Date
21/01/2024

Date / /

Question Bank (Module-4)

(1) Evaluate the following integrals -

(a) $\int_0^1\int_1^2 xy(1+x+y)\,dy\,dx$

Solution :→ Let $I = \int_{x=0}^{1}\left\{\int_{y=1}^{2}(xy + x^2y + xy^2)\,dy\right\}dx$

$$= \int_{x=0}^{1}\left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=1}^{2} dx$$

$$= \int_{x=0}^{1}\left[2x + 2x^2 + \frac{8x}{3} - \frac{x}{2} - \frac{x^2}{2} - \frac{x}{3}\right] dx$$

$$= \int_{x=0}^{1}\left(\frac{3x}{2} + \frac{3x^2}{2} + \frac{7x}{3}\right) dx$$

$$= \left[\frac{3}{2}\times\frac{x^2}{2} + \frac{3x^3}{2\times3} + \frac{7}{3}\times\frac{x^2}{2}\right]_{x=0}^{1}$$

$$= \frac{3}{4} + \frac{3}{6} + \frac{7}{6} - 0 - 0 - 0 \quad = \frac{9+20}{12} = \frac{29}{12}.\ \text{Ans}$$

(b) $\int_1^a\int_1^b \frac{dy\,dx}{xy}$

Solution :→ Let $I = \int_{x=1}^{a}\int_{y=1}^{b}\frac{dy\,dx}{xy} = \int_{x=1}^{a}\frac{1}{x}\left\{\int_{y=1}^{b}\frac{dy}{y}\right\}dx$

$$= \int_{x=1}^{a}\frac{1}{x}\left[\log y\right]_{y=1}^{b} dx$$

$$= \int_{x=1}^{a}\frac{1}{x}\cdot(\log b - \log 1)\,dx = \int_{x=1}^{a}\frac{1}{x}\cdot(\log b)\,dx$$

$$= \log b\left[\log x\right]_{x=1}^{a} = \log b\cdot(\log a - \log 1)$$

$(\because \log 1 = 0)$

$$= \log a\cdot\log b.\ \text{Ans}$$

Date / /

c) $\int_0^1\int_0^{x^2} e^{y/x}\,dy\,dx$

Solution :- Let $I = \int_{x=0}^{1}\left\{\int_{y=0}^{x^2} e^{y/x}\,dy\right\}dx$

$= \int_{x=0}^{1}\left[\frac{e^{y/x}}{1/x}\right]_{y=0}^{x^2} dx$

$= \int_{x=0}^{1}(x\cdot e^{x} - x\cdot e^{0})\,dx$

$= \int_{x=0}^{1}(xe^{x} - x)\,dx$

$= \left[xe^{x} - 1\cdot e^{x} - x^2/2\right]_{x=0}^{1}$

$= e^1 - e^1 - \frac{1}{2} + 0 + 0 + 1 = 1/2$ Ans.

d) $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta$

Solution → Let $I = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta = \int_{\theta=0}^{\pi}\cos\theta\left\{\int_{r=0}^{a(1+\cos\theta)} r^2\,dr\right\}d\theta$

$= \int_{\theta=0}^{\pi}\cos\theta\left[\frac{r^3}{3}\right]_{r=0}^{a(1+\cos\theta)^3} d\theta$

$= \int_{\theta=0}^{\pi}\cos\theta\left[\frac{a^3(1+\cos\theta)^3}{3}\right]d\theta$

$= \frac{a^3}{3}\int_{\theta=0}^{\pi}\cos\theta(1+\cos^3\theta + 3\cos\theta + 3\cos^2\theta)\,d\theta$

$= \frac{a^3}{3}\int_{\theta=0}^{\pi}(\cos^4\theta + 3\cos^3\theta + 3\cos^2\theta + \cos\theta)\,d\theta$

$I = \frac{a^3}{3}\int_{\theta=0}^{\pi}\left[\left(\frac{\cos2\theta+1}{2}\right)^2 + 3\left(\frac{\cos3\theta+3\cos\theta}{4}\right) + 3\left(\frac{\cos2\theta+1}{2}\right) + \cos\theta\right]d\theta$

$I = \frac{a^3}{3}\int_{\theta=0}^{\pi}\left[\frac{\cos^2 2\theta + 1 + 2\cos2\theta}{2} + \frac{3}{4}(\cos3\theta + 3\cos\theta) + \frac{3}{2}(\cos2\theta+1) + \cos\theta\right]d\theta$

Date / /

$$I = \frac{a^3}{3}\int_{\theta=0}^{\pi}\left[\frac{\cos 4\theta + 1 + 2 + 4\cos 2\theta}{4} + \frac{3\cos 3\theta}{4} + \frac{13\cos\theta}{4} + \frac{3}{2}\cos 2\theta + \frac{3}{2}\right]d\theta$$

$$= \frac{a^3}{3}\left[\frac{\sin 4\theta}{4\times 4} + \frac{3\theta}{4} + \frac{\sin 2\theta}{2} + \frac{3\sin 3\theta}{4\times 3} + \frac{13\sin\theta}{4} + \frac{3\sin 2\theta}{2\times 2} + \frac{3\theta}{2}\right]_{\theta=0}^{\pi}$$

$$= \frac{a^3}{3}\left[\frac{\sin 4\theta}{16} + \frac{\sin 3\theta}{4} + \frac{5\sin 2\theta}{4} + \frac{13}{4}\sin\theta + \frac{9\theta}{4}\right]_{\theta=0}^{\pi}$$

$$= \frac{a^3}{3}\left[0+0+0+0+\frac{9\pi}{4} - 0 - 0 - 0 - 0 - 0\right]$$

$$= \frac{a^3}{\cancel{3}} \times \frac{\cancel{9}\pi}{4} = \frac{3\pi a^3}{4}. \text{ Ans.}$$

Ques-(2) Evaluate the integrals $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{(1+x^2+y^2)}$.

Solution :→ Let $I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dy\,dx}{(1+x^2+y^2)}$

$\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$$\Rightarrow \quad I = \int_{x=0}^{1}\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_{y=0}^{\sqrt{1+x^2}} dx$$

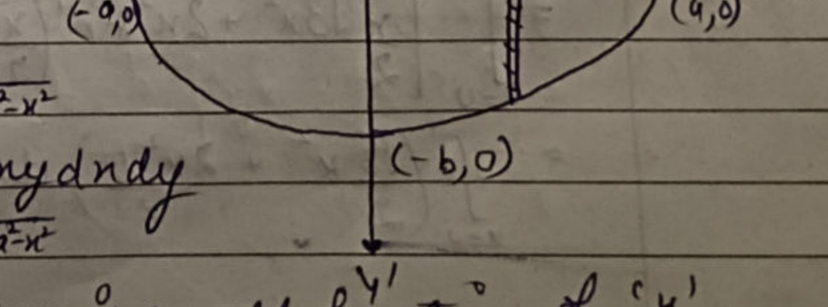
$$I = \int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}dx = \left[\log\left|x+\sqrt{1+x^2}\right|\right]_{x=0}^{1} = \log(1+\sqrt{2}). \text{ Ans.}$$

Ques-(3) Evaluate the integrals $\iint (x+y)^2\,dx\,dy$ over the area bounded by ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution :→ Let $I = \iint_R (x+y)^2\,dx\,dy$

Limits → $x = -a$ to $x = a$

$y = -\frac{b}{a}\sqrt{a^2-x^2}$ to $y = \frac{b}{a}\sqrt{a^2-x^2}$



$$I = \int_{x=-a}^{x=a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dy\,dx$$

$$= \int_{x=-a}^{x=a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx + \int_{x=-a}^{x=a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} 2xy\,dx\,dy$$

Since (x^2+y^2) is an even & $2xy$ is an odd function of 'y'.

$$\Rightarrow I = \int_{x=-a}^{a} 2\int_{y=0}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx + 0$$

$$= 2\int_{x=-a}^{a}\left[x^2y + \frac{y^3}{3}\right]_{y=0}^{b/a\sqrt{a^2-x^2}} dx$$

Date / /

Put ~~$x = a\sin\theta$~~ ~~$dx = a\cos\theta\,d\theta$~~

$$I = \int_{x=-a}^{a} 2\left[\frac{x^2 b\sqrt{a^2-x^2}}{a} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

∵ given function is an even function of 'x'

$$\Rightarrow I = 4\int_{x=0}^{a}\left(\frac{x^2 b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right)dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

$$I = 4\int_{\theta=0}^{\pi/2}\left[\frac{b}{a}\cdot a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta} + \frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]a\cos\theta\,d\theta$$

$$= 4a\int_{\theta=0}^{\pi/2}\left[a^2b\sin^2\theta\cos^2\theta + \frac{b^3}{3a^3}\times a^3(\cos^4\theta)\right]d\theta$$

$$= 4ab\int_{\theta=0}^{\pi/2}\left[\frac{a^2}{8}(1-\cos4\theta) + \frac{b^2}{24}(3+\cos4\theta+4\cos2\theta)\right]d\theta$$

$$= 4ab\left[\frac{a^2}{8}\left(\frac{\pi}{2}-0\right) + \frac{b^2}{24}\left(\frac{3\pi}{2}+0+0\right) - 0 - 0\right]$$

$$= 4ab\left(\frac{a^2}{8}\times\frac{\pi}{2} + \frac{b^2}{24}\times\frac{3\pi}{2}\right) \Rightarrow \frac{4ab\pi}{16}(a^2+b^2)$$

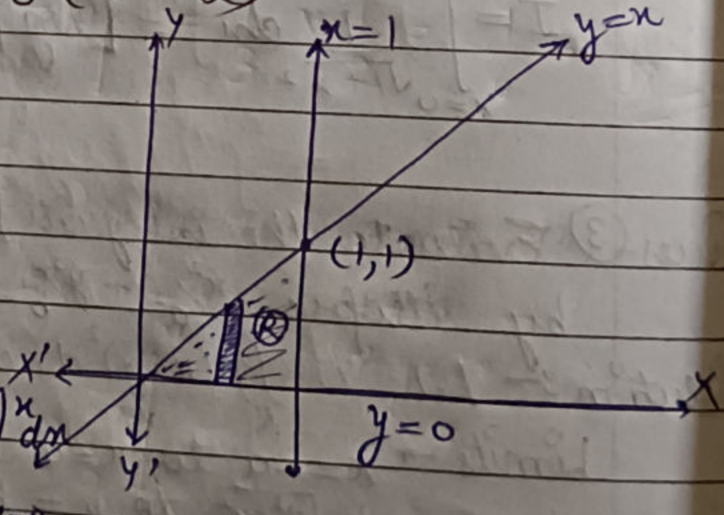
$$= \frac{\pi}{4}ab(a^2+b^2)$$ Ans

Ques-(4) When the region R of integration is the triangle given by $y=0$, $y=x$, and $x=1$, then prove that

$$\iint_R \sqrt{4x^2-y^2}\,dx\,dy = \frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

Solution→ Limits → $x=0$ to $x=1$

$y=0$ to $y=x$



Now,

$$I = \int_{x=0}^{1}\int_{y=0}^{x}\sqrt{4x^2-y^2}\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\frac{y}{2}\sqrt{4x^2-y^2} + \frac{4x^2}{2}\sin^{-1}\left(\frac{y}{2x}\right)\right]_{y=0}^{x}dx$$

$$= \int_{x=0}^{1}\left[\frac{x}{2}\sqrt{3x^2} + 2x^2\sin^{-1}\left(\frac{x}{2x}\right) - 0 - 0\right]dx \quad (\because \sin^{-1}(0)=0)$$

$$= \int_{x=0}^{1}\left(\frac{\sqrt{3}}{2}x^2 + \frac{2x^2\pi}{6}\right)dx = \int_{x=0}^{1}\left(\frac{\sqrt{3}}{2}x^2 + \frac{\pi x^2}{3}\right)dx$$

Date / /

$$I = \left[\frac{\sqrt{3}}{2}\times\frac{x^3}{3} + \frac{\pi}{3}\times\frac{x^3}{3}\right]_{x=0}^{1} = \frac{\sqrt{3}}{6} + \frac{\pi}{9} = \frac{1}{3}\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)\cdot \text{H.P.}$$

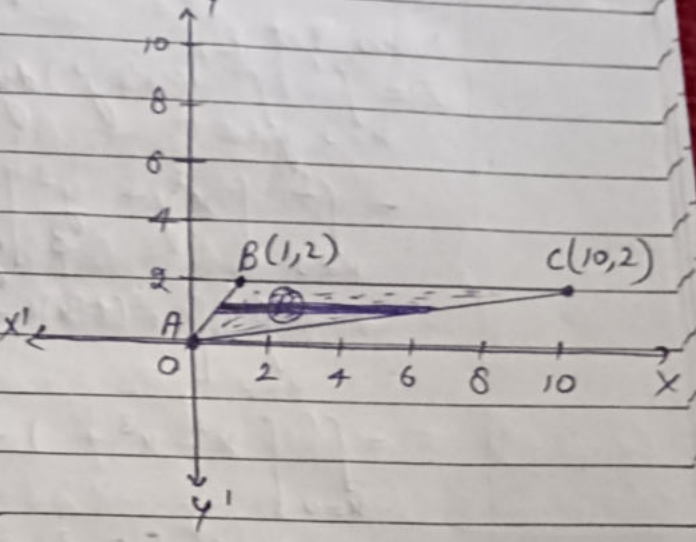
Ques-(5) Evaluate the following integration $\iint_R \sqrt{xy-y^2}\,dx\,dy$ where R is a triangle with vertices (0,0), (10,2) and (1,2).

Solution:→ Equation of lines:→

AB ⇒ $y = 2x$

BC ⇒ $y = 2$

AC ⇒ $5y = x$ or $y = x/5$



'Limits'→ $x = y/2$ to $x = 5y$

$y = 0$ to $y = 2$

Let $$I = \int_{y=0}^{2}\left\{\int_{x=y/2}^{5y}\sqrt{xy-y^2}\,dx\right\}dy$$

$$= \int_{y=0}^{2}\left[\frac{2(xy-y^2)^{3/2}}{3y}\right]_{x=y/2}^{5y} dy$$

$$= \int_{y=0}^{2}\frac{2}{3}\left[\frac{(5y^2-y^2)^{3/2}}{y} - \frac{(-y^2/2)^{3/2}}{y}\right]dy$$

$$= \frac{2}{3}\int_{y=0}^{2}\left[\frac{8y^3}{y} - \frac{\sqrt{-1}\,(y^3)}{2\sqrt{2}\,y}\right]dy = \frac{2}{3}\int_{y=0}^{2}\left(8y^2 - \frac{\sqrt{-1}\cdot y^2}{2\sqrt{2}}\right)dy$$

$$= \frac{2}{3}\left[8\frac{y^3}{3} - \frac{\sqrt{-1}}{2\sqrt{2}}\cdot\frac{y^3}{3}\right]_{y=0}^{2}$$

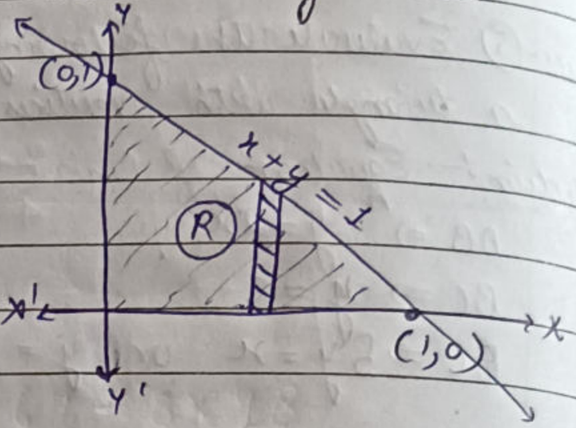
$$= \frac{2}{3}\left[\frac{8}{3}(2)^3 - \frac{\sqrt{-1}}{6\sqrt{2}}(2)^3 - 0 + 0\right]$$

$$= \frac{2}{3}\left\{\frac{64}{3} - \frac{\sqrt{-1}\times \cancel{8}\,\cancel{4}\,2\sqrt{2}}{3\,\cancel{6}\cancel{\sqrt{2}}}\right\} = \frac{2}{9}\left(64 - 2\sqrt{2}\,i\right)$$

$$= \frac{4}{9}\left(32 - \sqrt{2}\times\sqrt{-1}\right) = \frac{4}{9}\left(32 - \sqrt{-2}\right)$$ Ans

Date / /

Ques-(6) Evaluate the integral $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$.



Solution → Limits ⇒ $x=0$ to $x=1$

$y=0$ to $y=1-x$

Now, Let

$$I=\int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)\,dy\,dx$$

$$=\int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{1-x}dx$$

$$=\int_{x=0}^{1}\left[x^2(1-x)+\frac{(1-x)^3}{3}-0-0\right]dx$$

$$=\int_{x=0}^{1}\left[x^2-x^3+\frac{1-x^3-3x+3x^2}{3}\right]dx$$

$$=\frac{1}{3}\int_{x=0}^{1}\left[3x^2-3x^3+1-x^3-3x+3x^2\right]dx$$

$$=\frac{1}{3}\int_{x=0}^{1}\left[-4x^3+6x^2-3x+1\right]dx$$

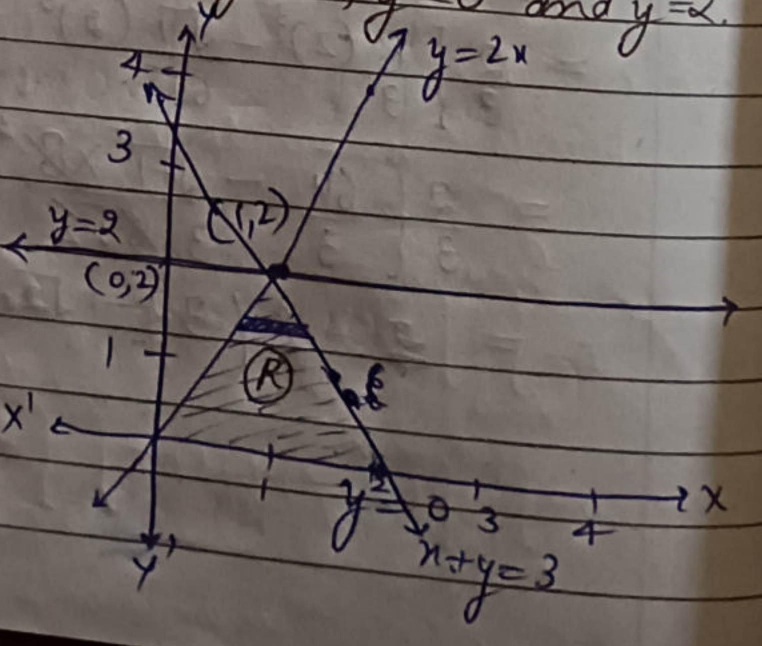
$$=\frac{1}{3}\left[-\frac{\cancel{4}x^4}{\cancel{4}}+\frac{\cancel{6}x^3}{\cancel{3}}-\frac{3x^2}{2}+x\right]_{x=0}^{1}$$

$$=\frac{1}{3}\left[\cancel{-1}+2-\frac{3}{2}\cancel{+1}-0\right]=\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}.\ \text{Ans.}$$

Ques-(7) Evaluate the integral $\iint (x^2+y^2)\,dx\,dy$ throughout the area enclosed by the curves $y=2x$, $x+y=3$, $y=0$ and $y=2$.

