
sage: to_orthonormal4cyl, e4cyl, Jacobians4cyl = R4.make_orthon_frames(R4.cyl_ch)
sage: Jacobians4sph[0].inverse()[:,R4.sph_ch]
[    cos(ph)*sin(th1)*sin(th2)  rh*cos(ph)*cos(th1)*sin(th2)  rh*cos(ph)*cos(th2)*sin(th1) -rh*sin(ph)*sin(th1)*sin(th2)]
[    sin(ph)*sin(th1)*sin(th2)  rh*cos(th1)*sin(ph)*sin(th2)  rh*cos(th2)*sin(ph)*sin(th1)  rh*cos(ph)*sin(th1)*sin(th2)]
[            cos(th2)*sin(th1)          rh*cos(th1)*cos(th2)         -rh*sin(th1)*sin(th2)                             0]
[                     cos(th1)                  -rh*sin(th1)                             0                             0]
sage: Jacobians4cyl[0].inverse()[:,R4.cyl_ch]
[   cos(phi)*sin(the1)  r*cos(phi)*cos(the1) -r*sin(phi)*sin(the1)                     0]
[   sin(phi)*sin(the1)  r*cos(the1)*sin(phi)  r*cos(phi)*sin(the1)                     0]
[            cos(the1)          -r*sin(the1)                     0                     0]
[                    0                     0                     0                     1]
```

As one can see, the Jacobian for the coordinate changes in Equations 10, 12, 14, 16, 18 was calculated (stop doing it by hand!).

25.0.2. *Euclidean metric g.* We'd want to now equip our Euclidean space as a smooth manifold with the usual Euclidean metric g, which exists (or is valid) everywhere (i.e. globally) on the manifold. Thus, we can think of the Euclidean space as a triple (with the standard topology $\mathcal{O}_{\text{std}}$ simply assumed and implicit):

$$(\mathbb{R}^2, \mathcal{A}_{\mathbb{R}^2}, g)$$
$$(\mathbb{R}^3, \mathcal{A}_{\mathbb{R}^3}, g)$$
$$(\mathbb{R}^n, \mathcal{A}_{\mathbb{R}^n}, g)$$

This is precisely implemented in Sage Math, using the `sagemanifolds` package, as follows:

```
sage: R2eg.equip_metric()
sage: R3eg.equip_metric()
sage: R4.equip_metric()
```

25.1. **Orthonormal coordinates, orthonormal (basis) vectors, and their problems.** Coordinate basis vectors, $\frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ that live on the chart domain U of chart $(U, \varphi) \in \mathcal{A}_M$ of a smooth manifold, need to be carefully distinguished from so-called orthonormal or orthogonal coordinates and their orthonormal basis vectors, $\mathbf{e}_i$. This becomes problematic because most of us have grown up, with physics and engineering textbooks, with orthonormal basis vectors $\mathbf{e}_i$ such as Cartesian basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, cylindrical basis vectors $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z$, and spherical basis vectors $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi$ without understanding how they are related to the metric on a manifold.

I'm not sure why textbooks I've read doesn't touch upon the difference between (local) coordinate basis vectors and orthonormal basis vectors more explicitly and lucidly. Frankel (2004) [4] touches upon this issue in Problem 2.1(2) of Chapter 2, calling the components of a vector field in these orthonormal basis vectors "physical" components. Bertschinger in his lectures notes for MIT Physics 8.962 Spring 1999 calls them "orthonormal non-coordinate bases" in Section 3.7 Example: Euclidean plane of "Introduction to Tensor Calculus for General Relativity" [2]. But from the manifold point of view, these non-coordinate bases seem to obfuscate what the components of a vector field should be.

We can agree upon a curve γ, $\begin{aligned} \gamma : \mathbb{R} &\to M \\ \gamma(\lambda) &\in M, \ \lambda \in \mathbb{R} \end{aligned}$, parametrized by $\lambda \in \mathbb{R}$ on the manifold, and it can be shown from a theorem in Lee (2012) [3] (from essentially existence and uniqueness theorems for ordinary differential equations), that there exists vector field $u \in \mathfrak{X}(M)$ such that $u(\lambda) = \dot{\gamma}(\lambda)$.

