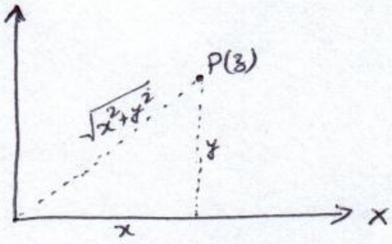
Complex Number :- The number $z = x+iy$ is called a complex number, where x and y are real numbers.



Again $\bar{z} = x - iy$

$\therefore \; z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$

or $\boxed{|z| = \sqrt{z\bar{z}}}$

Complex Variable :- $w = f(z) = u(x,y) + i\,v(x,y)$ is known as complex variable, where $z = x+iy$.

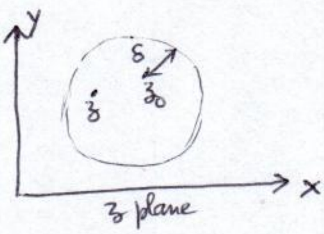
Note :- For one value of z we obtain only one value of w is known as single valued function. For one value of z we obtain more than one values of w is known as many valued function.

for example $w = z^2$ (single valued) & $w = z^{1/2}$ (many valued).

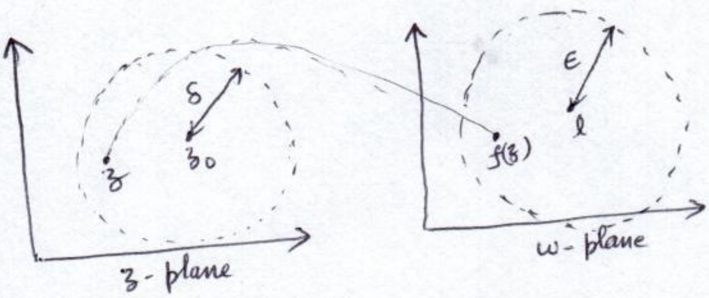
Neighbourhood of z_0 :- Let z_0 is a point in complex plane and let z be any set of points, then

$|z - z_0| < \delta$, where $\delta > 0$,

then z is called nbd of z_0.



Limit :- Let $f(z)$ be a single valued function at all points in same nbd at point z_0. Then $f(z)$ is said to be the limit l as z approches z_0 (ie. $\lim\limits_{z\to z_0} f(z) = l$) along any path such that



$|f(z) - l| < \epsilon$ & $|z - z_0| < \delta$ where $\epsilon > 0$ & $\delta > 0$.

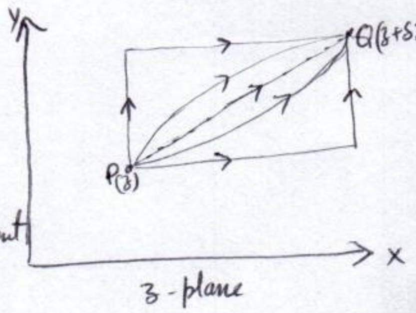
Continuity :- The function $f(z)$ is said to be continuous at $z = z_0$ if

$\lim\limits_{z\to z_0} f(z) = f(z_0)$ s.t. $|f(z) - f(z_0)| < \epsilon$ & $|z - z_0| < \delta$ when $\epsilon > 0$ & $\delta > 0$

Differentiability :- Let $f(z)$ be a single valued function then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent upon the path along which $\delta z \to 0$.



<u>Analytic Function</u> :— A function $f(z)$ is said to be analytic at pt. z_0, if it is differentiable not only at z_0 but at every point in the nbd of z_0.

<u>Singular Point</u> :— The point at which the function is not differentiable is called singular point.

<u>Entire Function</u> :— A function which is analytic every where is known as entire function.

<u>Cauchy Riemann Equation</u> :— The necessary and sufficient condition for a function $f(z) = u + iv$ be analytic at all the point in a region R, then

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ i.e. $u_x = v_y$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ i.e. $u_y = -v_x$

provided that u_x, u_y, v_x & v_y are exists.

<u>Proof</u> :— <u>Necessary Condition</u> :— Let $f(z)$ be analytic in region R.

Since $f(z) = u + iv$, hence $f(z+\delta z) = u + \delta u + i(v + \delta v)$

$$\text{Now } f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\cancel{u} + \delta u + i(\cancel{v} + \delta v) - (\cancel{u} + i\cancel{v})}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u + i\delta v}{\delta z}\right)$$

Since $\delta z \to 0$ along any path, hence now we consider

(i) <u>$\delta z \to 0$ along real axis (i.e. $y \to 0$)</u> :— $(z = x + iy,\ y \to 0 \ \therefore\ z = x$ i.e. $\delta z = \delta x.)$

$$\therefore f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u + i\delta v}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{— (A)}$$

(ii) Let <u>$\delta z \to 0$ along imaginary axis (i.e. $x \to 0$)</u> :— $(\because\ \delta z = i\delta y)$

$$\therefore f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u + i\delta v}{i\,\delta y}\right) = \lim_{\delta y \to 0}\left(\frac{1}{i}\frac{\delta u}{\delta y} + \frac{\delta v}{\partial y}\right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{— (B)}$$

comparing eq^n (A) & (B) we get $u_x = v_y$ & $u_y = -v_x$.

<u>Sufficient Condition</u> :— Let $u_x = v_y$ and $u_y = -v_x$, then u_x, u_y, v_x & v_y are continuous.

Now let $f(z) = u(x,y) + i\,v(x,y)$

$$\therefore f(z+\delta z) = u(x+\delta x, y+\delta y) + i\,v(x+\delta x, y+\delta y)$$

$$= u(x,y) + \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \cdots + i\left[v(x,y) + \frac{\partial v}{\partial x}\delta x + \frac{\delta v}{\partial y}\delta y \cdots\right]$$

(neglecting IInd and higher terms)

$$\left(f(a+h, b+k) = f(a,b) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{\lfloor 2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right] + \cdots\right)$$

$$= u(x,y) + i\,v(x,y) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

or $\quad f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(i^2 \frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\{\delta x + i\delta y\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\{z\}$$

$$\Rightarrow \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or } f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$\therefore$ $f(z)$ is differentiable it shows that $f(z)$ is continuous, hence u_x, u_y v_x and v_y are continuous in region R.

<u>C-R Equation in Polar Form</u> :- We know that

$$z = x+iy = r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta} \quad \text{①}$$

Now $\quad w = f(z) = u+iv$

or $\quad f(re^{i\theta}) = u+iv$ ——— ②

differentiating eqⁿ ② partially w.r.t. r and θ, we get

$$f'(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{③} \quad \text{and} \quad f'(re^{i\theta}).re^{i\theta}.i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \quad \text{④}$$

from eqⁿ ③ & ④, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

on comparing $\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$ and $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$ or $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$

<u>Que ①</u> :- Determine whether $\frac{1}{z}$ is analytic or not?

<u>Sol</u> :- Let $w = f(z) = u+iv = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$

$$\Rightarrow u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}$$

$$\therefore u_x = \frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2} \quad \text{and} \quad \frac{\partial v}{\partial x} = v_x = \frac{2xy}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{x^2+y^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) + y\cdot 2y}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$ i.e. CR eqⁿ are satisfied

Now $\frac{dw}{dz} = f'(z) = -\frac{1}{z^2}$ i.e. $f'(z)$ exists everywhere except at $z=0$.

Hence $\frac{1}{z}$ is analytic except at $z=0$.

Que 2 :- Using the CR equation show that $f(z)=|z|^2$ is not analytic at any pt.

Sol:- Let $w = f(z) = u+iv = |z|^2 = x^2+y^2$

$\Rightarrow u = x^2+y^2$ & $v = 0$

$\therefore u_x = 2x$, $u_y = 2y$, $v_x = 0$ & $v_y = 0$

$\Rightarrow u_x \neq v_y$ and $u_y \neq -v_x$ (except $x=0$ & $y=0$)

i.e. CR eqn is satisfied at $(0,0)$, hence function is differentiable at $(0,0)$.

Hence function is not analytic except at origin.

Que 3 :- Show that the function $e^x(\cos y + i \sin y)$ is an analytic function and also find its derivative.

Sol:- Let $w = f(z) = u+iv = e^x(\cos y + i\sin y)$

$\Rightarrow u = e^x \cos y$ & $v = e^x \sin y$

$\therefore u_x = e^x \cos y$, $u_y = -e^x \sin y$ & $v_x = e^x \sin y$, $v_y = e^x \cos y$

$\therefore u_x = v_y$ & $u_y = -v_x$ i.e. CR eqn are satisfied.

Again we know that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z$$

Ans

Que 4 :- Show that the function $w = \log z$ satisfy the CR equation when z is not zero. Also find its derivatives.

Sol:- $w = f(z) = u+iv = \log z = \log(x+iy) = \log(r\cos\theta + ir\sin\theta) = \log(re^{i\theta})$

$$= \log r + \log e^{i\theta} = \log r + i\theta = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$\Rightarrow u = \frac{1}{2}\log(x^2+y^2)$ and $v = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$, $\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$

Similarly $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

$\therefore u_x = v_y$ and $u_y = -v_x$ i.e. CR eqn are satisfied.

$\therefore w = f(z)$ is analytic except $z=0$ ($\because x^2+y^2=0 \Rightarrow x=0$ & $y=0 \Rightarrow z=0$)

Now $\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) = \frac{x-iy}{x^2+y^2} = \frac{(x-iy)}{(x-iy)(x+iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

Ans

Que (5) :- Find the values of c_1 & c_2 such that the function

$$f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy) \text{ is analytic, also find } f'(z).$$

Sol :- Let $w = f(z) = u + iv = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$

$$\Rightarrow u = x^2 + c_1 y^2 - 2xy \quad \& \quad v = c_2 x^2 - y^2 + 2xy$$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2y \qquad \& \quad \frac{\partial v}{\partial x} = 2c_2 x + 2y$$

$$\frac{\partial u}{\partial y} = 2c_1 y - 2x \qquad \& \quad \frac{\partial v}{\partial y} = -2y + 2x$$

By C R eq^n $\quad u_x = v_y \Rightarrow 2x - 2y = -2y + 2x$

& $\quad u_y = -v_x = 2c_1 y - 2x = -2c_2 x - 2y$

$\Rightarrow c_1 = -1$ & $c_2 = 1$ (equating the coefficient of x & y)

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

put $c_2 = 1$

$$= 2x - 2y + i(2x + 2y) = 2[x + i^2 y + i(x + y)]$$

$$= 2[x(1+i) + iy(1+i)]$$

$$= 2(1+i)(x + iy) = 2(1+i)z \qquad \textbf{Ans}$$

Que (6) :- Show that the function $f(z) = u + iv$, where $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

satisfies the CR eq^n at $z=0$. Is the function is analytic at $z=0$? Justify your answer.

Sol :-

$$w = f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{x^3 - y^3 - i(x^3 + y^3)}{x^2+y^2}$$

$$\Rightarrow u = \frac{x^3 - y^3}{x^2+y^2} \quad \& \quad v = \frac{x^3 + y^3}{x^2+y^2}$$

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1, \quad \frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1, \quad \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \frac{y - 0}{y} = 1$$

$\therefore u_x = v_y$ and $u_y = -v_x$ i.e. CR eq^n are satisfied at origin.

Again $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \left[\frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2+y^2} \cdot \frac{1}{x+iy}\right]$, Since $f(0) = 0$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x\to 0} \left[\frac{(\cancel{x^3} - \cancel{x^3}) + i(x^3 + x^3)}{x^2 + x^2} \cdot \frac{1}{x + ix}\right] = \lim_{x\to 0} \frac{\cancel{2} i x^3}{\cancel{2} x^2} \cdot \frac{1}{x(1+i)} = \frac{i}{1+i} \quad \text{—— (A)}$$

Again let $z \to 0$ along x axis i.e. $y = 0$, then

$$f'(0) = \lim_{x\to 0} \left[\frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x}\right] = 1 + i \quad \text{—— (B)}$$

, Hence we see that $f'(0)$ is not unique, $\therefore f(z)$ is not analytic at $z=0$.

Que ⑦ :- Examine the nature of the function $f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & , z \neq 0 \\ 0 & , z=0 \end{cases}$ is the region including the origin.

Sol :- $f(z) = u+iv = \dfrac{x^2y^5(x+iy)}{x^4+y^{10}} \Rightarrow u = \dfrac{x^3y^5}{x^4+y^{10}}$ and $v = \dfrac{x^2y^6}{x^4+y^{10}}$

At orgin $\dfrac{\partial u}{\partial x} = \lim\limits_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x} = \lim\limits_{x\to 0} \dfrac{0-0}{x} = 0$, $\dfrac{\partial u}{\partial y} = \lim\limits_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y} = \lim\limits_{y\to 0} \dfrac{0-0}{y} = 0$

$\dfrac{\partial v}{\partial x} = \lim\limits_{x\to 0} \dfrac{v(x,0)-v(0,0)}{x} = \lim\limits_{x\to 0} \dfrac{0-0}{x} = 0$, $\dfrac{\partial v}{\partial y} = \lim\limits_{y\to 0} \dfrac{v(0,y)-v(0,0)}{y} = \lim\limits_{y\to 0} \dfrac{0-0}{y} = 0$

i.e. $u_x = v_y$ and $u_y = -v_x$ i.e. CR eq^n are satisfied.

Now $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim\limits_{z\to 0} \left[\dfrac{x^2y^5(x+iy)}{x^4+y^{10}} \cdot \dfrac{1}{x+iy}\right] = \lim\limits_{z\to 0} \left[\dfrac{x^2y^5}{x^4+y^{10}}\right]$

Now let $z \to 0$ along $y = mx$, then

$$f'(0) = \lim_{x\to 0} \frac{m^5x^7}{x^4+m^{10}x^{10}} = \lim_{x\to 0} \frac{m^5x^3}{1+m^{10}x^6} = \frac{0}{1} = 0 \quad \text{—— (A)}$$

Again let $z \to 0$ along $y^5 = x^2$, then

$$f'(0) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2} \quad \text{—— (B)}$$

eq^n (A) & (B) shows that $f'(0)$ is not unique. Hence $f(z)$ is not analytic at origin although CR eq^n are satisfied.

Que ⑧ — Prove that the function $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$ and $f(0)=0$. is not analytic at $z=0$.

Hint :- Same as above question. ($z\to 0$ along $y=mx$ $f'(0)=0$, and $z\to 0$ along $y^2 = x$, $f'(0) = \frac{1}{2}$)

Que ⑨ :- If $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$, $z \neq 0$ and $f(0)=0$, Show that $\dfrac{f(z)-f(0)}{z} \to 0$, as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Sol :-
$$\frac{f(z)-f(0)}{z} = \left[\frac{x^3y(y-ix)}{x^6+y^2} - 0\right]\frac{1}{x+iy} = \frac{-i\,x^3y(x+iy)}{x^6+y^2(x+iy)} = \frac{-ix^3y}{x^6+y^2}$$

Let $z \to 0$ along $y = mx$, then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3(mx)}{x^6+(mx)^2} = \lim_{x\to 0} \frac{-imx^2}{x^4+m^2} = 0$$

Again let $z \to 0$ along $y = x^3$, then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3 \cdot x^3}{x^6+x^6} = -\frac{i}{2}$$

Hence $\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} \to 0$ along $y = mx$, But $\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} \not\to 0$ along any manner.

Proved

Que (10) :- Show that the function $f(z)=\sqrt{|xy|}$, satisfies CR equation at origin but not analytic at that point.

Sol :- $f(z)=u+iv=\sqrt{|xy|} \Rightarrow u=\sqrt{|xy|}$ & $v=0$

At origin $\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$, $\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$

$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$, $\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$

$\therefore$ $u_x=v_y$ and $u_y=-v_x$ i.e. CR eq^n are satisfied.

Now $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\left[\frac{\sqrt{|xy|}}{x+iy}\right]$

Let $z\to 0$ along the line $y=mx$, then

$f'(0)=\lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x(1+im)}=\frac{\sqrt{|m|}}{1+im}$

i.e. $f'(0)$ depending upon m, hence $f'(0)$ is not unique, $\therefore$ it is not analytic at $z=0$.

Que (11) :- Show that the function $f(z)=e^{-z^{-4}}$, $z\neq 0$ and $f(0)=0$ is not analytic at $z=0$, although CR eq^n are satisfied at that point.

Sol :- $f(z)=u+iv=e^{-z^{-4}}=e^{-(x+iy)^{-4}}=e^{-\frac{1}{(x+iy)^4}}$

$=e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}=e^{-\frac{[x^4+y^4-6x^2y^2-i\,4xy(x^2-y^2)]}{(x^2+y^2)^4}}=e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\cdot e^{i\left[\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]}$

$=e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}}\left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}+i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$

$\Rightarrow u=e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}}\cos\frac{4x(x^2-y^2)}{(x^2+y^2)^4}$ & $v=e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}}\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}$

At origin

$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\left[\frac{e^{-\frac{1}{x^4}}}{x}\right]=\lim_{x\to 0}\frac{1}{x\,e^{\frac{1}{x^4}}}=\lim_{x\to 0}\frac{1}{x\left(1+\frac{1}{x^4}+\frac{1}{L2}\cdot\frac{1}{x^8}+\cdots\right)}$

$=\frac{1}{0+\infty}=\frac{1}{\infty}=0$

Similarly $\frac{\partial u}{\partial y}=0$, $\frac{\partial v}{\partial x}=0$, and $\frac{\partial v}{\partial y}=0$, Hence CR eq^n are satisfied.