Solution → Limits → $x=y/2$ to $x=3-y$

$y=0$ to $y=2$



Let

$$I=\int_{y=0}^{2}\int_{x=y/2}^{3-y}(x^2+y^2)\,dx\,dy$$

$$=\int_{y=0}^{2}\left[\frac{x^3}{3}+xy^2\right]_{x=y/2}^{3-y}dy$$

Date / /

$$I = \int_{y=0}^{2}\left[\frac{(3-y)^3}{3} + (3-y)y^2 - \frac{(y/2)^3}{3} - (y/2)\times y^2\right]dy$$

$$= \int_{y=0}^{2}\left[\frac{27-y^3-27y+9y^2}{3} + 3y^2 - y^3 - \frac{y^3}{24} - \frac{y^3}{2}\right]dy$$

$$= \int_{y=0}^{2}\left[9 - \frac{y^3}{3} - 9y + 3y^2 + 3y^2 - y^3 - \frac{y^3}{24} - \frac{y^3}{2}\right]dy$$

$$= \int_{y=0}^{2}\left[9 - 9y + 6y^2 - \frac{8y^3 - 24y^3 - y^3 + 12y^3}{24}\right]dy$$

$$= \int_{y=0}^{2}\left[9 - 9y + 6y^2 - \frac{45}{24}y^3\right]dy$$

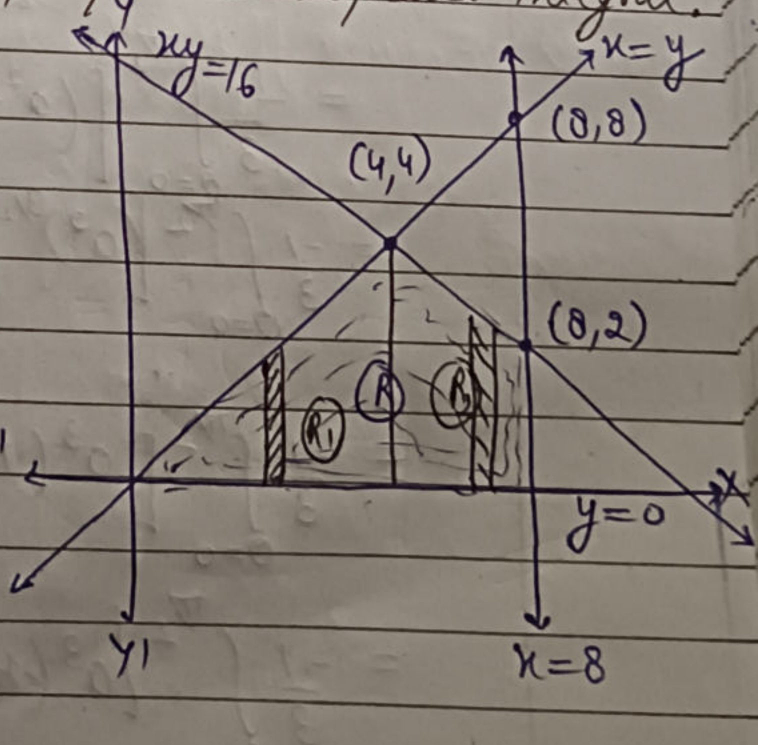
$$= \left[9y - \frac{9y^2}{2} + \frac{6y^3}{3} - \frac{\cancel{45}^{15}}{\cancel{24}_{8}}\times\frac{y^4}{4}\right]_{y=0}^{2}$$

$$= \left[9\times 2 - \frac{9(2)^2}{2} + 2(2)^3 - \frac{15}{32}(2)^4 - 0\right]$$

$$= \cancel{18} - \cancel{18} + 16 - \frac{15}{2} = \frac{32-15}{2} = \frac{17}{2} \text{ Ans.}$$

Ques-(8) Let D be the region in the first quadrant bounded by the curves $xy = 16$, $x = y$, $y = 0$ and $x = 8$. Sketch the region of integration of given integral $\iint_D x^2\,dx\,dy$ and evaluate it by expressing it as an appropriate repeated integral.

Solution :-



Limits :- $R_1 \rightarrow x = 0$ to $x = 4$

$y = 0$ to $y = x$

$R_2 \rightarrow x = 4$ to $x = 8$

$y = 0$ to $y = 16/x$

Let

$$I = \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dy\,dx + \int_{x=4}^{8}\int_{y=0}^{16/x} x^2\,dy\,dx$$

Date / /

$$I = \int_{x=0}^{4} \left[x^2 y\right]_{y=0}^{x} dx + \int_{x=4}^{8} x^2 \left[y\right]_0^{16/x} dx$$

$$I = \int_{x=0}^{4} x^3 dx + \int_{x=4}^{8} 16x \, dx$$

$$= \left[\frac{x^4}{4}\right]_{x=0}^{4} + 16\left[\frac{x^2}{2}\right]_{x=4}^{8}$$

$$= \frac{(4)^4}{4} + 8\left[8^2 - 4^2\right]$$

$$= 64 + 8\left[64 - 16\right] = 64 + 8 \times 48$$

$$= 64 + 384 = 448 \text{ Ans.}$$

Ques (9) Evaluate $\int_0^{\pi/2} \int_0^{a\cos\theta} r\sqrt{a^2 - r^2} \, dr \, d\theta$

Solution :→ Let

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} r\sqrt{a^2 - r^2} \, dr \, d\theta$$

$$\cancel{= \int_{\theta=0}^{\pi/2} \left[r\,(a \right.} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \frac{2r\sqrt{a^2 - r^2}}{2} \, dr \, d\theta$$

$$\because \int f'(x) f^n(x) \, dx = \frac{f^{n+1}(x)}{n+1}$$

$$\Rightarrow I = \int_{\theta=0}^{\pi/2} \frac{-1}{\cancel{2}} \left[\frac{(a^2 - r^2)^{3/2}}{3/\cancel{2}}\right]_{r=0}^{a\cos\theta} d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[(a^2 - a^2\cos^2\theta)^{3/2} - (a^2)^{3/2}\right] d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[(a^2)^{3/2} \{1 - \cos^2\theta\}^{3/2} - a^3\right] d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[a^3 (1 - \cos^2\theta)^{3/2} - a^3\right] d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[a^3 (\sin^2\theta)^{3/2} - a^3\right] d\theta$$

Date / /

$$\cancel{I = \frac{-1}{3}\int_{\theta=0}^{\pi/2}\left[a^3\left(1+\frac{\cos 2\theta+1}{2}\right)^{3/2} - a^3\right]d\theta}$$

$$\cancel{= \frac{-1}{3}\int_{\theta=0}^{\pi/2}\left[a^3\left[\frac{2+\cos 2\theta+1}{2}\right]^{3/2} - a^3\right]d\theta}$$

$$\cancel{= \frac{-1}{3}\left[\frac{a^3}{2}\left\{\frac{\sin 2\theta+3}{2}\right\} - a^3\right.}$$

$\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$I = \frac{-1}{3}\int_{\theta=0}^{\pi/2}\left[a^3\sin^3\theta - a^3\right]d\theta$$

$$= \frac{-1}{3}\int_{\theta=0}^{\pi/2}\left[a^3\left\{\frac{3\sin\theta-\sin 3\theta}{4}\right\} - a^3\right]d\theta$$

$$= \frac{-1}{3}\int_{\theta=0}^{\pi/2}\frac{a^3}{4}\left\{3\sin\theta - \sin 3\theta - 4\right\}d\theta$$

$$\cancel{= \frac{-a^3}{12}\left[3\sin\frac{\pi}{2} - \sin\frac{3\pi}{2} - 4 - 0 + 0 + 4\right]}$$

$$I = \frac{-a^3}{12}\left[-3\cos\theta + \frac{\cos 3\theta}{3} - 4\theta\right]_{\theta=0}^{\pi/2}$$

$$= \frac{-a^3}{12}\left[-3\cos\frac{\pi}{2} + \frac{\cos 3\pi/2}{3} - \frac{4\pi}{2} + 3\cos 0^\circ - \frac{\cos 0}{3} + 4\times 0\right]$$

$$= \frac{-a^3}{12}\left\{0 + 0 - \frac{4\pi}{2} + 3 - \frac{1}{3} + 0\right]$$

$$= \frac{-a^2}{12}\left(-\frac{4\pi}{2} + \frac{8}{3}\right)$$

$$= \frac{4a^2\pi}{24} - \frac{8a^3}{36}$$

$$= \frac{a^2\pi}{6} - \frac{2a^3}{9} \quad \text{Ans.}$$

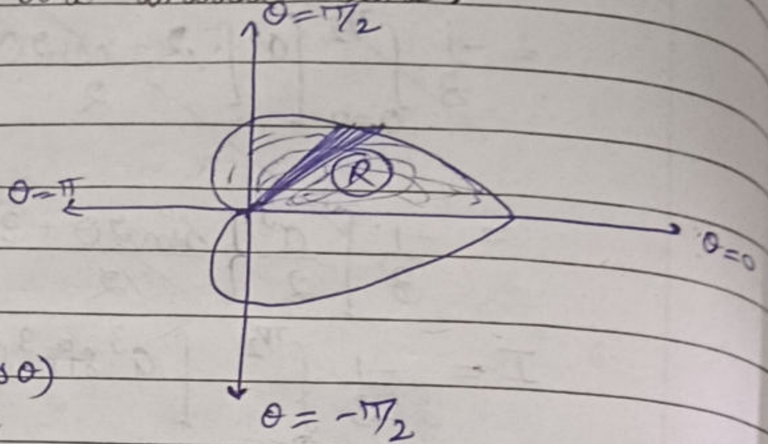
Date / /

Ques-(10) Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ above the initial line.

Solution:→ Limits ↦

$\theta = 0$ to $\theta = \pi$

$r = 0$ to $a(1+\cos\theta)$



Let

$$I = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_{r=0}^{a(1+\cos\theta)} d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{\sin\theta}{2} \times a^2(1+\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi} \sin\theta\,(1+\cos^{\cancel{2}}\theta)^2 \, d\theta$$

Put $1+\cos\theta = t$

$-\sin\theta \, d\theta = dt$

When $\theta = 0$, $t = 2$

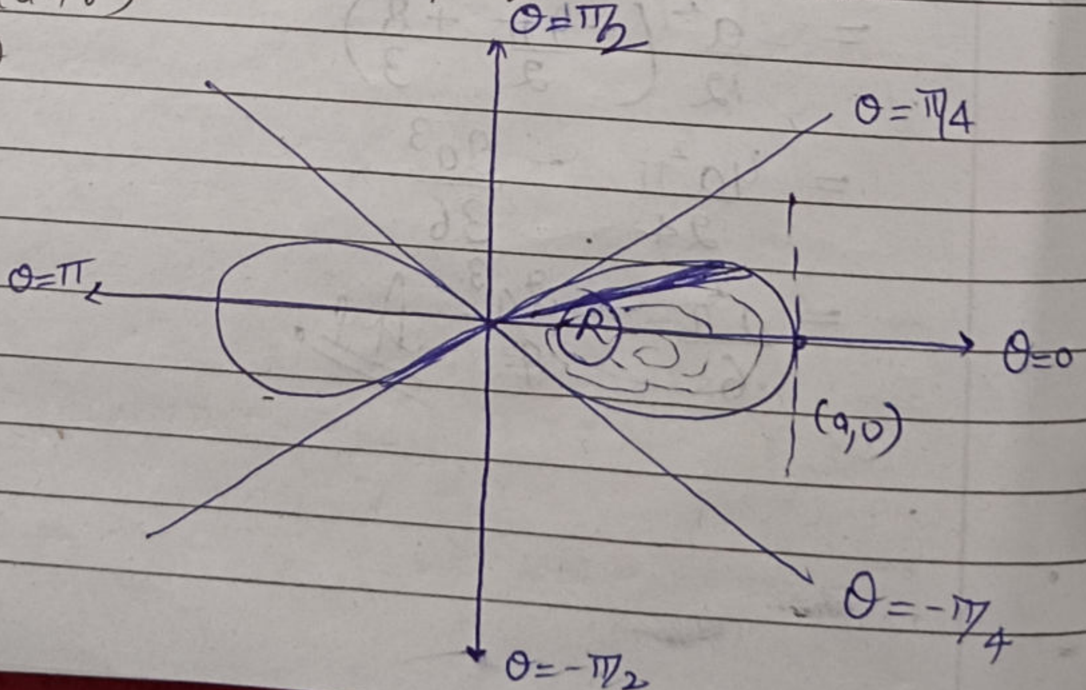
$\theta = \pi$, $t = 0$

$$I = -\frac{a^2}{2}\int_{t=2}^{0} t^2 \, dt = -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_{t=2}^{0}$$

$$= -\frac{a^2}{2} \times -\frac{2^3}{3} = \frac{8a^2}{6} = \frac{4a^2}{3} \text{ Ans.}$$

Ques-(11) Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{(a^2+r^2)}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$

Solution:→



Date / /

Limits: $\theta = -\pi/4$ to $\theta = \pi/4$

$r = 0$ to $r = a\sqrt{\cos 2\theta}$

$$I = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \frac{1}{2}\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{2r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$\because \int f'(x) f^n(x) = \frac{f^{n+1}(x)}{n+1}$

$$\Rightarrow I = \cancel{\frac{1}{2}}\int_{\theta=-\pi/4}^{\pi/4} \left[\frac{\sqrt{a^2+r^2}}{\cancel{1/2}}\right]_{r=0}^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} \left[(a^2 + a^2\cos 2\theta)^{1/2} - (a^2)^{1/2}\right] d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} \left\{a(1+\cos 2\theta)^{1/2} - a\right\} d\theta = \int_{\theta=-\pi/4}^{\pi/4} \left\{a(2\cos^2\theta)^{1/2} - a\right\} d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} (a\sqrt{2}\cos\theta - a)\,d\theta = \left[a\sqrt{2}\sin\theta - a\theta\right]_{\theta=-\pi/4}^{\pi/4}$$

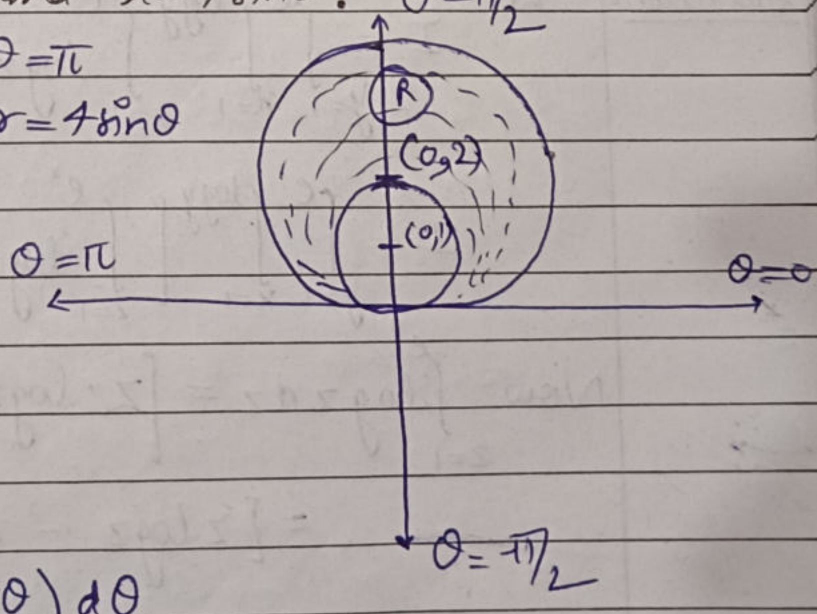
$$= a\sqrt{2}\times\sin\frac{\pi}{4} - \frac{a\pi}{4} + a\sqrt{2}\sin\frac{\pi}{4} - \frac{a\pi}{4}$$

$$= 2a\cancel{\sqrt{2}}\times\frac{1}{\cancel{\sqrt{2}}} - \frac{2a\pi}{4} = 2a - \frac{a\pi}{2}$$ Ans.

Ques-(12) → Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution :→ Limits :→ $\theta = 0$ to $\theta = \pi$

$r = 2\sin\theta$ to $r = 4\sin\theta$



$$I = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^4}{4}\right]_{r=2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{4^4\sin^4\theta - 2^4\sin^4\theta}{4}\right) d\theta$$

Date / /

$$I = \int_{\theta=0}^{\pi} 60 (\sin^4\theta)\, d\theta$$

$$= 60\int_{\theta=0}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= 60\int_{\theta=0}^{\pi} \left(\frac{1+\cos^2 2\theta - 2\cos 2\theta}{4}\right) d\theta$$

$$= \frac{30}{2}\int_{\theta=0}^{\pi} \left[1 + \frac{1-\cos 4\theta}{2} - 2\cos 2\theta\right] d\theta$$

$$= \frac{30}{2}\int_{\theta=0}^{\pi} \left(\frac{2+1-\cos 4\theta - 4\cos 2\theta}{2}\right) d\theta$$

$$= \frac{15}{2}\int_{\theta=0}^{2\pi} (3 - \cos 4\theta - 4\cos 2\theta)\, d\theta$$

$$= \frac{15}{2}\left[3\theta - \frac{\sin 4\theta}{4} - \frac{4\sin 2\theta}{2}\right]_{\theta=0}^{2\pi}$$

$$= \frac{15}{2}\left[3\times\pi - \frac{\sin 4\pi}{4} - 2\sin 2\pi + 0\right]$$

$$= \frac{15}{2}\times 3\pi = \frac{45\pi}{2} \text{ Ans}$$

Ques-(13) Evaluate the integral $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$.

