Engineering Mathematics - I

KAS - 103

Module - III
Differential Calculus - II

Lecture - 34

Contents: - Taylor's and Maclaurian's Theorem of one variable.

Taylor's Theorem and expansion of function for single variable: Let $f(x)$ be a function of x. If the function $f(x+h)$ can be expanded in positive powers of h, where x is independent of h, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

where h is a small distance.

Other Forms of Taylor's Theorem.

1) To expand a function about point $x = a$.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots$$

2) To expand a function in powers of $x-a$. replace h by $x-a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

3. To expand a function at $a = 0$ or about origin also known as Maclaurian's Theorem.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Remark: - To expand a function in powers of x. We use 3rd form or Maclaurian's Theorem.

$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

Working Process :- To expand a function $f(x)$ into Taylor's or Maclaurian's Series.

1). Consider $f(x)$
2). differentiate $f(x)$, number of times and find $f'(x), f''(x), f'''(x), \ldots$ and so on.
3. use appropriate form as required.
4. for Maclaurian's Series find $f(0), f'(0), f''(0)$
5. Substitute all the values into series.

Remark :- If not mentioned, by default we use Maclaurian's Theorem.

Q.1 Expand e^x in power of x.

To expand a function into powers of x, we use Maclaurian's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots \quad (1)$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1.$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1.$

$\vdots$

$f^n(x) = e^x \Rightarrow f^n(0) = e^0 = 1$

substitute these values in (1), we get

$$e^x = 1 + x \times 1 + \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times 1 + \cdots + \frac{x^n}{n!} \times 1 + \cdots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots}$$ Ans.

Q.2 Expand $\sin x$, $\cos x$ in powers of x.

Q.3 Expand a^x in powers of x.

Q.4 Expand $\log(1+x)$ in powers of x. Then find $\log\left(\frac{1+x}{1-x}\right)$ & hence calculate $\log\left(\frac{11}{9}\right)$

Sol:- $f(x) = \log(1+x)$ | $f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$ | $f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ | $f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ | $f'''(0) = 2$

$f''''(x) = \frac{-6}{(1+x)^4}$ | $f''''(0) = -6.$

using Maclaurin's Theorem, we know

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{----}$

$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \text{----}$

$$\boxed{\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6}{4!}x^4 + \text{------}}$$ Ans

changing x to $-x$, we get

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \text{------}$

$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{--}\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \text{--}\right]$

$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \text{--}\right]$

$\log\left(\frac{11}{9}\right)$ put $x = \frac{1}{10}$

$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3000} + \text{--}\right] = 0.20067$ Ans.

Q.5 Expand $2x^3+7x^2+x-6$ in powers of $(x-2)$ & calculate $f(2.1)$

Sol. using 2nd form of Taylor's theorem $a = 2$

$f(x) = 2x^3+7x^2+x-6.$ | $f(2) = 16+28+2-6 = 40$

$f'(x) = 6x^2+14x+1$ | $f'(2) = 24+28+1 = 53.$

$f''(x) = 12x+14.$ | $f''(2) = 24+14 = 38$

$f'''(x) = 12$ | $f'''(2) = 12$

Percentage Error: $\frac{\delta x}{x} \times 100$ is known as percen...

we get

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) \ldots$$

$$= 40 + 53(x-2) + \frac{38}{2!}(x-2)^2 + \frac{12}{3!}(x-2)^3$$

$$\boxed{\begin{array}{l} f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 \\ f(2.1) = 40 + 53(0.1) + 19(0.1)^2 + 2(0.1)^3 = 45.492 \end{array}}$$ Ans.

Q6. Imp Expand $\log(x+h)$ in powers of x.

Sol. To expand in powers of x, we consider x as constant & h as variable.

We know $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$

because we have to expand in powers of x interchanging x & h.

$$f(h+x) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ldots$$

$f(x+h) = \log(x+h)$

$f(x) = \log x$

$f(h) = \log h)$

$$\therefore \log(x+h) = \log h + x \times \frac{1}{h} + \frac{x^2}{2!}\left(\frac{-1}{h^2}\right) + \frac{x^3}{3!}\left(\frac{2}{h^3}\right) + \ldots$$

Ans.

⇒ Problems for Practice :-

Q1 find expansions of $\sin^{-1}x$, $e^x \cos x$, $\log(\sec x)$

Q2 Obtain $\tan^{-1}x$ in powers of $(x-1)$

Q3 Expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending powers of x.

Q4. Evaluate $\sqrt{25.17}$ using Taylor's Theorem.

we get
$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2)$.

Module-III
Differential Calculus-II

Lecture-35

Content:- Taylor's and Maclaurian's Theorem for function of two variables.

Taylor's Theorem for function of two Variables:-

Let $f(x,y)$ be a function of two variables x & y then $f(x+h, y+k)$ can be expanded in positive powers of h & k, where x & y are independent of h & k respectively and h, k are small distances.

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \frac{1}{3!}[h^3 f_{xxx} + 3h^2k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}].$$

Other forms of Taylor's Theorem of two variable

1) To expand function $f(x,y)$ about point $x=a, y=b$.

$$f(a+h, b+k) = f(a,b) + [h f_x(a,b) \cdot k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \text{----}$$

2. To expand a function in powers of $(x-a)$ & $(y-b)$ replace h by $(x-a)$ & k by $(y-b)$

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \text{----}] + \text{-}$$

3. To expand function $f(x,y)$ about $(0,0)$ or origin or in powers of x & y. we use Maclaurian's Theorem.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \text{----}$$

Working Process:- To expand a function $f(x,y)$ into Taylor's & Maclaurian's series.

Step I Consider $f(x,y)$

II Differentiate $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{yy}(x,y)$ and so on.

III use appropriate form as required.

IV for Maclaurian's series find $f(0,0)$, $f_x(0,0)$, $f_y(0,0)$, $f_{xx}(0,0)$, $f_{xy}(0,0)$, $f_{yy}(0,0)$

V Substitute all these values into series.

Q1 Expand $e^x \sin y$ in powers of x & y upto terms of 3rd degree.