Again $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{e^{-z^{-4}}}{z}$

Now let $z\to 0$ along $y=x$ i.e. put $z=re^{i\frac{\pi}{4}}$

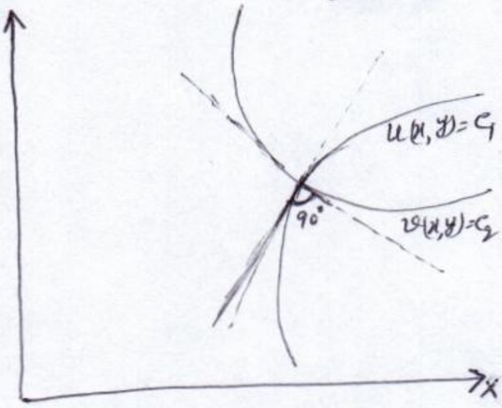
$f'(0)=\lim_{r\to 0}\frac{e^{-r^{-4}}\cdot e^{-\left[e^{\frac{i\pi}{4}}\right]^{-4}}}{re^{\frac{i\pi}{4}}}=\lim_{r\to 0}\frac{e^{-r^{-4}}\cdot e^{-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{-4}}}{re^{\frac{i\pi}{4}}}=\lim_{r\to 0}\frac{e^{-r^{-4}}\,e^{-\cos\pi}}{re^{\frac{i\pi}{4}}}$

$=\lim_{r\to 0}\frac{e^{r^{-4}}\cdot e}{re^{\frac{i\pi}{4}}}=\infty$

Hence $f'(z)$ does not exists at $z=0$. Hence $f(z)$ is not analytic at $z=0$.

Orthogonal Curves :— Two curves are said to be orthogonal to each other when they intersect at right angle at each of their point of intersection.

Theorem :— To show that if $w = f(z) = u + iv$, be analytic function, then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ form an orthogonal system.



Proof :— Since $u(x,y) = c_1$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{u_x}{u_y} = m_1$$

again $v(x,y) = c_2 \Rightarrow \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{v_x}{v_y} = m_2$

Now $m_1 \cdot m_2 = \left(-\frac{u_x}{u_y}\right)\cdot\left(-\frac{v_x}{v_y}\right) = \left(\frac{+\cancel{v_y}}{+\cancel{v_x}}\right)\left(\frac{-\cancel{v_x}}{\cancel{v_y}}\right) = -1$ i.e. $m_1 \perp m_2$.

Harmonic Function :— Any function which satisfies the Laplace's equation (i.e. $\nabla^2 u = 0$) is known as harmonic function.

Theorem :— If $f(z) = u + iv$, is an analytic function, then u and v both are harmonic function.

Proof :— Let $f(z) = u + iv$ be analytic function, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{——} (1) \qquad \& \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——} (2)$$

Differentiating eqn (1) w.r.t. x and eqn (2) w.r.t. y we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{——} (3) \qquad \& \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{——} (4)$$

$$(3)+(4) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \quad \text{or} \quad \boxed{\nabla^2 u = 0}$$

Similarly we can prove $\boxed{\nabla^2 v = 0}$.

Method to find the Conjugate function :—

Case I :— If u is given, to find v :—

we know that $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\Rightarrow v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy + C$$

Case II :— If v is given, to find u :—

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow u = \int \frac{\partial v}{\partial y}dx - \int \frac{\partial v}{\partial x}dy + C_1$$

Exact Differential Equation —
$Mdx + Ndy = 0$ is said to exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the sol is

$\int Mdx$ (y as const) $+ \int Ndx$ (ignor terms of x, i.e. $x=0$) $= c$

Que (12) :- Show that the function $u(x,y) = e^x \cos y$ is harmonic and find its conjugate and analytic function.

Sol :- $u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial^2 u}{\partial x^2} = e^x \cos y$, & $\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

Now. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$. i.e Laplace eq^n is satisfied.

Now $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$= -e^x \sin y \, dx + e^x \cos y \, dy$

$= d(e^x \sin y)$

$\Rightarrow v = e^x \sin y + c$

Now $f(z) = u + iv = e^x \cos y + i(e^x \sin y + c) = e^x(\cos y + i \sin y) + ic$

$= e^x e^{iy} + d = e^{x+iy} + d = e^z + d$ Ans

Que (13) :- If $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ both u & v satisfy Laplace equation but $f(z) = u + iv$ is not an analytic function.

Sol :- $u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x$ or $\frac{\partial^2 u}{\partial x^2} = 2$

& $\frac{\partial u}{\partial y} = -2y$ or $\frac{\partial^2 u}{\partial y^2} = -2$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 2 - 2 = 0$

$v = \frac{-y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 v}{\partial x^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$

$\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}$, $\frac{\partial^2 v}{\partial y^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

i.e. u & v both are satisfied the Laplace eq^n.

Again we see that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ i.e. CR eq^n not satisfied.

Que (14) :- Find the imaginary part of the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$.

Sol :- $u = x^3 - 3xy^2 + 3x^2 - 3y^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ and $\frac{\partial u}{\partial y} = -6xy - y$

we know that $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$= (6xy + 6y) dx + (3x^2 - 3y^2 + 6x) dy$

$\Rightarrow v = \int (6xy + 6y) dx + \int (3x^2 - 3y^2 + 6x) dy + c$

$= 3x^2y + 6xy - y^3 + C$ Ans

$M dx + N dy = 0$

$M = 6xy + 4y$, $N = 3x^2 - 3y^2 + 6x$

$\frac{\partial M}{\partial y} = 6x + 6$ $\frac{\partial N}{\partial x} = 6x + 6$

i.e. Exact diff eq^n

To Find $f(z)$ by Milne Thomson Method :—

We know that $z = x+iy$ & $\bar{z} = x-iy$, hence $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

Again $f(z) = u(x,y) + i\,v(x,y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = u(z,0) + i\,v(z,0)$
(put $\bar{z} = z$)

Case I :— If u is given :—

$$f(z) = u + iv$$
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad (\text{By CR eq}^n)$$
$$= \phi_1(x,y) - i\,\phi_2(x,y), \quad \text{put } \phi_1(x,y) = \frac{\partial u}{\partial x} \;\&\; \phi_2(x,y) = \frac{\partial u}{\partial y}$$

$$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$
$$\Rightarrow \boxed{f(z) = \int[\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c}$$

Case II :— If v is given :— $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$
$$f'(z) = \psi_1(x,y) + i\,\psi_2(x,y), \quad \text{where } \frac{\partial v}{\partial y} = \psi_1(x,y) \;\&\; \frac{\partial v}{\partial x} = \psi_2(x,y)$$
$$\Rightarrow \boxed{f(z) = \int[\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + c}$$

Que (15) :— If $u = e^x(x\cos y - y\sin y)$, find $f(z)$.

Sol :— $\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$

$$\therefore \phi_1(z,0) = e^z z + e^z = e^z(z+1)$$

and $\phi_2(x,y) = \frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$

$$\therefore \phi_2(z,0) = 0$$
$$\therefore f(z) = \int[\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$$
$$= \int \underset{\text{II}}{e^z}\underset{\text{I}}{(z+1)}\,dz + c = e^z(z+1) - \int 1\cdot e^z\,dz + c = e^z(z+1) - e^z + c = z e^z + c$$

Ans

Que (16) :— If $f(z) = u(x,y) + i\,v(x,y)$ is an analytic and $u(x,y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$ find $f(z)$.

Sol :— $\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\cdot 2\cos 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2} = \frac{2\cosh 2y\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2}$

$$\therefore \phi_1(z,0) = \frac{2}{1+\cos 2z} = \frac{2}{\cancel{1} + 2\cos^2 z - \cancel{1}} = \sec^2 z$$

again $\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{-2\sin 2x\sinh 2y}{(\cosh 2y + \cos 2x)^2}$ $\quad\therefore \phi_2(z,0) = 0$

Now $f(z) = \int[\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$
$$= \int \sec^2 z\,dz + c = \tan z + c$$

Ans

Que (17) :- If $u+v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$, find the analytic function $f(z)$.

Sol :- Differentiating above eqn w.r.t. x and y, we get

$$u_x + v_x = (x^2+4xy+y^2) + (x-y)(2x+4y)$$

or $$u_x - u_y = (x^2+4xy+y^2) + (x-y)(2x+4y) \quad \text{——— (1)}$$

and $$u_y + v_y = -(x^2+4xy+y^2) + (x-y)(4x+2y)$$

or $$u_y + u_x = -(x^2+4xy+y^2) + (x-y)(4x+2y) \quad \text{——— (2)}$$

$$(1)+(2) \Rightarrow 2u_x = (x-y)\{(2x+4y)+(4x+2y)\} = (x-y)(6x+6y) = 6(x^2-y^2)$$

or $$u_x = 3(x^2-y^2) = \phi_1(x,y) \quad \text{or} \quad \phi_1(z,0) = 3z^2 \quad \text{——— (3)}$$

$$(2)-(1) \Rightarrow 2u_y = -2(x^2+4xy+y^2) + (x-y)\{4x+2y-2x-4y\} = -2(\cancel{x^2}+4xy+\cancel{y^2}) + 2(\cancel{x^2}+\cancel{y^2}-2xy)$$

$$\Rightarrow u_y = -6xy = \phi_2(x,y) \quad \text{or} \quad \phi_2(z,0) = 0 \quad \text{——— (4)}$$

$$\therefore f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C$$

$$= \int 3z^2\,dz + C = z^3 + C \qquad \underline{\text{Ans}}$$

Que (18) :- Find the analytic function $f(z) = u+iv$ if $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Sol :-
$$u_x + v_x = \frac{(\cosh 2y - \cos 2x)\cdot 2\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} = \frac{2\cosh 2y \cos 2x - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

or $$u_x - u_y = \frac{2\cosh 2y \cos 2x - 2}{(\cosh 2y - \cos 2x)^2} \quad \text{——— (1)}$$

and $$u_y + v_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

or $$u_y + u_x = \text{- - - - - -} \quad \text{——— (2)}$$

$$(1)+(2) \Rightarrow u_x = \frac{\cosh 2y \cos 2x - 1 - \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \phi_1(x,y)$$

$$\therefore \phi_1(z,0) = \frac{\cos 2z - 1}{(1-\cos 2z)^2} = \frac{-1}{1-\cos 2z} = \frac{-1}{\cancel{1}-\cancel{1}+2\sin^2 z} = \frac{-1}{2}\operatorname{cosec}^2 z \quad \text{——— (3)}$$

$$(2)-(1) \Rightarrow u_y = \frac{-\sin 2x \sinh 2y - \cosh 2y \cos 2x + 1}{(\cosh 2y - \cos 2x)^2} = \phi_2(x,y)$$

$$\therefore \phi_2(z,0) = \frac{1-\cos 2z}{(1-\cos 2z)^2} = \frac{1}{1-\cos 2z} = \frac{1}{2}\operatorname{cosec}^2 z \quad \text{——— (4)}$$

$$\therefore f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C$$

$$= \int \left[-\tfrac{1}{2}\operatorname{cosec}^2 z - \tfrac{i}{2}\operatorname{cosec}^2 z\right] dz + C$$

$$= -\frac{(1+i)}{2}\int \operatorname{cosec}^2 z\,dz + C$$

$$= \frac{(1+i)}{2}\cot z + C \qquad \underline{\text{Ans}}$$

Que (19) :- If $f(z) = u + iv$ is an analytic function & $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$, find $f(z)$, subject to the condition $f\left(\frac{\pi}{2}\right) = 0$.

Sol :- $u - v = \frac{1}{2}\left[\frac{2\cos x - 2\sin x - 2e^{-y}}{2\cos x - e^{y} - e^{-y}} - 1 + 1\right] = \frac{1}{2}\left[1 + \frac{2\sin x + e^{y} - e^{-y}}{2\cos x - e^{y} - e^{-y}}\right] = \frac{1}{2}\left[1 + \frac{\sin x + \sinh y}{\cos x - \cosh y}\right]$

$$\Rightarrow u_x - v_x = \frac{1}{2}\left[\frac{\cos x(\cos x - \cosh y) + (\sin x + \sinh y)\sin x}{(\cos x - \cosh y)^2}\right]$$

$$\text{or } u_x + u_y = \frac{1}{2}\left[\frac{1 - \cos x\cosh y + \sin x\sinh y}{(\cos x - \cosh y)^2}\right] \quad \text{———— (A)}$$

$$\text{again } u_y - v_y = \frac{1}{2}\left[\frac{\cosh y(\cos x - \cosh y) + \sinh y(\sin x + \sinh y)}{(\cos x - \cosh y)^2}\right]$$

$$\Rightarrow u_y - u_x = \frac{1}{2}\left[\frac{\cosh y\cos x - \sinh y\sin x - 1}{(\cos x - \cosh y)^2}\right] \quad \text{———— (B)}$$

$$\text{(A)} - \text{(B)} \Rightarrow \quad u_x = \frac{1}{2\cdot\cancel{2}}\left[\frac{\cancel{2} - \cancel{2}\cosh y\cos x}{(\cos x - \cosh y)^2}\right] = \phi_1(x, y)$$

$$\text{or } \phi_1(z, 0) = \frac{1}{2}\frac{(1 - \cos z)}{(\cos z - 1)^2} = \frac{-1}{2(\cos z - 1)} = \frac{-1}{2(-2\sin^2\frac{z}{2} - 1)} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}$$

$$\text{(A)} + \text{(B)} \Rightarrow \quad u_y = \frac{1}{2}\left[\frac{\sinh y\sin x}{(\cos x - \cosh y)^2}\right] = \phi_2(x, y)$$

$$\text{or } \phi_2(z, 0) = 0$$

$$\therefore f(z) = \int[\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C = \int\frac{1}{4}\operatorname{cosec}^2\frac{z}{2}\,dz + C = -\frac{1}{2}\cot\frac{z}{2} + C$$

putting $z = \frac{\pi}{2}$, then $f\left(\frac{\pi}{2}\right) = 0 = -\frac{1}{2}\cot\frac{\pi}{4} + C \Rightarrow C = \frac{1}{2}\cdot 1 = \frac{1}{2}$

$$\therefore f(z) = -\frac{1}{2}\cot\frac{z}{2} + \frac{1}{2} = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right) \quad \text{Ans}$$

Que (20) — If $f(z)$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Sol :- Let $f(z) = u + iv \quad \therefore |f(z)|^2 = u^2 + v^2 = \phi \quad$ (Since $z = x + iy \quad |z| = \sqrt{x^2+y^2}$)

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{———— (1)}$$

Now $\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$ & $\frac{\partial^2\phi}{\partial x^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2}\right)\right]$ —— (2)

& $\frac{\partial \phi}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$ or $\frac{\partial^2\phi}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2}\right]$

$$= 2\left[\left(-\frac{\partial v}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial y^2}\right] \quad \text{—— (3)}$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + 2u\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{=0} + 2v\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{=0}$$

Since u & v are harmonic

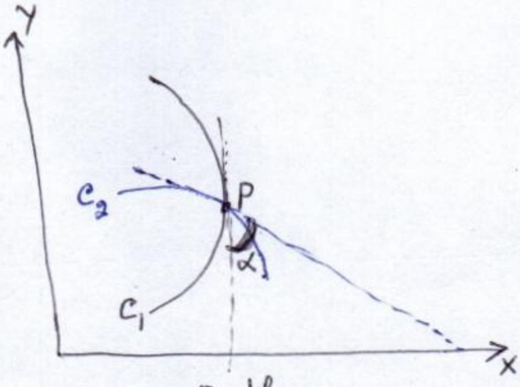
$$= 4\left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 \qquad \text{Since } x^2 + y^2 = |x + iy|^2 = |z|^2$$

$$= 4|f'(z)|^2 \quad \text{from eq}^n\ (1).$$

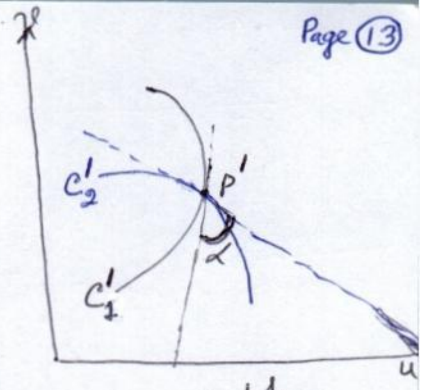
Proved

Conformal Mapping :-

Let two curves C_1 and C_2 in the z-plane intersect at the point P and the corresponding curve C_1' and C_2' in w-plane intersect at P'.



z-plane



w-plane

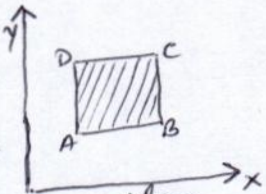
If the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curve at P' in magnitude as well as rotation, then the transformation is called conformal.