Solution:→ Let

$$I = \int_{y=1}^{e}\int_{x=1}^{\log y}\int_{z=1}^{e^x} \log z\, dz\, dx\, dy$$

$$I = \int_{y=1}^{e}\int_{x=1}^{\log y}\left[\int_{z=1}^{e^x} \log z\, dz\right] dx\, dy$$

Now, $\int_{z=1}^{e^x} \log z\, dz = \left[z\cdot\log z - \int \frac{1}{z}\, z\, dz\right]_{z=1}^{e^x}$

$$= [z\log z - z]_{z=1}^{e^x}$$

$$= e^x \log e^x - e^x - 1\cdot\log 1 + 1 = e^x\cdot x - e^x - 0 + 1$$

$$= xe^x - e^x + 1$$

Date / /

$$I = \int_{y=1}^{e}\int_{x=1}^{\log y} (xe^x - e^x + 1)\,dx\,dy$$

$$= \int_{y=1}^{e} \Big[xe^x - e^x - e^x + x\Big]_{x=1}^{\log y} dy$$

$$= \int_{y=1}^{e} \Big[\log y \cdot e^{\log y} - 2e^{\log y} + \log y - 1\cdot e^1 + 2e^1 - 1\Big] dy$$

$$= \int_{y=1}^{e} (y\cdot \log y - 2y + \log y + e - 1)\,dy$$

$$= \int_{y=1}^{e} \{\log y\,(y+1) - 2y + e - 1\}\,dy$$

$$= \left[\frac{(y+1)^2}{2}\log y - \int \frac{(y+1)^2}{2}\times\frac{1}{y} - \frac{2y^2}{2} + ey - y\right]_{y=1}^{e}$$

$$= \left[\frac{(y+1)^2}{2}\log y - \frac{1}{2}\int \frac{y^2}{y} + \frac{2y}{y} + \frac{1}{y} - y^2 + ey - y\right]_{y=1}^{e}$$

$$= \left[\frac{(y+1)^2}{2}\log y - \frac{1}{2}\left[\frac{y^2}{2} + 2y + \log y\right] - y^2 + ey - y\right]_{y=1}^{e}$$

$$= \Big[\frac{(e+1)^2}{2}\log e - \frac{1}{2}\left(\frac{e^2}{2} + 2e + \log e\right) - \cancel{e^2} + \cancel{e^2} - e$$

$$- 2\log 1 \cdot + \frac{1}{2}\left(\frac{1}{2} + 2 + 0\right) + 1 - e + 1\Big]$$

$$= \frac{(e+1)^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} + 2e + 1\right) - e + \frac{1}{2}\times\frac{5}{2} + 2 - e$$

$$= \frac{e^2}{2} + \cancel{\frac{1}{2}} + \cancel{\frac{2e}{2}} - \frac{e^2}{4} - \cancel{\frac{2e}{2}} - \cancel{\frac{1}{2}} - \cancel{e} + \frac{5}{4} + 2 - e$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

$$= \frac{e^2 - 8e + 13}{4}$$

Ans.

(14) $$\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)\,dz\,dy\,dx$$

Date: / /

$$=\int_{x=0}^{a}\int_{y=0}^{a-x}\left[zx+zy+\frac{z^2}{2}\right]_{z=0}^{a-x-y}dy\,dx$$

$$=\int_{x=0}^{a}\int_{y=0}^{a-x}\left[(a-x-y)x+(a-x-y)y+\frac{(a-x-y)^2}{2}\right]dy\,dx$$

$$=\int_{x=0}^{a}\int_{y=0}^{a-x}\left(-\frac{x^2}{2}-xy-\frac{y^2}{2}+\frac{a^2}{2}\right)dy\,dx$$

$$=\int_{x=0}^{a}\left[-\frac{x^2y}{2}-\frac{xy^2}{2}-\frac{y^3}{6}+\frac{a^2y}{2}\right]_{y=0}^{a-x}dx$$

$$=\int_{x=0}^{a}\left[-\frac{x^2}{2}(a-x)-\frac{x}{2}(a-x)^2-\frac{(a-x)^3}{6}+\frac{a^2(a-x)}{2}\right]dx$$

$$=\int_{x=0}^{a}\left(\frac{x^3}{6}-\frac{a^2x}{2}+\frac{a^3}{3}\right)dx$$

$$=\left[\frac{x^4}{24}-\frac{a^2x^2}{4}+\frac{a^3x}{3}\right]_{x=0}^{a}$$

$$=\frac{a^4}{24}-\frac{a^4}{4}+\frac{a^4}{3}=\frac{a^4-6a^4+8a^4}{24}$$

$$=\frac{3a^4}{24}=\frac{a^4}{8}$$ Ans

Ques - (15) Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$ where : $0 \le x \le 1$, $1 \le y \le 2$, $2 \le z \le 3$. area.

Solution:- Let $I = \int_{z=2}^{3}\int_{y=1}^{2}\int_{x=0}^{1} (x+y+z)\,dx\,dy\,dz$

$$I = \int_{z=2}^{3}\int_{y=1}^{2}\left[\frac{x^2}{2} + xy + xz\right]_{x=0}^{1} dy\,dz$$

$$I = \int_{z=2}^{3}\int_{y=1}^{2}\left[\frac{1}{2} + y + z\right] dy\,dz$$

$$= \int_{z=2}^{3}\left[\frac{1}{2}y + \frac{y^2}{2} + zy\right]_{y=1}^{2} dz$$

$$= \int_{z=2}^{3}\left[1 + 2 + 2z - \frac{1}{2} - \frac{1}{2} - z\right] dz$$

$$= \int_{z=2}^{3} (2+z)\,dz$$

$$= \left[2z + \frac{z^2}{2}\right]_{z=2}^{3} = 6 + \frac{9}{2} - 4 - \frac{4}{2}$$

$$= 6 + \frac{9}{2} - 6 = \frac{9}{2}$$ Ans.

Ques-(15) Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$ where: $0 \le x \le 1$, $1 \le y \le 2$, $2 \le z \le 3$. area.

Solution: Let $I = \int_{z=2}^{3}\int_{y=1}^{2}\int_{x=0}^{1} (x+y+z)\,dx\,dy\,dz$

$$I = \int_{z=2}^{3}\int_{y=1}^{2} \left[\frac{x^2}{2} + xy + xz\right]_{x=0}^{1} dy\,dz$$

$$I = \int_{z=2}^{3}\int_{y=1}^{2} \left[\frac{1}{2} + y + z\right] dy\,dz$$

$$= \int_{z=2}^{3} \left[\frac{1}{2}y + \frac{y^2}{2} + zy\right]_{y=1}^{2} dz$$

$$= \int_{z=2}^{3} \left[1 + 2 + 2z - \frac{1}{2} - \frac{1}{2} - z\right] dz$$

$$= \int_{z=2}^{3} (2+z)\,dz$$

$$= \left[2z + \frac{z^2}{2}\right]_{z=2}^{3} = 6 + \frac{9}{2} - 4 - \frac{4}{2}$$

$$= 6 + \frac{9}{2} - 6 = \frac{9}{2} \text{ Ans.}$$

Date / /

~~Evaluate~~ Ques-(15) Evaluate the integral $\iiint_R (x-2y+z)\,dz\,dy\,dx$ where R is the region determined by $0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le x+y$.

Solution :→ Let $I = \int_{x=0}^{1}\int_{y=0}^{x^2}\int_{z=0}^{x+y} (x-2y+z)\,dz\,dy\,dx$

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2}\left[xz - 2zy + \frac{z^2}{2}\right]_{z=0}^{x+y} dy\,dx$$

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2}\left[x(x+y) - 2(x+y)y + \frac{(x+y)^2}{2}\right] dy\,dx$$

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2}\left[x^2 + xy - 2xy - 2y^2 + \frac{(x+y)^2}{2}\right] dy\,dx$$

$$= \int_{x=0}^{1}\left[x^2y - \frac{xy^2}{2} - \frac{2y^3}{3} + \frac{(x+y)^3}{3\times 2}\right]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^{1}\left(x^4 - \frac{x^5}{2} - \frac{2x^6}{3} + \frac{(x+x^2)^3}{6} - \frac{x^3}{6}\right) dx$$

$$= \int_{x=0}^{1}\left[x^4 - \cancel{\frac{x^5}{2}} - \frac{2x^6}{3} + \cancel{\frac{x^3}{6}} + \frac{x^6}{6} + \frac{3x^4}{6} + \cancel{\frac{3x^5}{6}} - \cancel{\frac{x^3}{6}}\right] dx$$

$$= \int_{x=0}^{1}\left[\frac{-3x^6}{6} + \frac{9x^4}{6}\right] dx$$

$$= \left[\frac{-3}{6}\times\frac{x^7}{7} + \frac{9}{6}\times\frac{x^5}{5}\right]_{x=0}^{1}$$

$$= \frac{-3}{6}\times\frac{1}{7} + \frac{9}{6}\times\frac{1}{5} = \frac{-1}{2}\times\frac{1}{7} + \frac{3}{2}\times\frac{1}{5}$$

$$= \frac{-1}{14} + \frac{3}{10} = \frac{-5+21}{70}$$

$$= \frac{16}{70} = \frac{8}{35} \text{ Ans.}$$

Date / /

Ques-(16) Evaluate the integral $\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$.

Solution → Let $I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} r\left[z\right]_{z=0}^{\sqrt{a^2-r^2}} dr\,d\theta$

$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

$= \frac{-1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_{r=0}^{a\cos\theta} d\theta$

$= \frac{-1}{3}\int_{\theta=0}^{\pi/2}\left[a^3\sin^3\theta - a^3\right]d\theta$

$\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$= \frac{-a^3}{3}\int_{\theta=0}^{\pi/2}\left(\frac{3\sin\theta - \sin 3\theta - 4}{4}\right)d\theta$

$= \frac{-a^3}{12}\left[-3\cos\theta + \frac{\cos 3\theta}{3} - 4\theta\right]_{\theta=0}^{\pi/2}$

$= \frac{-a^3}{12}\left[-0+0-\frac{\pi}{2}\times 4 + 3 - \frac{1}{3} + 0\right]$

$= \frac{-a^3}{12}\left(-2\pi + \frac{8}{3}\right)$

$= \frac{a^3}{12}\left(\frac{-8}{3} + 2\pi\right) = \frac{a^3}{12}\left(\frac{6\pi-8}{3}\right)$

$= \frac{a^3(6\pi-8)}{36} = \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$ Ans.

(14)

Date / /

Ques (17) Define Gamma and Beta functions. Prove that:
$\gamma(n+1) = n\gamma(n)$ and $\gamma(n+1) = n!$

Solution:→ Gamma function → A definite integration in the form of

$$\gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \quad , n > 0$$

Beta function → A definite integration in the form of

$$\beta(m,n) = \int_0^1 x^{m+1} (1-x)^{n+1}\, dx \quad , m>0 ; n>0$$

$\gamma(n+1) = n\gamma(n)$

Proof:→ $LHS = \gamma(n+1) = \int_0^\infty e^{-x} x^{(n+1)-1} dx \quad , n+1 > 0$

$$= \int_0^\infty \underset{II}{e^{-x}} \underset{I}{x^n}\, dx \quad \text{--- (∵ Integrating by parts)}$$

$$= \left[-x^n e^{-x} + \int n x^{n-1} e^{-x}\, dx\right]_0^\infty$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= 0 + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= n\gamma(n) = RHS \quad \text{Hence Proved}$$

$\gamma(n+1) = n!$

Proof:→ $\gamma(n+1) = n\gamma n$

$$= n(n-1)\gamma(n-1)$$

$$= n(n-1)(n-2) \text{ ---- } 3\cdot 2\cdot 1\, \gamma 1 \quad (∵ \gamma 1 = 1)$$

$$= n(n-1)(n-2) \text{ ---- } 3\cdot 2\cdot 1\cdot 1$$

$$\gamma(n+1) = n!$$

Hence Proved

Date / /

Ques-(18) Prove that $\Gamma(n) = K^n \int_0^{\infty} e^{-Kx} x^{n-1}\, dx$ and hence calculate $\int_0^{\infty} e^{-3x} x^5\, dx$.

Solution:- RHS:→ $I = K^n \int_0^{\infty} \underset{II}{e^{-Kx}}\, \underset{I}{x^{n-1}}\, dx$

Integrating by parts

$$= K^n \left[\left[x^{n-1} \frac{e^{-Kx}}{-K} \right]_0^{\infty} - \int_0^{\infty} (n-1) x^{n-2} \cdot \frac{e^{-Kx}}{-K}\, dx \right]$$

$$= K^n \left[0 + \frac{(n-1)}{K} \int_0^{\infty} x^{n-2} e^{-Kx}\, dx \right]$$

$$= K^{n-1} (n-1) \int_0^{\infty} x^{n-2} e^{-Kx}\, dx$$

Put $Kx = t \Rightarrow K\,dx = dt \Rightarrow dx = dt/K$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$\Rightarrow I = K^n \int_0^{\infty} e^{-t} \left(t/K\right)^{n-1} \cdot \frac{dt}{K}$$

$$= K^n \int_0^{\infty} e^{-t} \frac{t^{n-1}}{K^{n-1} \cdot K}\, dt$$

$$= \frac{K^n}{K^{n-1+1}} \int_0^{\infty} e^{-t} t^{n-1}\, dt$$

$$= \frac{\cancel{K^n}}{\cancel{K^n}} \int_0^{\infty} e^{-t} t^{n-1}\, dt = \Gamma n = \text{LHS}.$$

H.P

$$\int_0^{\infty} e^{-3x} x^5\, dx = \frac{\Gamma 6}{3^6} = \frac{5!}{3\times3\times3\times3\times3\times3} = \frac{\cancel{8}}{\cancel{27}} = \frac{40}{3^5}.$$ Ans

Ques-(19) Prove that $\Gamma n = \frac{1}{n} \int_0^{\infty} e^{-x^{1/n}}\, dx$ and hence prove that $\Gamma \tfrac{1}{2} = 2 \int_0^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

Solution:- $\Gamma n = \frac{1}{n} \int_0^{\infty} e^{-x^{1/n}}\, dx$

Date / /

RHS :- $\frac{1}{n}\int_0^\infty e^{-x^{1/n}}\,dx$

Put $x^{1/n} = y$

$x = y^n$

$dx = ny^{n-1}\,dy$

$\Rightarrow \frac{1}{n}\int_0^\infty e^{-y}\,ny^{n-1}\,dy = \frac{1}{\cancel{n}}\times\cancel{n}\int_0^\infty e^{-y}y^{n-1}\,dy$

$= \Gamma n = LHS$ H.P.

Put $n = \frac{1}{2}$

$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-x^{1/\frac{1}{2}}}\,dx = 2\int_0^\infty e^{-x^2}\,dx$

Put $x^2 = t$

$x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}\,dt$

$\Rightarrow \Gamma\frac{1}{2} = \cancel{2}\int_0^\infty \frac{e^{-t}}{\cancel{2}\sqrt{t}}\,dt = \int_0^\infty e^{-t}\,t^{-1/2}\,dt$

Considering $\int_0^\infty e^{-t}t^{-1/2}\,dt = \sqrt{\pi}$

$\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$ Ans.

Ques (21) Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ and hence calculate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$.

Solution :- We know that

$\beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$

Put $y = \frac{1}{1+x}$

$dy = -\frac{1}{(1+x)^2}\,dx$

$y = 0,\ x = \infty$

$y = 1,\ x = 0$

Date / /

$$\beta(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+x}\right)^{m-1}\left(1-\frac{1}{1+x}\right)^{n-1}\left(\frac{-1}{(1+x)^2}\right)dx$$

$$= -\int_{\infty}^{0}\left(\frac{1}{1+x}\right)^{m-1}\left(\frac{x}{1+x}\right)^{n-1}\frac{1}{(1+x)^2}\,dx$$

$\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\Rightarrow \beta(m,n) = \int_0^{\infty}\left(\frac{1}{1+x}\right)^{m-1}\left(\frac{x}{1+x}\right)^{n-1}\frac{1}{(1+x)^2}\,dx$$

$$= \int_0^{\infty}\frac{x^{n-1}\,dx}{(1+x)^{m\cancel{-1}+n\cancel{-1}+\cancel{2}}}$$

$$= \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$(\because \beta(m,n) = \beta(n,m))$

$$\Rightarrow \boxed{\beta(m,n) = \int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx}$$ Hence Proved

$$\int_0^{\infty}\frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^{\infty}\frac{x^8}{(1+x)^{24}}\,dx - \int_0^{\infty}\frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^{\infty}\frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^{\infty}\frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(15,9) \quad [\because \beta(m,n)=\beta(n,m)]$$

$$= 0 \quad \text{Ans.}$$

Ques (21) Prove that $\beta(m,n) = \dfrac{\gamma m\,\gamma n}{\gamma(m+n)}$, $m>0$, $n>0$ and hence calculate $\int_0^{\infty}\frac{x^4(1+x^6)}{(1+x)^{15}}\,dx$.