Sol:- Let $f(x,y) = e^x \sin y$

To expand $f(x,y)$ in powers of x & y. we will use Maclaurian's Theorem.

$$f(x,y) = f(0,0) + (x) f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + ---- \quad —(1)$$

$f(x,y) = e^x \sin y$	$f(0,0) = 0$
$f_x(x,y) = e^x \sin y$	$f_x(0,0) = 0$
$f_y(x,y) = e^x \cos y$	$f_y(0,0) = 1$
$f_{xx}(x,y) = e^x \sin y$	$f_{xx}(0,0) = 0$
$f_{xy}(x,y) = e^x \cos y$	$f_{xy}(0,0) = 1$
$f_{yy}(x,y) = -e^x \sin y$	$f_{yy}(0,0) = 0.$
$f_{xxx}(x,y) = e^x \sin y$	$f_{xxx}(0,0) = 0$
$f_{xxy}(x,y) = e^x \cos y$	$f_{xxy}(0,0) = 1$
$f_{xyy}(x,y) = -e^x \sin y$	$f_{xyy}(0,0) = 0$
$f_{yyy}(x,y) = -e^x \cos y$	$f_{yyy}(0,0) = -1$

∴ Substituting these values in eq^n (1)

$$e^x \sin y = 0 + x(0) + y(1) + \frac{1}{2!}[x^2 \times 0 + 2xy \times 1 + y^2 \times 0] + \frac{1}{3!}[x^3 \times 0 + 3x^2 y \times 1 + 3xy^2(0) + y^3 \times (-1)] + \cdots$$

$$e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6}$$

[UPTU-2006, 14, 11, 12]

Q.2 Expand $\tan^{-1}\frac{y}{x}$ in neibhourhood of (1,1) upto and inclusive of second degree terms. Hence compute $f(1.1, 0.9)$

Sol. Let $f(x,y) = \tan^{-1}\frac{y}{x}$.

$$f_x(x,y) = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2} \Rightarrow \frac{-y}{x^2+y^2}$$

$$f_y(x,y) = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f_{xx}(x,y) = \frac{(-y)(-1)2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$f_{yy} = \frac{(x)(-1)2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_{xy} = \left[\frac{-1(x^2+y^2) + y(2y)}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f(1,1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$f_x(1,1) = \frac{-1}{2}$$

$$f_y(1,1) = \frac{1}{2}$$

$$f_{xx}(1,1) = \frac{2}{4} \Rightarrow \frac{1}{2}$$

$$f_{yy}(1,1) = \frac{-2}{4} = \frac{-1}{2}$$

$$f_{xy} = \frac{1-1}{4} = 0.$$

$$f(x,y) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1) + \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)] + \cdots$$

$$= \frac{\pi}{4} + (x-1)\left(\frac{-1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2!}\left[(x-1)^2\frac{1}{2} + 0 + (y-1)^2\left(\frac{-1}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{(x-1)}{2} + \frac{(y-1)}{2} + \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right] + \cdots$$

To calculate $f(1.1, 0.9) = f(1+0.1, 1-0.1)$ Ans.

replace x by 1.1 & $y = 0.9$

we get

$$= \frac{\pi}{4} + \frac{1}{2}(0.1) + \frac{1}{2}(-.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-.1)^2$$

$$= 0.6857 \text{ (approx)}$$

Q.3 Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$

Sol. $f(x,y) = x^2y + 3y - 2$

$f_x(x,y) = 2xy$

$f_y(x,y) = x^2 + 3$

$f_{xx}(x,y) = 2y$

$f_{xy}(x,y) = 2x$

$f_{yy}(x,y) = 0$

$f_{xxx} = 0$

$f_{xxy} = 2$

$f_{xyy} = 0$

$f_{yyy} = 0$

$f(1,-2) = -2 - 6 - 2 = -10$

$f_x(1,-2) = -4$

$f_y(1,-2) = 4$

$f_{xx}(1,-2) = -4$

$f_{xy}(1,-2) = 2$

$f_{yy}(1,-2) = 0$

$$\therefore f(x,y) = -10 + (x-1)(-4) + (y+2)4 + \frac{1}{2!}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)\right] + \frac{1}{3!}\left[0 + 3(x-1)^2(y+2)\times 2 + 0\right]$$

$$\boxed{f(x,y) = -10 - 4(x-1) + 4(y+2) + \left[-2(x-1)^2 + 2(x-1)(y+2)\right] + (x-1)^2(y+2) + \cdots}$$

Ans.

Q. ~~Problems for Practice~~

Expand following into Taylor's Series.

1) y^x in powers of $(x-1)$ & $(y-1)$

2) $e^x \tan^{-1} y$ in powers of $(x-1)$ and $(y-1)$.

3) $e^x \log(1+y)$ in powers of x & y as far as terms of 3rd degree.

4) Expand $e^x \cos y$ at $(1, \frac{\pi}{4})$

5) e^{xy} at $(1,1)$

$= \frac{\pi}{4} + \frac{1}{2}(0.1) + \frac{1}{2}(-.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-.1)^2 \ldots$

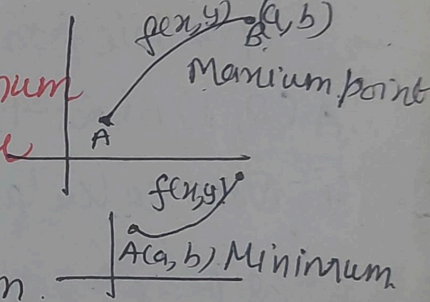
Engineering Mathematics-I
KAS-103 Module-III
Lecture - 3.7 Differential Calculus II

Content: Extrema of function of two variables.

Extrema of function of two Variables:- A maximum or minimum value of a function is called an extreme value or extremum.

Minimum Value:- Consider a function $f(x,y)$, then function has relative minimum at the point (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y).

Maximum Value:- Consider a function $f(x,y)$, then function has relative maximum value at point (a,b) if $f(x,y) \leq f(a,b)$ for all point (x,y)



The point where function is either minimum or maximum is called extreme value or extremum.

Saddle point:- The point where function is neither maximum nor minimum is known as saddle point.

Working Process:- To find minimum or maximum for function $f(x,y)$

1. Consider $f(x,y)$
2. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
3. Put $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, and solve these equation. to obtain points [known as stationary points]
4. for each solution compute $r = f_{xx}$, $s = f_{xy}$ & $t = f_{yy}$ and $rt - s^2$

1) if $rt - s^2 > 0$ we check further value of r.

$= 0.685$ (app...

Q.3 Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$

1. If $r > 0$ then point is minima.
2. If $r < 0$, then point of maxima.

2. If $rt - s^2 < 0$ then saddle point [neither max nor minima.

3. If $rt - s^2 = 0$, case is doubtful requires further ivestigation.

Q.1 find relative maximum and minimum values of $x^3 + y^3 - 3y - 12x + 20$.

Soln. 1) Consider $f(x,y) = x^3 + y^3 - 3y - 12x + 20$.

2. $\frac{\partial f}{\partial x} = 3x^2 - 12$, $\frac{\partial f}{\partial y} = 3y^2 - 3$.

3) put $3x^2 - 12 = 0$, $3y^2 - 3 = 0 \Rightarrow x = \pm 2, y = \pm 1$.

$\therefore$ possible points or stationary points are.

$A(2,1), B(2,-1), C(-2,1), D(-2,-1)$

4. Now calculating $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = 0$.
 r $\quad$ t $\quad$ s

$rt - s^2 = (6x)(6y) - 0 = 36xy$.

points	r	s	t	$rt - s^2$	Extreme value.
1. (2,1)	$12>0$	0	6	$72-0=72>0$	Minima at (2,1)
2. (2,−1)	$12>0$	0	−6	$-72<0$	Saddle point (2,−1)
3. (−2,1)	$-12<0$	0	6	$-72<0$	Saddle point (−2,1)
4. (−2,−1)	$-12<0$	0	−6	$72>0$	Maxima at (−2,−1).

Maximum value $\overset{(-2,-1)}{=}$ 38

Minimum value at (2,1) $8 + 1 - 3 - 24 + 20 = 2$

Q.2. Determine the points where function $f(x,y) = x^3 + y^3 - 3axy$ has maximum or minimum value.