1 - Translation i.e. $w = z + b$:— where $b = c + id$

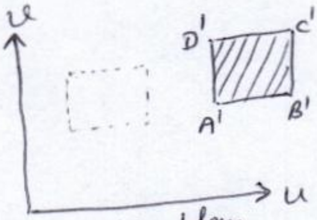
$\therefore u + iv = x + iy + c + id$

$\Rightarrow u = x + c$ & $v = y + d$

on putting the values of x & y in above eq^n then the equation is transformed in w-plane. i.e. The fig. in z-plane and its image in w-plane have the same shape, size and orientation.



z-plane



w-plane

2 - Rotation i.e. $w = z e^{i\theta}$;— where $z = x + iy$.

Que :- Consider a transformation $w = z e^{i\frac{\pi}{4}}$ and determine the region R' in w plane corresponding to the region R bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in z-plane.

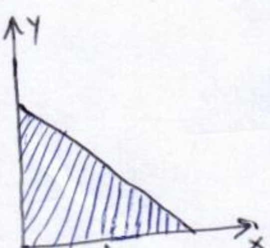
Sol :- $w = z e^{i}$ or $u + iv = (x+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = (x+iy)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$\Rightarrow u = \frac{1}{\sqrt{2}}(x-y)$ & $v = \frac{1}{\sqrt{2}}(x+y)$

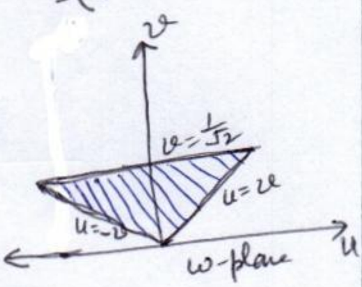
(i) put $x = 0$, $u = -\frac{y}{\sqrt{2}}$ & $v = \frac{y}{\sqrt{2}} \Rightarrow u = -v$

(ii) put $y = 0$, $u = \frac{x}{\sqrt{2}}$ & $v = \frac{x}{\sqrt{2}} \Rightarrow u = v$

(iii) put $x + y = 1$, $v = \frac{1}{\sqrt{2}}$



z plane



w-plane

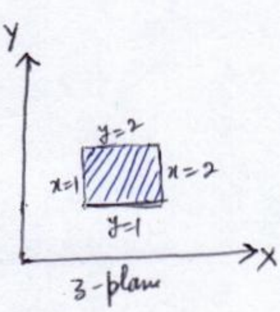
3 - Magnification i.e. $w = az$:- where a is real number.

Que :- Determine the region in w-plane on the transformation of rectangular region enclosed by $x = 1$, $y = 1$, $x = 2$ and $y = 2$ in the z-plane. The transformation is $w = 3z$.

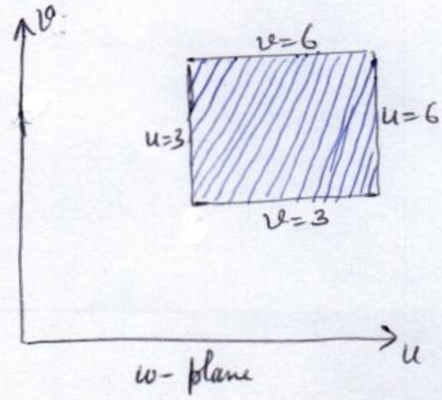
Sol :- $w = 3z$ or $u + iv = 3(x + iy)$

$\Rightarrow u = 3x$ & $v = 3y$

z plane		w plane	
x	y	$u = 3x$	$v = 3y$
1	1	3	3
2	2	6	6



z-plane



w-plane

(4) Magnification and Rotation i.e. $w = az$:— where $a = c + id$.

Let $a = ke^{i\alpha}$, $z = re^{i\theta}$, & $w = Re^{i\phi}$

$$\therefore Re^{i\phi} = (ke^{i\alpha})(re^{i\theta}) = kr\, e^{i(\alpha+\theta)}$$

$$\Rightarrow \boxed{R = kr} \quad \& \quad \boxed{\phi = \theta + \alpha}$$

i.e. magnification & rotation.

(5) Inverse and Reflection :— i.e. $w = \frac{1}{z}$:— where $z = re^{i\theta}$ and $w = Re^{i\phi}$

$$\therefore Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} \Rightarrow R = \frac{1}{r} \;\&\; \phi = -\theta$$

Hence the transformation is inverse of z and followed by reflection into real axis.

Example :— Find the image of $|z+1| = 1$ under the mapping $w = \frac{1}{z}$.

$$w = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2} \;\&\; v = -\frac{y}{x^2+y^2}$$

and $|z+1| = 1 \Rightarrow |x+iy+1| = 1$ or $|(x+1)+iy| = 1$

$$\Rightarrow (x+1)^2 + y^2 = 1 \text{ or } x^2+y^2+2x+\cancel{1} = \cancel{1}$$

or $x^2+y^2 = -2x$

or $1 = \frac{-2x}{x^2+y^2}$

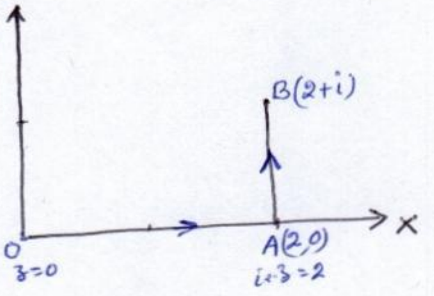
$$\Rightarrow \frac{1}{2} = \frac{-x}{x^2+y^2} = -u$$

$$\Rightarrow \underline{\underline{2u+1=0}}$$

Complex Integration :—

Que ① :— Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the real axis from $z=0$ to $z=2$ and then along a line parallel to y axis from $z=2$ to $z=2+i$.

Sol :—



$$\int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x-iy)^2 (dx+idy)$$

$$= \int_{OA,\ y=0,\ dy=0} (x-iy)^2(dx+idy) + \int_{AB,\ x=2,\ dx=0} (x-iy)^2(dx+idy)$$

$$= \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 (i\,dy)$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\int_0^1 (4-4iy-y^2)dy = \frac{8}{3} + i\left[4y - 2iy^2 - \frac{y^3}{3}\right]_0^1$$

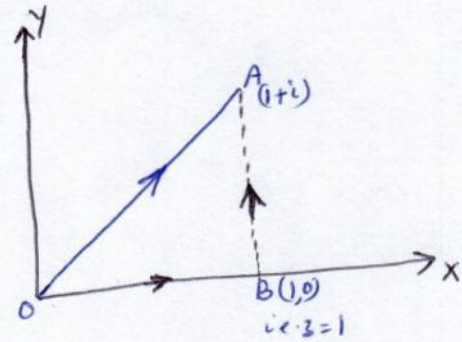
$$= \frac{8}{3} + i\left[4 - 2i - \frac{1}{3}\right] = \frac{8}{3} + i\left(\frac{11-6i}{3}\right) = \frac{8}{3} + \left(\frac{11i+6}{3}\right) = \frac{1}{3}(14+11i)$$ Ans

Que ② :— Find the value of the integral $\int_0^{1+i} (x-y+ix^2)dz$ along

(a) the straight line from $z=0$ to $z=1+i$

(b) the real axis from $z=0$ to $z=1$ and then along a line parallel to the imaginary axis from $z=1$ to $z=1+i$.

Sol :— (a) Straight line from $z=0$ to $z=1$ i.e. along the line OA. Hence eqn of OA is $y=x$



$$\therefore \int_0^{1+i} (x-y+ix^2)dz = \int_{OA,\ y=x,\ dy=dx} (x-y+ix^2)(dx+idy)$$

$$= \int_0^1 (\cancel{x}-\cancel{x}+ix^2)(dx+idx)$$

$$= \int_0^1 (ix^2)(1+i)\,dx = (1+i)i\left[\frac{x^3}{3}\right]_0^1 = \frac{i-1}{3}$$

(b)
$$\int_0^{1+i} (x-y+ix^2)dz = \int_{OB,\ y=0,\ dy=0} (x-y+ix^2)(dx+idy) + \int_{BA,\ x=1,\ dx=0} (x-y+ix^2)(dx+idy)$$

$$= \int_0^1 (x+ix^2)dx + i\int_0^1 (1-y+i)\,dy$$

$$= \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 + i\left[y - \frac{y^2}{2} + iy\right]_0^1$$

$$= \frac{1}{2} + \frac{i}{3} + i\left(1 - \frac{1}{2} + i\right)$$

$$= \frac{1}{2} + \frac{i}{3} + i\left(\frac{1}{2} + i\right)$$

$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1$$

$$= -\frac{1}{2} + \frac{5i}{6}$$ Ans

Que ③ :- Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along , (i) the line $y = \frac{x}{2}$

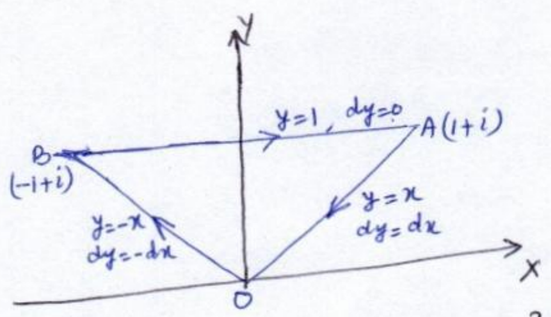
(ii) the real axis to 2 and then vertically to $2+i$.

Ans:- (i) $\frac{5}{3}(2-i)$, (ii) $\frac{1}{3}(14+i)$

Que ④ :- Evaluate $\int_0^{1+i} (x^2 - iy)\, dz$ along the path (i) $y = x$ (ii) $y = x^2$

Ans:- (i) $\frac{1}{6}(5-i)$, (ii) $\frac{1}{6}(5+i)$

Que ⑤ :- Evaluate $\int_C z^2 dz$, where C is the boundary of triangle with vertices 0, $1+i$, $-1+i$ clockwise.



Hint :- $\int_C z^2 dz = \int_{OB} z^2 dz + \int_{BA} z^2 dz + \int_{AO} z^2 dz$

$= 0$

Que ⑥ :- Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$ along the two paths, (i) $x = t+1$, $y = 2t^2 - 1$

(ii) the straight line joing $1-i$ and $2+i$.

Sol :- Since $x = t+1$ $\therefore dx = dt$ and $y = 2t^2 - 1$ $\therefore dy = 4t\, dt$

(i) $\int_{1-i}^{2+i} (2x+iy+1)(dx + i\,dy) = \int_0^1 \left[2(t+1) + i(2t^2-1) + 1\right](1 + 4i\,t)\,dt = 4 + \frac{25}{3} i$

(ii) The eq^n of the straight line joing $\underset{(x_1,\, y_1)}{(1, -1)}$ and $\underset{(x_2,\, y_2)}{(2, 1)}$ is

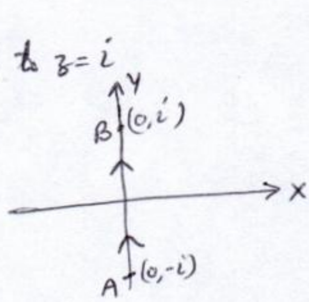
$y + 1 = \frac{1+1}{2-1}(x-1) \Rightarrow y + 1 = 2(x-1) \Rightarrow y = 2x - 3$ $\therefore dy = 2dx$

$\therefore \int_{1-i}^{2+i} (2x+iy+1)(dx+i\,dy) = \int_1^2 \left[2x + i(2x-3)\right](1+2i)\,dx = 4 + 8i$ Ans

Que ⑦ :- Evaluate the integral $\int_C |z|\, dz$, where C is the contour

(i) The straight line from $z = -i$ to $z = i$

(ii) The left of the unit circle $|z| = 1$ from $z = -i$ to $z = i$



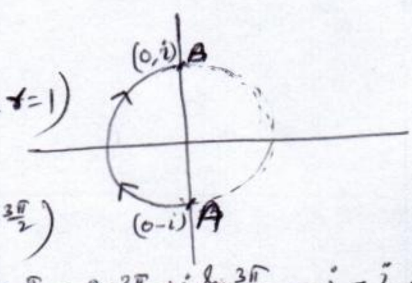
(i) $\int_C |z|\, dz = \int_{AB} (x^2 + y^2)^{1/2} (dx + i\,dy)$, $x = 0$ $\therefore dx = 0$

$= \int_{-1}^{1} y\,(i\,dy) = i\left[\frac{y^2}{2}\right]_{-1}^{1} = 0$

(ii) $|z| = 1 \Rightarrow \sqrt{x^2+y^2} = 1 \Rightarrow x^2 + y^2 = 1$

and $z = x + iy = r\cos\theta + ir\sin\theta = r e^{i\theta} = e^{i\theta}$ (Since $r = 1$)

$\therefore dz = i e^{i\theta} d\theta$



$\therefore \int_C |z|\, dz = \int_{3\pi/2}^{\pi/2} 1 \cdot i\, e^{i\theta} d\theta = \cancel{i}\left[\frac{e^{i\theta}}{\cancel{i}}\right]_{3\pi/2}^{\pi/2} = \left(e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}\right)$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = i - i = 0$ Ans

Continuous Arc:— If a point z on an arc is such that $z = x+iy = \phi(t) + i\psi(t)$ —— ①

If $\phi(t)$ and $\psi(t)$ are real continuous function in the range $\alpha \le t \le \beta$, then the arc is called a continuous arc.

Multiple Point:— If the eqⁿ ① are satisfied by more than one values of t in the given range, then the point $z(x,y)$ is called a multiple point of the arc.

Jordan Arc:— A continuous arc without multiple points is called a Jordan arc.

i.e. eqⁿ ① is single valued and $\phi(t)$ and $\psi(t)$ are continuous.

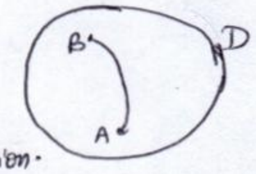
Regular Arc:— If $\phi'(t)$ and $\psi'(t)$ are continuous in the range $\alpha \le t \le \beta$, then the arc is called a regular arc.

Contour:— Contour means a Jordan curve consisting of continuous chain of a finite number of regular arcs.

If A be the starting point of a first arc and B is the end point of the the last arc, then integral along such a curve is written as $\int_{AB} f(z)\,dz$.

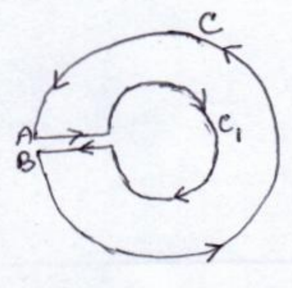
If contour is said to be closed if the starting point A coincides with the end point B. i.e. $\int_C f(z)\,dz$.

Connected Region:— A region is said to connected region if any two points of the region D can be connected by a curve which lies entirely within the region.

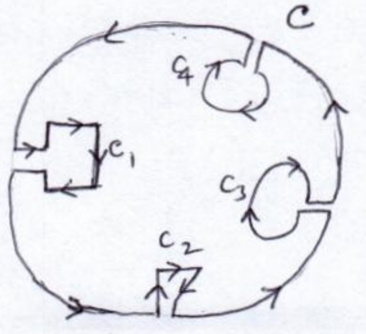


Simply Connected Region:— If all the points of the area bounded by any single closed curve C drawn in the region D are the points of the region D then the region D is said to be simply connected.

Multi Connected Region:— If all the points of the area bounded by two or more closed curves drawn in the region D, are the points of the region D, is said to multi connected region.



Simply Connected Region



Simply connected Region

Multi-Connected Region

Cauchy's Integral Theorem or Cauchy Goursat Theorem :—

If $f(z)$ is analytic and its derivatives $f'(z)$ is continuous at all points inside and on a closed curve c then,

$$\boxed{\int_C f(z)\,dz = 0}$$

Proof :– Let $f(z) = u + iv$ and $z = x + iy$ $\therefore$ $dz = dx + idy$

$$\therefore \int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (\underset{\phi}{u}\,dx - \underset{\psi}{v}\,dy) + i\int_C (\underset{\phi}{v}\,dx + \underset{\psi}{u}\,dy)$$

Now apply Green's theorem i.e. $\int_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

$$\therefore \int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

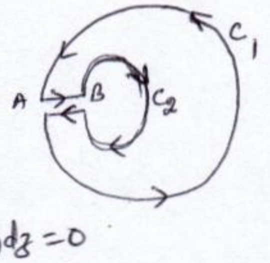
$$= \iint_R \left(-\cancel{\frac{\partial v}{\partial x}} + \cancel{\frac{\partial v}{\partial x}}\right) dx\,dy + i\iint_R \left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right) dx\,dy$$

Since $f(z)$ is analytic i.e. $u_x = v_y$ & $u_y = -v_x$.

$$= 0$$

Extension of Cauchy's Theorem to multiple connected region :— If $f(z)$ is analytic in the region R between two simple closed curves c_1 and c_2 then

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

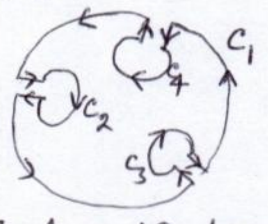


Proof :– We know that $\int_C f(z)\,dz = 0$

$$\Rightarrow \int_{AB} \cancel{f(z)}\,dz - \int_{C_2} f(z)\,dz + \int_{BA} \cancel{f(z)}\,dz + \int_{C_1} f(z)\,dz = 0$$

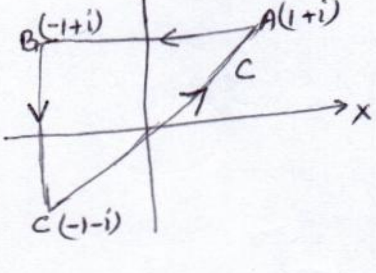
Since $\int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$.