Solution $\Rightarrow$ We have $\dfrac{\gamma n}{K^n} = \int_0^{\infty} e^{-Kx}x^{n-1}\,dx$

$$\gamma n = K^n\int_0^{\infty} e^{-Kx}x^{n-1}\,dx$$

Put $K = Z$

$$\gamma n = K^z\int_0^{\infty} e^{-zx}x^{n-1}\,dx$$

Multiplying $z^{m-1}e^{-z}$ both sides of above eq^n, we get

Date / /

$$\Gamma n \cdot e^{-z} z^{m-1} = z^n \cdot e^{-z} z^{m-1} \int_0^\infty e^{-zx} x^{n-1} dx$$

Integrating both sides wrt z from 0 to ∞

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-z-zx} z^{m-1} \cdot z^n \cdot dz \right\} dx$$

$$\Gamma n \, \Gamma m = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-z(1+x)} z^{m+n-1} dz \right\} dx$$

$$= \int_0^\infty x^{n-1} \times \frac{\Gamma m+n}{(1+x)^{m+n}} dx$$

$$\frac{\Gamma m \cdot \Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1} dx}{(1+x)^{m+n}}$$

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$ Hence Proved

$$\int_0^\infty \frac{x^4(1+x^6)}{(1+x)^{15}} dx = \int_0^\infty \frac{x^4}{(1+x)^{15}} dx + \int_0^\infty \frac{x^{10}}{(1+x)^{15}} dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}} dx + \int_0^\infty \frac{x^{11-1}}{(1+x)^{11+4}} dx$$

$$= \beta(5,10) + \beta(11,4)$$

$$= \frac{\Gamma 5 \cdot \Gamma 10}{\Gamma 5+10} + \frac{\Gamma 11 \cdot \Gamma 4}{\Gamma 5+10}$$

$(\because \Gamma n = (n-1)!)$

$$\Rightarrow \int_0^\infty \frac{x^4(1+x^6)}{(1+x)^{15}} dx = \frac{\Gamma 5 \, \Gamma 10 + \Gamma 11 \cdot \Gamma 4}{\Gamma 15}$$

$$= \frac{4! \cdot 9! + 10! \cdot 3!}{14!}$$

$$= \frac{9! \cdot 3! \, [4 + 10]}{14!}$$

$$= \frac{\cancel{3} \times \cancel{2} \times \cancel{14}}{\cancel{14} \times 13 \times \cancel{12}^{2} \times 11 \times 10}$$

$$= \frac{1}{2860}$$ Ans.

Date / /

Ques-(22) Prove that $\beta(l,m)\cdot\beta(l+m,n)\beta(l+m+n,p) = \dfrac{\gamma l\cdot\gamma m\cdot\gamma n\cdot\gamma p}{\gamma(l+m+n+p)}$

Solution:→ LHS:→ $\beta(l,m)\cdot\beta(l+m,n)\cdot\beta(l+m+n,p)$

$$= \frac{\gamma l\cdot\gamma m}{\cancel{\gamma l+m}}\cdot\frac{\cancel{\gamma l+m}\cdot\gamma n}{\cancel{\gamma l+m+n}}\cdot\frac{\cancel{\gamma l+m+n}\cdot\gamma p}{\gamma l+m+n+p}$$

$$= \frac{\gamma l\cdot\gamma m\cdot\gamma n\cdot\gamma p}{\gamma l+m+n+p} = \text{RHS.}$$ Hence Proved.

Ques(23) Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Solution:→ RHS:→ $\dfrac{\gamma m+1\cdot\gamma n}{\gamma m+n+1} + \dfrac{\gamma m\cdot\gamma n+1}{\gamma m+n+1}$

$$= \frac{m\gamma m\cdot\gamma n + \gamma m\, n\gamma n}{\gamma m+n+1} \qquad [\because \gamma m+1 = m\gamma m]$$

$$= \frac{\gamma m\cdot\gamma n\,\cancel{(m+n)}}{\cancel{(m+n)}\,\gamma m+n}$$

$$= \frac{\gamma m\cdot\gamma n}{\gamma m+n} = \beta(m,n)$$ Hence Proved

Ques(24) Prove that $\int_0^{\pi/2}\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta = \frac{1}{2}\beta(m,n)$ and hence show that $\int_0^{\pi/2}\sin^m\theta\cdot\cos^n\theta\,d\theta = \dfrac{\gamma\left(\frac{m+1}{2}\right)\gamma\frac{n+1}{2}}{2\gamma\left(\frac{m+n+2}{2}\right)}$.

Solution:→ We know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$x=0 \longrightarrow \theta = 0$

$x = 1 \longrightarrow \theta = \pi/2$

$$\Rightarrow \beta(m,n) = \int_0^{\pi/2}(\sin^2\theta)^{m-1}\cdot(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}(\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta$$

Date / /

$$\left[\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)\right]$$ Hence Proved

Put $2m-1 = m \rightarrow m \Rightarrow \frac{m+1}{2}$

$2n-1 = n \longrightarrow n = \frac{n+1}{2}$

$$\int_0^{\pi/2} \sin^{m}\theta \cdot \cos^{n}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+1+1}{2}}$$

$$= \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\cdot\Gamma\frac{m+n+2}{2}}.$$ Hence Proved

Ques-(25) Prove that $\int_0^{\pi/2} \sin^3 x \cdot \cos^{5/2} x \, dx = \frac{8}{77}$.

Solution $\rightarrow$ $$\int_0^{\pi/2} \sin^3 x \cdot \cos^{5/2} x \, dx = \frac{\Gamma\frac{3+1}{2} \; \Gamma\frac{5/2+1}{2}}{2\cdot\Gamma\frac{5/2+3+1+1}{2}}$$

$$= \frac{\Gamma 2 \cdot \Gamma 7/4}{2\cdot\Gamma 15/4} = \frac{1\cdot\cancel{3/4}\,\Gamma\cancel{3/4}}{2\cdot\cancel{11}\cdot\frac{3}{4}\cdot\cancel{7/4}\cancel{\frac{2}{4}}\,\Gamma\cancel{3/4}}$$

$$= \cancel{\frac{2}{3}\Gamma\frac{3/4}{1/4} = \frac{2}{3}\cdot\Gamma 3 = \frac{2\times2}{3} = \frac{4}{3}}.\ \cancel{\text{Ans}}$$

$$= \frac{8}{77}.\ \text{Ans}$$

Ques-(26) Prove following results-

(i) $$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\cdot d\theta = \pi/\sqrt{2}.$$

Date / /

Solution:→ $\int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta\, d\theta$

$$= \frac{\Gamma\frac{1/2+1}{2}\,\Gamma\frac{-1/2+1}{2}}{2\Gamma\frac{1/2-1/2+1+1}{2}} = \frac{\Gamma 3/4 \cdot \Gamma 1/4}{2\cdot\Gamma 1}$$

Here $n = 1/4$, $1-1/4 = 3/4$

$$\Rightarrow = \frac{\pi}{2\cdot\sin\pi/4} = \frac{\pi}{2\cdot\times\frac{1}{\sqrt{2}}} = \pi/\sqrt{2} \text{ H.P.}$$

$\int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta\, d\theta$

$$= \frac{\Gamma 1/4 \cdot \Gamma 3/4}{2\cdot\Gamma 1} = \pi/\sqrt{2} \text{ H.P.}$$

(ii) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} * \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

Solution→ $LHS = \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^0\theta\, d\theta * \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta\, d\theta$

$$= \frac{\Gamma\frac{-1/2+1}{2} \cdot \Gamma\frac{0+1}{2}}{2\cdot\Gamma\frac{-1/2+0+1+1}{2}} \times \frac{\Gamma\frac{1/2+1}{2}\,\Gamma\frac{0+1}{2}}{2\cdot\Gamma\frac{1/2+0+1+1}{2}}$$

$$= \frac{\Gamma 1/4 \cdot \Gamma 1/2 \times \cancel{\Gamma 3/4} \cdot \Gamma 1/2}{4\,\cancel{\Gamma 3/4} \cdot \Gamma 5/4}$$

$$= \frac{\cancel{\Gamma 1/4} \cdot \sqrt{\pi}\cdot\sqrt{\pi}}{\cancel{4}\times\frac{1}{\cancel{4}}\cdot\cancel{\Gamma 1/4}} = (\sqrt{\pi})^2 = \pi = RHS$$

H.P.

Ques (27) Using $\int_0^\infty \frac{x^{n-1}}{(1+x)}\, dx = \frac{\pi}{\sin n\pi}$, where $0 < n < 1$, prove that

$\Gamma n \cdot \Gamma \overline{n-1} = \pi/\sin n\pi$. Also deduce the following-

(i) $\Gamma 1/4\, \Gamma 3/4$ (ii) $\Gamma 1/3 \cdot \Gamma 2/3$.

Date / /

Solution :→ $\int_0^\infty \frac{x^{n-1}}{(1+x)} dx = \frac{\pi}{\sin n\pi}$

$$\frac{\pi}{\sin n\pi} = \int_0^\infty \frac{x^{n-1-n+2}}{(1+x)} dx = \int_0^\infty \frac{x^{n-1} dx}{(1+x)^{(n+(-n+2))}} + \int_0^\infty \frac{x^{-n+3-1}}{(1+x)^{(}} dx$$

$$\frac{\pi}{\sin n\pi} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+(-n+1)}}$$

$$= \beta(n, -n+1) = \frac{\Gamma(n) \cdot \Gamma(-n+1)}{\Gamma(n-n+1)}$$

$$= \frac{\Gamma(n) \cdot \Gamma(1-n)}{\Gamma(1)} = \Gamma(n) \cdot \Gamma(1-n)$$

$$\boxed{\frac{\pi}{\sin n\pi} = \Gamma(n) \cdot \Gamma(1-n)}$$ Hence Proved

(i) $\Gamma(\tfrac{1}{4}) \Gamma(\tfrac{3}{4}) = \Gamma(\tfrac{1}{4}) \Gamma(1-\tfrac{1}{4})$

$= \pi \operatorname{cosec} \pi/4 = \pi\sqrt{2}$.

(ii) $\Gamma(\tfrac{1}{3}) \cdot \Gamma(\tfrac{2}{3}) = \Gamma(\tfrac{1}{3}) \Gamma(1-\tfrac{1}{3})$

$= \pi \operatorname{cosec} \pi/3 = \frac{2\pi}{\sqrt{3}}$.

Date / /

Ques-(28) Prove the following—

(i) $\int_0^{\pi/2} \tan^n x\, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$

Solution:→ Let $\overset{LHS}{I} = \int_0^{\pi/2} \tan^n x\, dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx$

$$= \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{-n+1}{2}}{2\cdot\Gamma\frac{n+1-n+1}{2}} = \frac{\Gamma\frac{n+1}{2}\cdot\Gamma\frac{1-n}{2}}{2\cdot\Gamma 1}$$

$$= \frac{1}{2}\Gamma\frac{n+1}{2}\cdot\Gamma\frac{1-n}{2}$$

Now, $1 - \frac{(n+1)}{2} = \frac{2-n-1}{2} = \frac{1-n}{2}$

$$\Rightarrow \frac{1}{2}\Gamma\frac{n+1}{2}\,\Gamma\frac{1-(n+1)}{2}$$

$\because \Gamma n \cdot \Gamma 1-n = \frac{\pi}{\sin n\pi}$ $(0<n<1)$

$$\Rightarrow \frac{1}{2}\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$\because \sin\left(\frac{n+1}{2}\right)\pi = \sin\left\{\frac{n\pi}{2} + \frac{\pi}{2}\right\} = \cos\frac{n\pi}{2}$

$$= \frac{1}{2}\frac{\pi}{\cos n\pi/2}$$

$$= \frac{\pi}{2}\sec\frac{n\pi}{2} = RHS \quad \text{H.P}$$

(ii) $\int_0^1 x^5(1-x^3)^{10}\, dx = \frac{1}{396}$

Solution:→ $LHS = \int_0^1 x^5(1-x^3)^{10}\, dx$

Put $x^3 = t$ $\qquad x=0,\ t=0$

$3x^2\, dx = dt$ $\qquad x=1,\ t=1$

$x^2\, dx = dt/3$

Date / /

$$\int_0^1 x^2 \cdot x^3 (1-x^3)^{10} dx = \int_0^1 \frac{t(1-t)^{10} dt}{3}$$

$$= \frac{1}{3}\int_0^1 t^{2-1}(1-t)^{11-1} dt$$

$$\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$\Rightarrow \frac{1}{3}\beta(2,11) = \frac{1}{3}\frac{\Gamma 2\,\Gamma 11}{\Gamma 2+11}$$

$$= \frac{1}{3} \times \frac{1 \times 10!}{12!} = \frac{1}{3} \times \frac{\cancel{10!}}{11 \times 12 \times \cancel{10!}}$$

$$= \frac{1}{396} \therefore = \text{RHS. } \underline{\underline{\text{HP}}}$$

(iii) $\int_0^2 x(8-x^3)^{1/3} dx = 16\pi/9\sqrt{3}$

Solution :→ $\text{LHS} = \int_0^2 x(8-x^3)^{1/3} dx$

Put $x^3 = 8y$ $\qquad x = 0 \longrightarrow y = 0$

$x = 2y^{1/3}$ $\qquad x = 2 \longrightarrow y = 1$

$$dx = \frac{2}{3} y^{\frac{1}{3}-1} dy$$

$$dx = \frac{2\,dy}{3y^{2/3}}$$

$$\Rightarrow \int_0^1 2y^{1/3} \cdot (8-8y)^{1/3} \times \frac{2}{3} y^{-2/3} dy$$

$$= \int_0^1 \frac{4}{3} \times 2(1-y)^{1/3} y^{-1/3} dy$$

$$= \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{\frac{4}{3}-1} dy$$

$$= \frac{8}{3} \times \frac{\Gamma 2/3 \cdot \Gamma 4/3}{\Gamma 6/3} = 8\frac{\Gamma 2/3 \cdot 1/3\,\Gamma 1/3}{3\,\Gamma 2}$$

$$= \frac{8}{3 \times 3}\Gamma\frac{2}{3}\,\Gamma\frac{1}{3} = \frac{8}{9}\Gamma\frac{1}{3} \cdot \Gamma\left(1-\frac{1}{3}\right)$$

Date / /

$$= \frac{8\cancel{2}}{9} \cdot \frac{\pi}{\sin\frac{\pi}{3}} = \frac{8\cancel{2}\pi}{9\sqrt{3}/\cancel{2}} = \frac{16\pi}{9\sqrt{3}} = RHS \quad \text{Hp}$$

(iv) $\int_0^1 \frac{dx}{\sqrt{1+x^4}} \cancel{dx} = \frac{1}{8\sqrt{\pi}} (\gamma\tfrac{1}{4})^2$

Solution :- $LHS = \int_0^1 (1+x^4)^{-1/2} dx$

~~Put $x^4 = y$, $x = y^{1/4}$, $dx = \frac{1}{4} y^{-3/4} dy$~~ ~~$x = 0 \to y = 0$, $x = 1 \to y = 1$~~

$\Rightarrow$ Put $x^2 = \tan\theta \Rightarrow x = \tan^{1/2}\theta$

$$dx = \frac{1}{2} \tan^{-1/2}\theta \sec^2\theta \, d\theta$$

$$= \frac{1}{2} \sin^{-1/2}\theta \frac{\cos^{1/2}\theta}{\cos^2\theta} d\theta$$

$$= \frac{1}{2} \sin^{-1/2}\theta \cdot \cos^{-3/2}\theta \, d\theta$$

When $x=0 \to \theta = 0$, $x = 1 \to \theta = \pi/4$

$$I = \int_0^{\pi/4} (1+\tan^2\theta)^{-1/2} \left(\tfrac{1}{2} \sin^{-1/2}\theta \cos^{-3/2} d\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/4} \sec^{-1}\theta \cdot (\sin^{-1/2}\theta \cos^{-3/2}\theta \, d\theta)$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin^{-1/2}\theta \cdot \cos^{-1/2}\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{-1/2} d\theta$$

$$= \frac{1}{2} \times \frac{1}{2^{-1/2}} \int_0^{\pi/4} (\sin 2\theta)^{-1/2} d\theta$$

Put $2\theta = \phi \Rightarrow d\theta = \frac{1}{2} d\phi$ (When $\theta = 0$, $\phi = 0$; $\theta = \pi/4$, $\phi = \pi/2$.

$$= \frac{\sqrt{2}}{2 \times 2} \int_{\phi=0}^{\pi/2} \sin^{-1/2}\phi \, d\phi \quad (m = -1/2, n = 0)$$

$$= \frac{\sqrt{2}}{4} \left[\frac{\gamma\frac{-1/2+1}{2} \, \gamma\frac{1}{2}}{2 \cdot \gamma\frac{-1/2+0+2}{2}} \right] = \frac{1}{2\sqrt{2}} \frac{\gamma\frac{1}{4} \sqrt{\pi}}{2\gamma\frac{3}{4}}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\gamma\frac{1}{4} \cdot \gamma\frac{3}{4}}{(\gamma\frac{3}{4})^2} \cdot \quad \text{Ans}$$

Date / /

(29) Prove that $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$

Solution:→ $LHS = \int_0^\infty y^{1/2} e^{-y^2} dy \times \int_0^\infty y^{-1/2} e^{-y^2} dy$

Put $y^2 = t$

$y = \sqrt{t}$

$dy = \frac{1}{2\sqrt{t}} dt$

$\Rightarrow \int_0^\infty (\sqrt{t})^{1/2} e^{-t} \frac{dt}{2\sqrt{t}} \times \int_0^\infty (\sqrt{t})^{-1/2} e^{-t} \frac{dt}{2\sqrt{t}}$

$= \frac{1}{4}\int_0^\infty t^{1/4-1/2} e^{-t} dt \times \int_0^\infty t^{-1/4-1/2} e^{-t} dt$

$= \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4} dt \times \cancel{\frac{1}{4}}\int_0^\infty e^{-t} t^{-3/4} dt$

$= \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{3}{4}-1} dt \times \int_0^\infty e^{-t} t^{\frac{1}{4}-1} dt$

$= \frac{1}{4}\Gamma\frac{3}{4} \cdot \Gamma\frac{1}{4} = \frac{1}{4}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)$

$= \frac{1}{4}\frac{\pi}{\sin \pi/4} = \frac{1}{4} \times \frac{\pi}{1/\sqrt{2}} = \frac{\sqrt{2}\pi}{4} = \frac{\pi}{2\sqrt{2}} = RHS.$ H.P.