Sol. Let $f(x,y) = x^3 + y^3 - 3axy$.

$f_x = 3x^2 - 3ay = 0$

$f_y = 3y^2 - 3ax = 0$.

$3x^2 - 3ax = 0$

$x(x-a) = 0$

$\Rightarrow x = 0$ and $x = a$

$\therefore$ stationary points are $(0,0)$ and (a,a)

$r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = f_{yy} = 6y$.

$rt - s^2 = 36xy - 9a^2$

At $(0,0) = 0 - 9a^2 = -9a^2$ for all values of a

< 0. $\therefore$ saddle point.

At $(a,a) = 36a^2 - 9a^2 = 27a^2 > 0$.

$r = 6a$ $\quad \forall a > 0 \; r = 6a > 0 \Rightarrow$ point of minima if $a > 0$

$\forall a < 0 \cdot r = 6a < 0 \cdot$ point of maxima if $a < 0$.

$$\boxed{\begin{array}{l}\text{Max value } a^3 \\ \text{Min value } -a^3\end{array}}$$

Q3 Discuss Maxima and minima of function.

$f(x,y) = \cos x \cos y \cos(x+y)$

Sol. $f_x(x,y) = \cos y\,[-\sin x \cos(x+y) - \cos x \sin(x+y)]$

$= -\cos y \sin(2x+y)$.

$f_y(x,y) = \cos x\,[-\sin y \cos(x+y) - \sin(x+y)\cos y]$

$= -\cos x \sin(2y+x)$

For stationary values. $\quad f_x = 0 \quad \cos y \sin(2x+y) = 0$ — (1)

$f_y = 0 \quad \cos x \sin(2y+x) = 0$. — (2)

(1) $\Rightarrow$ either $\cos y = 0$ or $\sin(2x+y) = 0$

$y = \pi/2$. $\qquad 2x + y = 0 \,|\, \pi \,|\, 2\pi$

$2x + y = (-1)^n n\pi$.

from (2) $x = \pi/2$ $\qquad 2y + x = (-1)^n n\pi$

At $y = \pi/2$ $\quad 2x + y = 0$ $\quad \Rightarrow x = -\pi$

$x + 2y = 0$

$-3y = 0$

$y = 0$, $x = 0$

we get

when $2x+y=\pi$, $x+2y=\pi$ we get $(\pi/3, \pi/3)$

$\therefore$ Stationary points are $(0,0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{2\pi}{3}, \frac{2\pi}{3})$

$r = f_{xx} = -2\cos y \cos(2x+y)$

$s = f_{xy} = -\cos(2y+2x)$

$t = f_{yy} = -2\cos x \cos(2y+x)$

$rt - s^2 = 4\cos x \cos y \cos(2x+y)\cos(2y+x) - \cos^2(2y+2x)$

$= 4 - 1 = 3 > 0$ at $(0,0)$

$r = -2 < 0$ $\therefore$ $(0,0)$ is point of maxima.

at $(\frac{\pi}{2}, \frac{\pi}{2})$ $rt - s^2 = 0 - 0 = 0$ $\therefore$ case is doubtful.

at $(\frac{\pi}{3}, \frac{\pi}{3})$ $rt - s^2 = 4 \times \frac{1}{2} \times \frac{1}{2} \cos\pi \cos\pi + \cos^2\frac{\pi}{3}$

$= 1 + \frac{1}{4} = \frac{5}{4} > 0.$

$r = -2 \times \frac{1}{2} \cos\pi = 1 > 0$ $\therefore$ point of minima.

at $(\frac{2\pi}{3}, \frac{2\pi}{3})$ $rt - s^2 = 4 \times -\frac{1}{2} \times -\frac{1}{2} (1)(1) + (\frac{1}{4})$

$\Rightarrow$ $1 + 1 = 2 > 0.$

$r = -2(-\frac{1}{2})(1) = 1 > 0$ pt. of minima.

Problems for Practice!

2011 Q.1 find minimum value of $f(x,y) = x^2 + y^2$

2012 Q.2 find minimum value of $x^2 + y^2 + 6x + 12 = 0$.

2014 Q.3 find stationary points of $5x^2 + 10y^2 + 12xy - 4x - 6y + 1$

201 Q.4 Test the function $f(x,y) = x^3y^2(6-x-y)$ for maxima and minima for points not at the origin.

13 Q.5. Show that minimum value of $xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$

Engineering Mathematics - I
KAS-103.
Module - III
Differential Calculus-II
Lecture - 38

Content: Lagrange's Method of Multipliers for Maxima and Minima of function.

Lagrange's Method of Multipliers: To find extreme value of function of several variable (more than two), when variables are connected by a relation. Lagrange's Method is used.

Working Process:-

1) Consider a function $u = f(x, y, z)$ where x, y, z are connected by a relation $\phi(x, y, z) = 0$ to be maximised or minimised.

$F = f + \lambda\phi$

2). $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad \text{— (1)}$$

& $$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \quad \text{— (2)}$$

Multiplying (2) by λ (unknown multiplier and adding (1) we get

$$du = (f_x + \lambda\phi_x)dx + (f_y + \lambda\phi_y)dy + (f_z + \lambda\phi_z)dz$$

3). put $du = 0$ for stationary points.

or. $$\left.\begin{aligned} f_x + \lambda\phi_x &= 0 \\ f_y + \lambda\phi_y &= 0 \\ f_z + \lambda\phi_z &= 0 \end{aligned}\right]$$ eq^ns gives value of λ, x, y, z.

4) substitute these values which gives extreme value of u.

Remark:-

$r = -2 < 0 \quad \therefore (0,0)$ is point of maxima

Q.1 Find the minimum values of $x^2+y^2+z^2$ when $x+y+z=3a$.

Sol. Let $f(x,y,z) = x^2+y^2+z^2$ —(1) & $\phi(x,y,z) = x+y+z-3a$

Let $F = f + \lambda\phi$

put $dF = (f_x + \lambda\phi_x)dx + (f_y+\lambda\phi_y)dy + (f_z+\lambda\phi_z)dz$

For maximum or minimum value we put $dF=0$.

$f_x + \lambda\phi_x = 0 \qquad 2x+\lambda = 0 \qquad \Rightarrow x = -\lambda/2$

$f_y + \lambda\phi_y = 0 \qquad 2y+\lambda = 0 \qquad \Rightarrow y = -\lambda/2$

$f_z + \lambda\phi_z = 0 \qquad 2z+\lambda = 0 \qquad z = -\lambda/2$.

we get $x = y = z$. substituting in eq^n (2)

$3x = 3a \Rightarrow x = a$.

$\therefore$ Minimum value at (a,a,a) is. $3a^2$ Ans.

2014

Q.2 Divide 24 into three parts such that continued product of first, square of second and cube of third may be maximum.

