Fourier Series

Periodic Function :— A function $f(x)$ is said to be periodic if there exists a possitive real number T such that

$$f(x) = f(x+T) = f(x+2T) = \cdots$$

where T is the time period of the function.

For example $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are the periodic functions with time period 2π and $\tan x$, $\cot x$ are also the periodic functions with time period π.

Some Usefull Integrals :—

1. $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, where n is an integer.

2. $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \text{, i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$$

3. $\int_0^{2\pi} \sin nx\,dx = 0$, 4. $\int_0^{2\pi} \cos nx\,dx = 0$, 5. $\int_0^{2\pi} \sin^2 nx\,dx = \pi$

6. $\int_0^{2\pi} \cos^2 nx\,dx = 0$, 7. $\int_0^{2\pi} \sin nx \sin mx\,dx = 0$, 8. $\int_0^{2\pi} \cos nx \cos mx\,dx = 0$

9. $\int_0^{2\pi} \sin nx \cos mx\,dx = 0$. 10. $\int_0^{2\pi} \sin nx \cos nx\,dx = 0$

11. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$. 12. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

13. $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where $v_1 = \int v\,dx$, $v_2 = \int v_1\,dx \ \ldots$ and $u' = \frac{du}{dx}$, $u'' = \frac{du'}{dx}$ ---

Fourier Series :— Let the function $f(x)$ (in desired rage) can be expressed in form

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots a_n\cos nx + \cdots b_1\sin x + b_2\sin 2x + \cdots b_n\sin nx + \cdots \quad (1)$$

or $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad (2)$$

The above series is known as Fourier series and $a_0, a_1, a_2, \ldots a_n \cdots$ $b_1, b_2 \cdots b_n \cdots$ are called Fourer coefficients.

Determination of Fourier Coefficients or Euler's Formulae :—

① **To find a_0** :— Integrate both sides of equation ① between the limit 0 to 2π, we get

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1\int_0^{2\pi} \cos x\,dx + a_2\int_0^{2\pi}\cos 2x\,dx + \cdots\cdots b_1\int_0^{2\pi}\sin x\,dx + b_2\int_0^{2\pi}\sin 2x\,dx + \cdots$$

$$= \frac{a_0}{2}\left[x\right]_0^{2\pi} = a_0\pi$$

or $$\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx}$$

② **To find a_n** :— Multiply both sides by $\cos nx$ in equation ① and integrate between the limit 0 to 2π, we get

$$\int_0^{2\pi} f(x)\cos nx\,dx = a_n\int_0^{2\pi}\cos^2 nx\,dx = a_n\pi$$

or $$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx}$$

③ **To find b_n** :— Multiply both sides by $\sin nx$ in equation ① and integrate between the limit 0 to 2π, we get

$$\int_0^{2\pi} f(x)\sin nx\,dx = b_n\int_0^{2\pi}\sin^2 nx\,dx = b_n\pi$$

or $$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx}$$

Dirichlet Condition :— If a function $f(x)$ defined on interval $(-\pi, \pi)$ is said to satisfy Dirichlet condition if

① $f(x)$ is periodic, single valued and bounded.

② $f(x)$ has a finite number of finite discontinuities in any one period.

③ $f(x)$ has a finite number of maxima and minima in a defined interval.

Note :— If a function $f(x)$ is discontinuous at a point $x = c$, then

$$f(c) = \frac{f(c+0) + f(c-0)}{2}$$

Que (1) :- Find the Fourier series of $f(x) = x$, $0 < x < 2\pi$.

Sol :- Let the Fourier Series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ —— (1)

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$ —— (2)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos 2n\pi}{n^2} - 0 - \frac{1}{n^2}\right] = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0 \quad (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\right] = \frac{-2}{n}$ —— (4), hence from eqⁿ (1)

$$f(x) = x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

Ans

Que (2) :- Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Sol :- Let the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ —— (1)

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{1}{\pi}(1 - e^{-2\pi})$ —— (2)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\cdot\frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]_0^{2\pi}$$

$$= \frac{1 - e^{-2\pi}}{\pi(1+n^2)} \quad (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx = \frac{1}{\pi}\cdot\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)_0^{2\pi}$$

$$= \frac{n(1 - e^{-2\pi})}{\pi(1+n^2)} \quad (4)$$

Hence from equation (1)

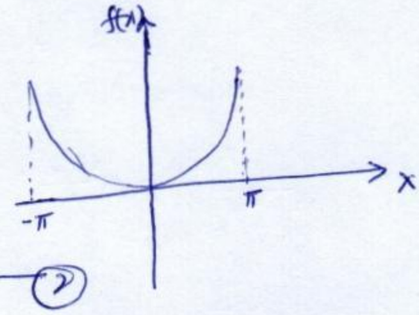
$$f(x) = e^{-x} = \frac{1 - e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\left\{\frac{(1-e^{-2\pi})}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx\right\}$$

$$= \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left\{\frac{\cos nx}{1+n^2} + \frac{n\sin nx}{1+n^2}\right\}\right]$$

Ans

Que ③ :- Obtain the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$. Sketch the graph of $f(x)$ and show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Sol :- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— ①



Since $f(x) = x^2$ is even function, hence $b_n = 0$.