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



Corollary :– $\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz}$

Que ① :— Verify Cauchy's Theorem for the function $f(z) = e^{iz}$ along the boundary of the triangle with vertices at the points $1+i$, $-1+i$ and $-1-i$.



Sol :–

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$$

$$= \int_{1+i}^{-1+i} e^{iz}\,dz + \int_{-1+i}^{-1-i} e^{iz}\,dz + \int_{-1-i}^{1+i} e^{iz}\,dz$$

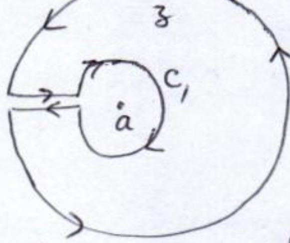
$$= \left[\frac{e^{iz}}{i}\right]_{1+i}^{-1+i} + \left[\frac{e^{iz}}{i}\right]_{-1+i}^{-1-i} + \left[\frac{e^{iz}}{i}\right]_{-1-i}^{1+i}$$

$$= \frac{1}{i}\left[\cancel{e^{i(-1+i)}} - \cancel{e^{i(1+i)}} + \cancel{e^{i(-1-i)}} - \cancel{e^{i(-1+i)}} + \cancel{e^{i(1+i)}} - \cancel{e^{i(-1-i)}}\right] = 0.$$

Que ② :– Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ along the square with vertices $1 \pm i$, $-1 \pm i$.

Cauchy's Integral Formula:— If $f(z)$ is analytic within and on a closed contour c, and a is any point within c, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z-a}}$$



Proof:– Consider the function $\frac{f(z)}{z-a}$, which is analytic at all points within c, except $z=a$. Now we draw a small circle c_1 having radius r and centre is a.

Hence by cauchy's integral theorem for multiple connected region, we have

$$\int_c \frac{f(z)}{z-a}\,dz = \int_{c_1}\frac{f(z)}{z-a}\,dz = \int_{c_1}\frac{f(z)-f(a)+f(a)}{z-a}\,dz = \int_{c_1}\frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_{c_1}\frac{dz}{z-a} \quad \text{--- (1)}$$

Now $$\int_{c_1}\frac{f(z)-f(a)}{z-a}\,dz = \int_0^{2\pi}\frac{f(a+re^{i\theta})-f(a)}{\cancel{r}\cancel{e^{i\theta}}}\,i\cancel{r}\cancel{e^{i\theta}}\,d\theta \qquad \left(\text{Since } \begin{array}{l}|z-a|=r\\ \therefore z-a=re^{i\theta}\\ \therefore dz = ire^{i\theta}d\theta\end{array}\right)$$

$$= \int_0^{2\pi}[f(a+re^{i\theta})-f(a)]\,d\theta$$

$$= 0 \quad , \quad \text{where } r\to 0.$$

Again $$\int_{c_1}\frac{dz}{z-a} = \int_0^{2\pi}\frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = \int_0^{2\pi} i\,d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

Hence from eqⁿ (1)

$$\int_c \frac{f(z)}{(z-a)}\,dz = 0+(2\pi i)f(a) \Rightarrow \boxed{f(a) = \frac{1}{2\pi i}\int_c\frac{f(z)}{z-a}\,dz}$$

Cauchy's Integral Formula for the Derivative:— If $f(z)$ is analytic in a region R, then its derivative at any point $z=a$ of R is also analytic in R, and is given by

$$\boxed{f'(a) = \frac{1}{2\pi i}\int_c\frac{f(z)}{(z-a)^2}\,dz}$$

Proof:– We know by Cauchy integral formula $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$

Now differentiating w.r.t. a, we get

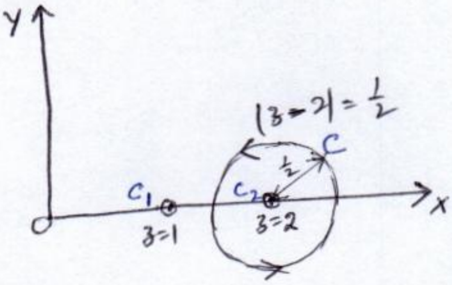
$$f'(a) = \frac{1}{2\pi i}\int_c f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)dz = \frac{1}{2\pi i}\int_c\frac{f(z)}{(z-a)^2}\,dz$$

Similarly $$\boxed{f''(a) = \frac{\lfloor 2}{2\pi i}\int_c\frac{f(z)}{(z-a)^3}\,dz}$$

$$\boxed{f^n(a) = \frac{\lfloor n}{2\pi i}\int_c\frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Que ① :- Use Cauchy's integral formula to evaluate $\int_C \frac{z}{(z^2-3z+2)} dz$, where c is the circle $|z-2| = \frac{1}{2}$.

Sol :- For finding the poles, putting denominator equal to zero i.e.



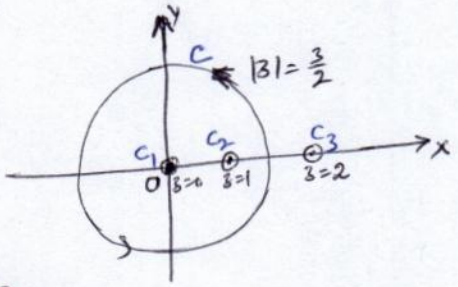
$$z^2-3z+2=0 \Rightarrow (z-1)(z-2)=0$$
$$\text{or } z = 1, 2$$

So, there are two poles $z=1$, and $z=2$, but only one pole at $z=2$ lies inside C. Hence

$$\int_C \frac{z}{(z^2-3z+2)} dz = \int_C \frac{z}{(z-1)(z-2)} dz = \int_{C_2} \frac{\frac{z}{z-1}}{z-2} dz$$

$$= 2\pi i \left[\frac{z}{z-1}\right]_{z=2} = 2\pi i \left(\frac{2}{2-1}\right) = 4\pi i \quad \underline{\text{Ans}}$$

Que ② :- Using Cauchy's integral formula solve $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$, where c is the circle $|z| = \frac{3}{2}$.



Sol :- The poles are $z=0$, $z=1$ and $z=2$. But $z=0$ and $z=1$ are lies within c, hence

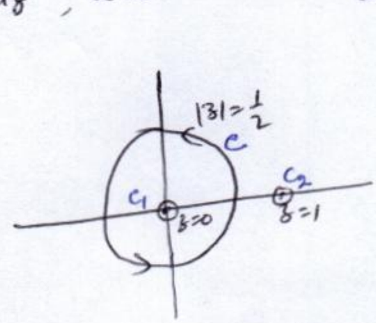
$$\int_C \frac{4-3z}{z(z-1)(z-2)} dz = \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z} dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1} dz$$

$$= 2\pi i \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i \left[\frac{4}{(0-1)(0-2)}\right] + 2\pi i \left[\frac{4-3}{1(1-2)}\right]$$

$$= 2\pi i [2] + 2\pi i (-1) = 2\pi i \quad \underline{\text{Ans}}$$

Que ③ :- Using Cauchy's integral formula calculate $\int_C \frac{2z+1}{z^2+z} dz$, where c is $|z| = \frac{1}{2}$



Sol :- The poles are $z^2+z=0$ or $z=0, 1$, But only $z=0$ lies within c, hence

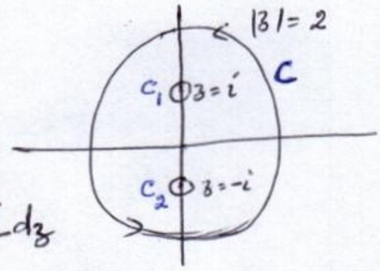
$$\int_C \frac{2z+1}{z^2+z} dz = \int_C \frac{2z+1}{z(z+1)} dz = \int_{C_1} \frac{\frac{2z+1}{z+1}}{z} dz$$

$$= 2\pi i \left(\frac{2z+1}{z+1}\right)_{z=0}$$

$$= 2\pi i \quad \underline{\text{Ans}}$$

Que ④ :- Evaluate $\int_C \frac{e^z}{z^2+1}\,dz$ over the circular path $|z| = 2$.

Sol :- The poles are $z^2+1=0 \Rightarrow z = \pm i$, Both the poles $z = i$ and $z=-i$ lies within C, hence



$$\int_C \frac{e^z}{z^2+1}\,dz = \int_C \frac{e^z}{(z+i)(z-i)}\,dz = \int_{C_1} \frac{\frac{e^z}{z+i}}{z-i}\,dz + \int_{C_2} \frac{\frac{e^z}{z-i}}{z+i}\,dz$$

$$= 2\pi i \left[\frac{e^z}{z+i}\right]_{z=i} + 2\pi i \left[\frac{e^z}{z-i}\right]_{z=-i}$$

$$= 2\pi i \left(\frac{e^i}{2i}\right) + 2\pi i\left(\frac{e^{-i}}{-2i}\right) = 2\pi i \left(\frac{e^i - e^{-i}}{2i}\right) = 2\pi i \sin 1$$

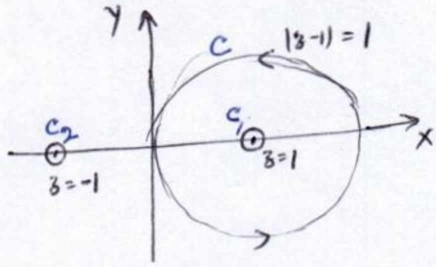
Ans

Que ⑤ :- Evaluate $\int_C \frac{e^z}{(z-1)(z-4)}\,dz$, where C is $|z|=2$, by Cauchy integral formula.

Ans :- $-\frac{2\pi i e}{3}$

Que ⑥ :- Find the value of $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is circle $|z-1| = 1$.

Sol :- The poles are $z^2-1=0 \Rightarrow z = \pm 1$, but only $z = 1$ lies within C, hence



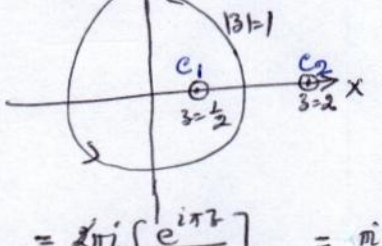
$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \int_C \frac{3z^2+z}{(z-1)(z+1)}\,dz = \int_{C_1} \frac{\frac{3z^2+z}{z+1}}{z-1}\,dz$$

$$= 2\pi i \left[\frac{3z^2+z}{z+1}\right]_{z=1} = 4\pi i \quad \text{Ans}$$

Que ⑦ :- Evaluate $\int_C \frac{\exp(i\pi z)}{2z^2-5z+2}\,dz$, where C is the unit circle with centre at origin.

Sol :- The poles are $2z^2-5z+2=0$

or $(2z-1)(z-2) = 0 \Rightarrow z = \frac{1}{2}, 2$ But only pole $z=\frac{1}{2}$ lies within C, hence



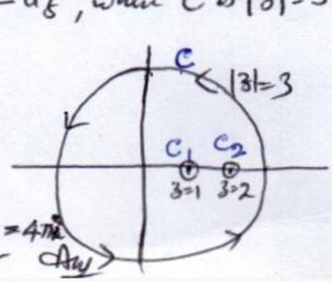
$$\int_C \frac{e^{i\pi z}}{2z^2-5z+2}\,dz = \int_C \frac{e^{i\pi z}}{(2z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{e^{i\pi z}}{z-2}}{2(z-\frac{1}{2})}\,dz = \frac{2\pi i}{2}\left[\frac{e^{i\pi z}}{z-2}\right]_{z=\frac{1}{2}} = \pi i \frac{e^{i\frac{\pi}{2}}}{\left(-\frac{3}{2}\right)}$$

$$= -\frac{2\pi i}{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = -\frac{2\pi i}{3}(0+i) = \frac{2\pi}{3}$$

Ans

Que ⑧ :- Use Cauchy integral formula, evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$, where C is $|z|=3$.

Hint :- The both poles $z=1$ and $z=2$ lies within C, hence

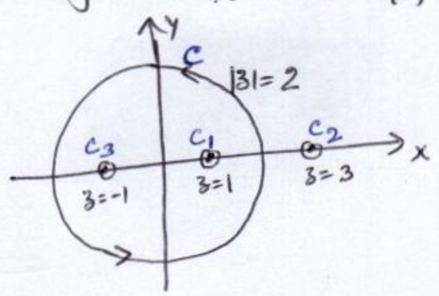


$$\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{\sin\pi z^2+\cos\pi z^2}{z-2}}{z-1}\,dz + \int_{C_2} \frac{\frac{\sin\pi z^2+\cos\pi z^2}{z-1}}{z-2}\,dz$$

$$= 2\pi i\left[\sin\pi z^2 + \cos\pi z^2\right]_{z=1} + 2\pi i\left[\sin\pi z^2+\cos\pi z^2\right]_{z=2} = 4\pi i$$

Ans

<u>Que (9)</u> :- Evaluate $\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$, by Cauchy's integral formula when C is $|z|=2$



<u>Sol</u> :- The poles are $(z^2-1)(z+3)=0 \Rightarrow z = \pm 1, 3$

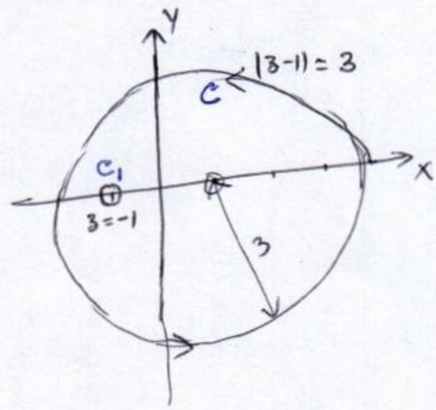
The poles $z = \pm 1$ lies within C, hence

$$\int_C \frac{3z^2+z+1}{(z-1)(z+1)(z+3)}\,dz = \int_{C_1} \frac{\frac{3z^2+z+1}{(z+1)(z+3)}}{z+1}\,dz + \int_{C_3} \frac{\frac{3z^2+z+1}{(z+1)(z+3)}}{z-1}\,dz$$

$$= 2\pi i \left[\frac{3z^2+z+1}{(z+1)(z+3)}\right]_{z=1} + 2\pi i \left[\frac{3z^2+z+1}{(z-1)(z+3)}\right]_{z=-1}$$

$$= 2\pi i\left(\frac{5}{8}\right) + 2\pi i\left(\frac{3}{-4}\right) = \frac{-\pi i}{4} \quad \text{Ans}$$

<u>Que (10)</u> - Evaluate $\oint_C \frac{e^z}{(z+1)^2}\,dz$, where c is the circle $|z-1| = 3$.



<u>Sol</u> :- The poles are $z = -1, -1$. i.e. the poles $z = -1$ of order two lies within c, hence

$$\oint_C \frac{e^z}{(z+1)^2}\,dz = \int_{C_1} \frac{e^z}{(z+1)^2}\,dz = 2\pi i \left[\frac{d}{dz} e^z\right]_{z=-1}$$

$$= 2\pi i \left[e^z\right]_{z=-1}$$

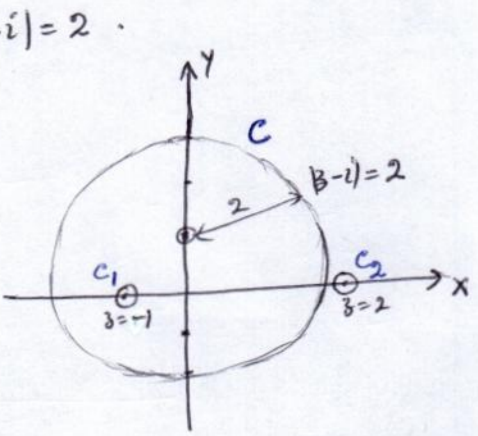
$$= 2\pi i\, e^{-1} \quad \text{Ans}$$

<u>Que (11)</u> :- By Cauchy's integral formular evaluate $\int_C \frac{e^{3z}}{(z+1)^4}\,dz$, where c is $|z|=2$.