Sol. $x+y+z=24$. where x,y,z are three parts of 24.

function to be maximised. $f(x,y,z) = xy^2z^3$

Consider $F = f + \lambda\phi$

$F = xy^2z^3 + \lambda[x+y+z-24]$ —(1)

for Max value $dF=0$

$f_x+\lambda\phi_x=0 \qquad y^2z^3+\lambda=0$ —(2)

$f_y+\lambda\phi_y=0 \qquad 2xyz^3+\lambda=0$ —(3)

$f_z+\lambda\phi_z=0 \qquad 3xy^2z^2+\lambda=0$ —(4)

(2) − (3) gives. $y^2z^3 - 2xyz^3 = 0 \quad yz^3[y-2x]=0$.

(3)−(4) gives. $2xyz^3 - 3xy^2z^2 = 0 \quad xyz^2[2z-3y]=0$

(4)−(2) gives $3xy^2z^2 - y^2z^3 = 0 \quad y^2z^2[3x-z]=0$

$y = 2x \qquad 2z = 3y \qquad$ or $z = 3x$.

Hence $x + 2x + 3x = 124$

$x = 24$

$y = 2x = 48, \quad z = 3x = 24 \times 3 = 82.$

$$\boxed{x = 24,\ y = 48,\ z = 82}$$ Ans.

Q.3 A wire of length b is cut into 2 small parts which are ~~bent~~ in the form of a square and circle. Find least value of sum of areas so found using Lagrange's Method of multipliers.

Sol. Let $x + y = b$, x is side of square. y is perimeter of circle.

$x = 4a$ $2\pi r = y$

or $a = x/4$ $r = \frac{y}{2\pi}$

$\therefore$ Area of square $= \frac{x^2}{16}$

Area of circle $= \frac{y^2}{4\pi}$

$$f = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda[x + y - b]$$

$df = 0$

$f_x + \lambda \phi_x = 0 \quad = \quad \frac{2x}{16} + \lambda = 0$ (1)

$f_y + \lambda \phi_y = 0 \quad \frac{2y}{4\pi} + \lambda = 0$ (2)

(1) − (2) gives $\frac{x}{8} = \frac{y}{2\pi} \Rightarrow x = \frac{4}{\pi} y$. −(3)

substitute this value in $x + y = b$

$\frac{4}{\pi} y + y = b.$

$(4 + \pi) y = b\pi$

$y = \frac{b\pi}{4 + \pi}$

$x = \frac{4}{\pi}\left[\frac{b\pi}{4 + \pi}\right]$

$\Rightarrow \frac{4b}{4 + \pi}$

$$\boxed{\text{Area} = \frac{x^2}{16} + \frac{y^2}{4\pi} = \frac{b^2}{4(4+\pi)}}$$ Ans.

Problems for Practice:-

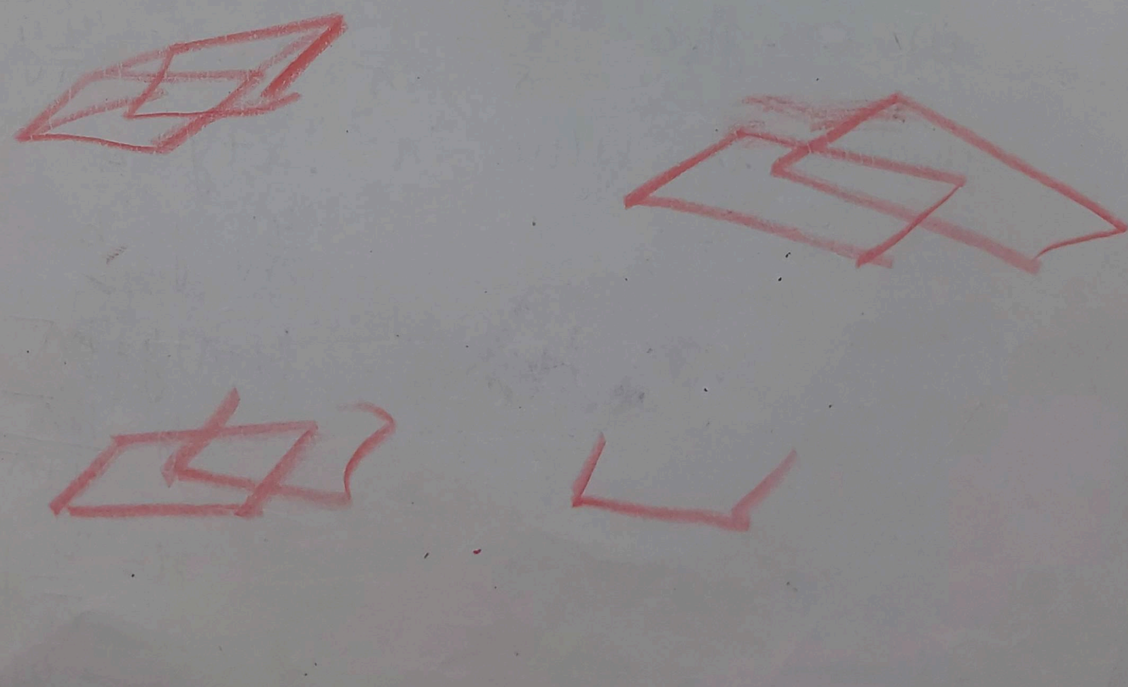
2009,12 Q.1 Find maximum and minimum distance of pt. (3,4,12) from the sphere $x^2+y^2+z^2=1$

2010 Q.2 find the volume of largest rectangular parallel-o-piped that can be inscribed in ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Q3 Find the minimum distance from the point
2018 (1,2,0) to the cone $z^2 = x^2+y^2$.

Q.4 The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.

2012 find the maximum and minimum distances from the origin to the curve $x^2+4xy+6y^2=140$.



$t = f_{yy} = -2\cos x \cos(2y+x)$

Engineering Mathematics-I

KAS-103

Module-III

Lecture-38 Differential Calculus-II

Content:- Problems based on Lagrange's Method and Maxima and Minima.

9,10,14,10,15

2014 Q. 2012 Q. 2014 Q. 2001 Q. 2013 Q5.

Q1 Find the dimensions of rectangular box of maximum capacity whose surface area is given when
(1) Box is open at the top.
(2) Box is closed.

Sol. Let x, y, z be dimensions of box. So volume $V = xyz$.

Total Surface area is (open at the top) $S = xy + 2yz + 2zx$ —(1)

$F = xyz + \lambda[xy + 2yz + 2zx - S]$

$dF = 0$

$yz + \lambda[y + 2z] = 0$ —(2)

$xz + \lambda[x + 2z] = 0$ —(3)

$xy + \lambda[2y + 2x] = 0$ —(4)

$xyz + \lambda[xy + 2zx] = 0$

$xyz + \lambda[xy + 2yz] = 0$

$xyz + \lambda[2yz + 2zx] = 0$

subtract

$\lambda[xy + 2zx - xy - 2yz] = 0$

$2\lambda z[x - y] = 0$

$\lambda[xy - 2zx] = 0$

or $\lambda x[y - 2z] = 0$.