Schuller (2015) [3] defines the *speed* $s(\lambda)$ to be

$$s(\lambda) := \sqrt{g(u,u)}$$

Furthermore, Schuller argues, by equipping the manifold with a metric, that $s(\lambda)$ carries the physical value of speed with units $\frac{[L]}{[T]}$, with L representing the physical dimensional unit for length, and T representing the physical dimensional unit for time.

Consider $M = \mathbb{R}^2$ and the vector field on $U \subset \mathbb{R}^2$, $\frac{\partial}{\partial \varphi} \in \mathfrak{X}(\mathbb{R}^2)$. Now

$$g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \right) = g_{\varphi\varphi} = r^2 \Longrightarrow s(\lambda) = r$$

Clearly $s(\lambda) = r$ carries the physical dimension of length, represented by r. However, prior, $\frac{\partial}{\partial \varphi}$ makes no connection with the metric g of $\mathbb{R}^2$. So, prior, the (local) coordinate basis vectors carry no physical dimensions.

However, consider these non-coordinate basis vectors for $\mathbb{R}^2$:

$$\mathbf{e}_r = \frac{\partial}{\partial r}$$
$$\mathbf{e}_\varphi = \frac{1}{r} \frac{\partial}{\partial \varphi}$$

Now

$$g(\mathbf{e}_r, \mathbf{e}_r) = 1$$
$$g(\mathbf{e}_\varphi, \mathbf{e}_\varphi) = 1$$

[2]Edmund Bertschinger. MIT Physics 8.962 Spring 1999, Introduction to Tensor Calculus for General Relativity gr1.pdf

[3]Frederic P. Schuller. "Lecture 10: Metric Manifolds (International Winter School on Gravity and Light 2015)". *The WE-Heraeus International Winter School on Gravity and Light.* https://youtu.be/0NCZNwKswn4

and so the speed, for instance, of a curve flowing along the orthonormal vector $\mathbf{e}_\varphi$ would be

$$s(\lambda) = \sqrt{g(\mathbf{e}_\varphi, \mathbf{e}_\varphi)}$$

and so $\mathbf{e}_r, \mathbf{e}_\varphi$ have a norm of 1 with respect to the metric g and can be considered as "unit vectors" since the speed, while flowing along them, $s(\lambda)$ is 1.

On the other hand, I would argue that using these non-coordinate basis vectors is complicated because we'll have to define them, in general, as such:

$$\mathbf{e}_i := \frac{1}{\sqrt{g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i})}} \frac{\partial}{\partial x^i} = \frac{1}{\sqrt{g_{ii}}} \frac{\partial}{\partial x^i} \tag{20}$$

The non-coordinate basis vectors are dependent upon the metric and it's unclear if $\mathbf{e}_i \in \mathfrak{X}(M)$. Possibly, one could say that it is a valid vector field on the valid domain if g is valid on there.

Nevertheless, these orthonormal non-coordinate basis vectors are implemented in `Rn.sage` with the method `.make_orthon_frames` for each of the classes `R2,R3,Rn`. One runs or instantiates the method and the second output is the new orthonormal frame. Thus, the orthonormal frame or orthonormal non-coordinate basis vectors are calculated:

```
sage: e2[1].display( R2eg.sph_ch.frame(), R2eg.sph_ch)
e_1 = d/dr
sage: e2[2].display( R2eg.sph_ch.frame(), R2eg.sph_ch)
e_2 = 1/r d/dph
sage: for i in range(1,3+1):
....:     e3sph[i].display( R3eg.sph_ch.frame(), R3eg.sph_ch )
....:
e_1 = d/drh
e_2 = 1/rh d/dth
e_3 = 1/(rh*sin(th)) d/dph
sage: for i in range(1,3+1):
    e3cyl[i].display( R3eg.cyl_ch.frame(), R3eg.cyl_ch )
....:
e_1 = d/dr
e_2 = 1/r d/dphi
e_3 = d/dzc
sage: for i in range(1,4+1):
    e4sph[i].display( R4.sph_ch.frame(), R4.sph_ch )
....:
e_1 = d/drh
e_2 = 1/rh d/dth1
e_3 = 1/(rh*sin(th1)) d/dth2
e_4 = 1/(rh*sin(th1)*sin(th2)) d/dph
sage: for i in range(1,4+1):
    e4cyl[i].display( R4.cyl_ch.frame(), R4.cyl_ch )
....:
e_1 = d/dr
e_2 = 1/r d/dthe1
e_3 = 1/(r*sin(the1)) d/dphi
e_4 = d/dz
```

Part 4. My notes

Let's following Taubes (2011) [5].