(30) Prove that $\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{(2)^{2m-1}} \Gamma(2m)$, where m is +ve.

Also, show that $\beta(m,m) = 2^{1-2m}\beta(m, 1/2)$

Solution:→ We have
$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m\, \Gamma n}{2\,\Gamma(m+n)} \quad \text{---} (1)$$

Put $n = 1/2$,

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^0\theta \, d\theta = \frac{\Gamma m \cdot \Gamma \frac{1}{2}}{2\,\Gamma\left(m+\frac{1}{2}\right)}$$

Date / /

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\sqrt{\pi}\,\Gamma m}{2\cdot\Gamma m+\frac{1}{2}} \quad \text{—} (2) \quad (\because \Gamma\tfrac{1}{2} = \sqrt{\pi})$$

Put $n = m$ in eq^n, (1)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \cdot \Gamma m}{2\cdot \Gamma m+m}$$

$$\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\cdot\Gamma 2m}$$

$$\int_0^{\pi/2} \left(\frac{2}{2}\sin\theta\cos\theta\right)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\cdot\Gamma 2m}$$

$$\int_0^{\pi/2} \frac{(\sin 2\theta)^{2m-1}}{2^{2m-1}} d\theta = \left(\frac{(\Gamma m)^2}{2\cdot\Gamma 2m}\right)$$

$$\frac{1}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1}2\theta \, d\theta = \frac{(\Gamma m)^2}{2\cdot\Gamma 2m}$$

Put $2\theta = \phi$

$2d\theta = d\phi$

$\theta = 0 \longrightarrow \phi = 0$

$\theta = \pi/2 \longrightarrow \phi = \pi$

$$\frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi \, \frac{d\phi}{\cancel{2}} = \frac{(\Gamma m)^2}{\cancel{2}\cdot\Gamma 2m}$$

$$\frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi \, d\phi = \frac{(\Gamma m)^2}{\Gamma 2m}$$

$$\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \qquad [\because f(2a-x) = f(x)]$$

$$\Rightarrow \frac{1}{2^{2m-1}} \times 2 \int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi = \frac{(\Gamma m)^2}{\Gamma 2m}$$

$$\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi = \frac{2^{2m-1}(\Gamma m)^2}{2\cdot\Gamma 2m} \quad \text{—} (3)$$

$$\because \int_0^{a} f(x)\,dx = \int_0^{a} f(y)\,dy$$

from (2) and (3), $\dfrac{\sqrt{\pi}\,\Gamma m}{\cancel{2}\cdot\Gamma m+\frac{1}{2}} = \dfrac{2^{2m-1}(\Gamma m)^2}{\cancel{2}\cdot\Gamma 2m}$

Date / /

$$\boxed{\Gamma m \cdot \Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m}$$ H.P.

$\beta(m,m) = 2^{1-2m}\beta(m,\tfrac{1}{2})$

RHS $\rightarrow 2^{1-2m}\beta(m,\tfrac{1}{2}) = 2^{1-2m}\dfrac{\Gamma m \cdot \Gamma \tfrac{1}{2}}{2\cdot \Gamma m+\tfrac{1}{2}}$

$$= \frac{2^{1-2m} \cdot \sqrt{\pi}\, \Gamma m\, \Gamma m}{2 \cdot \Gamma m \cdot \Gamma m+\tfrac{1}{2}}$$

$$= \frac{2^{1-2m} \cancel{\sqrt{\pi}} (\Gamma m)^2}{2 \cdot \dfrac{\cancel{\sqrt{\pi}}\, \Gamma 2m}{2^{2m-1}}}$$

$$= \frac{2^{1-2m} (\Gamma m)^2 \cdot 2^{2m-1}}{2 \cdot \Gamma 2m}$$

$$= \frac{2^{1-\cancel{2m}+\cancel{2m}-\cancel{1}} (\Gamma m)^2}{2 \cdot \Gamma 2m}$$

$$= 2^0 \cdot \frac{\Gamma m \cdot \Gamma m}{2 \cdot \Gamma m+m}$$

$= 1 \times \beta(m,m) = \beta(m,m) = \text{LHS}$ H.P.

Ques-(32) Evaluate the following integrals :→

(i) $\int_0^\infty e^{-\sqrt{x}} x^{1/4} dx$

Solution:→ We have, Let $I = \int_0^\infty e^{-\sqrt{x}} \cdot x^{1/4} dx$

Put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

$x = 0 \rightarrow t = 0$

$x = \infty \rightarrow t = \infty$

$I = \int_0^\infty e^{-t} (t^2)^{1/4} \times 2t\,dt = 2\int_0^\infty e^{-t}\, t^{\frac{1}{2}+1} dt$

$n-1 = 3/2$

Date / /

$$I = 2\int_0^\infty e^{-t}\, t^{3/2}\, dt = 2\int_0^\infty e^{-t}\, t^{5/2-1}\, dt$$

$$= 2\,\Gamma\tfrac{5}{2} = \cancel{2}\times\frac{3}{2}\times\frac{1}{\cancel{2}}\,\Gamma\tfrac{1}{2} = \frac{3\sqrt{\pi}}{2} \quad \text{Ans.}$$

(ii) $\displaystyle\int_0^1 \left(\frac{x^3}{(1-x^3)}\right)^{1/3} dx$

Solution :→ Let $\displaystyle I = \int_0^1 x\,(1-x^3)^{-1/3}\, dx$

Put $x^3 = y$

$x = y^{1/3}$

$dx = \frac{1}{3}\, y^{-2/3}\, dy$

$$I = \int_0^1 y^{1/3}(1-y)^{-1/3} \times \frac{1}{3}\, y^{-2/3}\, dy$$

$$= \frac{1}{3}\int_0^1 y^{-1/3}(1-y)^{-1/3}\, dy$$

$$= \frac{1}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{\frac{2}{3}-1}\, dy$$

$$= \frac{1}{3}\,\beta\left(\frac{2}{3},\frac{2}{3}\right) \quad \cancel{= \frac{1}{3}\times 2^{1-2\times 2/3}\,\beta}$$

$$= \frac{1}{3}\,\frac{\Gamma\frac{2}{3}\cdot\Gamma\frac{2}{3}}{\Gamma\frac{4}{3}} = \frac{\frac{1}{\cancel{3}}\,\Gamma\frac{2}{3}\cdot\Gamma\frac{2}{3}}{\frac{1}{\cancel{3}}\,\Gamma\frac{1}{3}}$$

$$= \frac{\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\frac{1}{3}} \quad \text{Ans.}$$

Date / /

Ques-(33) Prove that $\dfrac{\Gamma\frac{1}{3} \cdot \Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = (2)^{1/3}\sqrt{\pi}$

Solution :→ LHS :→ $\dfrac{\Gamma\frac{1}{3} \cdot \Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = \dfrac{\Gamma\frac{1}{3} \cdot \Gamma\frac{5}{6}\,\Gamma\frac{2}{3}}{(\Gamma\frac{2}{3})^2}$

$= \dfrac{\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right)\Gamma\frac{5}{6}}{(\Gamma\frac{2}{3})^2} = \dfrac{\pi \operatorname{cosec}\frac{\pi}{3}\,\Gamma\frac{5}{6}}{(\Gamma\frac{2}{3})^2}$

LHS → $\dfrac{\Gamma\frac{1}{3}\Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = \dfrac{\Gamma\frac{1}{3}\,\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)}{\Gamma\frac{2}{3}} = \dfrac{\sqrt{\pi}\,\Gamma\frac{2}{3}}{2^{2/3-1}\,\Gamma\frac{2}{3}} = \dfrac{\sqrt{\pi}}{2^{-1/3}}$

$= (2)^{1/3}\sqrt{\pi} = RHS$

Hp

V. Imp

Ques-(34) Assuming $\Gamma n \cdot \Gamma 1-n = \pi$ cosec...

V. Imp
Ques-(34) Assuming $\Gamma n \cdot \Gamma 1-n = \pi \operatorname{cosec} n\pi$, $0<n<1$, show that $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, $0<p<1$.

Solution:→ $\Gamma n \cdot \Gamma 1-n = \pi \operatorname{cosec} n\pi$ —— (1)

We have

$$\beta(n,m) = \frac{\Gamma n \cdot \Gamma m}{\Gamma n+m} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}}\,dx$$

Put $m+n=1 \longrightarrow m = 1-n$

$$\frac{\Gamma n \cdot \Gamma 1-n}{\Gamma \cancel{n}+1-\cancel{n}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{\cancel{n}+1-\cancel{n}}}\,dx$$

$$\Gamma n \cdot \Gamma 1-n = \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx \quad \text{—— (2)}$$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \pi \operatorname{cosec} n\pi$$

Put $n=p$.

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$$ H.P.

Date / /

Ques (37) Evaluate the integral $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ where x, y, z are all positive but limited by the condition $(x/a)^p + (y/b)^q + (z/c)^r \le 1$.

Solution:→ Let

$$I = \iiint_R x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$R: x \ge 0,\ y \ge 0,\ z \ge 0,\ \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$$

Put $\left(\frac{x}{a}\right)^p = u \Rightarrow x = (u)^{1/p} a$

$$dx = a \times \frac{1}{p} u^{\frac{1}{p}-1} du$$

Put $\left(\frac{y}{b}\right)^q = v \Rightarrow y = (v)^{1/q} b$

$$dy = b \times \frac{1}{q} \cdot v^{\frac{1}{q}-1} dv$$

Put $\left(\frac{z}{c}\right)^r = w \Rightarrow z = (w)^{1/r} c$

$$dz = c \times \frac{1}{r} w^{\frac{1}{r}-1} dw$$

$$\Rightarrow I = \iiint_{R'} (u^{1/p} a)^{l-1} (v^{1/q} b)^{m-1} (w^{1/r} c)^{n-1} \frac{a}{p} u^{\frac{1}{p}-1} \times \frac{b}{q} \cdot v^{\frac{1}{q}-1} \times \frac{c}{r} w^{\frac{1}{r}-1}\, du\, dv\, dw$$

$R': u \ge 0,\ v \ge 0,\ w \ge 0,\ u + v + w \le 1$

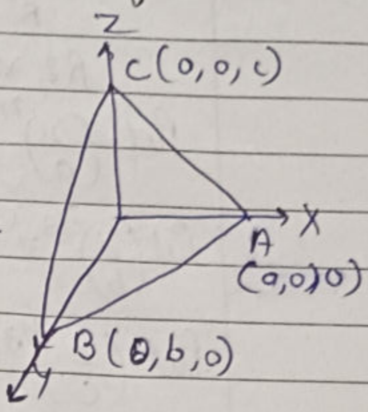
$$I = \iiint \frac{a^l b^m c^n}{pqr} u^{l/p-1} v^{m/q-1} w^{n/r-1}\, du\, dv\, dw$$

$\because\ l = l/p,\ m = m/q,\ n = n/r$

$$I = \frac{a^l b^m c^n}{pqr} \frac{\Gamma l/p \cdot \Gamma m/q\ \Gamma n/r}{\Gamma\left(l/p + m/q + n/r + 1\right)}$$ Ans

Ques-(38) The plane $x/a + y/b + z/c = 1$ meets the axes in A, B and C, Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also, find the mass if the density at any point is $kxyz$.

Solution:→



$$V = \iiint_R dx\,dy\,dz$$

$R: x \geq 0, y \geq 0, z \geq 0, x/a + y/b + z/c \leq 1$

Put $\frac{x}{a} = u \Rightarrow x = au$

$dx = a\,du$

$\frac{y}{b} = v \Rightarrow y = bv \Rightarrow dy = b\,dv$

$\frac{z}{c} = w \Rightarrow z = cw \Rightarrow dz = c\,dw$

$$V = \iiint_{R'} (a\,du)(b\,dv)(c\,dw)$$

$R': u \geq 0, v \geq 0, w \geq 0, u + v + w \leq 1$

$$V = abc \iiint u^{1-1} v^{1-1} w^{1-1}\,du\,dv\,dw$$

$$= abc \frac{\Gamma 1 \cdot \Gamma 1 \cdot \Gamma 1}{\Gamma 4} = \frac{abc}{6} \text{ Ans.}$$

$$\text{Mass} = \iiint_R Kxyz\,dx\,dy\,dz$$

$$= \iiint_{R'} K(au)(bv)(cw)(a\,du)(b\,dv)(c\,dw)$$

$$= Ka^2b^2c^2 \iiint u^{2-1} v^{2-1} w^{2-1}\,du\,dv\,dw$$

$$= Ka^2b^2c^2 \frac{\Gamma 2 \cdot \Gamma 2 \cdot \Gamma 2}{\Gamma 7} = \frac{Ka^2b^2c^2 \times 1 \times 1 \times 1}{6!}$$

$$= \frac{K(abc)^2}{720} \text{ Ans.}$$

Date / /

Ques-(39) Find the volume of the solid surrounded by the surface $(x/a)^{2/3}+(y/b)^{2/3}+(z/c)^{2/3}=1$

Solution: $\to$ $V=\iiint_R dx\,dy\,dz$

$R: x\geq 0, y\geq 0, z\geq 0, (x/a)^{2/3}+(y/b)^{2/3}+(z/c)^{2/3}\leq 1$

Put $\left(\frac{x}{a}\right)^{2/3}=u \Rightarrow x=(u)^{3/2}\cdot a \Rightarrow dx = a\times\frac{3}{2}u^{1/2}du$

Put $\left(\frac{y}{b}\right)^{2/3}=v \Rightarrow y=(v)^{3/2}b \Rightarrow dy = b\times\frac{3}{2}v^{1/2}dv$

Put $\left(\frac{z}{c}\right)^{2/3}=w \Rightarrow z=(w)^{3/2}c \Rightarrow dz = c\times\frac{3}{2}w^{1/2}dw$

$I=\iiint_{R'}\left(\frac{3a}{2}u^{1/2}du\right)\left(\frac{3b}{2}v^{1/2}dv\right)\left(\frac{3}{2}c\,w^{1/2}dw\right)$

$R': u\geq 0, v\geq 0, w\geq 0, u+v+w\leq 1$

$I=\frac{27abc}{8}\iiint u^{\frac{3}{2}-1}v^{\frac{3}{2}-1}w^{\frac{3}{2}-1}du\,dv\,dw$

$=\frac{27abc}{8}\frac{\gamma 3/2\cdot\gamma 3/2\cdot\gamma 3/2}{\gamma 11/2}$

$=\frac{\cancel{27}abc}{2\,\cancel{8}}\times\frac{1}{\cancel{8}}\frac{(\sqrt{\pi})^{\cancel{3}\,2}}{\cancel{\frac{9}{2}}\times\cancel{\frac{7}{2}}\times\cancel{\frac{5}{2}}\times\cancel{\frac{3}{2}}\times\cancel{\frac{1}{2}}\cancel{\sqrt{\pi}}}$

$=\frac{abc\,(\sqrt{\pi})^2}{2\times 7\times 5}=\frac{\pi abc}{70}$ Ans.

Ques-(40) Evaluate $\iiint x^2yz\,dx\,dy\,dz$ throughout the volume bounded by the planes $x=0, y=0, z=0$ and $x/a+y/b+z/c=1$.

Solution: $\to$ ~~Let~~ $= V=\iiint_R x^2yz\,dx\,dy\,dz$

$R: x\geq 0, y\geq 0, z\geq 0, x/a+y/b+z/c\leq 1$

Date / /

Put $\frac{x}{a} = u \Rightarrow x = au \Rightarrow dx = a\,du$

Put $\frac{y}{b} = v \Rightarrow y = bv \Rightarrow dy = b\,dv$

Put $\frac{z}{c} = w \Rightarrow z = cw \Rightarrow dz = c\,dw$

$$V = \iiint_{R'} (au)^2 (bv)(cw)(a\,du)(b\,dv)(c\,dw)$$

R' : $u \geq 0,\ w \geq 0,\ v \geq 0,\ u + v + w \leq 1$

$$V = \iiint a^3 b^2 c^2 u^2 v\, w\, du\, dv\, dw$$

$$= a^3 b^2 c^2 \iiint u^{3-1} v^{2-1} w^{2-1}\, du\, dv\, dw$$

$$= \frac{a^3 b^2 c^2\, \Gamma 3\, \Gamma 2\, \Gamma 2}{\Gamma 8} = \frac{2 a^3 b^2 c^2}{7!} \quad \text{Ans.}$$

Ques- (41) Evaluate $\iiint x^2 yz\, dx\, dy\, dz$, the integral being taken throughout the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

Solution :→ $V = \iiint_R x^2 yz\, dx\, dy\, dz$: $R \Rightarrow x \geq 0, y \geq 0, z \geq 0;$ $x+y+z \leq 1$

$$= \iiint_R x^{3-1} y^{2-1} z^{2-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma 3 \cdot \Gamma 2 \cdot \Gamma 2}{\Gamma 8} = \frac{2!}{7!} = \frac{2}{7!}$$

Ques - (42) Evaluate $\iiint_V e^{-(x+y+z)}\, dx\, dy\, dz$ where the region of integration is bounded by the planes $x=0, y=0, z=0$ and $x+y+z=a$, $a>0$ using Liouville's theorem.

Solution :→ Let $I = \iiint_V e^{-(x+y+z)}\, dx\, dy\, dz$

V: $x \geq 0,\ y \geq 0,\ z \geq 0,\ 0 < x+y+z < a$

Date / /

$$I = \iiint_V x^{1-1} y^{1-1} z^{1-1} e^{-(x+y+z)} \, \cancel{dxdydz} \, dxdydz$$

$l=1, m=1, n=1, h_1=0, h_2=a$

$t = x+y+z$

$$I = \frac{\gamma_1 \cdot \gamma_1 \cdot \gamma_1}{\gamma_3} \int_0^a e^{-t} t^{1+1+1-1} dt$$

$$= \frac{1}{2}\int_0^a e^{-t} t^2 dt$$

$$= \frac{1}{2}\left[t^2(-e^{-t}) - 2t(e^{-t}) + 2(-e^{-t})\right]_0^a$$

$$= \frac{1}{2}\left[-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right]_0^a$$

$$= \frac{1}{2}\left[-a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2e^0\right]$$

$$= \frac{1}{2}\left[-a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2\right]$$

$$= \frac{1}{2}\left[2 - e^{-a}(a^2+2a+2)\right] \text{ Ans.}$$

Ques-(43) Prove that $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real.