$\lambda y[2z - x] = 0$.

i.e. $x = y = 4K$

$y = 2z = 2k$

$x = 2z = 2k$ Let $z = k$

Substituting in (1)

$x^2 + 2(2z)z + 2z(2z) = S$

$4z^2 + 4z^2 + 4z^2 = S$

$z^2 = \frac{S}{12}$

$z = \frac{1}{2}\frac{S}{\sqrt{3}}$ $y = \frac{S}{\sqrt{3}}$ $x = \frac{S}{\sqrt{2}}$

for closed

$S = 2xy + 2yz + 2zx$

$x = y = z = \sqrt{\frac{S}{6}}$

Q.2 If $u = ax^2 + by^2 + cz^2$ and $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$. prove that stationary points satisfy eqn.

$$\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$$

Sol. $F = ax^2 + by^2 + cz^2 + d_1(x^2 + y^2 + z^2 - 1) + d_2(lx + my + nz)$

$2ax + 2d_1x + ld_2 = 0 \quad (1)$

$2by + 2d_1y + md_2 = 0 \quad (2)$

$2cz + 2d_1z + nd_2 = 0 \quad (3)$

Multiplying (1) by x, (2) by y, (3) by z.

$2[ax^2 + by^2 + cz^2] + 2d_1[x^2 + y^2 + z^2] + d_2[lx + my + nz] = 0$

$2d_1 = -2u \quad$ or $\boxed{d_1 = -u}$

substituting in eqns (1), (2) and (3)

$2ax - 2xu + ld_2 = 0 \qquad x = -\frac{d_2 l}{2(a-u)} = \frac{d_2 l}{2(u-a)}$

$2by - 2yu + md_2 = 0 \qquad y = \frac{d_2 m}{2(u-b)}$

$2cy - 2zu + nd_2 = 0$

$z = \frac{d_2 n}{2(u-c)}$

$lx + my + nz = 0$

$$d_2\left[\frac{l^2}{2(u-a)} + \frac{m^2}{2(u-b)} + \frac{n^2}{2(u-c)}\right] = 0.$$

$$\boxed{\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0}$$ Hence proved.

Lecture-39

Content: Jacobian's and Properties of Jacobians.

Jacobians:- Jacobian is a functional determinant, useful in transformation of variables from cartesian to polar, cylindrical and spherical polar co-ordinates in multiple integrals.

Jacobian for two functions of two variables:-
Let u and v are functions of two independent variables x & y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called Jacobian of u & v with respect to x & y. $u(x,y)$, $v(x,y)$

Notation: The notation used for Jacobian is

$$J \text{ or } \frac{\partial(u,v)}{\partial(x,y)} \text{ or } J(u,v)$$

Q.1. If $u = x^2$, $v = y^2$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol. $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy \text{ Ans.}$$

Q.2 If $u = x(1-y)$ and $v = xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol. $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x - \cancel{xy} + \cancel{xy}$$
$$= x \text{ Ans}$$

Jacobian for three functions of three Variables:-
Let u, v and w are functions of three independent variables x, y, z then the determinant denoted by $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Partial Derivatives Using Jacobians:- We can find partial derivatives of implicit functions.

Let ...

$$\frac{J(u,v,w)}{\partial} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$ is called Jacobian of u, v and w with respect to x, y and z respectively.

Remark:- 1) $\frac{\partial(u,v)}{\partial(u,v)} = 1$

2) The formula can be extended for n independent variable

3) The Jacobian is determinant ∴ no. of functions = no. of variables.

2007-08

Q.1 Calculate $J(u,v,w)$ of $u = x+2y+z$, $v = x+2y+3z$ and $w = 2x+3y+5z$.

Sol. $$J(u,v,w) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 2.$$

Q.2 If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ find $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$

Sol. $$J(y_1,y_2,y_3) = \begin{vmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix} = \frac{1}{x_1 x_2 x_3} \begin{vmatrix} -\frac{x_2 x_3}{x_1} & x_3 & x_2 \\ x_3 & -\frac{x_3 x_1}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1 x_2}{x_3} \end{vmatrix}$$

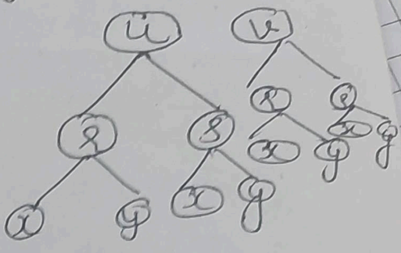
$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_1 x_3 & x_1 x_2 \\ x_3 x_2 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix} = \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 4$ Ans.

Properties of Jacobians.

Chain Rule: - If u and v are functions of r, s and r, s are functions of x, y.

then $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)}$



Similarly if u, v, w are functions of r, s, t and r, s, t are functions of x, y, z.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$$

2. If $J_1 = \dfrac{\partial(u,v)}{\partial(x,y)}$ and $J_2 = \dfrac{\partial(x,y)}{\partial(u,v)}$

then $J_1 \cdot J_2 = 1$ i.e. $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1$.

3. If $u_1 = f(x_1)$
$u_2 = f(x_1, x_2)$
$u_3 = f(x_1, x_2, x_3)$
$\vdots$
$u_n = f(x_1, x_2, x_3, \ldots x_n)$

then $J = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\ \vdots & & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$

$$J(u_1, u_2, \ldots u_n) = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

v.v. Imp.

Q.1 If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ and $\dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

Sol. $J(x,y,z) = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} r\sin\theta\cos\phi & r\cos\theta\cos\phi \\ r\sin\theta\sin\phi & \\ \cos\theta & \end{vmatrix}$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi\,[r^2\sin^2\theta\cos\phi] + r^2\cos^2\theta\cos\phi\sin\theta\cos\phi - r\sin\theta\sin\phi\,[-r\sin^2\theta\sin\phi - r\cos^2\theta\cos\phi]$

$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta$

$= r^2\sin\theta$

$J' = \dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$ using property II $\quad J' = \dfrac{1}{J}$

$$\boxed{J' = \frac{1}{r^2\sin\theta}}$$

⇒ Problems for Practice:-

Q.2 If $u = xyz$, $v = xy + yz + zx$ and $w = x + y + z$ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Q.3 If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ & $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ find $J(x,y,z)$

Q.4 If $y_1 = \cos x_1$, $y_2 = \sin x_1\cos x_2$, $y_3 = \sin x_1\sin x_2\sin x_3$ find $\dfrac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = -\sin^3 x_1\sin^2 x_2\sin x_3$. using property III.

Q.5 If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Q.6 If $u = x(1-r^2)^{-1/2}$, $v = y(1-r^2)^{-1/2}$, $w = z(1-r^2)^{-1/2}$ and $r^2 = x^2 + y^2 + z^2$ then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (1-r^2)^{-5/2}$

Q.7 verify chain rule.
2008
$x = u$, $y = u\tan v$, $z = w$

Q.8 $x = uv$, $y = \dfrac{u+v}{u-v}$ find $\dfrac{\partial(u,v)}{\partial(x,y)}$

KAS-103 Module III
Lecture-40 Differential Calculus-II

Contents: Jacobians for Implicit functions & Functional Relationship, partial derivative using Jacobian.

Implicit function:- A function or relation in which the dependent variable is not isolated on one side of the equation. Thus $x^2+xy-y^2=1$ represents an implicit relation.