25.1.1. *Torsion free covariant derivatives on T^*M (Sec. 15.2 of Taubes (2011) [5]).*

$$\nabla : \Gamma(T^*M) \to \Gamma(T^*M) \otimes \Gamma(T^*M) = \Gamma(T^*M \otimes T^*M)$$
$$\nabla \alpha = d\alpha - \Gamma^k{}_i \alpha_k e^i$$

Locally, for $\alpha = \alpha_j e^j$,

$$d\alpha = \frac{\partial \alpha_j}{\partial x^k} dx^k \wedge e^j = \frac{1}{2}\left[\frac{\partial \alpha_j}{\partial x^k} dx^k \wedge e^j + \frac{\partial \alpha_k}{\partial x^j} dx^j \wedge e^k \right] = \frac{1}{2}\left[\frac{\partial \alpha_j}{\partial x^k} - \frac{\partial \alpha_k}{\partial x^j} \right] dx^k \wedge e^j$$

$$\Gamma^k{}_i \alpha_k e^i = \Gamma^k{}_{ij} \alpha_k e^i \otimes e^j \overset{[ij]}{\mapsto} \frac{1}{2}\left(\Gamma^k{}_{ij}\alpha_k - \Gamma^k{}_{ij}\alpha_k \right) e^i \wedge e^j$$

where the antisymmetrizing operation, $[\cdot\cdot]$, s.t.

$$[\cdot\cdot] : \otimes^2 T^*M \to \Lambda^2 T^*M = \Omega^2(M)$$
$$T_{ij} e^i \otimes e^j \overset{[ij]}{\mapsto} \frac{1}{2}(T_{ij} - T_{ji}) e^i \wedge e^j$$

is always possible with tensors, being well-defined for each local expression in the coordinate chart, or local frame.

Then define

Definition 55.

$$d_\nabla := [\cdot\cdot] \circ \nabla$$
$$T_\nabla : T^*M \to \Omega^2(M)$$
$$T_\nabla = d_\nabla - d$$

with

$$T_\nabla \in \Gamma(Hom(T^*M, \Lambda^2 T^*M)) = \Gamma(Hom(T^*M, \Omega^2(M)))$$

Suppose $f \in C^\infty(M)$

$$d(f\alpha) = df \wedge \alpha + f d\alpha$$
$$d_\nabla(f\alpha) = [\cdot\cdot] \circ \nabla(f\alpha) = [\cdot\cdot] \circ ((df) \wedge \alpha + f d\alpha - f\Gamma^k{}_i \alpha_k e^i) = df \wedge \alpha + f d_\nabla \alpha$$
$$T_\nabla(f\alpha) = f T_\nabla \alpha$$

By Lemma 11.1 of Taubes (2011) [5] (Suppose $\begin{matrix} E \to M \\ E' \to M \end{matrix}$ vector bundle ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and $\mathcal{L}$ is $\mathbb{R}$ or $\mathbb{C}$-linear map s.t. $\mathcal{L} : \Gamma(E) \to \Gamma(E')$. Suppose $\mathcal{L}(fs) = f\mathcal{L}(s) \quad \forall f \in C^\infty(M)$. Then $\exists!\, L \in \Gamma(\text{Hom}(E; E'))$ s.t. $\mathcal{L}(\cdot) = L(\cdot)$).

Thus, by Lemma 11.1, T_∇ exists and is unique; as $T_\nabla \in \Gamma(\text{Hom}(T^*M; \Omega^2(M)))$.

∇ torsion free if $T_\nabla = 0$.

<u>Claim:</u> Given any covariant derivative ∇, $\exists$ torsion-free $\nabla' = \nabla + A$ covariant derivative.