<u>Hint</u> :- The pole $z = -1$ of orther four lies within c, hence

$$\int_C \frac{e^{3z}}{(z+1)^4}\,dz = \int_{C_1} \frac{e^{3z}}{(z+1)^4}\,dz = \frac{2\pi i}{\underline{|3}} \left[\frac{d^3}{dz^3} e^{3z}\right]_{z=-1} = \frac{\pi i}{3}\left[27 e^{3z}\right]_{z=-1} = 9\pi i\, e^{-3}$$

<u>Que (12)</u> - Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, where c is $|z-i| = 2$.



<u>Sol</u> :- The poles are $z = -1, -1, 2$ But $z = -1$ of order two lies within c, hence

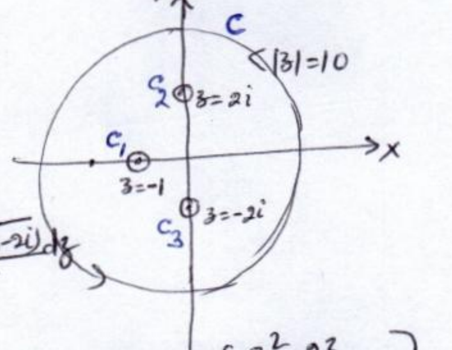
$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = \int_{C_1} \frac{\frac{z-1}{z-2}}{(z+1)^2}\,dz$$

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]_{z=-1}$$

$$= 2\pi i \left[\frac{(z-2)\cdot 1 - (z-1)\cdot 1}{(z-2)^2}\right]_{z=-1} = 2\pi i \left[\frac{-1}{(z-2)^2}\right]_{z=-1} = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Ans

Que (13) :- Evaluate $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$, where C is the circle $|z|=10$.



Sol :- The poles $z=-1$ (of order two) and $z=\pm 2i$, are lies within C, hence

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = \int_{C_1} \frac{\frac{z^2-2z}{z^2+4}}{(z+1)^2}\,dz + \int_{C_2} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{z-2i}\,dz + \int_{C_3} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{z+2i}\,dz$$

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)\right]_{z=-1} + 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i} + 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i\left[\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}\right]_{z=-1} + 2\pi i\left[\frac{-4-4i}{(2i+1)^2(4i)}\right] + 2\pi i\left[\frac{-4+4i}{(2i+1)^2(-4i)}\right]$$

$$= 2\pi i\left[\frac{5(-4)-(3)(-2)}{25}\right] + 2\pi i\left[\frac{-1-i}{(2i+1)^2 i}\right] + 2\pi i\left[\frac{1-i}{(2i+1)i}\right]$$

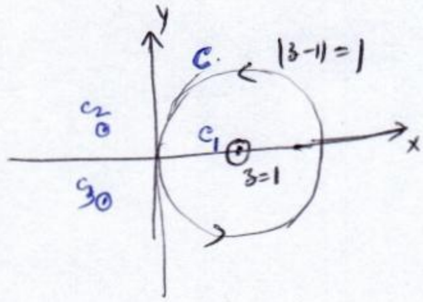
$$= 2\pi i\left(\frac{14}{25}\right) + \frac{2\pi i}{(4i^2+4i+1)i}\left[\cancel{-1}-i+\cancel{1}-i\right]$$

$$= \frac{28\pi i}{25} + \frac{2\pi i}{(-4+4i+1)\cancel{i}}\left[-2\cancel{i}\right]$$

$$= \frac{28\pi i}{25} - \frac{4\pi i}{(4i-3)}\frac{(4i+3)}{(4i+3)} = \frac{28\pi i}{25} - \frac{(-16\pi+12\pi i)}{-16-9}$$

$$= \frac{28\pi i}{25} + \frac{(-16\pi+12\pi i)}{25} = \frac{40\pi i-16\pi}{25} \quad \underline{\text{Ans}}$$

Que (14) :- Evaluate $\int_C \frac{1}{(z^3-1)^2}\,dz$ where C is $|z-1|=1$.



Sol :- The poles are $(z^3-1)^2=0$ or $(z-1)^2(z^2+z+1)^2=0$

or $z=1,1$ & $z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

Only $z=1$ (of order two) poles lies within C, hence

$$\int_C \frac{1}{(z^3-1)^2}\,dz = \int_C \frac{1}{(z-1)^2(z^2+z+1)^2}\,dz = \int_{C_1} \frac{\frac{1}{(z^2+z+1)^2}}{(z-1)^2}\,dz$$

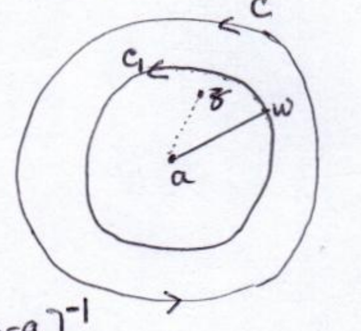
$$= 2\pi i\left[\frac{d}{dz}\frac{1}{(z^2+z+1)^2}\right]_{z=1}$$

$$= 2\pi i\left[\frac{-2(2z+1)}{(z^2+z+1)^3}\right]_{z=1}$$

$$= 2\pi i\left[\frac{-2(3)}{(1+1+1)^3}\right] = -\frac{4\pi i}{9} \quad \underline{\text{Ans}}$$

Taylor's Theorem :- If a function $f(z)$ is analytic at all points inside a circle C, with its centre at the point a and radius R, then at each point z inside C

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \frac{(z-a)^3}{\lfloor 3} f'''(a) + \cdots$$



Proof :- Take any point z inside C. Draw a circle C_1 with centre a and enclosed the point z. Let w be a point on circle C_1, then

$$|z-a| < |w-a| \quad \text{or} \quad \frac{|z-a|}{|w-a|} < 1$$

Now
$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3 + \cdots\right]$$

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \frac{(z-a)^3}{(w-a)^4} + \cdots$$

Multiplying by $\frac{f(w)}{2\pi i}$ and integrating w.r.t. w along C_1 we get

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a}\,dw + \frac{(z-a)}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2}\,dw + \frac{(z-a)^2}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^3}\,dw + \frac{(z-a)^3}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^4}\,dw + \cdots$$

or
$$\boxed{f(z) = f(a) + (z-a) f'(a) + (z-a)^2 \frac{f''(a)}{\lfloor 2} + (z-a)^3 \frac{f'''(a)}{\lfloor 3} + \cdots}$$

Que (1) :- Find the Taylor series expansion of the function $f(z) = \frac{1}{(z-1)(z-3)}$ about the point $z = 4$.

Sol :- $f(z) = \frac{1}{(z-1)(z-3)}$, $\quad f(4) = \frac{1}{(4-1)(4-3)} = \frac{1}{3}$

or $f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$, By partial fractions

$\therefore f'(z) = \frac{1}{2}\left[-\frac{1}{(z-3)^2} + \frac{1}{(z-1)^2}\right]$, $\quad f'(4) = \frac{1}{2}\left[-\frac{1}{(4-3)^2} + \frac{1}{(4-1)^2}\right] = -\frac{4}{9}$

$f''(z) = \frac{1}{2}\left[\frac{2}{(z-3)^3} - \frac{2}{(z-1)^3}\right]$, $\quad f''(4) = \frac{1}{2}\left[\frac{2}{(4-3)^3} + \frac{2}{(4-1)^3}\right] = \frac{26}{27}$

$$\therefore f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{3} + (z-4)\left(-\frac{4}{9}\right) + \frac{(z-4)^2}{\lfloor 2}\left(\frac{26}{27}\right) + \frac{(z-4)^3}{\lfloor 3}\left(\frac{-80}{27}\right) + \cdots$$

Ans

Que (2) :- For the function $f(z) = \frac{4z-1}{z^4-1}$, find Taylor series about the centre zero.

Sol :- Same as above question.

Que (3) :- Expand the function $\sin^{-1} z$ in powers of z.

Sol :- Let $f(z) = \sin^{-1} z \quad , \quad f(0) = 0$

$$f'(z) = \frac{1}{\sqrt{1-z^2}} = (1-z^2)^{-\frac{1}{2}} \quad , \quad f'(0) = 1$$

$$f''(z) = \left(-\frac{1}{\cancel{2}}\right)(1-z^2)^{-\frac{3}{2}}(-\cancel{2}z) = z(1-z^2)^{-3/2} \quad , \quad f''(0) = 0$$

$$f'''(z) = z\left(-\frac{3}{2}\right)(1-z^2)^{-5/2} + 1 \cdot (1-z^2)^{-3/2} \quad , \quad f'''(0) = 1$$

Hence by Taylor series

$$f(z) = f(0) + z f'(0) + \frac{z^2}{\lfloor 2} f''(0) + \frac{z^3}{\lfloor 3} f'''(0) + \text{------}$$

$$\sin^{-1} z = 0 + z \cdot 1 + \frac{z^2}{\lfloor 2} \cdot 0 + \frac{z^3}{\lfloor 3} \cdot 1 + \frac{z^4}{\lfloor 4} \cdot 0 + \frac{z^5}{\lfloor 5}(9) + \text{- - -}$$

or $$\sin^{-1} z = z + \frac{z^3}{\lfloor 3} + \frac{9z^5}{\lfloor 5} + \text{- - -}$$ **Ans**

Que (4) :- Expand the function $f(z) = \tan^{-1} z$ in powers of z.

Hint :- Same as above question.

Que (5) :- Expand the function $\frac{\sin z}{z-\pi}$ about $z = \pi$.

Sol :- Let $f(z) = \frac{\sin z}{z-\pi} = \frac{\sin(z-\pi+\pi)}{z-\pi} = -\frac{\sin(z-\pi)}{z-\pi}$

$$= -\frac{1}{z-\pi}\left[(z-\pi) - \frac{(z-\pi)^3}{\lfloor 3} + \frac{(z-\pi)^5}{\lfloor 5} - \text{- - - -}\right]$$

$$= -1 + \frac{(z-\pi)^2}{\lfloor 3} - \frac{(z-\pi)^4}{\lfloor 5} + \text{- - - - -}$$ **Ans**

Que (6) :- Expand the function $\frac{e^z}{(z-1)^2}$ about $z = 1$.

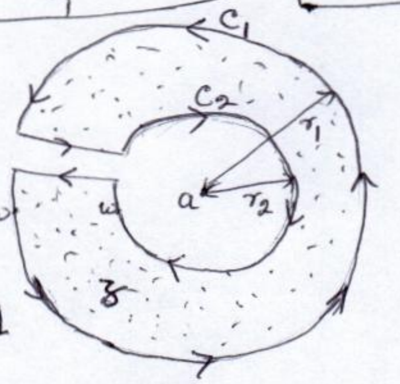
Sol :- Let $f(z) = \frac{e^{z-1+1}}{(z-1)^2} = e \cdot \frac{e^{z-1}}{(z-1)^2}$

$$= \frac{e}{(z-1)^2}\left[1 + \frac{(z-1)}{\lfloor 1} + \frac{(z-1)^2}{\lfloor 2} + \frac{(z-1)^3}{\lfloor 3} + \text{------}\right]$$

$$= e\left[\frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{\lfloor 2} + \frac{(z-1)}{\lfloor 3} + \text{- - -}\right]$$ **Ans**

Laurent's Theorem:— If a function $f(z)$ is analytic in a annular (ring shaped region) between two concentric circles C_1 and C_2 with centre at point a and radii r_1 and r_2, then at each point z in the annular

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}}, \text{ where } \boxed{a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw} \ \& \ \boxed{b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{(w-a)^{-n+1}}}$$



Proof :— By Cauchy integral formula for multi connected region, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{w-z} \quad \text{——— (1)}$$

If w lies on C_1 then $|z-a| < |w-a|$ or $\frac{|z-a|}{|w-a|} < 1$

$$\therefore \frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

$$= \frac{1}{w-a}\left[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots\right]$$

$$= \frac{1}{w-a} + \frac{(z-a)}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots$$

multiply by $\frac{f(w)}{2\pi i}$ and integrate both sides w.r.t. w along the C_1, then

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)\,dw}{w-a} + \frac{(z-a)}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2}\,dw + \frac{(z-a)^2}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^3}\,dw + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{—— (2)}, \text{ where } a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)\,dw}{(w-a)^{n+1}}$$

If w lies on C_2 then $|w-a| < |z-a|$ or $\frac{|w-a|}{|z-a|} < 1$, hence

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)} = \frac{1}{-(z-a)}\left[1 - \frac{w-a}{z-a}\right]^{-1}$$

$$= \frac{1}{-(z-a)}\left[1 + \left(\frac{w-a}{z-a}\right) + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \cdots\right]$$

multiplying by $-\frac{f(w)}{2\pi i}$ on both sides and integrate w.r.t. w along the C_2 then

$$-\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\cdot\frac{1}{z-a}\int_{C_2} f(w)\,dw + \frac{1}{(z-a)^2}\cdot\frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{(w-a)^{-1}} + \frac{1}{(z-a)^3}\cdot\frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{(w-a)^{-2}}$$

$$+ \frac{1}{(z-a)^4}\cdot\frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{(w-a)^{-3}} + \cdots$$

$$= \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \text{—— (3)}, \text{ where } b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{(w-a)^{-n+1}}$$

Putting the values from eq^n (2) & (3) in eq^n (1) we get the result.

Que ① :- Obtain the Taylor or Laurent series for the function $f(z) = \frac{z+3}{z(z^2-z-2)}$ when (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

Sol :- $f(z) = \frac{z+3}{z(z+1)(z-2)} = -\frac{3}{2z} + \frac{2}{3(z+1)} + \frac{5}{6(z-2)}$ (By partial fraction)

(i) If $|z|<1$, then $\frac{|z|}{2}<1$, hence

$$f(z) = -\frac{3}{2z} + \frac{2}{3}(1+z)^{-1} - \frac{5}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{3}{2z} + \frac{2}{3}\left(1 - z + z^2 - z^3 + \cdots\right) - \frac{5}{12}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right)$$ Ans

(ii) If $1<|z|<2 \Rightarrow 1<|z|$ & $|z|<2$ or $\frac{1}{|z|}<1$ & $\frac{|z|}{2}<1$, hence

$$f(z) = -\frac{3}{2z} + \frac{2}{3z}\left[1+\frac{1}{z}\right]^{-1} - \frac{5}{12}\left[1-\frac{z}{2}\right]^{-1}$$

$$= -\frac{3}{2z} + \frac{2}{3z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right] - \frac{5}{12}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$ Ans

(iii) If $|z|>2$ or $2<|z|$ or $\frac{2}{|z|}<1$, hence $\frac{1}{|z|}<1$

$$f(z) = -\frac{3}{2z} + \frac{2}{3z}\left[1+\frac{1}{z}\right]^{-1} + \frac{5}{6z}\left[1-\frac{2}{z}\right]^{-1}$$

$$= -\frac{3}{2z} + \frac{2}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] + \frac{5}{6z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right]$$ Ans

Que ② :- Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in Laurent series valid for (i) $|z-1|>1$ and (ii) $0<|z-2|<1$

Sol :- $f(z) = \frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$, By partial fraction

(i) If $|z-1|>1$ or $1<|z-1|$ or $\frac{1}{|z-1|}<1$, hence

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2} = \frac{1}{z-1} - \frac{2}{(z-1)-1} = \frac{1}{z-1} - \frac{2}{z-1}\left[1-\frac{1}{z-1}\right]^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left[1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots\right]$$ Ans

(ii) $0<|z-2|<1$, hence

$$f(z) = \frac{1}{z-2+1} - \frac{2}{z-2} = \frac{1}{z-2}\left[1+\frac{1}{z-2}\right]^{-1} - \frac{2}{z-2}$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right] - \frac{2}{z-2}$$ Ans

Que ③ :- Find four terms of the Laurent series expansion valid in the region $0<|z-1|<1$ for the function $f(z) = \frac{2z+1}{z^3+z^2-2z}$

Ans :- $-\frac{1}{2}\left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z-1}{3} + \frac{(z-1)^2}{9} - \frac{(z-1)^3}{27} + \cdots\right] + \frac{1}{z-1}$

Que ④ :- Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1 < |z| < 3$ (ii) $|z| > 3$ (iii) $0 < |z+1| < 2$ (iv) $|z| < 1$

Sol :- $f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\cdot\frac{1}{(z+1)} + \frac{1}{2}\cdot\frac{1}{z+3}$, By partial fraction

(i) If $1 < |z| < 3$ i.e. $1 < |z|$ and $|z| < 3$ or $\frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$, hence

$$f(z) = \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} + \frac{1}{6}\left[1+\frac{z}{3}\right]^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] + \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$ Ans

(ii) If $|z| > 3$ i.e. $3 < |z|$ or $\frac{3}{|z|} < 1$, hence $\frac{1}{|z|} < 1$

$$f(z) = \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} + \frac{1}{2z}\left[1+\frac{3}{z}\right]^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] + \frac{1}{2z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$ Ans

(iii) If $0 < |z+1| < 2$ i.e. $\frac{|z+1|}{2} < 1$, hence

$$f(z) = \frac{1}{2(z+1)} + \frac{1}{2(z+1+2)} = \frac{1}{2(z+1)} + \frac{1}{4}\left[1+\frac{z+1}{2}\right]^{-1}$$

$$= \frac{1}{2(z+1)} + \frac{1}{4}\left[1 - \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 - \left(\frac{z+1}{2}\right)^3 + \cdots\right]$$ Ans

