Engineering Mathematics-I

KAS-103

Lecture: 52

Module-V

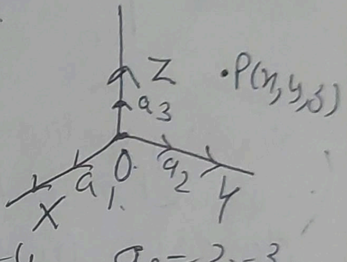
Vector Calculus.

Content: Introduction of vectors and important properties.

Vector calculus is important concept for all Engineering branches. It is used in all Engineering applications.

Vector:- A vector is a quantity which has magnitude and direction. Consider (x, y, z) in a cartesian co-ordinate system in a space.

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a given vector with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$



then difference $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$ are called components of a vector with respect to co-ordinate system.

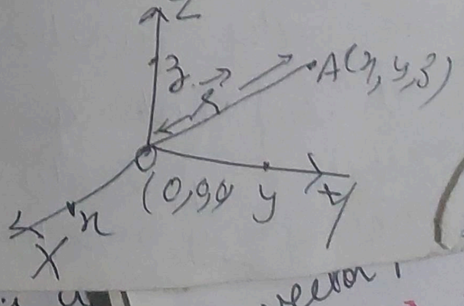
$|\vec{PQ}| = |\vec{a}|$ = Magnitude of vector.

$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$ [distance formula

$= \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Position Vector:- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the vector with initial point $O(0,0,0)$ & terminal point (x, y, z). denoted as $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is called position vector.

$\vec{r}$ has components x, y, z.



Properties of Vector:-

1) **Dot Product:-** Dot-product of two vectors is a scalar quantity. Consider two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

as. $[\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1, \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0]$

2) **Cross product:-** Cross product of two vectors is always a vector quantity. consider two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar function:- Scalar function $f(x, y, z)$ is a function defined at each point in certain domain D in space. Its value is real and depends only on the point $P(x, y, z)$ eg.

$f = xyz, \quad x + y + z = 1, \quad x^2 + y^2 + z^2 = 1.$

Vector function:- If to each value of variable t, there corresponds a value of vector $\vec{r}$, then $\vec{r}$ is called vector function of variable t. denoted by $\vec{r} = \vec{r}(t)$ or $\vec{r} = \vec{f}(t)$

eg. position vector $(\vec{r})$ of moving particle is a vector function of time t.

Any vector function can be represented in its components as $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ denotes unit vectors along axis of x, y, z respectively.

Gradient of Scalar field:- Let $\phi(x,y,z)$ be a scalar point function and differential operator be $\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$ then gradient of function $\phi(x,y,z)$ is denoted as $\nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x,y,z) = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial y}$$

Thus ∇ converts scalar point function to a vector point function. gradϕ is normal to surface.

Remark:-

\# $\nabla\phi = \text{grad}\,\phi$

\# $|\text{grad}\,\phi|$ is rate of change of $\phi(x,y,z)$ in direction of normal.

Properties:-

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$
2. $\nabla(f \cdot g) = g\nabla f + f\nabla g$
3. $\nabla(f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.
4. Gradient of a constant function is always 0 i.e. $\nabla C = 0$, if C is a constant function.

2007

Q.1 find gradϕ if $\phi(x,y,z) = 3x^2y - y^3z^2$. Also find grad ϕ at $(1,-2,1)$

Sol. Let $\phi(x,y,z) = 3x^2y - y^3z^2$.

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} - 2y^3z\,\hat{k}$$

$$[\nabla\phi]_{(1,-2,1)} = -12\hat{i} + [3-12]\hat{j} - (-8)\hat{k} = -12\hat{i} - 9\hat{j} - 16\hat{k}$$ Ans

2015, 13 Q: If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, prove that grad u, grad v and grad w are co-planar vector

Sol. To prove vectors are co-planar we have to prove $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$.

$\vec{a} = \text{grad } u = \hat{i}+\hat{j}+\hat{k}$.