Proof. Observe that

$$\text{End}(T^*M) = T^*M \otimes TM$$

Consider $A \in \Gamma(\text{End}(T^*M) \otimes T^*M) = \Gamma(\Omega^1(M; \text{End}(T^*M))) = \Gamma((T^*M \otimes TM) \otimes T^*M) = \Gamma((T^*M \otimes T^*M) \otimes TM)$ with automorphism

$$(T^*M \otimes TM) \otimes T^*M \to (T^*M \otimes T^*M) \otimes TM$$
$$(u \otimes u') \otimes v \mapsto (u \otimes v) \otimes u'$$

Observe also that

$$T_\nabla \in \Gamma(\text{Hom}(T^*M; \Lambda^2 T^*M) = \Gamma(\Lambda^2 T^*M \otimes TM) \xrightarrow{(\cdot\cdot)} \Gamma(\otimes^2 T^*M \otimes TM)$$

and so for

$$T_\nabla = d_\nabla - d = [\cdot\cdot] \circ \nabla - d$$

then

$$T_{\nabla'} = [\cdot\cdot] \circ \nabla' - d = [\cdot\cdot] \circ (\nabla + A) - d = T_\nabla + A = 0$$

where $A = -T_\nabla$. ∎

Claim: Suppose ∇ torsion free, ∇ covariant derivative on $T*M$.
Then so is $\nabla + A$ iff $A \in \Gamma((T^*M \otimes T^*M) \otimes T^*M)$ s.t. A symmetric with respect to automormorphism

$$(T^*M \otimes TM) \otimes T^*M \to (T^*M \otimes TM) \otimes T^*M$$
$$(u \otimes u') \otimes v \mapsto (v \otimes u') \otimes u$$
$$\Gamma^k_{\ ij} \mapsto \Gamma^k_{\ ji}$$

From Subsection 15.3 The Levi-Civita connection/covariant derivative of Taubes (2011) [5],

Theorem 9 (Levi-Civita Thm.). *$\exists ! \nabla$ on orthonormal frame bundle s.t. $\nabla : \Gamma(T^*M) \to \Gamma(\otimes^2 T^*M)$ torsion free, i.e. $\exists ! \nabla : \Gamma(T^*M) \to \Gamma(\otimes^2 T^*M)$ s.t. ∇ metric compatible and torsion free.*

Proof. Choose orthonormal frame $\{e^1 \dots e^n\}$ for $T*M$ over some coordinate chart $U \subset M$.

$$\nabla e^k = A^k_{\ i} e^i = A^k_{\ ij} e^i \otimes e^j$$

∇ metric compatible iff $A^k_{\ i} = -A^i_{\ k}$ ($A^k_{\ i} = -\overline{A^i_{\ k}}$ Hermitian case)

I mention moving coframe here from Choquet-Bruhat (2009) [6] because I either skipped over or can't find in Taubes' book where he came up with the relation for de^j on pp. 211, Subsection 15.3 The Levi-Civita connection/covariant derivative of Taubes (2011) [5].

From Chapter 6 Local Cauchy Problem, Section 2 Moving frame formulae, Subsection 2.1 Frame and coframe, pp. 143, Choquet-Bruhat (2009) [6],

Suppose

$$[e_a, e_b] = \frac{1}{2} C^c_{ab} e_c \tag{21}$$

where $[e_a, e_b] := e_a(e_b) - e_b(e_a)$, so e_a's $\in$ Lie algebra $\mathfrak{g}$, a vector space $\mathfrak{g}$, with (orthonormal frame) basis, equipped with Lie Bracket $[\, ,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and C^c_{ab} are **structure coefficients**.

Now for the inner product on this $\mathfrak{g}$ vector space, $\langle e^a, e_b \rangle \equiv e^a(e_b) \equiv e^a \cdot e_b = \delta^a_{\ b}$. *Don't confuse* this with the inner product related to the metric g of (M, g).

One can start from

$$de^c := \frac{-1}{2} C^c_{ab} e^a \wedge e^b$$

and then obtain $[e_a, e_b] = \frac{1}{2} C^c_{ab} e_c$, Eq. (21) (cf. pp. 143, Eq. (2.3) of Ch. VI Local Cauchy Problem, Subsection 2.1 Frame and coframe, Choquet-Bruhat (2009) [6], EY : 20160319 Choquet-Bruhat doesn't show this explicitly; please someone help me show this explicitly).