(iv) $|z| < 1$, then $\frac{|z|}{3} < 1$, hence

$$f(z) = \frac{1}{2(z+1)} + \frac{1}{3(z+3)} = \frac{1}{2}[1+z]^{-1} + \frac{1}{9}\left[1+\frac{z}{3}\right]^{-1}$$

$$= \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] + \frac{1}{9}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$ Ans

Que ⑤ :- Find the Laurent expansion for $f(z) = \frac{7z-2}{z^3-z^2-2z}$ in the region

(i) $0 < |z+1| < 1$ (ii) $1 < |z+1| < 3$ (iii) $|z+1| > 3$

Sol :- $f(z) = \frac{7z-2}{z(z^2-z-2)} = \frac{7z-2}{z(z+1)(z-2)} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$, (By Partial fraction)

or $f(z) = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$

(i) If $|z+1| < 1$ then $\frac{|z+1|}{3} < 1$, hence

$$f(z) = \frac{1}{-1}\left[1-(z+1)\right]^{-1} - \frac{3}{z+1} + \frac{2}{-3}\left[1 - \frac{z+1}{3}\right]^{-1}$$

$$= -\left[1 + (z+1) + (z+1)^2 + \cdots\right] - \frac{3}{z+1} - \frac{2}{3}\left[1 + \left(\frac{z+1}{3}\right) + \left(\frac{z+1}{3}\right)^2 + \cdots\right]$$ Ans

(ii) If $1 < |z+1| < 3$ i.e. $1 < |z+1|$ and $|z+1| < 3$ or $\frac{1}{|z+1|} < 1$ & $\frac{|z+1|}{3} < 1$, then

$$f(z) = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3} = \frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{3}{z+1} - \frac{2}{3}\left[1-\frac{z+1}{3}\right]^{-1}$$

$$= \frac{1}{z+1}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\cdots\right] - \frac{3}{z+1} - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

Ans

(iii) If $|z+1| > 3$ i.e. $3 < |z+1|$ or $\frac{3}{|z+1|} < 1$, Hence $\frac{1}{|z+1|} < 1$, the

$$f(z) = \frac{1}{(z+1)}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{3}{z+1} + \frac{2}{(z+1)}\left[1-\frac{3}{z+1}\right]^{-1}$$

$$= \frac{1}{(z+1)}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\cdots\right] - \frac{3}{z+1} + \frac{2}{(z+1)}\left[1+\left(\frac{3}{z+1}\right)+\left(\frac{3}{z+1}\right)^2+\cdots\right]$$

Ans

Que 6 :- Expand $\frac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ in $3 < |z+2| < 5$

Sol :- Let $f(z) = \frac{z^2-6z-1}{(z-1)(z+2)(z-3)} = \frac{1}{z-1} + \frac{1}{z+2} - \frac{1}{z-3}$

$$= \frac{1}{(z+2)-3} + \frac{1}{z+2} - \frac{1}{(z+2)-5}$$

given $3 < |z+2| < 5$ i.e. $\frac{3}{|z+2|} < 1$ and $\frac{|z+2|}{5} < 1$, Hence

$$f(z) = \frac{1}{(z+2)}\left[1-\frac{3}{z+2}\right]^{-1} + \frac{1}{z+2} + \frac{1}{5}\left[1-\frac{(z+2)}{5}\right]^{-1}$$

$$= \frac{1}{(z+2)}\left[1+\frac{3}{z+2}+\left(\frac{3}{z+2}\right)^2+\cdots\right] + \frac{1}{z+2} + \frac{1}{5}\left[1+\left(\frac{z+2}{5}\right)+\left(\frac{z+2}{5}\right)^2+\cdots\right]$$

$$= \frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{3^2}{(z+2)^3} + \cdots + \frac{1}{5}\left[1+\frac{(z+2)}{5}+\frac{(z+2)^2}{5^2}+\cdots\right]$$

Ans

Que 7 :- If $f(z) = \frac{z+4}{(z+3)(z-1)^2}$, find Laurent's series expansion in $0 < |z-1| < 4$ and $|z| > 4$.

Hint :- $f(z) = \frac{1}{16(z+3)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

$$= \frac{1}{16[(z-1)+4]} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

Ans :- (i) $f(z) = \frac{1}{64}\left[1-\frac{z-1}{4}+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+\cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

(ii) $f(z) = \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$

Singularities :— A point at which a function $f(z)$ is not analytic is known as singular point or singularity of the function. For example the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$ have the singular points $z=1$ & $z=2$.

Isolated Singularities :— If $z=a$ is the singularity of $f(z)$ and if there is no othere singularity within a small circle surrounding the point $z=a$, then $z=a$ is said to be an isolated singularity of the function $f(z)$, othere wise it is called non isolated singularities.

Let the function $f(z) = \frac{1}{\sin\frac{\pi}{z}}$

Now $\sin\frac{\pi}{z} = 0 = \sin n\pi \Rightarrow \frac{1}{z} = n$ or $z = \frac{1}{n}$ or $z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\cdots = 0$

i.e. $z=0$ is the non-isolated singularity.

Essential Singularity — In the Laurent series the number of the terms of negative powers is infinite then $z=a$ is called an essential singularity. For example

(i) $$f(z) = \sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{\lfloor 3}\frac{1}{(z-a)^3} + \frac{1}{\lfloor 5}\cdot\frac{1}{(z-a)^5} - \cdots\cdots$$

i.e. $f(z)$ has infinite number of terms in the -ive power of $z-a$, hence $f(z)$ has essential singularity at $z=a$.

(ii) Let $$f(z) = \frac{e^{\frac{1}{z}}}{z^2} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{\lfloor 2 z^2} + \frac{1}{\lfloor 3 z^3} + \cdots\right)$$

i.e. $f(z)$ has essential singularity at $z=0$.

Residue at Pole :— In Laurent series the coefficient of $\frac{1}{z-a}$ is known as residue of $f(z)$ at the pole $z=a$

i.e. Residue (at $z=a$) $= b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$

Residue at Infinity :— The residue of $f(z)$ at $z=\infty$ is defined as

$-\frac{1}{2\pi i}\int_C f(z)\,dz$ i.e. $-b_1$ i.e. the coefficient of $\left(-\frac{1}{z}\right)$.

Formula for finding the Residue of $f(z)$ (at $z=\infty$) $= \lim_{z\to\infty}\left[-z f(z)\right]$

Formula for Finding Residues at Poles :—

(i) If $f(z)$ has a simple pole at $z=a$, then

$$\text{Residue of } f(z) \text{ (at } z=a) = \lim_{z\to a} (z-a) f(z)$$

(ii) If $f(z)$ has a pole at $z=a$ of orter n, then

$$\text{Residue of } f(z) \text{ (at } z=a \text{ of order } n) = \lim_{z\to a}\left[\frac{1}{\lfloor n-1}\left\{\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)\right\}\right]$$

(iii) If $f(z) = \dfrac{\phi(z)}{\psi(z)}$, where $\psi(a)=0$ and $\phi(a)\neq 0$, then

$$\text{Residue of } f(z) \text{ (at } z=a) = \frac{\phi(a)}{\psi'(a)}$$

Que (1) :— Find the residue of the following function at its poles

(i) $\dfrac{1-2z}{z(z-1)(z-2)}$ (ii) $\dfrac{z^2}{(z-1)(z-2)^2}$ (iii) $\dfrac{1-e^{2z}}{z^4}$ (iv) $\cot z$

Sol :— (i) The poles are given by $z(z-1)(z-2)=0 \Rightarrow z=0,1,2$

$$\text{Residue of } f(z) \text{ (at } z=0) = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{\cancel{z}(1-2z)}{\cancel{z}(z-1)(z-2)} = \frac{1}{2}$$

$$\text{Res. of } f(z) \text{ (at } z=1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{\cancel{(z-1)}(1-2z)}{z\cancel{(z-1)}(z-2)} = 1$$

$$\text{Res. of } f(z) \text{ (at } z=2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{\cancel{(z-2)}(1-2z)}{z(z-1)\cancel{(z-2)}} = -\frac{3}{2}$$

Ans

(ii) Let $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$, The poles are $(z-1)(z-2)^2 \Rightarrow z=1,\ z=2,2$

$$\therefore \text{Res of } f(z) \text{ (at } z=1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{\cancel{(z-1)}z^2}{\cancel{(z-1)}(z-2)^2} = 1$$

$$\text{Res of } f(z) \text{ (at } z=2, \text{ of order two}) = \lim_{z\to 2}\left[\frac{d}{dz}(z-2)^2 f(z)\right]$$

$$= \lim_{z\to 2}\left[\frac{d}{dz}\cancel{(z-2)^2}\cdot\frac{z^2}{(z-1)\cancel{(z-2)^2}}\right]$$

$$= \lim_{z\to 2}\left[\frac{(z-1)\,2z - z^2\cdot 1}{(z-1)^2}\right]$$

$$= \frac{4-4}{1} = 0$$

Ans

(iii) Let $f(z) = \dfrac{1-e^{2z}}{z^4}$, the poles are $z=0$ of order 4. Then

Residue of $f(z)$ (at $z=0$ of order 4) $= \lim_{z\to 0}\left[\dfrac{1}{\lfloor 3}\dfrac{d^3}{dz^3}\left\{(z-0)^4 \cdot \dfrac{1-e^{2z}}{z^4}\right\}\right]$

$= \lim_{z\to 0}\left[\dfrac{1}{\lfloor 3}\left(-8e^{2z}\right)\right] = -\dfrac{4}{3}$ Ans

(iv) Let $f(z) = \cot z = \dfrac{\cos z}{\sin z}$, the pole are given by

$\sin z = 0 \Rightarrow \sin z = \sin 0 \Rightarrow z = 0$

$\therefore$ Res (at $z=0$) $= \dfrac{\phi(a)}{\psi'(a)} = \dfrac{\cos 0}{\cos 0} = 1$ Ans

Que ② :- Find the residue at $z=0$ of the following functions —

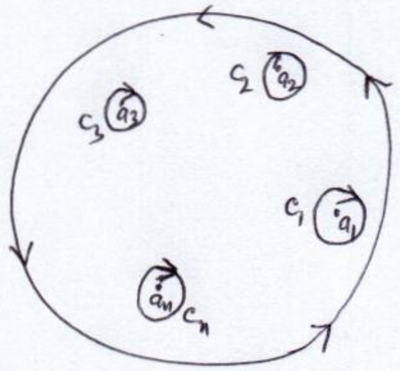
(i) $\dfrac{1+e^z}{\sin z + z\cos z}$ (ii) $z\cos\frac{1}{z}$

Sol :- (i)
Res (at $z=0$) $= \lim_{z\to 0}\left[\dfrac{z(1+e^z)}{\sin z + z\cos z}\right]$, ($\frac{0}{0}$ form, hence apply L'Hospital rule)

$= \lim_{z\to 0}\left[\dfrac{1+ze^z+e^z}{\cos z + \cos z - z\sin z}\right] = \dfrac{1+1}{1+1} = 1$ Ans

(ii) $f(z) = z\cos\frac{1}{z} = z\left[1 - \dfrac{1}{\lfloor 2\, z^2} + \dfrac{1}{\lfloor 4\, z^4} - \cdots\right] = z - \dfrac{1}{2z} + \dfrac{1}{24z^3} - \cdots$

Hence Res (at $z=0$) = the coefficient of $\frac{1}{z}$ $= -\frac{1}{2}$ Ans



Cauchy's Residue Theorem :- If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then

$\int_C f(z)\,dz = 2\pi i$ (Sum of residues at the poles within C)

Proof :- Let $C_1, C_2, C_3 \cdots C_n$ be the non intersecting circles with centres at $a_1, a_2, a_3 \cdots a_n$ and lies within C.
Then by Cauchy integral theorem for multiconnected region

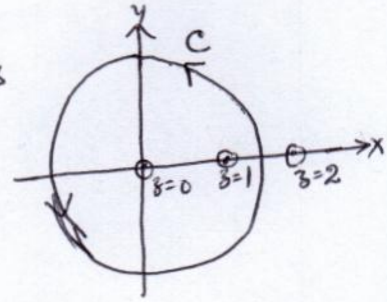
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots \int_{C_n} f(z)\,dz$$

$$= 2\pi i\left[\text{Res } f(a_1) + \text{Res } f(a_2) + \text{Res } f(a_3) + \cdots \text{Res } f(a_n)\right]$$

$$= 2\pi i \sum R^+$$

Proved

Que ① Evaluate the integral $\int_C \frac{1-2z}{z(z-1)(z-2)}\,dz$, where C is the circle $|z|=1.5$



Sol :- The poles are $z=0,1,2$. The pole $z=0,1$ lies within C, but $z=2$ outside C, hence

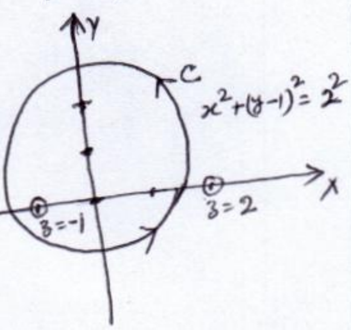
$$R_1 = \text{Residue (at } z=0) = \lim_{z\to 0} \frac{\cancel{(z-0)}(1-2z)}{\cancel{z}(z-1)(z-2)} = \frac{1}{2}$$

$$R_2 = \text{Residue (at } z=1) = \lim_{z\to 1} \frac{\cancel{(z-1)}(1-2z)}{z\cancel{(z-1)}(z-2)} = 1$$

Hence by Cauchy Residue theorem

$$\int_C \frac{1-2z}{z(z-1)(z-2)}\,dz = 2\pi i \sum R^+ = 2\pi i\left(\frac{1}{2}+1\right) = 3\pi i \quad \underline{\text{Ans}}$$

Que ② :- Evaluate the integral $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, where C is $|z-i|=2$



Sol :- The poles are $z=2$ and $z=-1$ of order two. only $z=-1$ lies within C, then

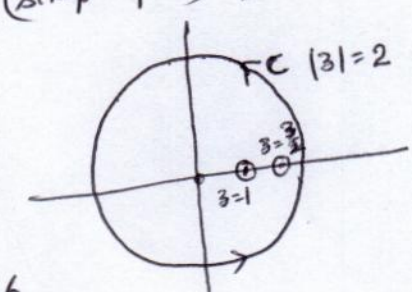
$$\text{Residue of } f(z) \text{ (at } z=-1 \text{ of order two)} = \lim_{z\to -1} \frac{d}{dz}\left[\frac{\cancel{(z+1)^2}(z-1)}{\cancel{(z+1)^2}(z-2)}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)-(z-1)}{(z-2)^2}\right] = \frac{-1}{9}$$

Hence by Cauchy Residue theorem

$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = 2\pi i\,(\text{Sum of the residues within } C) = \frac{-2\pi i}{9} \quad \underline{\text{Ans}}$$

Que ③ :- Evaluate $\int_C \frac{12z-7}{(z-1)^2(2z-3)}\,dz$, where C is the circle $|z|=2$.