Solution:-

$$I = \iiint_R \frac{dxdydz}{\sqrt{1-(x^2+y^2+z^2)}}$$

$R: x>0, y>0, z>0, 0 < x^2+y^2+z^2 < 1$

Put $x^2 = u \Rightarrow dx = \frac{1}{2\sqrt{u}} du$

$y^2 = v \Rightarrow dy = \frac{1}{2\sqrt{v}} dv$

$z^2 = w \Rightarrow dz = \frac{1}{2\sqrt{w}} dw$

Date / /

$$I = \iiint_{R'} \frac{1}{8} \frac{u^{-1/2} v^{-1/2} w^{-1/2}}{\sqrt{1-(u+v+w)}} \, du\, dv\, dw \quad : 0 < u+v+w < 1$$

$$= \frac{1}{8} \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{3}{2}} \int_0^1 \frac{1}{\sqrt{1-t}} t^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1} dt$$

$$= \frac{1}{\cancel{8}\,4} \times \frac{(\sqrt{\pi})^2 \cancel{\Gamma\frac{1}{2}}}{\cancel{\frac{1}{2}}\cancel{\Gamma\frac{1}{2}}} \int_0^1 t^{1/2} (1-t)^{-1/2} dt$$

$$= \frac{\pi}{4} \int_0^1 t^{\frac{3}{2}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{\pi}{4} \beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\pi}{4} \frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma 2}$$

$$= \frac{\pi}{4} \times \frac{1}{2} (\sqrt{\pi})^2$$

$$= \frac{\pi^2}{8} \text{ Ans.}$$

Ques-(44) Evaluate $\iiint \sqrt{\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}} \, dx\,dy\,dz$ integral being taken over all positive values of x, y, z such that $x^2+y^2+z^2 < 1$.

Solution:→ ~~Given~~ Let $I = \iiint_R \sqrt{\frac{1-(x^2+y^2+z^2)}{1+x^2+y^2+z^2}} \, dx\,dy\,dz$

$R: x>0, \; y>0, \; z>0, \quad 0 < x^2+y^2+z^2 < 1$

Put $x^2 = u \Rightarrow dx = \frac{1}{2} u^{-1/2} du$

$y^2 = v \Rightarrow dy = \frac{1}{2} v^{-1/2} dv$

$z^2 = w \Rightarrow dz = \frac{1}{2} w^{-1/2} dw$

$$I = \iiint_{R'} \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}} \left(\tfrac{1}{2} u^{-1/2} du\right)\left(\tfrac{1}{2} v^{-1/2} dv\right)\left(\tfrac{1}{2} w^{-1/2} dw\right)$$

$$= \frac{1}{8} \iiint \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}} \cdot u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1} \, du\,dv\,dw$$

Date / /

$l=\frac{1}{2},\ m=\frac{1}{2},\ n=\frac{1}{2},\ h_0=0,\ h_1=1,$

$t=u+v+w$

$$I = \frac{1}{8}\frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{\Gamma\frac{3}{2}}\int_0^1\sqrt{\frac{1-t}{1+t}}\,t^{\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1\right)}dt$$

$$= \frac{1\cdot(\sqrt{\pi})^3}{8\times\frac{1}{2}\Gamma\frac{1}{2}}\int_0^1\sqrt{\frac{1-t}{1+t}}\cdot t^{\frac{1}{2}}dt$$

$$= \frac{\pi}{4}\int_0^1\sqrt{\frac{1-t}{1+t}}\cdot t^{\frac{1}{2}}dt$$

Put $t=\cos\theta$ $\qquad t=0 \longrightarrow \theta=\pi/2$

$dt=-\sin\theta\,d\theta$ $\qquad t=1 \longrightarrow \theta=0$

$$I=\frac{\pi}{4}\int_0^1\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(\cos\theta)^{\frac{1}{2}}(-\sin\theta)\,d\theta$$

$$=\frac{\pi}{4}\int_{\pi/2}^{0}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\,(\cos\theta)^{\frac{1}{2}}(-\sin\theta)\,d\theta$$

$$=\frac{\pi}{4}\int_0^{\pi/2}\frac{\sin\theta/2}{\cancel{\cos\theta/2}}\,(\cos\theta)^{\frac{1}{2}}\left(2\sin\theta/2\,\cancel{\cos\theta/2}\right)d\theta$$

$$=\frac{\pi}{4}\int_0^{\pi/2}\left(2\sin\theta/2\right)^2(\cos\theta)^{\frac{1}{2}}d\theta$$

$$=\frac{\pi}{4}\int_0^{\pi/2}(1-\cos\theta)^2(\cos\theta)^{\frac{1}{2}}d\theta$$

$$=\frac{\pi}{4}\left[\left\{\int_0^{\pi/2}(\cos\theta)^{\frac{1}{2}}d\theta\right\}-\left\{\int_0^{\pi/2}(\cos\theta)^{\frac{3}{2}}d\theta\right\}\right]$$

$$=\frac{\pi}{4}\left[\frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{2\cdot\Gamma\frac{5}{4}}-\frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{4}}{2\cdot\Gamma\frac{7}{4}}\right]$$

$$=\frac{\pi}{4}\left[\frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{\cancel{2}\cdot\times\frac{1}{\cancel{4}_2}\Gamma\frac{1}{4}}-\frac{\Gamma\frac{1}{2}\,\cancel{\Gamma\frac{5}{4}}\,\frac{1}{4}\Gamma\frac{1}{4}}{\cancel{2}\times\frac{3}{\cancel{4}_2}\Gamma\frac{3}{4}}\right]$$

Date / /

$$= \frac{\pi}{4}\frac{\Gamma\frac{1}{2}}{\frac{1}{2}}\left[\frac{\Gamma\frac{3}{4}}{\Gamma\frac{1}{4}} - \frac{\Gamma\frac{1}{4}}{12\cdot\Gamma\frac{3}{4}}\right]$$

$$= \frac{\pi\sqrt{\pi}}{2}\left[\frac{\Gamma\frac{3}{4}\cdot\Gamma\frac{1}{4}}{(\Gamma\frac{1}{4})^2} - \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{12\cdot(\Gamma\frac{3}{4})^2}\right]$$

$$= \frac{\pi\sqrt{\pi}}{2}\left[\frac{\pi\,\mathrm{cosec}\,\pi/4}{(\Gamma\frac{1}{4})^2} - \frac{\pi\,\mathrm{cosec}\,\pi/4}{12\cdot(\Gamma\frac{3}{4})^2}\right]$$

$$= \frac{\pi\sqrt{\pi}}{2}\times\frac{\pi\sqrt{2}}{\cancel{1}}\left[\frac{1}{(\Gamma\frac{1}{4})^2} - \frac{1}{12(\Gamma\frac{3}{4})^2}\right]$$

$$= \frac{\pi^{5/2}}{\sqrt{2}}\left[\frac{1}{(\Gamma\frac{1}{4})^2} - \frac{1}{12\cdot(\Gamma\frac{3}{4})^2}\right] \text{ Ans.}$$

Ques-(31) Evaluate $\int_0^a \frac{x^2}{\sqrt{a-x}}\,dx$.

Solution:→ Let $I = \int_0^a \frac{x^2}{\sqrt{a-x}}\,dx$

Put $x = ay$ $\qquad x=0,\ y=0$

$dx = a\,dy$ $\qquad x=a;\ y=1$

$$I = \int_0^1 \frac{a^2y^2}{\sqrt{a(1-y)}}\,a\,dy$$

$$= \int_0^1 a^{3-1/2}\,y^2(1-y)^{-1/2}\,dy$$

$$= a^{5/2}\int_0^1 y^{3-1}(1-y)^{\frac{1}{2}-1}\,dy$$

$$= a^{5/2}\,\beta(3,\tfrac{1}{2})$$

$$= a^{5/2}\frac{\Gamma 3\,\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}} = \frac{2a^{5/2}\,\cancel{\Gamma\frac{1}{2}}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\cancel{\Gamma\frac{1}{2}}}$$

$$= \frac{16}{15}a^{5/2}. \text{ Ans.}$$

Date / /

Ques- (45) Evaluate the following integral by changing the order of integration $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$.

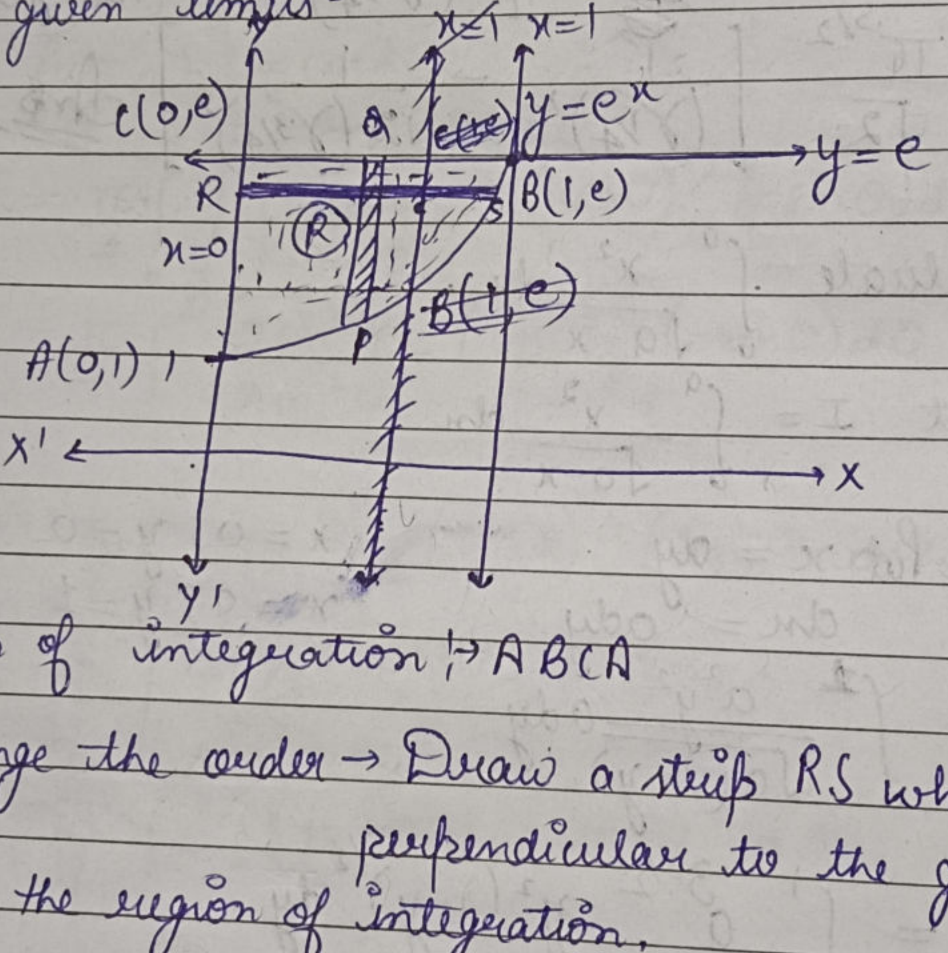
Solution:→ We have $I = \int_{x=0}^{1} \int_{y=e^x}^{e} \frac{dy\,dx}{\log y}$

Integrating first w.r.t 'y' and then w.r.t 'x';

Given limits → $x=0$ to $x=1$

$y=e^x$ to $y=e$

Draw the Region of integration with the help of given limits -



Region of integration :→ ABCA

To change the order → Draw a strip RS which is perpendicular to the given strip PQ in the region of integration.

New limits → $y=1$ to $y=e$

$x=0$ to $x=\log y$

Now, $I = \int_{y=1}^{e} \int_{x=0}^{\log y} \frac{dy\,dx}{\log y}$

$= \int_{y=1}^{e} (\log y)^{-1} \left[x\right]_{x=0}^{\log y} dy$

Date / /

$$I = \int_{y=1}^{e} (\log y)^{-1+1}\, dy = \int_{y=1}^{e} dy$$
$$= [y]_{y=1}^{e} = (e-1) \text{ Ans.}$$

Ques- (46) Evaluate the integral by changing the order of integration: $\int_0^1 \int_{2y}^{2} e^{x^2}\, dx\, dy$.

Solution. We have $I = \int_{y=0}^{1} \int_{x=2y}^{2} e^{x^2}\, dx\, dy$

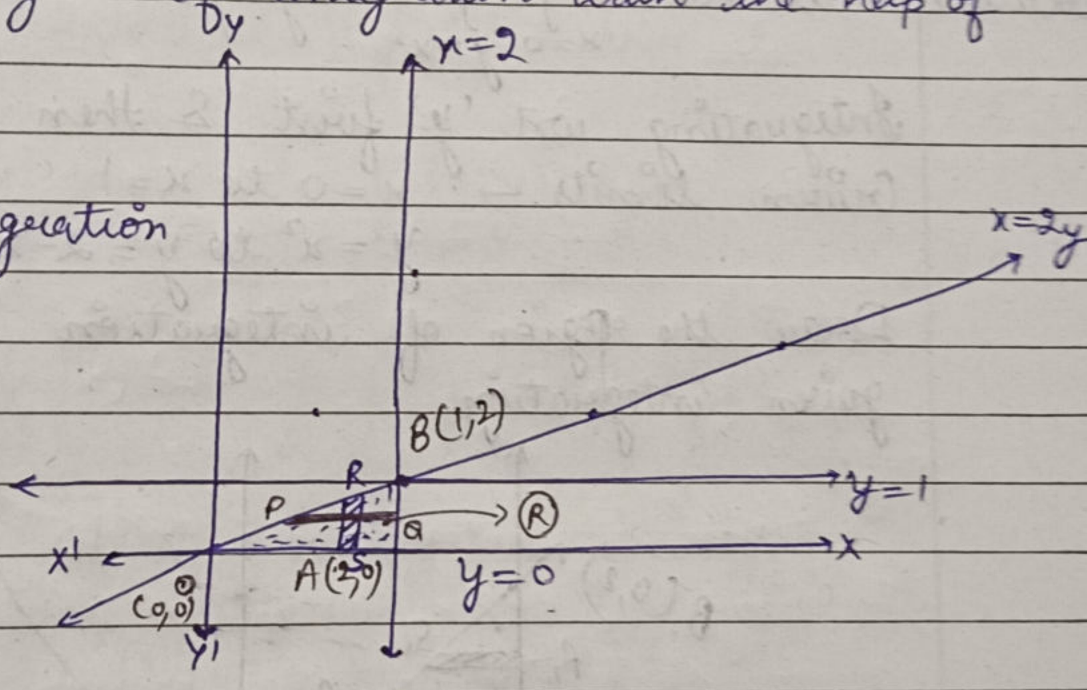
Integrating first wrt 'x' and then wrt 'y'.

Given limits : $x = 2y$ to $x = 2$

$y = 0$ to $y = 1$

Draw the region of integration with the help of given limits

Region of Integration $= OABO$



To change the limit, draw a strip RS which is $\perp$ to the given strip PQ in the region of integration

New limits $\rightarrow$ $x = 0$ to $x = 2$

$y = 0$ to $y = x/2$

Now, $I = \int_{x=0}^{2} \int_{y=0}^{x/2} e^{x^2}\, dx\, dy = \int_{x=0}^{2} e^{x^2} [y]_{y=0}^{x/2}\, dx$

$= \frac{1}{2} \int_{x=0}^{2} x e^{x^2}\, dx = \frac{1}{4} \int_{x=0}^{2} 2x\, e^{x^2}\, dx$

Date / /

Put $x^2 = y \Rightarrow x = \sqrt{y}$

$dx = \frac{1}{2\sqrt{y}}\,dy$

$x = 0 \rightarrow y = 0$

$x = 2 \rightarrow y = 4$

$$I = \frac{1}{2}\int_0^4 \sqrt{y}\,\frac{e^y}{2\sqrt{y}}\,dy$$

$$= \frac{1}{4}\left[e^y\right]_0^4 = \frac{1}{4}(e^4 - e^0)$$

$$= \frac{1}{4}(e^4 - 1) \text{ Ans.}$$

Ques → (47) Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$ and evaluate the same.