$\vec{b} = \text{grad } v = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$\vec{c} = \text{grad } w = (z+y)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$.

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & x+y \end{vmatrix}$$

$R_1+R_2+R_3$
$$2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ z+y & z+x & x+y \end{vmatrix} = 0 \qquad [\because R_1 = R_2]$$

$\therefore$ vectors are co-planar.

Hence grad u, grad v, grad w are co-plana.

2014
Q. find a unit vector normal to surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$

Sol. Normal vector to surface $= \nabla\phi$

Here $\phi = x^3+y^3+3xyz-3$.

$$\nabla\phi = (3x^2+3yz)\hat{i}+(3y^2+3xz)\hat{j}+3xy\hat{k}$$

$$[\nabla\phi]_{(1,2,-1)} = (3-6)\hat{i}+[12-3]\hat{j}+6\hat{k}$$

$$= -3\hat{i}+9\hat{j}+6\hat{k}.$$

unit normal vector $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{9+81+36}}$

$$= \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{3\sqrt{14}} = \frac{-\hat{i}+3\hat{j}+2\hat{k}}{\sqrt{14}}$$

Engineering Mathematics-I
KAS-103
Lecture-53

Module-V
Vector Calculus

Content: Gradient, Normal unit vector, Tangent, Tangent and Normal, Directional Derivative.

Gradient:-

1) Prove that $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

$$\nabla\left(\frac{f}{g}\right) = \nabla(fg^{-1})$$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](fg^{-1})$$

$$= \hat{i}\frac{\partial}{\partial x}(fg^{-1}) + \hat{j}\frac{\partial}{\partial y}(fg^{-1}) + \hat{k}\left(\frac{\partial}{\partial z}\right)(fg^{-1})$$

$$= \hat{i}\left[f\left(-1g^{-2}\frac{\partial g}{\partial x}\right) + g^{-1}\frac{\partial f}{\partial x}\right] + \hat{j}\left[f\left(-1g^{-2}\frac{\partial g}{\partial y}\right) + g^{-1}\frac{\partial f}{\partial y}\right]$$
$$+ \hat{k}\left[f\left(-1g^{-2}\frac{\partial g}{\partial z}\right) + g^{-1}\frac{\partial f}{\partial z}\right]$$

$$= \frac{g\nabla f - f\nabla g}{g^2}, \quad g \neq 0$$

Normal Unit Vector:- Let $\phi(x,y,z)$ be scalar point function, then unit vector denotes as $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ is called normal unit vector $\because |\hat{n}| = 1$.

Remark If ϕ is a scalar point function $\nabla\phi$ is a vector point funct. If $\nabla\phi$ is given to determine ϕ we have to use relation $d\phi = \vec{F}\cdot d\vec{r}$.

Where $\vec{F} = \nabla\phi$.

Q. If $\nabla\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z$

find ϕ.

Sol. Let $\vec{F} = \nabla\phi$.

$$\vec{F}\cdot d\vec{r} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz$$

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial y}dz = d\phi$$ [Total Derivative]

$$d\phi = (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2y$$

Integrating with respect to 'x', treating y, z as c

$$\phi = xy^2 - x^2yz^3 + c_1$$

$$\phi = 3y + xy^2 - x^2z^3y + c_2 \quad [\text{w.r.to 'y'}]$$

$$\phi = \frac{6z^4}{4} - x^2yz^3 + c_3 \quad [\text{w.r.to 'z'}]$$

$$\boxed{\therefore \phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + C}$$, C is constant.

2011 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. show that grad $r = \frac{\vec{r}}{r}$

Sol. $$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

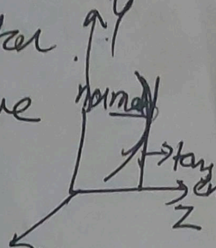
$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence grad $r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{\vec{r}}{r}$ Hence proved.

Tangent and Normal plane



The gradient vector $\nabla f(x_0, y_0, z_0)$ is althogonal to the level curve $f(x,y,z)=k$ at the point (x_0, y_0, z_0).

eq^n of tangent plane to the surface $f(x,y,z)=k$ at point (x_0, y_0, z_0) is.