Jacobian of implicit functions: If $u_1, u_2, u_3, \ldots u_n$ are implicit functions of variables $x_1, x_2, \ldots x_n$ then

$$F_1(u_1, u_2, \ldots u_n, x_1, x_2 \ldots x_n) = 0$$
$$F_2(u_1, u_2, \ldots u_n, x_1, x_2 \ldots x_n) = 0$$
$$F_n(u_1, u_2, \ldots u_n, x_1, x_2 \ldots x_n) = 0.$$

$$\frac{\partial(u_1, u_2, u_3, \ldots u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)}}{\frac{\partial(F_1, F_2, \ldots F_n)}{\partial(u_1, u_2, \ldots, u_n)}}$$

Q1 If $x+y+z=u$, $y+z=uv$, $z=uvw$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(F_1, F_2, F_3)}{\partial(u,v,w)}}{\frac{\partial(F_1, F_2, F_3)}{\partial(x,y,z)}}$$

$F_1 = x+y+z-u=0$
$F_2 = y+z-uv=0$
$F_3 = z-uvw=0$

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u,v,w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} = -u^2v$$

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

$\boxed{\text{Hence } J = u^2v}$

Ans.

variable. ... $f(x+h)$ can be ...
of h, where x is independent of h

Q2 If $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^3+y^3+z^3$, $u+v+w = x^2+y^2+z^2$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Q3 If λ,μ,ν are roots of eq^n $\frac{x}{a+k}+\frac{y}{b+k}+\frac{z}{c+k}=1$ in k, prove that $\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = \frac{-(\nu-\lambda)(\lambda-\mu)(\mu-\nu)}{(a-b)(b-c)(c-a)}$

Sol. consider eq^n.

$$x(b+k)(c+k) + y(a+k)(c+k) + z(a+k)(b+k) = (a+k)(b+k)(c+k)$$

$$= x[bc+k^2+(b+c)k] + y[ac+k^2+(a+c)k] + z[ab+(a+b)k+k^2]$$

$$= abc+kab+(a+b)ck+(a+b)k^2+k^2c+k^3.$$

$$= k^3+[a+b+c-x-y-z]k^2+k[ab+bc+ca-(b+c)x-(a+c)y-(a+b)z] + abc-xbc-acy-abz=0.$$

sum of roots $= \frac{-b}{a}$

product of roots taken two at a time $= \frac{c}{a}$

product of roots taken three at a time $= -\frac{d}{a}$

of eq^n $ax^3+bx^2+cx+d=0$

$$\therefore \lambda+\mu+\nu = x+y+z-a-b-c.$$

$$\lambda\mu+\mu\nu+\nu\lambda = ab+bc+ca-(b+c)x-(a+c)y-(a+b)z.$$

$$\lambda\mu\nu = xbc+acy+abz-abc.$$

$$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = (-1)^3 \frac{\partial(F_1,F_2,F_3)}{\partial(\lambda,\mu,\nu)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}$$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(\lambda,\mu,\nu)} = (\lambda-\mu)(\lambda-\nu)(\mu-\nu)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = -(a-b)(b-c)(c-a)$$

$$J = \frac{-(\alpha-\mu)(\mu-\nu)(\nu-\alpha)}{(a-b)(b-c)(c-a)}$$

Problems for practice!

Q.1 If u, v, w are roots of eqn $(\alpha-x)^3 + (\alpha-y)^3 + (\alpha-z)^3 = 0$ and then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Q.2 If $x^2+y^2+u^2-v^2=0$ and $uv+xy=0$ then prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2-y^2}{u^2+v^2}$

Q.3 If $u^3+v^3 = x+y$, $u^2+v^2 = x^3+y^3$ then show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2-x^2}{2uv(u-v)}$$

Functional Relationship: Let $u_1, u_2 \ldots u_n$ be functions of $x_1, x_2, \ldots, x_n$. Then necessary condition for the existence of relation of for $F(u_1, u_2 \ldots u_n) = 0$ is that Jacobian $\frac{\partial(u_1, u_2 \ldots, u_n)}{\partial(x_1, x_2 \ldots, x_n)} = 0$ or vanish identically.

i.e. variables are related to each other and we can find relation between them by hit and trial.

Remark! - If $J=0$, functions are linearly dependent else linearly Independent.

Q.1 If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$. Verify that they are functionally dependent and if so find relation between them.

Sol.
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1-xy+y(x+y)}{(1-xy)^2} & \frac{(1-xy)+x(x+y)}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0.$$

$\therefore$ u & v are functional dependent.

$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} u$ or $\boxed{u = \tan v}$

Q.2 Determine functional dependence & find relation.

$u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$.

Sol. $\frac{\partial u}{\partial x} = \frac{x-y-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$\frac{\partial u}{\partial y} = \frac{2x}{(x-y)^2}$, $v_x = \frac{-y(x+y)}{(x-y)^3}$, $v_y = \frac{x(x+y)}{(x-y)^3}$

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{-y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix} = \frac{1}{(x-y)^5}\, 2(x+y) \begin{vmatrix} -y & x \\ -y & x \end{vmatrix}.$$

$\therefore$ Functional relationship exists $= 0$ between u and v.

$u = \frac{x+y}{x-y}$

$u+1 = \frac{x+\cancel{y}+x-\cancel{y}}{x-y} = \frac{2x}{x-y}$.

$u-1 = \frac{x+y-x+y}{x-y} = \frac{2y}{x-y}$.

$\frac{x}{y} = \frac{u+1}{u-1}$

$v = \frac{xy}{(x-y)^2} \Rightarrow \frac{\cancel{y^2}\left(\frac{x}{y}\right)}{\cancel{y^2}\left[\frac{x}{y}-1\right]^2} \Rightarrow \frac{\frac{u+1}{u-1}}{\left[\frac{u+1}{u-1}-1\right]^2} = \frac{\frac{u+1}{u-1}}{\frac{4}{(u-1)^2}}$

$\Rightarrow \frac{u+1}{u-1}\,\frac{(u-1)^2}{4}$

$4v = (u+1)(u-1)$

or $\boxed{u^2 - 4v - 1 = 0}$ Ans.

Problems for practice!

Q.1 Show that $u = y+z$, $v = x+2z^2$, $w = x-4yz-2y^2$ are not independent.

Q.2 Show $u = x^2+y^2+z^2$, $v = x+y+z$, $w = yz+zx+xy$ are not independent. If not find relation between them.

$u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2zx$ & $w = x^3+y^3+z^3-3xyz$ show that they are functionally related, hence find relation between them.

Partial Derivatives Using Jacobians:- We can find partial derivatives of implicit functions using Jacobians.