Nevertheless,

$$T_\nabla e^k = (d_\nabla - d) e^k = [\cdot\cdot] \circ \nabla e^k - de^k = [\cdot\cdot] \circ A^k_{\ ij} e^i \otimes e^j + \frac{1}{2} C^k_{\ ij} e^i \wedge e^j = \frac{1}{2}((A^k_{\ ij} - A^k_{\ ji}) + C^k_{ij}) e^i \wedge e^j$$
$$= 0 \text{ if } \nabla \text{ torsion free (i.e. } T_\nabla = 0)$$
$$\Longrightarrow C^k_{ij} = -(A^k_{\ ij} - A^k_{\ ji})$$

Now *in the specific case* of $\nabla : \Gamma(T*M) \to \Gamma(T^*M \otimes T^*M)$, and not in general, $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$, then $i, j, k \in 1 \dots d$ for $\dim M = d$ (whereas, possibly, $\dim E = n$ and $k \in 1 \dots n$, in general). It might be interesting to consider the torsion on E^* in general, and if $\exists !$ a Levi-Civita connection for this general principal G-bundle (probably not, but I don't have a formal proof).

By construction off the *structure constants* C^k_{ij}, $A^k_{\ ij}$ connection 1-forms uniquely determined (and exist) since

$$C^k_{\ ij} - C^j_{\ ik} - C^i_{\ jk} = -(A^k_{\ ij} - A^k_{\ ij}) + A^j_{\ ik} - A^k_{\ ki} + A^i_{\ jk} - A^i_{\ kj} = -A^k_{\ ij} + A^k_{\ ji} + -A^i_{\ jk} + A^k_{\ ji} + A^i_{\ jk} + A^k_{\ ij} =$$
$$= 2A^k_{\ ji}$$

$$\boxed{A^k_{\ ij} = \frac{1}{2}[C^k_{\ ji} - C^i_{\ jk} - C^k_{\ ik}]}$$

where the skew-symmetry property of $A^i_{\ k} = -A^k_{\ i}$, which applies for $\mathbb{K} = \mathbb{R}$ (i.e. real valued manifolds), arising from *metric compatibility* was exploited several times above.

Taubes (2011) [5] remarked that this was the only proof he could think of, a proof that pushes around indices in index calculations, exploiting a particular basis choice, and cyclic permutation, which wasn't very deep to him. I take it on his word that there's no other way to prove this theorem. □

cf. Subsection 15.4 A formula for the Levi-Civita connection of Taubes (2011) [5]

$$\nabla e^k = \Gamma^k_{\ i} e^i$$
$$\nabla s = ds - \Gamma^k_{\ i} s_k e^i$$
$$\nabla e_j = \Gamma^k_{\ j} e_k$$

$$T_\nabla dx^k = (d_\nabla - d)(dx^k) = [\cdot\cdot] \circ \nabla dx^k = [\cdot\cdot] \cdot (-\Gamma^k_{\ i} dx^i) = [\cdot\cdot] \cdot (-\Gamma^k_{\ ij} dx^i \otimes dx^j) = \left(\frac{-1}{2} \right) (\Gamma^k_{\ ij} - \Gamma^k_{\ ji}) dx^i \wedge dx^j$$
$$= 0 \Longrightarrow \Gamma^k_{\ ij} = \Gamma^k_{\ ji}$$

But keep in mind that this symmetry in the bottom 2 indices is true, for sure, in terms of the coordinates and coordinate bases.

If $T_\nabla = 0$, i.e. ∇ torsion free, necessarily, $\Gamma^k_{\ ij} = \Gamma^k_{\ ji}$ in the coordinate frame.

$$\nabla g = \nabla (g_{ij} dx^i \otimes dx^j) = \left(\frac{\partial g_{ij}}{\partial x^k} - g_{lj} \Gamma^l_{\ ik} - g_{il} \Gamma^l_{\ jk} \right) (dx^i \otimes dx^j) \otimes dx^k$$
$$\frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ik}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} \right) - \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} \right) = 0 \qquad \text{formally, by cyclic permutation}$$
$$\Longrightarrow g_{lj} \Gamma^l_{\ ik} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

since $\nabla g = 0$ (metric compatible).