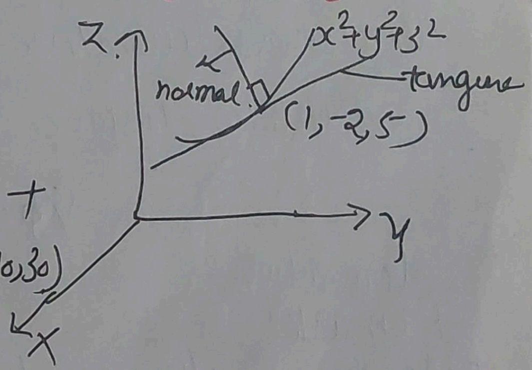
$$(x-x_0)\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0,z_0)} + (y-y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0,z_0)} + (z-z_0)\left(\frac{\partial f}{\partial z}\right)_{(x_0,y_0,z_0)} = 0$$

eq^n of Normal plane to the surface is.

$$\frac{(\partial f/\partial x)_{(x_0,y_0,z_0)}}{x-x_0} = \frac{\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0,z_0)}}{y-y_0} = \frac{\left(\frac{\partial f}{\partial z}\right)_{(x_0,y_0,z_0)}}{z-z_0}$$

a. $\vec{r}(t) = (x_0, y_0, z_0) + t\,\nabla f(x_0, y_0, z_0)$

Q. find the tangent plane and normal line to $x^2+y^2+z^2=30$ at $(1,-2,5)$



Sol. eq^n of tangent plane is

$$(x-x_0)\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0,z_0)} + (y-y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0,z_0)} + (z-z_0)\left(\frac{\partial f}{\partial z}\right)_{(x_0,y_0,z_0)} = 0$$

$$= (x-1)(2x)_{(1,-2,5)} + (y+2)\,2y_{(1,-2,5)} + (z-5)(2z)_{(1,-2,5)} = 0$$

$\Rightarrow +2(x-1) - 4(y+2) + 10(z-5) = 0$ Ans.

$= 2x - 4y + 10z - 2 - 8 - 50 = 0$

$2x - 4y + 10z - 60 = 0$ or.

$x - 2y + 5z - 30 = 0.$

Engineering Mathematics-I

KAS-103

Unit-V

Vector Calculus

Lecture 5/4

Content: Divergence and Solenoidal vector

Divergence: - Divergence is a vector operator that measures the magnitude of a vector field's source at a given point.

Divergence represents the volume density of the outward flux of a vector field.

If $\vec{V} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$ be a vector point function then $\operatorname{div} \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \vec{V}$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Remark:

1. $\operatorname{div} \vec{V}$ is a scalar quantity.
2. div. Converts a vector point function into scalar function
3. $\vec{V} \cdot \nabla \neq \nabla \cdot \vec{V}$
4. If $\vec{V}$ is constant vector, then $\nabla \cdot \vec{V} = 0$
5. If c is constant $\operatorname{div}(c\vec{V}) = c \operatorname{div} \vec{V}$

Solenoidal vector: - If $\operatorname{div} \vec{V} = 0$, then $\vec{V}$ is called solenoidal [incompressible] vector function.

2014.

Q.1 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that $\operatorname{div} \vec{r} = 3$.

Sol. $\operatorname{div} \vec{r} = \nabla \cdot \vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3 \text{ Ans.}$$

If $\vec{f}$ is a constant vector then $\nabla \times \vec{f} = 0$

2016 Q. find divergence of vector function $\vec{F}(x,y,z)$ $e^{xyz}(xy^2\hat{i}+yz^2\hat{j}+zx^2\hat{k})$ at the point (1, 2, 3)

Sol. $\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

when $F_1 = e^{xyz}(xy^2)$, $F_2 = e^{xyz}(yz^2)$, $F_3 = e^{xyz}$

$\therefore \text{div}\,\vec{F} = y^2[e^{xyz} + xyz\,e^{xyz}] + z^2[e^{xyz} + yxz\,e^{xyz}] + x^2[e^{xyz} + xyz\ldots$