Now $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$ —— ②

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\left(\frac{\cos n\pi}{n^2}\right)\right] = \frac{4}{n^2}(-1)^n \text{ —— ③}$$

$$\therefore f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right] \text{ —— ④}$$

Now putting $x = 0$ in equation ④, we have

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) \Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Proved

Que ④ :- Obtain the Fourier series for $f(x) = \frac{1}{4}(\pi - x)^2$ in interval $0 \le x \le 2\pi$. Hence obtain the relations — (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$. (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol :- Let the Fourier Series $f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— ①

$$\therefore a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4}(\pi - x)^2 = \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\, dx = \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) + 2(\pi - x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi} = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \sin nx\, dx = \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) + 2(\pi - x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} = 0$$

$$\therefore f(x) = \frac{1}{4}(\pi - x)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots \text{ —— ②}$$

In ② putting $x = 0$, we get $\frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ —— ③

In ② putting $x = \pi$, we get $0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$ —— ④

Adding ③ & ④ we get

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Proved

Que 3 Find the Fourier series for $f(x) = x - x^2$ in the interval $-\pi < x < \pi$ and deduce that

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ (ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Sol:- Let the Fourier series $f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— (1)

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= 0 - \frac{2}{\pi}\int_0^{\pi} x^2\,dx = -\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = -\frac{2\pi^2}{3} \quad \text{—— (2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= 0 - \frac{2}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = -\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[2\pi \frac{\cos n\pi}{n^2}\right] = \frac{-4(-1)^n}{n^2} \quad \text{—— (3)}$$

$$\therefore a_1 = \frac{4}{1^2},\ a_2 = \frac{-4}{2^2},\ a_3 = \frac{4}{3^2},\ a_4 = \frac{-4}{4^2} \cdots \text{ etc.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \frac{-2(-1)^n}{n} \quad \text{—— (4)}$$

$$\therefore b_1 = \frac{2}{1},\ b_2 = \frac{-2}{2},\ b_3 = \frac{2}{3},\ b_4 = \frac{-2}{4} \ \cdots \text{ etc.}$$

Hence from eqn (1)

$$x - x^2 = -\frac{\pi^2}{3} + 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right) \quad \text{—— (5)}$$

putting $x = 0$ in equation (5) we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

or $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Now again putting $x = \pi$ and $x = -\pi$ in equation (5), we get

$$\pi - \pi^2 = \frac{-\pi^2}{3} - 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) \quad \text{—— (6)}$$

and $$-\pi - \pi^2 = \frac{-\pi^2}{3} - 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) \quad \text{—— (7)}$$

again adding equation (6) and (7) we get

$$-2\pi^2 = \frac{-2\pi^2}{3} - 8\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Proved

Que (6) :- Expand $f(x) = x\sin x$, $0 < x < 2\pi$ as a Fourier series.

Sol :- Let the Fourier series $f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ —— (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\Big[x(-\cos x) + \sin x\Big]_0^{2\pi} = -2 \quad \text{—— (2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\big[\sin(n+1)x - \sin(n-1)x\big]dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\cdot\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, \quad n \neq 1 \quad \text{—— (3)}$$

Now $a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{2\pi} = -\frac{1}{2}$ —— (4)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\sin nx)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx\sin x)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n-1)x - \cos(n+1)x\}dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)}{n+1}\right\} - \left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] \quad \text{—— (5)}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0, \quad \text{but } n \neq 1$$

Now $b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos 2x\,dx$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\left(\frac{\sin 2x}{2}\right) + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\cdot 2\pi^2 - \frac{1}{2\pi}\left[\frac{1}{4} - \frac{1}{4}\right] = \pi \quad \text{—— (6)}$$

Hence from equation (1)

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2\cos nx}{n^2-1}$$

Ans

Que 7 :- Expand $f(x) = x \sin x$, $-\pi < x < \pi$ as a Fourier series. Hence deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi - 2}{4} .$$

Hint :- Given function is even function, hence $b_n = 0$. Now find a_0, a_n & a_1 according to question number 6. we get

$$a_0 = 2 \quad , \quad a_n = \frac{2(-1)^{n-1}}{n^2-1} \text{ and } a_1 = -\frac{1}{2}$$

Now putting $x = \frac{\pi}{2}$, we get the result.