Let $f_1(x,y,z,u,v,w)=0$

$f_2(x,y,z,u,v,w)=0$

$f_3(x,y,z,u,v,w)=0$

$f_n(x_1,x_2,x_3,\ldots x_n, u_1,u_2,\ldots u_n)=0$

$$\frac{\partial x}{\partial u} = (-1)\frac{\partial(f_1,f_2,\ldots,f_n)/\partial(u,y,z)}{\partial(f_1,f_2,\ldots,f_n)/\partial(x,y,z)}$$

Remark:- first we write Jacobian in the denominator then we write Jacobian in numerator by replacing x by u.

$$\frac{\partial u}{\partial x} = (-1)\frac{\partial(f_1,f_2,\ldots,f_n)/\partial(x,v,w)}{\partial(f_1,f_2,\ldots,f_n)/\partial(u,v,w)}$$

Q: Given $x=u+v+w$, $y=u^2+v^2+w^2$, $z=u^3+v^3+w^3$ find $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = (-1)\frac{\partial(f_1,f_2,f_3)/\partial(x,v,w)}{\partial(f_1,f_2,f_3)/\partial(u,v,w)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2v & 2w \\ 0 & 3v^2 & 3w^2 \end{vmatrix} = -6vw\begin{vmatrix} 1 & 1 \\ v & w \end{vmatrix} = 6vw(v-w)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ 2u & 2v & 2w \\ 3u^2 & 3v^2 & 3w^2 \end{vmatrix} = 6(v-u)(w-u)(w-v)$$

$$\boxed{\therefore \frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)}}$$ Ans.

2) To expa...
$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$

Q: find $\left(\frac{\partial u}{\partial x}\right)_v$ if $u^2 + xv^2 - xy = 0$
$u^2 + xyv + v^2 = 0$

Sol. $f_1 = u^2 + xv^2 - xy$, $f_2 = u^2 + xyv + v^2$.

$$\frac{\partial u}{\partial x} = -\frac{\partial(f_1, f_2)/\partial(x,v)}{\partial(f_1,f_2)/\partial(u,v)}$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} 2u & 2xv \\ 2u & xy+2v \end{vmatrix} = 2uxy + 4uv - 4uxv$$

$$\frac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} v^2-y & 2xv \\ yv & xy+2v \end{vmatrix} = -xyv^2 + 2v^3 - xy^2 - 2yv$$

$$\left(\frac{\partial u}{\partial x}\right)_v = xyv^2 - 2v^3 + xy^2 + 2yv / 2uxy + 4uv - 4uxv$$

– treating v as constant.

Problems for Practice:-

Q1 If $u = x + y^2$, $v = y + z^2$, $w = z + x^2$
find $\frac{\partial x}{\partial u}$.

Q2 If $u = x^2 + y^2 + z^2$, $v = xyz$ find $\frac{\partial x}{\partial u}$.

Q3 If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$. find $\frac{\partial x}{\partial u}$.

Content:- Ap...
Approxima...
Working...
1. Cons...
2. Re...
3. C...
4.
5

Engineering Mathematics-I
KAS-103
Module-III
Differential Calculus-II
Lecture - 4.1

Content:- Approximation.

Approximation:- using error approximation the approximate values of expressions can be determined.

Working Process:- To approximate value of expressions.

1. Consider the given expression
2. Round of each term to nearest integer. eg. $(27.1)^{1/3} = (27)^{1/3}$
3. Consider the given expression as function $y = f(x) = x^{1/3}$.
   Consider the approximate numbers as variable.
4. Substitute these approximate numbers as take $x = 27$.
5. The difference in original value and estimated value is termed as error $\delta x = 27.3 - 27 = 0.3$.
6. $\delta y = \frac{\partial f}{\partial x} \delta x$.
7. $f(x+\delta x) = y + \delta y$.

Q1- Find approximate value of $(27.2)^{1/3}$.

Sol. 1) Round off to nearest $(27)^{1/3}$ where $x = 27$.

2) Let $y = f(x) = x^{1/3}$

3) $\delta x = 27.2 - 27 = 0.2$.

4) $\delta y = \frac{\partial f}{\partial x} \delta x =$

$$\delta y = \frac{1}{3} x^{-2/3} . 0.2$$
$$= \frac{1}{3} (27)^{-2/3} \times 0.2$$
$$= \frac{1}{27} \times 0.2$$

$$\delta y = 0.007.$$

5) Hence $f(x+\delta x) = y + \delta y$
$$= 3 + 0.007$$
$$f(27.2) = 3.007$$

Q2 Find approximate value of $[(3.82)^2 + 2(2.1)^3]^{1/5}$

Sol Let $f(x,y) = [x^2+2y^3]^{1/5}$

when $x=4$, $y=2$

$\delta x = -4+3.82$, $\quad \delta y = 2.1-2 = 0.1$

$= -0.18$

$$\frac{\partial f}{\partial x} = \frac{1}{5}[x^2+2y^3]^{-4/5} \times 2x$$

$$= \frac{1}{5}[16+16]^{-4/5} \times 8 = \frac{1}{5} \times \frac{8}{2^4} = \frac{1}{10}$$

$$\frac{\partial f}{\partial y} = \frac{1}{5}(x^2+2y^3)^{-4/5}\, 6y^2$$

$$= \frac{1}{5}[16+16]^{-4/5} \times 6 \times 2^2 = \frac{1}{5} \times \frac{2^3 \times 3}{2^4} = \frac{3}{10}$$

By total differentiation

$$df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$= \frac{1}{10}(0.18) + \frac{3}{10}(0.1) = 0.012$$

$$\therefore\ f(x+\delta x, y+\delta y) = f(x,y) + \delta f$$

$$f(3.82, 2.1) = f(4,2) + 0.012 = 2.012$$

Q.3 Compute $(1.99)^2 \times (1.02)^{1/10}$

Sol:- $f(x,y) = x^2 \times y^{1/10}$ where $x=2$, $\delta x = -0.01$

$y=1$, $\delta y = 0.02$

$$f_x(x,y) = 2x\,y^{1/10} = 4$$

$$f_y(x,y) = \frac{1}{10}x^2 y^{-9/10} = \frac{4}{10}$$

$$\delta f = f_x\,\delta x + f_y\,\delta y$$

$$4 \times -0.01 + \frac{4}{10} \times 0.02$$

$$= -0.04 + 0.008$$

$$= -0.392$$

$f(x+\delta x, y+\delta y) = f(x,y) + \delta f$

$f(1.99, 3.01) = f(2,1) + \delta f$

$= 2^2 \times 1^{1/10} + (-0.392)$

$= 4 - 0.392$

$\boxed{f(1.99, 3.01) = 3.608}$ Ans.

Q.4 find approximate values of $\sqrt{(0.98)^2 + (2.01)^2 + (1.94)^2}$

Sol. $x = 1, \; y = 2, \; z = 2.$

$\delta x = -0.02 \quad \delta y = 0.01, \quad \delta z = -0.06.$

$f(x,y,z) = \sqrt{x^2+y^2+z^2}.$

$f_x = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\delta f = f_x \delta x + f_y \delta y + f_z \delta z$

$= \frac{1}{3}(-0.02) + \frac{2}{3}(0.01) + \frac{2}{3}(-0.06)$

$= \frac{1}{3}[-0.02 + 0.02 - 0.12] = -0.04$

$f(x+\delta x, y+\delta y, z+\delta z) = f(x,y,z) + \delta f$

$\boxed{f(0.98, 2.01, 1.94) = 3 - 0.04 = 2.96}$ Ans.