$= (1+xyz)(x^2+y^2+z^2)\,e^{xyz} = 98e^6$ at (1, 2, 3)

Q. Prove that $\text{div}(\text{grad}\, r^n) = n(n+1)r^{n-2}$ where $r^2 = x^2+y^2+z^2$.

Sol. $\text{grad}\, r^n = \nabla r^n$

$= \left[\hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n\right]$

$= \left[\hat{i}\,n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\,n r^{n-1}\frac{\partial r}{\partial z}\right]$

$\Rightarrow n r^{n-1}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right] = n r^{n-2}\vec{r}$

$\therefore \text{div}(\text{grad}\, r^n) = \text{div}(n r^{n-2}\vec{r})$

$\Rightarrow n r^{n-2}\cdot \nabla\cdot\vec{r}$

$= \frac{\partial}{\partial x}(n r^{n-2}x) + \frac{\partial}{\partial y}(n r^{n-2}y) + \frac{\partial}{\partial z}(n r^{n-2}z)$

$= \sum n r^{n-2} + n(n-2)r^{n-3}x\frac{\partial r}{\partial x}$

$= \sum n r^{n-2} + n(n-2)r^{n-3}\frac{x^2}{r}$

$= 3n r^{n-2} + n(n-2)\frac{r^{n-3}}{r}(x^2+y^2+z^2) = [3n + n^2 - 2n]r^{n-2}$

$= n(n+1)r^{n-2}$

Q. Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is solenoidal.

Sol. To prove vector is solenoidal we have to show $\operatorname{div} \vec{F} = 0$

Here $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2 = 0.$$ Hence proved.

Q. Find the value of p such that $\vec{V} = (x + py)\hat{i} + (py - 3z)\hat{j} + (x - 2z)\hat{k}$ is solenoidal.

Sol. given $\vec{V}$ is solenoidal $\Rightarrow \nabla \cdot \vec{V} = 0$.

$$\frac{\partial}{\partial x}(x + py) + \frac{\partial}{\partial y}(py - 3z) + \frac{\partial}{\partial z}(x - 2z) = 0$$

$$1 + p - 2 = 0 \qquad p = 1.$$ Ans.

Q. Show that $\nabla^2\left(\frac{x}{r^3}\right) = 0$ where r is magnitude of position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. We have $r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x} - \frac{6x}{r^5} + \frac{15}{r^6}x^2\frac{\partial r}{\partial x}$$

$$= -\frac{3x}{r^5} - \frac{6x}{r^5} + \frac{15x^3}{r^7} = -\frac{9x}{r^5} + \frac{15x^3}{r^7}.$$

similarly $$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = -3x\frac{\partial}{\partial y}\left(\frac{y}{r^5}\right) = -\frac{3x}{r^5} + \frac{15xy^2}{r^7}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = \frac{-3x}{r^5} + \frac{15xz^2}{r^7}$$

$$\therefore \nabla^2\left(\frac{x}{r^3}\right) = -\frac{15x}{r^5} + \frac{15x}{r^7}(x^2+y^2+z^2)$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^5} = 0$$

Important results

① $\text{div}(\text{grad}\,\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial y^2}$

② $\text{div}(\text{curl}\,\vec{\phi}) = 0$

③ $\text{curl}(\text{grad}\,\phi) = 0$

proof

Engineering Mathematics-I
KAS-103.

Lecture 5 — Unit-V Vector Calculus.

1

Content: Curl, Irrotational Vector.

Curl:- Let $\vec{v}$ be a vector point function that curl $\vec{v}$ is defined as.

$$\text{curl}\,\vec{v} = \nabla \times \vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Curl. $\vec{v}$ the vector point function $\vec{v}$ is again a vector point function.

Curl $\vec{v}$ is vector point function.

Curl $\vec{v}$ is used to represent rotation of a field.