Que 8 :- Prove that in the interval $-\pi < x < \pi$

(i) $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2-1} \sin x$

(ii) $x\left(\frac{\pi^2 - x^2}{12}\right) = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + - - -$

Que 9 :- Find the Fourier series of $f(x) = \begin{cases} x & , 0 \le x \le \pi \\ 2\pi - x & , \pi \le x \le 2\pi \end{cases}$. and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - - = \frac{\pi^2}{8}$$

Sol :- Let the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ——— (1)

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right] = \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right] = \pi$ ——— (2)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right] + \frac{1}{\pi n^2}\left[-1 + (-1)^n\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & , \text{if } n \text{ is even} \\ \frac{-4}{\pi n^2} & , \text{if } n \text{ is odd} \end{cases}$$ ——— (3)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi\right] + \frac{1}{\pi}\left[0 + \frac{\pi}{n}\cos n\pi\right] = 0$$ ——— (4)

hence from equation (1), we get

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + - - -\right)$$ ——— (5)

Now putting $x = 0$ in equation (5), hence $f(0) = 0$.

$$\therefore \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - -\right)$$

or $\quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - -$

Proved

Que (10) :- Find the Fourier series of the function $f(x)=\begin{cases}1 & , \text{ for } 0\le x\le\pi \\ 2 & , \text{ for } \pi\le x\le 2\pi\end{cases}$

and deduce that $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+ \cdots$

Sol :- Let the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ——— (1)

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} 1\cdot dx + \int_{\pi}^{2\pi} 2\cdot dx\right] = \frac{1}{\pi}[x]_0^{\pi} + \frac{1}{\pi}[2x]_{\pi}^{2\pi} = 3$ (2)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx + 2\int_{\pi}^{2\pi}\cos nx\,dx\right] = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$

$$= 0 \quad \text{——— (3)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx + 2\int_{\pi}^{2\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi} = \frac{1}{\pi n}\left[-(-1)^n + 1 - 2 + 2(-1)^n\right] = \frac{1}{\pi n}\left[(-1)^n - 1\right]$$

$$= \begin{cases}0 & , \text{ when } n \text{ is even} \\ \frac{-2}{\pi n} & , \text{ when } n \text{ is odd}\end{cases} \quad \text{——— (4)}$$

Hence from equation (1), we get

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \quad \text{——— (5)}$$

Now putting $x = \frac{\pi}{2}$ in equation (5), we get

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{3}{2} - \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$$

Ans

Que (11) :- Find the Fourier series to represent the function $f(x) = \begin{cases}-k & , -\pi < x < 0 \\ k & , 0 < x < \pi\end{cases}$

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

Hint :- $a_0 = 0$, $a_n = 0$ and $b_n = \frac{2k}{\pi n}\left[1-(-1)^n\right] = \begin{cases}0 & , \text{ when } n \text{ is even} \\ \frac{4k}{\pi n} & , \text{ when } n \text{ is odd}\end{cases}$

$$\therefore f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Now putting $x = \frac{\pi}{2}$, we get $f\left(\frac{\pi}{2}\right) = k$, hence we find the result.

Que (12) :- Find the Fourier expansion for $f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$, hence deduce that

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol :- Let the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— (1)

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx \right]$

$$= \frac{-\pi}{\pi}\left[x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = -\left[0+\pi\right] + \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$ —— (2)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\, dx + \int_0^{\pi} x \cos nx\, dx\right]$$

$$= -\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & , \text{if } n \text{ is even} \\ \frac{-2}{\pi n^2} & , \text{if } n \text{ is odd} \end{cases}$$ —— (3)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\, dx + \int_0^{\pi} x \sin nx\, dx\right]$$

$$= \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{n}\left[1-(-1)^n\right] + \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}(-1)^n}{n}\right] = \frac{1}{n}\left[1 - 2(-1)^n\right] = \begin{cases} \frac{3}{n} & , \text{when } n \text{ is odd} \\ -\frac{1}{n} & , \text{when } n \text{ is even} \end{cases}$$ —— (4)

$\therefore$ from equation (1), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$
$$- \left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \cdots\right)$$ —— (5)

Now putting $x = 0$ in equation (6)

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$ —— (6)

Since $f(x)$ is discontinuous at $x=0$, hence $f(0) = \frac{f(0+0) + f(0-0)}{2} = \frac{0-\pi}{2} = \frac{-\pi}{2}$

$$\therefore \; -\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

or $$-\frac{\pi}{4} = -\frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

or $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Proved

Que (13) :- Find the Fourier series of $f(x) = \begin{cases} x , & -\pi < x < 0 \\ -x , & 0 < x < \pi \end{cases}$, hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Hint :- Same as question number (12). Here $a_0 = -\pi$, $a_n = \begin{cases} 0 , & n \text{ is even} \\ \frac{4}{\pi n^2} , & n \text{ is odd} \end{cases}$ and $b_n = 0$

Hence Fourier Series $f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right)$

Now put $x = 0$ in above series and $f(0) = \frac{f(0+0) + f(0-0)}{2} = 0$, we get the result.