Sol :- The poles are $z=1$ (of order two) and $z=\frac{3}{2}$ (simple pole), both the poles lie inside C

$$R_1 = \text{Res (at } z=1 \text{ of order two)} = \lim_{z\to 1}\left[\frac{d}{dz}\,\frac{\cancel{(z-1)^2}(12z-7)}{\cancel{(z-1)^2}(2z-3)}\right]$$

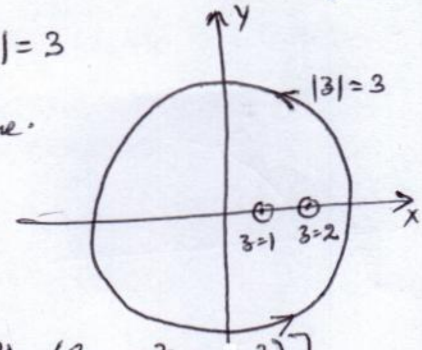
$$= \lim_{z\to 1}\left[\frac{(2z-3)\cdot 12-(12z-7)\cdot 2}{(2z-3)^2}\right] = 26$$

$$R_2 = \text{Res (at } z=\tfrac{3}{2}) = \lim_{z\to \frac{3}{2}}\left[\left(z-\frac{3}{2}\right)\cdot\frac{(12z-7)}{(z-1)^2(2z-3)}\right] = \lim_{z\to\frac{3}{2}}\left[\frac{1}{2}\cdot\frac{12z-7}{(z-1)^2}\right] = 22$$

∴ By Cauchy Residue theorem

$$\int_C \frac{12z-7}{(z-1)^2(2z-3)}\,dz = 2\pi i\,(26+22) = 96\pi i \quad \underline{\text{Ans}}$$

Que 4 :- Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, where $C : |z| = 3$



Sol :- The poles are $z = 1$ of order two and $z = 2$ of order one.
Both the poles are lies within C, hence

$$R_1 = \text{Res}(\text{at } z=1) = \lim_{z\to 1} \frac{d}{dz}\left[\frac{\cancel{(z-1)^2}(\sin \pi z^2 + \cos \pi z^2)}{\cancel{(z-1)^2}(z-2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]$$

$$= \frac{(1-2)(2\pi \cos \pi - 2\pi \sin \pi) - (\sin \pi + \cos \pi)}{1}$$

(Since $\cos \pi = -1$ and $\sin \pi = 0$)

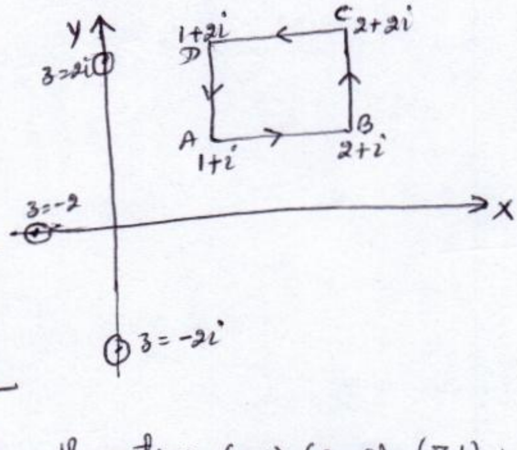
$$= (1 + 2\pi)$$

$$R_2 = \text{Res}(\text{at } z=2) = \lim_{z\to 2}\left[\frac{\cancel{(z-2)}(\sin \pi z^2 + \cos \pi z^2)}{(z-1)^2 \cancel{(z-2)}}\right] = \frac{\sin 4\pi + \cos 4\pi}{(2-1)^2} = 1$$

$\therefore$ By Cauchy Residue theorem

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz = 2\pi i (1 + 2\pi + 1) = 4\pi i (1+\pi)$$ Ans

Que 5 :- Evaluate $\int_C \frac{2z^2+5}{(z+2)^3 (z^2+4)} dz$ where c is the square with the vertices at $1+i$, $2+i$, $2+2i$, $1+2i$.

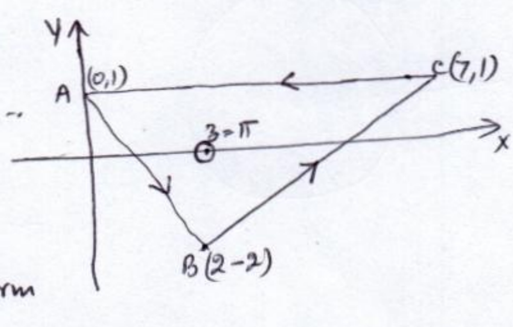


Sol :- The poles are $z = -2$ (pole of order 3)
and $z = \pm 2i$ (simple pole)

Since no poles lies within c, hence by C-R T

$$\int_C \frac{2z^2+5}{(z+2)^3 (z^2+4)} dz = 0$$ Ans

Que 6 :- Evaluate $\int_C \frac{dz}{z^2 \sin z}$, where c is triangle with vertices $(0,1)$ $(2,-2)$, $(7,1)$.



Sol :- The poles are $z^2 \sin z = 0 \Rightarrow z = 0$ (of order 2)
and $\sin z = 0 = \sin n\pi \Rightarrow z = n\pi = 0, \pi, 2\pi \cdots$

Only the pole $z = \pi$ lies within c, hence

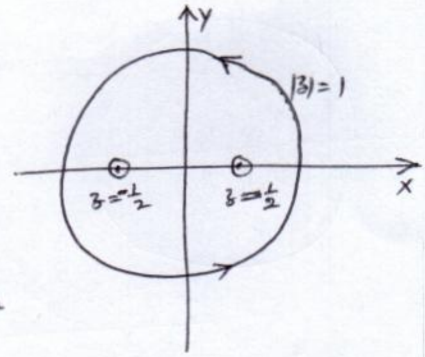
$$\text{Residue}(\text{at } z=\pi) = \lim_{z\to\pi}\left[(z-\pi)\cdot\frac{1}{z^2 \sin z}\right], \quad \frac{0}{0} \text{ form}$$

$$= \lim_{z\to\pi}\left[\frac{1}{2z \sin z + z^2 \cos z}\right] = \frac{1}{-\pi^2}$$

Hence by Cauchy Residue Theorem

$$\int_C \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{-\pi^2}\right) = -\frac{2i}{\pi}$$ Ans

Que 7 :- Evaluate $\int_C \frac{e^z}{\cos \pi z}\,dz$, where C is the unit circle $|z| = 1$.



Sol :- The poles are $\cos \pi z = 0 = \cos\left(\frac{2n+1}{2}\right)\pi$

$$\Rightarrow z = \pm\left(\frac{2n+1}{2}\right) \Rightarrow z = \pm\frac{1}{2}, \pm\frac{3}{2} \cdots$$

The poles $z = \pm\frac{1}{2}$ lies within C, then

$$R_1 = \text{Residue (at } z = \tfrac{1}{2}) = \lim_{z \to \frac{1}{2}} \frac{(z-\frac{1}{2})\, e^z}{\cos \pi z}, \quad \frac{0}{0} \text{ form}$$

$$= \lim_{z \to \frac{1}{2}} \frac{(z-\frac{1}{2})\, e^z + e^z}{-\pi \sin \pi z}, \quad \text{(By L' Hospital Rule)}$$

$$= \frac{e^{\frac{1}{2}}}{-\pi}$$

Similarly $R_2 = \text{Residue (at } z = -\frac{1}{2}) = \frac{e^{-\frac{1}{2}}}{\pi}$, Hence By Cauchy Residue Theorem

$$\int_C \frac{e^z}{\cos \pi z}\,dz = 2\pi i\left[\frac{e^{\frac{1}{2}}}{-\pi} + \frac{e^{-\frac{1}{2}}}{\pi}\right] = \frac{4\pi i}{-\pi}\left[\frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2}\right] = -4i \sinh\tfrac{1}{2} \quad \underline{\text{Ans}}$$

Que 8 :- Evaluate $\int_C \frac{dz}{z^2 \sinh z}$, where C is the circle $|z-1| = 2$

Sol :- $$f(z) = \frac{1}{z^2 \sinh z} = \frac{1}{z^2\left(z + \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} + \cdots\right)} = \frac{1}{z^3}\left[1 + \left(\frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} + \cdots\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} + \cdots\right) + \left(\frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} + \cdots\right)^2 + \cdots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)z^4 + \cdots\right]$$

$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \cdots$$

$\therefore$ Residue of $f(z)$ (at the pole $z = 0$) = Coefficient of $\frac{1}{z} = -\frac{1}{6}$

Now the pole $z = 0$ lies within C, hence by Cauchy Residue theorem

$$\int_C \frac{dz}{z^2 \sinh z} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3} \quad \underline{\text{Ans}}$$

Que 9 :- Evaluate $\int_C \frac{dz}{\sinh z}$, where C is the circle $|z| = 4$.

Sol :- Method I :- Solve same as above question.

Method II :- The poles are $\sinh z = 0 \Rightarrow \frac{e^z - e^{-z}}{2} = 0 \Rightarrow e^{2z} - 1 = 0$

or $e^{2z} = 1 = \cos 2n\pi + i \sin 2n\pi = e^{2n\pi i} \Rightarrow \boxed{z = n\pi i} = 0, \pm\pi i \cdots$

The poles $z = 0$ and $z = \pm\pi i$ lies within C hence

$\therefore R_1 = 1$ & $R_2 = -1$ & $R_3 = -1$ $\quad$ Ans :- $-2\pi i$

Application of Residue Theorem For Evaluation of Integrals:—

(i) Evaluation $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$ Type:— We consider a contour c, consists of the unit circle $|z| = 1$.

hence $z = e^{i\theta}$ or $dz = i e^{i\theta} d\theta = i z\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\therefore \int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta = \int_0^{2\pi} f\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}, \frac{e^{i\theta} + e^{-i\theta}}{2}\right) d\theta = \int_c f\left\{\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right\} \frac{dz}{iz}$$

$$= \int_c F(z)\,dz$$

Now we find the poles of $F(z)$ and apply Cauchy Residue Theorem.

Que ①:– Evaluate $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

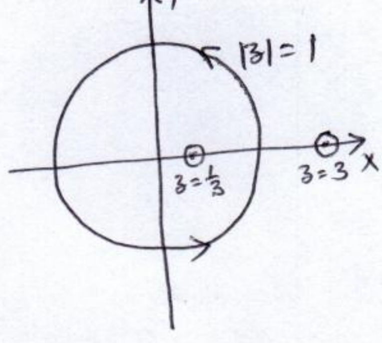
Sol:– Let C be a unit circle $|z| = 1$, hence $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5 - 3\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)} = \int_c \frac{1}{5 - \frac{3}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \int_c \frac{dz}{10z - 3z^2 - 3}\left(\frac{2}{i}\right)$$

$$= \frac{2}{-i}\int_c \frac{dz}{3z^2 - 10z + 3} = \frac{2i}{3}\int_c \frac{dz}{\left(z - \frac{1}{3}\right)(z - 3)} = \frac{2i}{3}\int_c f(z)\,dz$$

The poles of $f(z)$ are $z = \frac{1}{3}, 3$. But only the pole $z = \frac{1}{3}$ lies within c, then



$$\text{Residue of } f(z)\left(\text{at } z = \tfrac{1}{3}\right) = \lim_{z \to \frac{1}{3}} \left(z - \tfrac{1}{3}\right) \frac{1}{\left(z - \frac{1}{3}\right)(z - 3)}$$

$$= \lim_{z \to \frac{1}{3}} \frac{1}{(z - 3)} = \frac{-3}{8}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \frac{2i}{3}\int_c f(z)\,dz = \frac{2i}{3}(2\pi i)\left(-\frac{3}{8}\right) = \frac{\pi}{2}$$

Ans

Que ② Evaluate (i) $\int_0^{2\pi} \frac{4\,d\theta}{5 + 4\sin\theta}$ (ii) $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$, Ans:– (i) $\frac{8\pi}{3}$, (ii) $\frac{2\pi}{\sqrt{3}}$

Hint:– (solve same as above question)

Que ③:– Show that $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2} = \frac{2\pi}{1-a^2}$, $0<a<1$

Sol:– Let c be a unit circle $|z|=1$, hence $z=e^{i\theta}$ or $d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2a\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)+a^2} = \int_C \frac{1}{1-\frac{a}{i}\left(z-\frac{1}{z}\right)+a^2}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{1}{1-\frac{a}{i}\left(\frac{z^2-1}{z}\right)+a^2}\cdot\frac{dz}{iz} = \int_C \frac{dz}{iz - a(z^2-1)+a^2 iz}$$

$$= \int_C \frac{dz}{-az^2+i(1+a^2)z+a} = \int_C \frac{dz}{-a\left[z^2-i\left(a+\frac{1}{a}\right)-1\right]}$$

$$= \frac{1}{-a}\int_C \frac{dz}{(z-ia)\left(z-\frac{i}{a}\right)}$$

$$= -\frac{1}{a}\int_C f(z)\,dz$$

$$z = \frac{i\left(a+\frac{1}{a}\right) \pm \sqrt{-\left(a+\frac{1}{a}\right)^2+4}}{2} = \frac{i\left(a+\frac{1}{a}\right) \pm \sqrt{-\left(a^2+\frac{1}{a^2}-2\right)}}{2} = \frac{i\left(a+\frac{1}{a}\right) \pm i\left(a-\frac{1}{a}\right)}{2} = ai\,,\ \frac{i}{a}$$

The poles of $f(z)$ are $z=ai, \frac{i}{a}$. But only $z=ai$ lies within c, Since $a<1$.

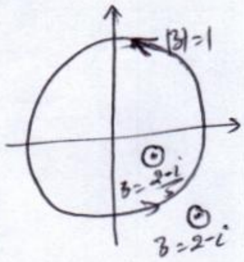
$\therefore$ Residue of $f(z)$ (at $z=ai$) $= \lim_{z\to ia} \frac{(z-ia)}{(z-ia)\left(z-\frac{i}{a}\right)}$

$$= \frac{1}{ia-\frac{i}{a}} = \frac{a}{i(a^2-1)}$$

Hence by Cauchy Residue Theorem

$$\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2} = -\frac{1}{a}\int_C f(z)\,dz = -\frac{1}{a}(2\pi i)\left(\frac{a}{i(a^2-1)}\right) = \frac{2\pi}{1-a^2}$$

Proved



Que ④:– Show that $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \pi$

Hint:– $\int_C \frac{1}{3-\frac{2}{2}\left(z+\frac{1}{z}\right)+\frac{1}{2i}\left(z-\frac{1}{z}\right)}\,\frac{dz}{iz} = \int_C \frac{2\,dz}{6iz-(z^2+1)2i+(z^2-1)}$

$$= \int_C \frac{2\,dz}{(1-2i)z^2+6iz-(1+2i)} = \frac{2}{1-2i}\int_C \frac{dz}{z^2+\left(\frac{6i}{1-2i}\right)z-\frac{(1+2i)}{(1-2i)}}$$

$$= \frac{2}{1-2i}\int_C \frac{dz}{z^2+\frac{(-12+6i)}{5}z+\frac{(3-4i)}{5}} = \frac{10}{1-2i}\int_C \frac{dz}{5z^2+(6i-12)z+(3-4i)} = \frac{10}{1-2i}\int_C f(z)dz$$

The poles are $z=2-i$ and $z=\frac{2-i}{5}$, only $z=\frac{2-i}{5}$ lies with in c

$\therefore$ Res of $f(z)$ (at $z=2-i$) $= \frac{1}{-8+4i}$

$$\therefore\ \frac{10}{1-2i}\int_C f(z)dz = \frac{10}{1-2i}\cdot\frac{1}{(-8+4i)}\cdot 2\pi i = \pi$$

$$= \text{Real part of } \frac{1}{-2i}\int_C \frac{z^3}{z^2-\frac{5}{2}z+1}\,dz$$

$$= \text{Real part of } \frac{i}{2}\int_C \frac{z^3}{(z-\frac{1}{2})(z-2)}\,dz$$

$$= \text{Real part of } \frac{i}{2}\int_C f(z)\,dz$$

The poles of $f(z)$ are $z=\frac{1}{2}, 2$. But only $z=\frac{1}{2}$ lies inside c, hence

Residue of $f(z)$ (at $z=\frac{1}{2}$) $= \lim_{z\to\frac{1}{2}} (z-\frac{1}{2})\,f(z) = \lim_{z\to\frac{1}{2}} (z-\frac{1}{2})\frac{z^3}{(z-\frac{1}{2})(z-2)} = -\frac{1}{12}$

Hence by Cauchy Residue theorem

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \text{Real part of } \frac{i}{2}(2\pi i)\left(\frac{1}{-12}\right) = \frac{\pi}{12}$$ Ans

Que ⑧ :- Apply calculus of residue to prove that $\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{\pi}{6}$

Hint :- Same as above question.

Que ⑨ :- Using contour integration, evaluate $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$

Sol :- $$\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi}\frac{1+2e^{i\theta}}{5+4\cos\theta}\,d\theta$$

$$= \text{Real part of } \frac{1}{2}\int_C \frac{1+2z}{5+2(z+\frac{1}{z})}\left(\frac{dz}{iz}\right)$$, where c is $|z|=1$.

$$= \text{Real part of } \frac{1}{2i}\int_C \frac{1+2z}{(2z+1)(z+2)}\,dz$$

$$= \text{Real part of } \frac{1}{2i}\int_C \frac{1}{z+2}\,dz = \text{Real part of } \frac{1}{2i}\int_C f(z)\,dz$$

The pole is $z=2$ lies out side the c, hence by Cauchy Residue theorem

$$\therefore \int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \text{Real part of } \frac{1}{2i}(2\pi i)(0) = 0$$ Ans

Que ⑩ :- Using calculus of residue prove that (i) $\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}$, $a^2<1$

(ii) $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$, $0<b<a$

Que 5 :- Using contour integration, evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, where $a>|b|$

Sol :- Let the integration round a unit circle c, i.e. $|z|=1 \Rightarrow z=e^{i\theta}$ or $d\theta = \frac{dz}{iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+\frac{b}{2}(e^{i\theta}+e^{-i\theta})} = \int_C \frac{1}{a+\frac{b}{2}(z+\frac{1}{z})}\left(\frac{dz}{iz}\right)$$

$$= \int_C \frac{2z}{bz^2+2az+b}\left(\frac{dz}{iz}\right) = \frac{2}{ib}\int_C \frac{dz}{z^2+\frac{2a}{b}z+1} = \frac{2}{ib}\int_C \frac{dz}{(z-\alpha)(z-\beta)}$$

$$= \frac{2}{ib}\int_C f(z)\,dz$$

where $\alpha = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}$ and $\beta = -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}$

The poles of $f(z)$ are $z=\alpha, \beta$. But only $z=\alpha$ lies inside c because $a>|b|$, hence $|\beta|>1$ and since $\alpha\beta = 1$, hence $|\alpha|<1$.

$$\therefore \text{Residue of } f(z) (\text{at } z=\alpha) = \lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z\to\alpha} (z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$

$$= \frac{1}{\alpha-\beta} = \frac{b}{2\sqrt{a^2-b^2}}$$

Hence by Cauchy Residue theorem

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{ib}\int_C f(z)\,dz = \frac{2}{ib}\cdot 2\pi i \cdot \frac{b}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$ Ans

Que 6 :- By contour integration prove that $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, where $a>|b|$

Hint :- Same as above question.