$\gamma : I \subset \mathbb{R} \to M$

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \left(\dot{\gamma}^j \frac{\partial \dot{\gamma}^k}{\partial x^j} + \Gamma^k_{\ ij} \dot{\gamma}^i \dot{\gamma}^j \right) \frac{\partial}{\partial x^k} = \ddot{\gamma}^k + \Gamma^k_{\ ij} \dot{\gamma}^i \dot{\gamma}^j = 0 \text{ if } \ddot{\gamma}^k = -\Gamma^k_{\ ij} \dot{\gamma}^i \dot{\gamma}^j$$

cf. Subsection 15.5 Covariantly constant sections of Taubes (2011) [5]

Suppose vector bundle $E \to M$, connection ∇ on E.

$$\begin{tikzcd} \gamma^* E & E \arrow[l, "\gamma^*"'] \\ I \subset \mathbb{R} \arrow[u, "\gamma^*(\pi^{-1} \circ \gamma)"] \arrow[r, "\gamma"] & M \arrow[u, "\pi^{-1}"'] \end{tikzcd}$$

$s \in \Gamma(E)$ covariantly constant if $\nabla s = 0$.

If $\nabla s = 0$, $d_\nabla \nabla s = 0$.

$$d_\nabla \nabla s = [\cdot\cdot] \circ \nabla^2 s = Fs = 0 \Longrightarrow F \in \Gamma(\text{End}(E) \otimes \Lambda^2 T^*M) = \Gamma(\Omega^2(M; \text{End}E))$$

Suppose $\gamma : I \subset \mathbb{R} \to M$

$\gamma^* \nabla s = 0$ covariantly constant with respect to $\gamma^* \nabla$.

Fix isomorphism $\Phi : \gamma^* E \to I \times V \qquad (V = \mathbb{R}^n \text{ or } \mathbb{C}^n)$

$\Phi(s) = (t, s_\Phi(t))$

$$\Phi((\gamma^* \nabla) s) = t \mapsto (t, ((\gamma^* \nabla) s)_\Phi dt) \in \Gamma((I \times V) \otimes T^* I)$$
$$((\gamma^* \nabla) s)_\Phi = \frac{d s_\Phi}{dt} + (A_\Phi(t))^k_{\ i} s^i e_k \equiv \frac{ds}{dt} + (A(t)) s$$

$$(\gamma^* \nabla)s = 0 \Longrightarrow \frac{ds}{dt} + (A(t))s = 0$$

Then by Ch.8.3 vector field theorem from Taubes (2011) [5], or by ODE theorem, this system of ODEs can be solved, given initial conditions.

$s \in \Gamma(\gamma^* E)$ *parallel transported along* γ or *parallel along* γ

References

[1] Eric Poisson, **A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics**, Cambridge University Press, 2004. ISBN 0 521 83091 5
EY's references:

[2] Klaus Jänich (Author), S. Levy (Translator), **Topology** (Undergraduate Texts in Mathematics), Springer; 1st ed. 1984. 2nd Corr. printing 1994 edition (January 1, 1995). ISBN-13: 978-0387908922

[3] John M. Lee. **Introduction to Smooth Manifolds**. 2012. Springer. ISBN-10: 0-387-95495-3 (hardcover), ISBN-13: 978-0387-95448-6

[4] T. Frankel, **The Geometry of Physics**, Cambridge University Press, Second Edition, 2004.

[5] Clifford Henry Taubes, **Differential Geometry: Bundles, Connections, Metrics and Curvature** (Oxford Graduate Texts in Mathematics, Vol. 23) Oxford University Press; 1 edition (December 1, 2011). ISBN-13: 978-0199605873

[6] Yvonne Choquet-Bruhat. **General Relativity and the Einstein Equations** (Oxford Mathematical Monographs) 1st Edition. Oxford Mathematical Monographs (February 4, 2009) ISBN-13: 978-0199230723

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