Bilinear Transformation / Mobius Transformation

1. Find the bilinear transformation which maps the points $z = 1, -i, -1$ to the points $w = i, 0, -i$ respectively. Show also that transformation maps the region outside the circle $|z| = 1$ into the half planes $Re(w) \geq 0$

Ans:- $w = \frac{iz-1}{iz+1}$, $z = i\left(\frac{w+1}{w-1}\right)$

2. Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also find the image of unit circle $|z| = 1$.

Ans:- $w = \frac{z+1}{z-i}$ $\quad z = \frac{iw+1}{w-1}$

3. Find the bilinear transformation which maps the points $i, -i, 1$ of the z-plane into $0, 1, \infty$ of the w-plane respectively.

Ans:- $w = \frac{(i-1)z + (i+1)}{-2z+2}$

Q:-4. Show that the transformation $w = i\left(\frac{1-z}{1+z}\right)$ transforms the circle $|z| = 1$ onto the real axis of the w-plane and the interior of the circle into the upper half of the w-plane.

Ans:- $z = \frac{i-w}{i+w}$, $|z| = 1 \Rightarrow v = 0$, $|z| < 1 \Rightarrow v > 0$

Q:-5. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane. What is the image of the circle $|z| = 1$ under this transformation.

Ans:- $u = \frac{x(1-x) - y^2}{(1-x)^2 + y^2}$, $v = \frac{y}{(1-x)^2 + y^2}$, $v > 0 \Rightarrow y > 0$, $|z| = 1 \Rightarrow 2u + 1 = 0$

Milne Thomson's Method:-

1. Determine the analytic function $f(z)$ interms of z whose real part is

(i) $\frac{1}{2}\log(x^2+y^2)$ (ii) $\cos x \cosh y$

(iii) $e^{-x}(x\cos y + y\sin y)$ (iv) $\frac{\sin 2x}{\cosh 2y + \cos 2x}$

Ans:- (i) $\log z + c$ (ii) $\cos z + c$ (iii) $1 + ze^{-z}$

(iv) $\tan z + c$

2. Find the regular function $f(z)$ interms of z whose imaginary part is

(i) $\frac{x-y}{x^2+y^2}$ (ii) $6xy - 5x + 3$ (iii) $e^x(x\sin y + y\cos y)$

(iv) $\frac{x}{x^2+y^2} + \cosh x \cos y$

Ans:- (i) $\frac{1+i}{z} + c$ (ii) $3z^2 - 5iz + c$ (iii) $ze^z + c$

(iv) $\frac{i}{z} + i\cosh z + c$

3. If $f(z) = u + iv$ is an analytic function, find $f(z)$ interms of z if

(i) $u - v = e^x(\cos y - \sin y)$ **Ans:-** $e^z + c$

(ii) $u + v = \frac{x}{x^2+y^2}$ **Ans:-** $f(z) = \frac{1}{1+i}\left(\frac{i}{z} + 1\right)$

(iii). $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, **Ans:-** $f(z) = \cot\frac{z}{2} + \frac{1}{2}(1-i)$

(iv) $u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ **Ans:-** $f(z) = \frac{1}{2}(1+i)\cot z +$