If $\vec{f}$ is a constant vector then $\nabla \times \vec{f} = 0$.

Irrotational vector:- If curl $\vec{v} = 0$ then $\vec{v}$, vector point function is said to be irrotational.

2014 Q.1 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove curl $\vec{r} = 0$

Sol.
$$\text{curl}\,\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(0-0)$$

$$\boxed{\nabla \times \vec{r} = 0.}$$ Hence proved.

2007 Q. If $\vec{F}(x,y,z) = x^2y^2\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$ find curl of vector.

Sol. curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y^2 & 2xy & (y^2-xy) \end{vmatrix}$

$= \hat{i}[2y-x-0] - \hat{j}[-y+0] + \hat{k}[2y-2x \ldots]$

$\nabla\times\vec{F} = (2y-x)\hat{i} + y\hat{j} + 2y(1-x^2)\hat{k}$ Ans

Q. Prove that $\vec{a}\times(\nabla\times\vec{r}) = \nabla(\vec{a}\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r}$, where $\vec{a}$ is a constant vector.

Sol. LHS $\vec{a}\times(\nabla\times\vec{r}) = \vec{a}\times\vec{0} = \vec{0}$.

RHS $\nabla(\vec{a}\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r}$

$= \nabla(a_1x+a_2y+a_3z) - \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$

$= a_1\hat{i}+a_2\hat{j}+a_3\hat{k} - a_1\hat{i} - a_2\hat{j} - a_3\hat{k} = 0$ Hence proved

Q.1 Prove that $\vec{F} = 2xy\hat{i} + (x^2+2yz)\hat{j} + (y^2+1)\hat{k}$ is irrotational.

Sol. A vector is irrotational if $\nabla\times\vec{F} = 0$

$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & (x^2+2yz) & (y^2+1) \end{vmatrix}$

$= \hat{i}(2y-2y) - \hat{j}(0-0) + \hat{k}(2x-2x)$

$= \vec{0}$. Hence proved.

Scalar Potential: $d\phi = \nabla\phi\cdot d\vec{r}$ is called total derivative of ϕ.

as $d\phi = \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$ —(1)

where ϕ is a scalar, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, $d\vec{r} = dx\hat{i}+dy\ldots$ function ϕ is ... or scalar poten... evaluating ϕ ... potential.

Q. Show that $\vec{A}$ ... irrotational ... $\vec{A} = \nabla\phi$ ... $\nabla\times$ ...

Sol.

where φ is a scalar point function
$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ [position vector
$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$.
function φ is called potential field or scalar potential.
evaluating φ from relation ①. gives scalar potential φ.

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Q. Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. find velocity potential φ such that $\vec{A} = \nabla\varphi$.

Ans.
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \hat{i}(-1+1) - \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x)$$
$$= \vec{0}.$$
$\Rightarrow \vec{A}$ is irrotational.

Given $\vec{A} = \nabla\varphi$.
to determine scalar potential. φ.
consider $d\varphi = \nabla\varphi \cdot dr$
$$d\varphi = (6xy + z^3)dx + (3x^2 - z)dy + (3xz^2 - y)dz$$
To determine φ integrate w.r.to 'x', 'y', 'z' resp.

$$\varphi = 3x^2y + xz^3 + C_1$$
$$\varphi = 3x^2y - yz + C_2$$
$$\varphi = xz^3 - yz + C_3$$

$$\boxed{\text{Hence } \varphi = 3x^2y - yz + xz^3 + C}$$ Ans.

Find constants a, b, c such that $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+(y+2z)\hat{k}$ is irrotational, find φ if $\vec{F} = \nabla\varphi$.

Sol.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \hat{i}[c+1] - \hat{j}[4-a] + \dots = \vec{0}$$

$\Rightarrow c=-1, \quad a-4=0 \Rightarrow a=4, \quad b=2$

Hence $a=4, b=2, c=-1$

$$\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}.$$

φ is scalar potential

$$d\varphi = \nabla\varphi \cdot d\mathbf{r} = (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$

$$= \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + C.$$