Problems for Practice:-

1) find approximate value when $x = 1.99, y = 3.01, z = 0.98$ for $f(x,y,z) = x^2 y^2 z^{1/10}$. 2010-11

2) find approximate value of $(2.4)^{3.1}$

3) find approximate value of $(x)^y$ where $x = 4.01$ & $y = 2.99$.

4) find approximate value of $[(3.98)^2 + (2.01)^2 + 3(1.94)^2]^{1/5}$ 2009-10

Ranjeet Kumar

Engineering Mathematics I
KAS-103

Lecture-42

Module-III
Differential Calculus II

Content:- Errors

Errors:- When one quantity or variable is related to several others by a functional relationship it is possible to estimate the percentage change in it caused by given percentage change in the other variable.

When input variables are measured and measurement are in error, due to limits on precision of measurement, then we can estimate the effect these errors have on the output.

Error determination:- Consider a function $z = f(x,y)$

if δx and δy are small increments in variables x & y. then an approximate change or error in z denoted by δz can be measured as.

$$\boxed{\delta z = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y}$$

i.e. Let $z = f(x,y)$ —(1)

$z + \delta z = f(x+\delta x, y+\delta y)$ —(2)

(2)-(1) gives -

$\delta z = f(x+\delta x, y+\delta y) - f(x,y)$

or $\delta z = \cancel{f(x,y)} + \delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y} + \dots - \cancel{f(x,y)}$

$$\delta z = \delta x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\delta y$$

$$\boxed{\text{or } dz = f_x dx + f_y dy}$$

Absolute error: δx is known as absolute error in x.

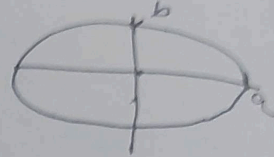
Relative Error: $\delta x/x$ is known as relative error in x!

Percentage Error: $\frac{\delta x}{x} \times 100$ is known as percentage error in x.

Q1 Find percentage error in area of ellipse when error of 1 percent is made in measuring major and minor axis.

Sol: Consider ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $2a$ be major and $2b$ be minor axis



Area of ellipse $= \pi ab$

$A = \pi ab$

Taking log $A = \log\pi + \log a + \log b$.

Total Differentiating both sides.

$$\frac{1}{A}\delta A = 0 + \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{\delta A}{A}\times 100 = \frac{\delta a}{a}\times 100 + \frac{\delta b}{b}\times 100$$

$$\frac{\delta A}{A}\times 100 = 1+1.$$

$\Rightarrow$ percentage error in Area $= 2\%$ Ans.

Q2 Find percentage error in area of rectangle when an error of 1 percent is made in measuring the sides.

Q3 The period T of simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$ find maximum percentage error in T due to possible error of 2% in l and 5% in g.

Imp

Sol.
$$T = 2\pi\sqrt{\frac{l}{g}}$$

$\boxed{3.5\%}$ Ans.

Taking log both sides

$$\log T = \log 2\pi + \frac{1}{2}[\log l - \log g]$$

Differentiating

$$\frac{1}{T}\delta T = 0 + \frac{1}{2}\left[\frac{1}{l}\delta l - \frac{1}{g}\delta g\right]$$

$$\frac{1}{T}\delta T\times 100 = \frac{1}{2}\left[\frac{1}{l}\delta l\times 100 - \frac{1}{g}\delta g\times 100\right]$$

$$= \frac{1}{2}[2 \pm 5]$$

The power 'P' required to propel a steamer of length 'l' at a speed 'u' is given by $P = \lambda u^3 l^3$ where λ is constant. If u is increased by 3% & l is decreased by 1% find Increasing

Sol. $P = \lambda u^3 l^3$

$\log P = \log \lambda + 3\log u + 3\log l$

$\frac{1}{P}\delta P = 0 + \frac{3}{u}\delta u + \frac{3}{l}\delta l$

$\because$ decreased $\boxed{\frac{\delta l}{l} \times 100 = -1}$

$\frac{\delta P}{P} \times 100 = 3 \times 3 - 3(1)$.

$\boxed{\frac{\delta P}{P} \times 100 = 6\%}$ Ans.

Q.5 In estimating the number of bricks in a pile measured ($5m \times 10m \times 5m$), count of bricks is 100 bricks per m^3.

2004 2003. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks Rs 2000 per 1000 bricks.

Sol:- volume of pile $= l \times b \times h$

$\log V = \log l + \log b + \log h$

$\frac{\delta V}{V} \times 100 = \frac{\delta l}{l} \times 100 + \frac{\delta b}{b} \times 100 + \frac{\delta h}{h} \times 100$

$\frac{\delta V}{V} \times 100 = 2 + 2 + 2$

$\delta V = \frac{6V}{100}$

$\delta V = \frac{6 \times (5 \times 10 \times 5)}{100} = 15 m^3$.

No. of bricks in $\delta V = 15 \times 100 = 1500$.

Cost of 1 brick $= \frac{Rs. 2000}{1000} = Rs. 2$.

$\boxed{\text{Error in cost} = Rs 2 \times 1500 = Rs. 3000}$ Ans.

Q.6 The Diameter and height of right circular cylinder are found by measurement to be 8.0 cm and 12.5 cm respectively with possible error 0.05 in each. Find maximum approximate error in volume.

$V = \pi r^2 h = \frac{\pi D^2 h}{4}$

$\frac{\delta V}{V} = \frac{2}{D}\delta D + \frac{\delta h}{h}$

$\frac{\delta v}{v} = \frac{2}{8} \times 0.05 + \frac{1}{12.5} 0.05$

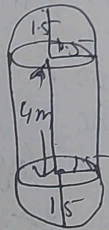
$\delta V = 0.05 \left[\frac{1}{4.0} + \frac{1}{12.5}\right] \times V$

$= 0.05 [12.5 + 4.0] \times \pi 4 \times 4 \times 12.5$

$3.3\pi \, cm^3$ Ans.

Q.7 A ballon is in the form of right circular cylinder of
2006
$r = 1.5$ and length $= 4\,m$ and surmounted by hemispheric ends. If radius is increased by $0.01\,m$ & length is increased by 0.05 find percentage change in volume of b...

$r = 1.5, \; h = 4\,m$



Sol. $V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$

$V = \pi r^2 h + \frac{4}{3}\pi r^3$

$\delta V = 0 + 2\pi r h \delta r + \pi r^2 \delta h + \frac{4}{3} \times 3\pi r^2 \delta r$

$\delta V = 2 \times 1.5 \times 4 \times \delta r + \pi (1.5)^2 \delta h + 4\pi (1.5)^2 \delta r$

$= 8 \times 1.5 \times 0.01 + \pi [1.5]^2 [0.05 + 4 \times 0.01]$

$= 1.5 \times 0.08 + \pi (2.25)(0.09)$

$\frac{\delta v}{V} = \frac{1.5 \times 0.08 + \pi (2.25)(0.09)}{\pi r^2 h + \frac{4}{3}\pi r^3}$

$\Rightarrow 2.389\%$

Problems for Practice

2011-12
1) If radius & height of cone are 4 cm and 8 cm with a possible error of 0.04 and 0.08 inches respectively. Find percentage error in volume of cone.