Que 7 :- Evaluate by contour integration $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$

Sol :- $$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \text{Real part of} \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta$$

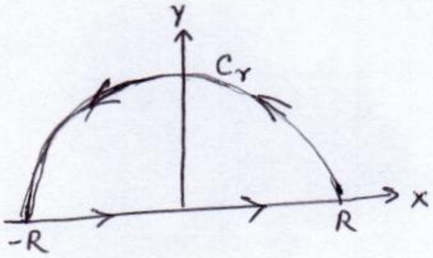
$$= \text{Real part of} \int_0^{2\pi} \frac{e^{3i\theta}}{5-\frac{4}{2}(e^{i\theta}+e^{-i\theta})}\,d\theta$$

$$= \text{Real part of} \int_C \frac{z^3}{5-2(z+\frac{1}{z})}\left(\frac{dz}{iz}\right)$$ where c is circle $|z|=1$

$$= \text{Real part of } \frac{1}{i}\int_C \frac{z^3}{5z-2z^2-2}\,dz$$

Evaluation of $\int_{-\infty}^{\infty} f(x)dx$:- where $f(x) = \frac{f_1(x)}{f_2(x)}$, and $f_2(x)$ has no real roots, i.e. $f(x)$ has no poles on real axis.

Consider $\int_C f(z)\,dz$ taken round a closed curve c consisting of upper half circumference C_r of the circle $|z| = R$ and part of the real axis from $-R$ to R.



There are no poles of $f(z)$ on the real axis, but possibly some poles lies within C, then by Cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{Sum of the residues of } f(z) \text{ at the poles within } C$$

or $$\int_{-R}^{R} f(x)\,dx + \int_{C_r} f(z)\,dz = 2\pi i \times \text{Sum of the residues}$$

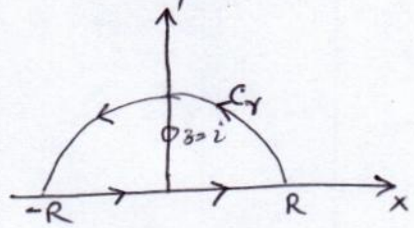
or $$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = -\lim_{R\to\infty}\int_{C_r} f(z)\,dz + 2\pi i \times \text{Sum of the residues}$$

or $$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \times \text{Sum of the residues}}$$ $\left(\text{Since } \lim_{R\to\infty}\int_{C_r} f(z)\,dz = \int_0^{\pi} f(Re^{i\theta})\,iRe^{i\theta}d\theta = 0\right)$

Que ① :- By contour integration prove that $\int_0^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}$

Sol :- Consider the integral $\int_C f(z)\,dz$, where $f(z) = \frac{1}{1+z^2}$

taken round a closed contour C consisting of upper half of a large circle $|z| = R$ and real axis from $-R$ to R.



The poles of $f(z)$ are $1+z^2 = 0$ or $z = \pm i$

Only $z = i$ lies within C, hence

Residue of $f(z)$ (at $z=i$) $= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}$

Hence by Cauchy residue theorem

$\int_C f(z)dz = 2\pi i \times$ Sum of the residues of $f(z)$ lies within c

$$\Rightarrow \int_{-R}^{R} f(x)dx + \int_{C_r} f(z)dz = 2\pi i \times \frac{1}{2i}$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_r} f(z)\,dz = \pi$$

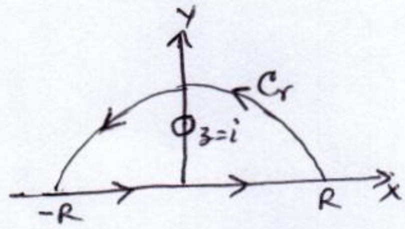
$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi \quad , \left(\text{Since} \left|\int_{C_r} f(z)dz\right| \le \int_{C_r} \frac{|dz|}{|1+z^2|} , \quad |z| = R \right.$$

$$< \int_0^{\pi} \frac{R\,d\theta}{R^2\left(1+\frac{1}{R^2}\right)} \to 0 \text{ as } R\to\infty$$

$$\text{or} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

$$\text{or} \quad \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} \quad , \text{(Since } f(x) \text{ is even function.)}$$

Que (2) :- Evaluate $\int_0^{\infty} \frac{\cos mx}{1+x^2}\,dx$



Sol :- Consider $\int_C f(z)dz$, when $f(z) = \frac{e^{imz}}{1+z^2}$, taken round the closed contour c consisting of a upper half of a large circle $|z| = R$ and the real axis from $-R$ to R

The poles of $f(z)$ are , $z = \pm i$, only $z = i$ lies within c, hence

$$\text{Residue of } f(z) \text{ (at } z=i) = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} (z-i) \frac{e^{imz}}{(z-i)(z+i)} = \frac{e^{-m}}{2i}$$

Hence by Cauchy residue theorem

$\int_C f(z)dz = 2\pi i \times$ Sum of the residues

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{C_r} f(z)\,dz = 2\pi i \times \frac{e^{-m}}{2i}$$

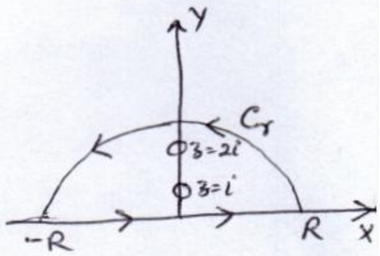
$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{C_r} f(z)dz = \pi e^{-m}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{1+x^2}\,dx = \pi e^{-m}$$

equating real parts on both sides

$$\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2}\,dx = \pi e^{-m} \quad \text{or} \quad \int_0^{\infty} \frac{\cos mx}{1+x^2} = \frac{\pi e^{-m}}{2} \quad \text{Ans}$$

Que (3) :- Using the complex variable evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$



Sol :- Consider $\int_C f(z)dz$ where $f(z)= \frac{z^2}{(z^2+1)(z^2+4)}$, taken round the closed contour C consisting of a upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.

The poles of $f(z)$ are, $z = \pm i, \pm 2i$; only $z = i, 2i$ lies within C hence

$$\text{Residue of } f(z) \text{ (at } z=i) = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} (z-i) \frac{z^2}{(z-i)(z+i)(z^2+4)} = \frac{-1}{6i}$$

$$\text{Residue of } f(z) \text{ (at } z=2i) = \lim_{z\to 2i} (z-2i) f(z) = \lim_{z\to 2i} (z-2i) \frac{z^2}{(z^2+1)(z+2i)(z-2i)} = \frac{1}{3i}$$

Hence by cauchy residue theorem

$$\int_C f(z)dz = 2\pi i \times \text{Sum of the residues} = 2\pi i \left(-\frac{1}{6i} + \frac{1}{3i}\right) = \frac{\pi}{3}$$

$$\Rightarrow \int_{-R}^{R} f(x)dx + \int_{C_r} f(z)dz = \frac{\pi}{3}$$

Now taking $\lim_{R\to\infty}$ on both sides, hence

$$\int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty} \int_{C_r} f(z)\,dz = \frac{\pi}{3}$$

or $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx + 0 = \frac{\pi}{3}$$

(Since $\left|\int_{C_r} f(z)dz\right| \leq \int_{C_r} \frac{|z^2|\,|dz|}{|z^2+1|\,|z^2+4|}$

$\leq \int_0^{\pi} \frac{R^2 \cdot R\,d\theta}{R^2(1+\frac{1}{R^2})R^2(1+\frac{4}{R^2})}$, $|z|=R$

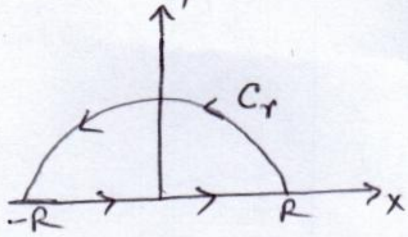
$\to 0$ as $R\to\infty$)

or $$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$ Ans

Que (4) Using contour integration, show that $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$

Hint :— The poles $z=ai$ of order two lies within C, hence $\text{Res (at } z=ai) = \frac{1}{4a^3 i}$ and sove same as above question.

Que (5) Show that $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$, $a>0$



Sol:- Consider $\displaystyle\int_C f(z)\,dz$, where $f(z) = \frac{1}{z^4+a^4}$, taken round the closed contour C consisting of a upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.

The poles of $f(z)$ are $z^4+a^4=0$ or $z^4 = a^4(-1)$

or $z^4 = a^4(\cos\pi + i\sin\pi) = a^4[\cos(2n+1)\pi + i\sin(2n+1)\pi] = a^4 e^{(2n+1)\pi i}$

or $z = a e^{(2n+1)\frac{\pi}{4}i}$, Now putting $n = 0, 1, 2$ and 3

when $n=0$, $z = a e^{\frac{\pi}{4}i} = a\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = a\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)$

when $n=1$, $z = a e^{\frac{3\pi}{4}i} = a\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = a\left(-\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)$

when $n=2$, $z = a e^{\frac{5\pi}{4}i} = a\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = a\left(-\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right)$

when $n=3$, $z = a e^{\frac{7\pi}{4}i} = a\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = a\left(\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right)$

Since $a>0$, then only two poles $z = ae^{\frac{\pi}{4}i}$, $ae^{\frac{3\pi}{4}i}$ lies within C. hence

Residue of $f(z)$ (at $z = ae^{\frac{\pi}{4}i}$) $= \displaystyle\lim_{z\to ae^{\frac{\pi}{4}i}} \left(\frac{z - ae^{\frac{\pi}{4}i}}{z^4+a^4}\right) = \lim_{z\to ae^{\frac{\pi}{4}i}} \left(\frac{1}{4z^3}\right)$

$$= \frac{1}{4a^3 e^{\frac{3\pi}{4}i}} = \frac{1}{4a^3} e^{-\frac{3\pi}{4}i} = \frac{1}{4a^3}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4a^3}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

Similarly Residue of $f(z)$ (at $z = ae^{\frac{3\pi}{4}i}$) $= \displaystyle\lim_{z\to ae^{\frac{3\pi}{4}i}} \left(\frac{z - ae^{\frac{3\pi}{4}i}}{z^4+a^4}\right) = \lim_{z\to ae^{\frac{3\pi}{4}i}} \left(\frac{1}{4z^3}\right)$

$$= \frac{1}{4a^3 e^{\frac{9\pi}{4}i}} = \frac{1}{4a^3}\left[\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right] = \frac{1}{4a^3}\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

Now Sum of the residues $= \dfrac{-i}{2a^3\sqrt{2}}$

Hence by Cauchy Residue theorem

$\displaystyle\int_C f(z)\,dz = 2\pi i \times$ Sum of the residues $= 2\pi i \times \dfrac{-i}{2a^3\sqrt{2}} = \dfrac{\pi}{a^3\sqrt{2}}$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{C_r} f(z)\,dz = \frac{\pi}{a^3\sqrt{2}}$$

Now taking $R\to\infty$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} + \lim_{R\to\infty} \int_{C_1} f(z)\,dz = \frac{\pi}{a^3\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{a^3\sqrt{2}}, \quad \text{Since } \left|\int_{C_1} f(z)dz\right| \le \int_{C_1} \frac{|dz|}{|z^4+a^4|}, \ |z|=R$$

$$\le \int_0^{\pi} \frac{R\,d\theta}{R^4\left(1+\frac{a^4}{R^4}\right)} \to 0 \text{ as } R\to\infty$$

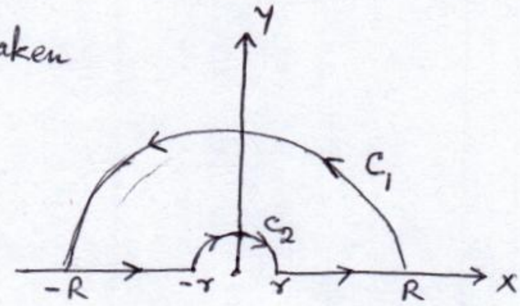
$$\text{or } 2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{a^3\sqrt{2}}$$

$$\text{or } \int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{2a^3\sqrt{2}} = \frac{\pi\sqrt{2}}{4a^3}$$

Proved

Que (6) :- By Contour integration prove that $\int_0^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$

Sol :- Consider $\int_C f(z)\,dz$, where $f(z) = \frac{e^{imz}}{z}$, taken round a closed contour C of a large semi circle $|z| = R$ indented at $z=0$, let r be the radius of indentation.



Since there is no singularity within the given contour, hence by Cauchy residue theorem

$$\int_C f(z)\,dz = 0$$

$$\text{i.e. } \int_{-R}^{-r} f(x)\,dx + \int_{C_2} f(z)\,dz + \int_r^R f(x)\,dx + \int_{C_1} f(z)\,dz = 0$$

$$\text{or } \int_{-R}^{-r} \frac{e^{imx}}{x}\,dx + \int_{C_2} \frac{e^{imz}}{z}\,dz + \int_r^R \frac{e^{imx}}{x}\,dx + \int_{C_1} \frac{e^{imz}}{z}\,dz = 0$$

now substituting $-x$ for x in the first integral and combining it with third integral, we get

$$\int_r^R \frac{e^{imx} - e^{-imx}}{x}\,dx + \int_{C_2} \frac{e^{imz}}{z}\,dz + \int_{C_1} \frac{e^{imz}}{z}\,dz = 0$$

$$\text{or } 2i\int_r^R \frac{\sin mx}{x}\,dx + \int_{C_2} \frac{e^{imz}}{z}\,dz + \int_{C_1} \frac{e^{imz}}{z}\,dz = 0 \quad \text{——(1)}$$

$$\text{Now } \int_{C_2} \frac{e^{imz}}{z}\,dz = \int_{C_2} \frac{1}{z}\,dz + \int_{C_2} \frac{e^{imz}-1}{z}\,dz \quad \text{——(2)}$$

On C_2 i.e. $|z| = r \Rightarrow z = re^{i\theta}$ $\therefore dz = ire^{i\theta}d\theta$

$$\therefore \int_{C_2} \frac{1}{z}dz = \int_{\pi}^{0} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_{\pi}^{0} i\,d\theta = -i\pi$$

Also $$\left|\int_{C} \frac{e^{imz}-1}{z}dz\right| \leq M\int_{C_2}\frac{|dz|}{|z|} = \pi M \to 0 \text{ as } r\to 0$$

where M is the maximum value on C_2 of $|e^{imz}-1| = |e^{imr(\cos\theta + i\sin\theta)}-1|$

hence from eqn ② $$\int_{C_2} \frac{e^{imz}}{z}dz = -i\pi \quad \text{——— ③}$$

On C_1 i.e. $|z| = R \Rightarrow z = Re^{i\theta}$

$$\therefore \int_{C_1} \frac{e^{imz}}{z}dz = \int_{0}^{\pi} \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} Re^{i\theta}\cdot i\,d\theta = i\int_{0}^{\pi} e^{imR\cos\theta}\cdot e^{-mR\sin\theta}d\theta$$

Since $|e^{-imR\cos\theta}| \leq 1$, hence

$$\left|\int_{C_1} \frac{e^{imz}}{z}dz\right| \leq \int_{0}^{\pi} e^{-mR\sin\theta}d\theta = 2\int_{0}^{\pi/2} e^{-mR\sin\theta}d\theta$$

Also $\frac{\sin\theta}{\theta}$ continuously decreases from 1 to $\frac{2}{\pi}$ as θ increases from 0 to $\frac{\pi}{2}$

$\therefore$ for $0 \leq \theta \leq \frac{\pi}{2}$, $\frac{\sin\theta}{\theta} \geq \frac{2}{\pi}$ or $\sin\theta \geq \frac{2\theta}{\pi}$

$$\therefore \left|\int_{C_1} \frac{e^{imz}}{z}dz\right| \leq 2\int_{0}^{\pi/2} e^{-2mR\theta/\pi}d\theta = -\frac{\pi}{mR}\left[e^{-2mR\theta/\pi}\right]_{0}^{\pi/2} = \frac{\pi}{mR}\left[1-e^{-mR}\right]$$

As $R\to\infty$, $\frac{\pi}{mR}(1-e^{-mR}) \to 0$, hence

$$\int_{C_1} \frac{e^{imz}}{z}dz = 0 \quad \text{——— ④}$$

Putting the values of ③ & ④ in eqn ① and taking $R\to\infty$ and $r\to 0$, we get

$$2i\int_{0}^{\infty} \frac{\sin mx}{x}dx - i\pi = 0$$

or $$\int_{0}^{\infty} \frac{\sin mx}{x}dx = \frac{\pi}{2}$$

Proved