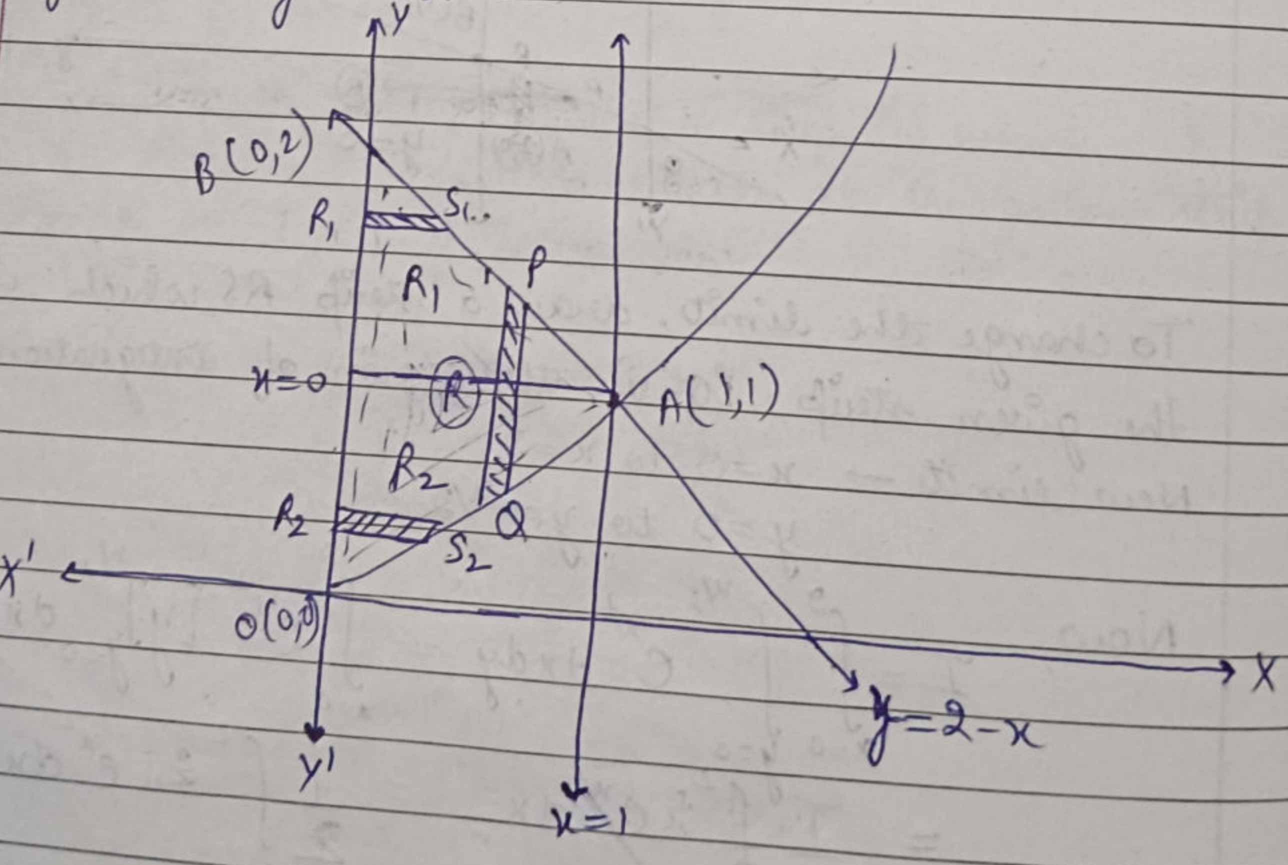
Solution → We have $I = \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\,dx$

Integrating wrt 'y' first & then wrt 'x'.

Given limits → $x = 0$ to $x = 1$

$y = x^2$ to $y = 2-x$

Draw the region of integration with the help of given integration.



Region of integration: OABO.

Here, $R = R_1 + R_2$

To change the order, draw strips $\perp$ to the given strip PQ in the region of integration

New limits $\rightarrow$ $R_1 \Rightarrow y=1$ to $y=2$

$x=0$ to $x=2-y$

$R_2 \Rightarrow y=0$ to $y=1$

$x=0$ to $x=\sqrt{y}$

Now,

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} dy + \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_{x=0}^{2-y} dy$$

$$= \int_{y=0}^{1} \frac{y^2}{2}\, dy + \int_{y=1}^{2} y\,\frac{(2-y)^2}{2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_{y=0}^{1} + \frac{1}{2}\int_{y=1}^{2} y(4+y^2-4y)\, dy$$

$$= \frac{1}{6}\left[1^3-0\right] + \frac{1}{2}\int_{y=1}^{2}(4y+y^3-4y^2)\, dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{\cancel{4}^{2} y^2}{\cancel{2}} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_{y=1}^{2}$$

$$= \frac{1}{6} + \frac{1}{2}\left[8+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{10+16-128-3}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{120-115}{12}\right) = \frac{1}{6} + \frac{1}{2}\times\frac{5}{12}$$

$$= \frac{4+5}{24} = \frac{9}{24} \text{ Ans.}$$

Date / /

Ques-(10) Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y} \, dy \, dx$ by changing the order of integration:

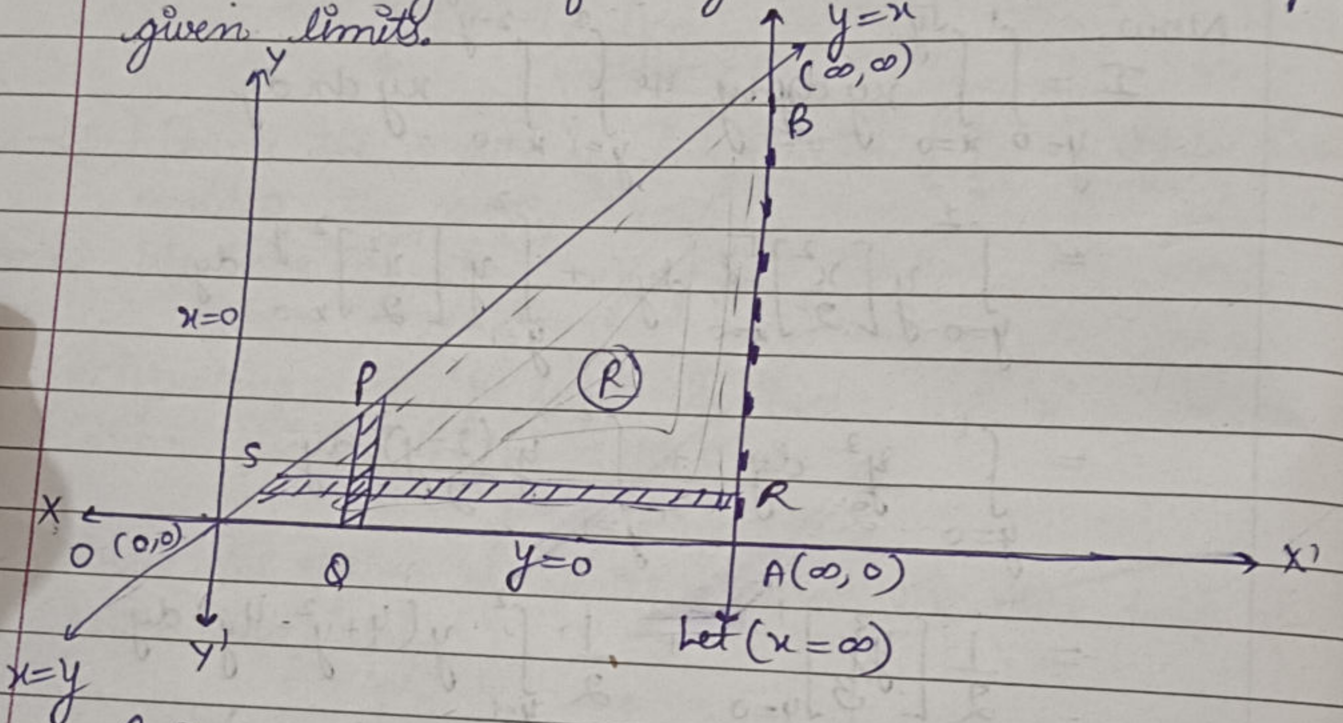
Solution:-> We have $I = \int_{x=0}^{\infty} \int_{y=0}^{x} x e^{-x^2/y} \, dy \, dx$

Integrating first w.r.t 'y' and then w.r.t. 'x':-

Given limits → $x=0$ to $x=\infty$

$y=0$ to $y=x$

Draw the region of integration with the help of given limits.



Region of integration: OABO

To change the order, draw a strip RS ⊥ to the given strip PQ in the region of integration.

New ~~strut~~ limits =

$y=0$ to $y=\infty$

$x=y$ to $x=\infty$

Now,

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-x^2/y} \, dx \, dy$$

$$= \int_{y=0}^{\infty} \frac{-y}{2} \int_{x=y}^{\infty} \frac{-2x}{y} e^{-x^2/y} \, dx \, dy$$

Date / /

$\because \int f'(x)\, e^{f(x)} = e^{f(x)}$

$$\Rightarrow I = \frac{-1}{2}\int_{y=0}^{\infty} y\left[e^{-x^2/y}\right]_{x=y}^{\infty}$$

$$= \frac{-1}{2}\int_{y=0}^{\infty} y\left(e^{-\infty} - e^{-y}\right) dy$$

$$= \frac{-1}{2}\int_{y=0}^{\infty} y\left(0 - e^{-y}\right) dy$$

$$= \frac{1}{2}\int_{y=0}^{\infty} y e^{-y}\, dy$$

$$= \frac{1}{2}\left[y(-e^{-y}) - 1\cdot e^{-y}\right]_{y=0}^{\infty}$$

$$= \frac{1}{2}\left[\infty(-e^{-\infty}) - e^{-\infty} - 0 + e^{0}\right]$$

$$= \frac{1}{2}(0 - 0 - 0 + 1) = \frac{1}{2} \text{ Ans.}$$

Ques-(49) Change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} f(x,y)\,dy\,dx$.

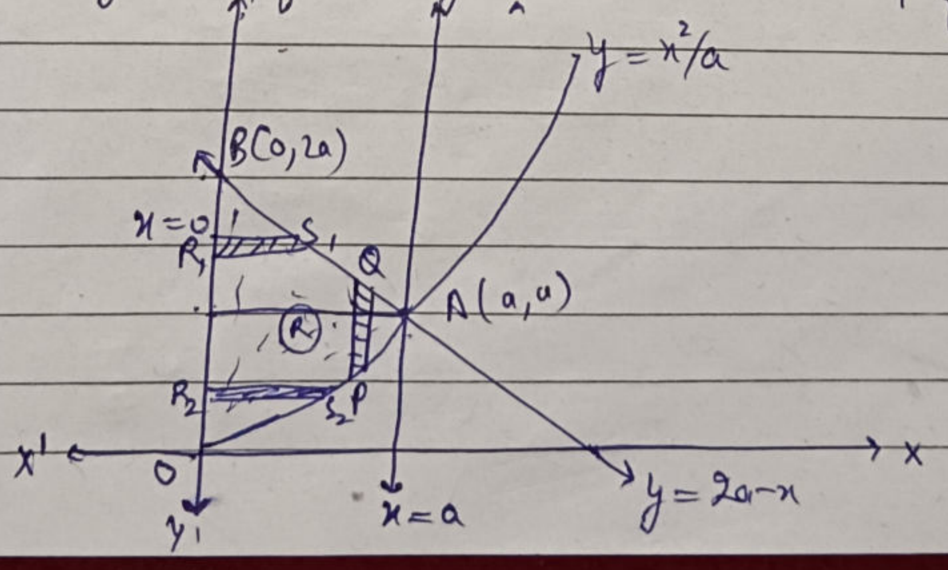
Solution :- We have $I = \int_{x=0}^{a}\int_{y=x^2/a}^{2a-x} f(x,y)\,dy\,dx$

Integrating first wrt 'y' & then w.r.t. 'x'

Given limits :- $x=0$ to $x=a$

$y = x^2/a$ to $y = 2a - x$

Draw the region of integration with the help of given limits :-



Date / /

Region of integration = OABOA

To change the order, draw strips ⊥ to given strip PQ in the Region of integration.

R : $R_1 + R_2$

New limits :→ $R_1 \Rightarrow$ $y=0$ to $y=a$

$x=0$ to $x=\sqrt{ay}$

$R_2 \Rightarrow$ $y=a$ to $y=2a$

$x=0$ to $x=2a-y$

Now,

$$I = \int_{y=0}^{a}\int_{x=0}^{\sqrt{ay}} f(x,y)\,dx\,dy + \int_{y=a}^{2a}\int_{x=0}^{2a-y} f(x,y)\,dx\,dy.$$

Ans

Ques-(50) By changing the order of integration, evaluate the

Ques-(50) By changing the order of integration, evaluate the following integration $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy$, and hence show that $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$

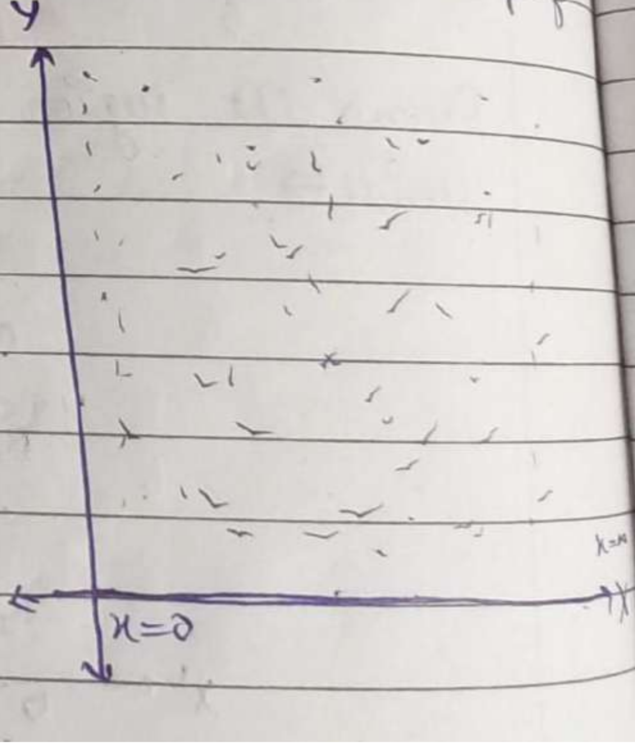
Solution → We have $I = \int_{y=0}^{\infty} \int_{x=0}^{\infty} e^{-xy} \sin px \, dx \, dy$

Integrating w.r.t 'x' at first and then w.r.t 'y',

Given limits → $x = 0$ to $x = \infty$

$y = 0$ to $y = \infty$

Draw the region of integration with the help of given limits



Region of intergration = 1st quadrant

Now,

$$I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-xy} \sin px \, dy \, dx$$

Date / /

$$I = \int_{x=0}^{\infty} \sin px \left(\frac{-e^{-xy}}{x} \right)_0^{\infty} dx = \int_{x=0}^{\infty} \sin px \left[\frac{-e^{\infty}}{x} + \frac{e^{0}}{x} \right] dx$$

$$= \int_{x=0}^{\infty} \sin px \times \frac{e^{0}}{x} \, dx = \int_{x=0}^{\infty} \frac{\sin px}{x} \, dx \quad \text{——— (1)}$$

$$\cancel{= \left[\sin px \log x \right]_0^{\infty} - \int -\cos px \log x}$$

Again $I = \int_{y=0}^{\infty} \int_{x=0}^{\infty} e^{-xy} \sin px \, dx \, dy$

$$= \int_{y=0}^{\infty} \left[\frac{-e^{-xy}}{p^2+y^2} (p \cos px + y \sin px) \right]_0^{\infty} dy$$

$$= \int_0^{\infty} \frac{-1}{p^2+y^2} \times (-p) \, dy = \int_0^{\infty} \frac{p}{p^2+y^2} \, dy$$

$$= \left[\tan^{-1}(y/p) \right]_0^{\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \text{——— (2)}$$

from (1) & (2),

$$\int_0^{\infty} \frac{\sin px \, dx}{x} = \pi/2$$ Hence Proved

24/1/24 C.W Date / /

Ques-5). Change into polar co-ordinates and evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\,dx$. Hence show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Imp

Solution → Let $I = \int_0^\infty e^{-x^2} dx$ ——— (1)

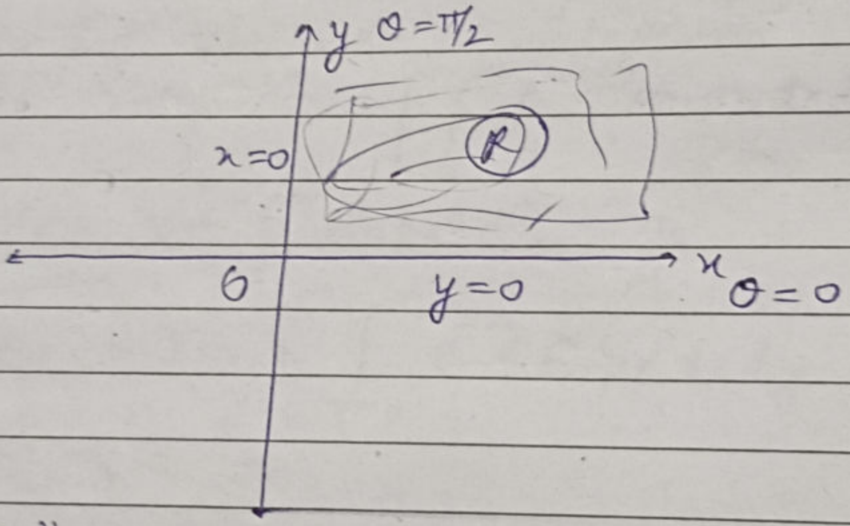
$$I = \int_0^\infty e^{-y^2} dy \text{ ——— (2)}$$

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\,dx \text{ — (3)}$$

Put $x = r\cos\theta$, $y = r\sin\theta \Rightarrow x^2+y^2 = r^2$

and $dx\,dy = |J|\,dr\,d\theta$ where $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$

$= r\,dr\,d\theta$



"Limits" → $\theta = 0$ to $\theta = \pi/2$

$\theta = 0$ to $r \to \infty$

$$I^2 = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} \cdot r\,dr\,d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{\infty} -2r e^{-r^2} dr \right] d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} \left[e^{-r^2} \right]_{\theta=0}^{\infty} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} d\theta = \frac{1}{2} \times \frac{\pi}{2} = \pi/4 \text{ Ans.}$$

Date / /

$\therefore I = \int_0^\infty e^{-x^2} dx = \sqrt{\pi/4} = \sqrt{\pi}/2$ H.P.

"Note" :- Prove that $\gamma_{1/2} = \sqrt{\pi}$

$\because \gamma_n = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx$

Put $n = 1/2$

$\gamma_{1/2} = 2\int_0^\infty e^{-x^2} dx$

$\because \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

$\Rightarrow \gamma_{1/2} = \frac{2 \times \sqrt{\pi}}{2} = \sqrt{\pi}$ H.P.

Ques (52) Evaluate the following $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx dy$ over the positive quadrant of circle $x^2+y^2=1$.

Solution :- Let $I = \iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}$

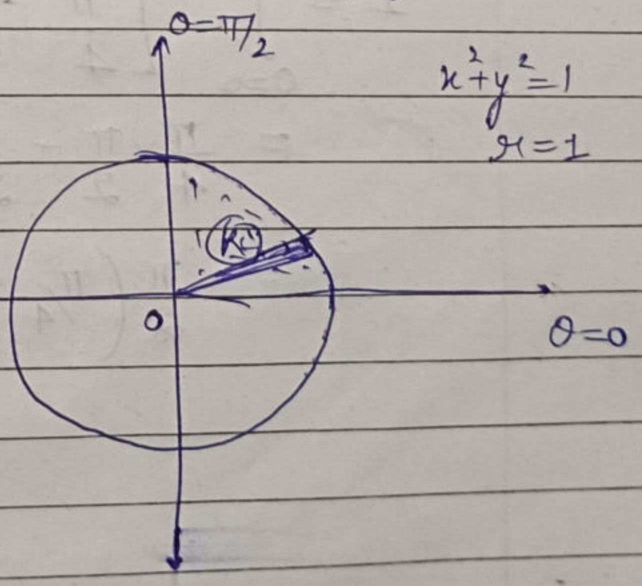
Put $x = r\cos\theta,\ y = r\sin\theta$

$x^2+y^2 = r^2$

$dx dy = |J|\, dr d\theta = r dr d\theta \qquad \because J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$

Limits :- $\theta = 0$ to $\theta = \pi/2$

$r = 0$ to $r = 1$



Now,

$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\, r dr d\theta$

$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\sqrt{\frac{(1-r^2)^2}{1-r^4}}\, dr d\theta$

$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \frac{r\cdot(1-r^2)}{\sqrt{1-r^4}}\, dr d\theta$

Date / /

$$I = \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{1} \frac{r(1-r^2)}{\sqrt{1-r^4}}\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{1}{2}\int_{r=0}^{1}\frac{2r}{\sqrt{1-r^4}}\,dr \quad \frac{-1}{(-4)}\int_{r=0}^{1}\frac{-4r^3}{\sqrt{1-r^4}}\,dr\right]d\theta \text{ —— (i)}$$

$$I_1 = \frac{1}{2}\int_{r=0}^{1}(2r)\frac{1}{\sqrt{1-r^4}}\,dr \quad ; \text{ Put } r^2 = t \Rightarrow 2r\,dr = dt$$

$$= \frac{1}{2}\int_{t=0}^{1}\frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\left[\sin^{-1}t\right]_{r=0}^{1} = \frac{1}{2}\left[\pi/2 - 0\right) = \frac{\pi}{4} \text{ —— (ii)}$$

$$I_2 = \frac{-1}{4}\int_{r=0}^{1}(-4r^3)\frac{1}{\sqrt{1-r^4}}\,dr \quad ; \text{ Put } (1-r^4) = t^2 \Rightarrow -4r^3\,dr = 2t\,dt$$

When $r = 0 \rightarrow t = 1$

$r = 1 \rightarrow t = 0$

$$= \frac{-1}{2\cancel{4}}\int_{t=1}^{0}\frac{\cancel{2t}\,dt}{\cancel{\sqrt{t^2}}} = \frac{-1}{2}\left[t\right]_{t=1}^{0}$$

$$= \frac{-1}{2}\left[0-1\right] = \frac{1}{2} \text{ —— (iii)}$$

from (i), (ii) & (iii)

$$I = \int_{\theta=0}^{\pi/2}\left[\frac{\pi}{4} - \frac{1}{2}\right]d\theta = \left[\pi/4\,\theta - \frac{1}{2}\theta\right]_{\theta=0}^{\pi/2}$$

$$= \frac{\pi}{4}\times\frac{\pi}{2} - \frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{\pi}{2}\left(\pi/4 - 1/2\right) \text{ Ans.}$$

Date / /

Ques-53 Evaluate $\iint \sqrt{(a^2-x^2-y^2)}\, dx\, dy$, over the semi-circle $x^2+y^2=ax$ in the positive quadrant.

Solution :→ Let $I = \iint_R \sqrt{a^2-x^2-y^2}\, dx dy = \iint_R \sqrt{a^2-(x^2+y^2)}\, dx dy$

Put $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

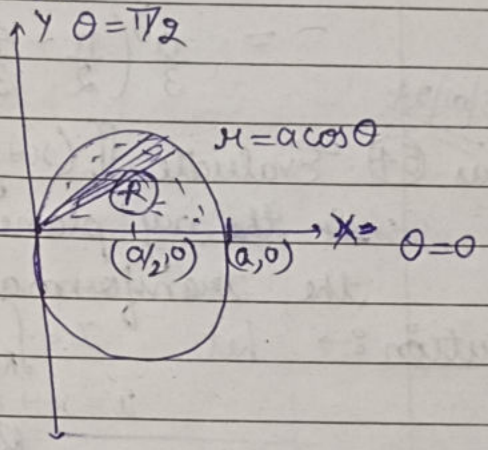
$dx dy = |J|\, dr d\theta \Rightarrow J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$

$\Rightarrow dx dy = r\, dr d\theta$ $\qquad J = r$

$R :\to x^2+y^2 = ax$

$x^2 - ax + \left(\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

$\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$



Limits :→ $\theta = 0$ to $\theta = \pi/2$

$r = 0$ to $r = a\cos\theta$

Now, $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\, dr d\theta$

$= \int_{\theta=0}^{\pi/2} \left[\frac{-1}{2} \int_{r=0}^{a\cos\theta} -2r\,(a^2-r^2)^{1/2}\, dr d\theta \right.$

$\because \int f'(x)\, f^n(x)\, dx = \dfrac{f^{n+1}(x)}{n+1}$

$\Rightarrow I = \int_{\theta=0}^{\pi/2} \frac{-1}{\cancel{2}} \left[\frac{(a^2-r^2)^{3/2}}{3/\cancel{2}} \right]_{r=0}^{a\cos\theta}$

$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[(a^2 - a^2\cos^2\theta)^{3/2} - (a^2)^{3/2} \right] d\theta$

$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[(a^2)^{3/2} (\sin^2\theta)^{3/2} - a^3 \right] d\theta$

$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} a^3 \left[\sin^3\theta - 1 \right] d\theta$

Date / /

$$I = -\frac{a^3}{3}\left[\int_{\theta=0}^{\pi/2} \sin^3\theta\, d\theta - \int_{\theta=0}^{\pi/2} d\theta\right]$$

$$= -\frac{a^3}{3}\left[\frac{\Gamma 4/2\, \Gamma 1/2}{\Gamma 5/2} - [\theta]_{\theta=0}^{\pi/2}\right]$$

$$= -\frac{a^3}{3}\left[\frac{\Gamma 2 \cdot \cancel{\Gamma 1/2}}{3/2 \cdot 1/2 \cancel{\Gamma 1/2}} - \pi/2\right]$$

$$= \frac{-a^3}{3}\left(\frac{4}{3} - \frac{\pi}{2}\right)$$

$$= \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{4}{3}\right) \text{ Ans.}$$

25/01/24

Ques-(54) Evaluate $\iint_R (x+y)^2 dx\,dy$, where R is the parallelogram in the xy plane with vertices $(1,0), (3,1), (2,2), (0,1)$ using the transformation $u = x+y$ and $v = x-2y$.

Solution :→ Let $I = \iint_R (x+y)^2 dx\,dy$

$u = x+y$ and $v = x-2y$

$x = \frac{(2u+v)}{3}$ & $y = \frac{(u-v)}{3}$

Now, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

$$= \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3}$$

Now,

$$I = \iint_R (x+y)^2 dx\,dy = \iint_{R'} u^2 |J|\, du\,dv$$

$$= \frac{1}{3}\iint_{R'} u^2 du\,dv$$

Region of Integration -

Date / /

$u = x + y$: $x=1, y=0 \Rightarrow u=1$

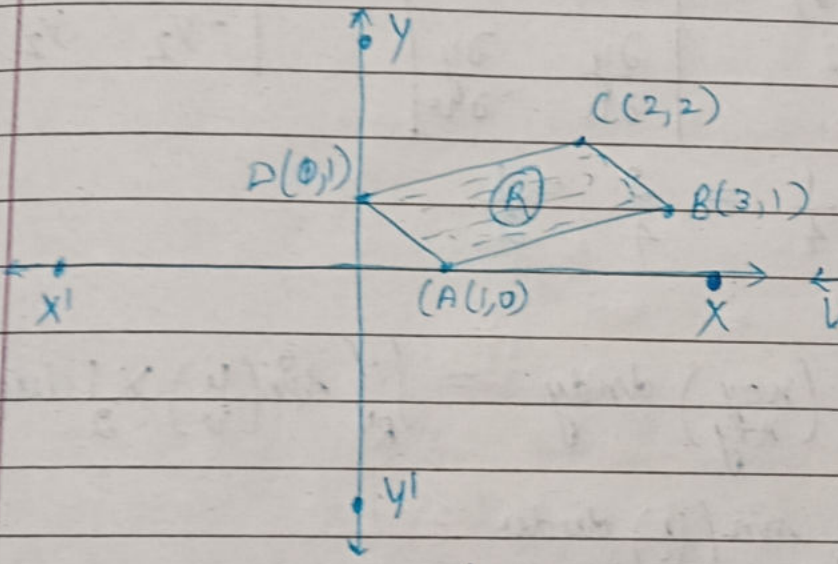
$x=3, y=1 \Rightarrow u=4$

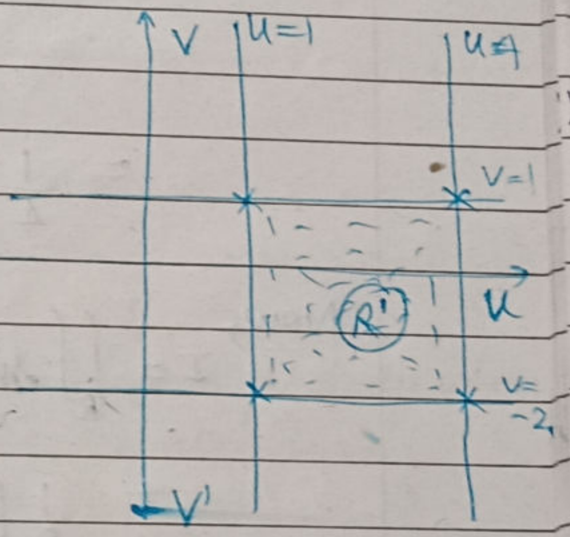
$v = x - 2y$: $x=2, y=2 \Rightarrow v=-2$

$x=3, y=1 \Rightarrow v=1$

OR

Coordinate	xy-plane		u-v-plane	
	x	y	$u = x+y$	$v = x-2y$
A(1,0)	1	0	$u=1$	$v=1$
B(3,1)	3	1	$u=4$	$v=1$
C(2,2)	2	2	$u=4$	$u=-2$
D(0,1)	0	1	$u=1$	$v=-2$





Now, $I = \int_{v=-2}^{1} \int_{u=1}^{4} \frac{u^2}{3}\, du\, dv = \frac{1}{3} \int_{v=-2}^{1} \left[\frac{u^3}{3}\right]_{u=1}^{4} dv$

$= \frac{1}{3} \int_{v=-2}^{1} \left(\frac{64-1}{3}\right) dv$

$= \frac{1}{3} \times \frac{63}{3} \int_{v=-2}^{1} dv = \frac{63}{9} [v]_{v=-2}^{1}$

$= \frac{63}{9} [1+2] = \frac{63 \times \cancel{3}}{\cancel{9}}$

$= \frac{63}{3} = 21$ Ans

Date / /

Ques-(55) Using the transformation $x-y=u$, $x+y=v$, show that $\iint_R \sin\left(\frac{x-y}{x+y}\right) dx\,dy = 0$, where R is bounded by the co-ordinate axes and $x+y=1$ in the first quadrant.

Solution:→ Let $I = \iint_R \sin\left(\frac{x-y}{x+y}\right) dx\,dy = 0$

$$x-y=u \quad , x+y=v$$

$$x=\frac{u+v}{2} \quad , y=\frac{v-u}{2}$$

Now, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix}$

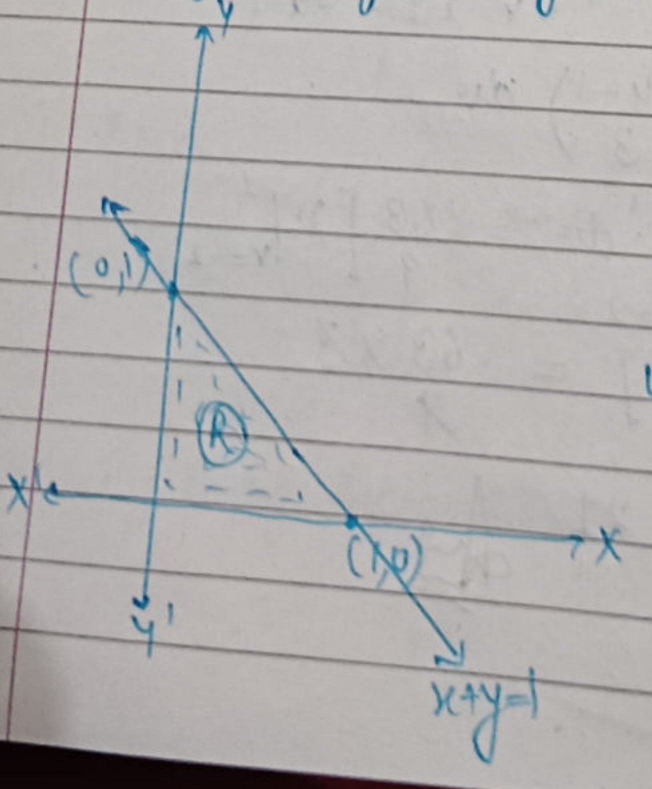
$$= \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

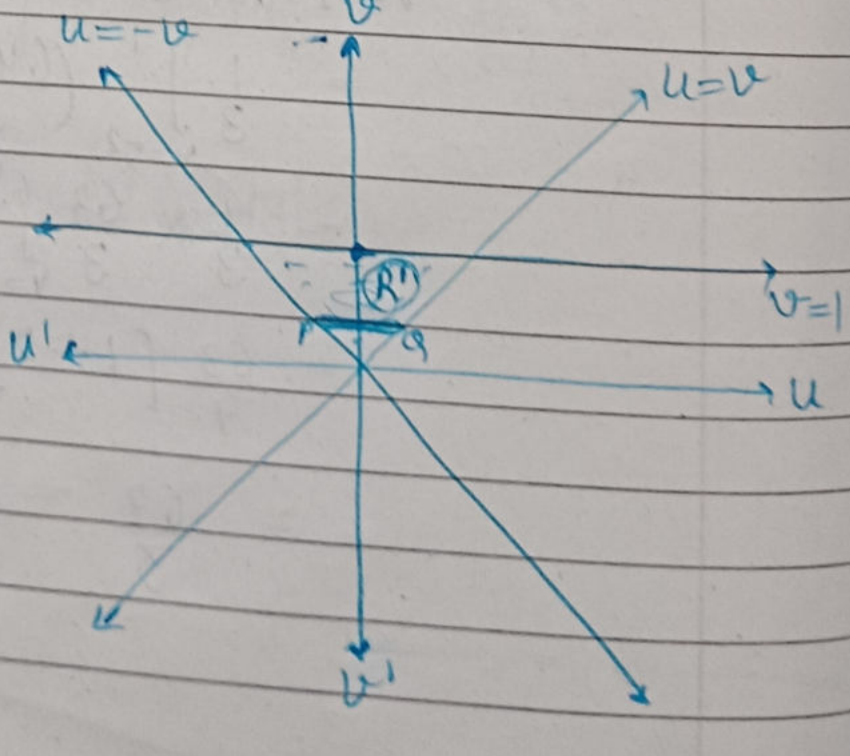
Now,

$$I = \iint_R \sin\left(\frac{x-y}{x+y}\right) dx\,dy = \iint_{R'} \sin\left(\frac{u}{v}\right) \times \frac{1}{2}\, du\,dv$$

$$= \frac{1}{2}\iint_{R'} \sin\left(\frac{u}{v}\right) du\,dv$$

Region of integration:





Date / /

(i) $x=0 \Rightarrow u=-v$

(ii) $y=0 \Rightarrow u=v$

(iii) $x+y=1 \Rightarrow v=1$

Now, $$I = \int_{v=0}^{1} \frac{1}{2}\int_{u=-v}^{v} \sin\left(\frac{u}{v}\right) du\, dv$$

$$\cancel{= \frac{1}{2}\int_{v=0}^{1} \left[\frac{-\cos\left(\frac{u}{v}\right)}{1/v} \right]_{u=-v}^{v} dv}$$

$$\cancel{= \frac{1}{2}\int_{v=0}^{1} -v\left[\right.}$$

$\because \sin\left(\frac{u}{v}\right)$ is an odd function w.r.t. 'u'

$$\therefore \int_{u=-v}^{v} \sin(u/v)\, du = 0$$

$$\Longrightarrow \quad I = 0 \quad \text{Ans}$$

Ques (56) Evaluate by changing the variable $\iint_R (x+y)^2 dx\,dy$ where R is the region bounded by the parallelogram $x+y=0$, $x+y=2$, $3x-2y=0$ and $3x-2y=3$.

Solution :- Let $I = \iint_R (x+y)^2 dx\,dy$

$R:$ $x+y=0$, $x+y=2$, $3x-2y=0$ and $3x-2y=3$

Let $x+y=u$ and $3x-2y=v$

$$x = \frac{1}{5}(2u+v), \quad y = \frac{1}{5}(3u-v)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2/5 & 1/5 \\ 3/5 & -1/5 \end{vmatrix}$$

$$= -\frac{2}{25} - \frac{3}{25} \Rightarrow \frac{-1}{5}.$$

Date / /

~~Limits~~ Now,

Now, $\iint_R (x+y)^2 \, dx\, dy = \int_{R'}\int u^2 |J| \, du\, dv$

$= \frac{1}{5} \int_{R'}\int u^2 \, du\, dv$

Limits → $u=0$ to $u=2$

$v=0$ to $v=3$

$$I = \frac{1}{5}\int_{v=0}^{v=3} \int_{u=0}^{2} u^2 \, du\, dv$$

$$= \frac{1}{5}\int_{v=0}^{3} \left[\frac{u^3}{3}\right]_{u=0}^{2} dv$$

$$= \frac{1 \times 8}{5 \times 3} \int_{v=0}^{3} dv$$

$$= \frac{8}{15} \left[v\right]_{v=0}^{3}$$

$$= \frac{8}{15} \times 3 = \frac{8}{5} \text{ Ans.}$$

Question Bank Completed