Que (14) :- Obtain the Fourier series for the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi} , & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} , & 0 < x < \pi \end{cases}$

hence prove that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

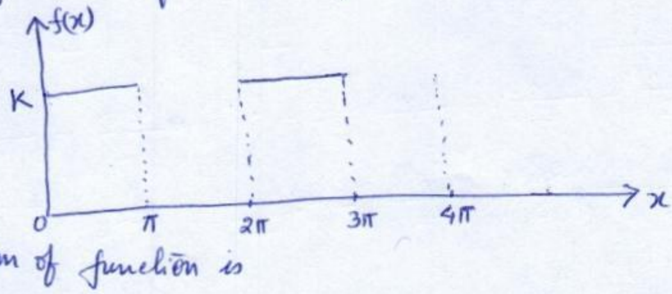
Hint :- Here $a_0 = 0$, $a_n = \begin{cases} 0 , & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2} , & \text{if } n \text{ is odd} \end{cases}$ and $b_n = 0$, hence Fourier Series

$$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right)$$

Now putting $x = 0$, in above equation and $f(0) = \frac{f(0+0)+f(0-0)}{2} = \frac{1+1}{2} = 1$

hence we get the result.

Que (15) :- Obtain the Fourier series for the square waveform as shown in the following figure :-



Hint :- The given figure in the form of function is

$$f(x) = \begin{cases} K , & 0 < x < \pi \\ 0 , & \pi < x < 2\pi \end{cases}$$

Here $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} K\,dx + \int_{\pi}^{2\pi} 0\,dx \right] = \frac{K}{\pi}\left[x\right]_0^{\pi} = K$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{K}{\pi}\int_0^{\pi} \cos nx\,dx = \frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{K}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{-K}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-K}{\pi n}\left[(-1)^n - 1\right] = \begin{cases} 0 , & \text{even} \\ \frac{2K}{\pi n} , & \text{odd} \end{cases}$$

Hence the Fourier series is

$$f(x) = \frac{K}{2} + \frac{2K}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right)$$

Ans

Functions Having Arbitrary Period OR Change of Interval $(0, 2c)$ or $(-c, c)$:—

Let the function $f(x)$ is defined in the interval $(0, 2c)$, that is time period is $2c$. Now we want to change the function into the period 2π.

$\because$ $2c$ is the time period for the variable x

$\therefore$ 1 " " " , $= \frac{x}{2c}$

$\therefore$ 2π " " " , $= \frac{x}{2c} \cdot 2\pi = \frac{\pi x}{c} = z$ or $x = \frac{cz}{\pi}$

Hence the function $f(x)$ having period $2c$ is changed into $f\left(\frac{cz}{\pi}\right)$ or $F(z)$ of period 2π. Hence by Fourier series $F(z)$ can be expaned

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where $a_n = \frac{1}{\pi} \int_0^{2\pi} F(z) \cos nz \, dz$

$= \frac{1}{\pi} \int_0^{2\pi} f\left(\frac{cz}{\pi}\right) \cos nz \, dz$, put $z = \frac{\pi x}{c}$ $\therefore$ $dz = \frac{\pi}{c} dx$

$$\boxed{a_n = \frac{1}{c} \int_0^{2c} f(x) \cos \frac{n\pi x}{c} \, dx}$$

Similarly $\boxed{a_0 = \frac{1}{c} \int_0^{2c} f(x) \, dx}$ and $\boxed{b_n = \frac{1}{c} \int_0^{2c} f(x) \sin \frac{n\pi x}{c} \, dx}$

Que (16) :— Find the Fourier series for the function $f(x) = \frac{\pi - x}{2}$, where $0 < x < 2$.

Sol :— Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c}\right]$ —— (1)

here $c = 1$, hence

$$a_0 = \frac{1}{c} \int_0^{2c} f(x) dx = \int_0^{2} \left(\frac{\pi - x}{2}\right) dx = \frac{1}{2}\left[\pi x - \frac{x^2}{2}\right]_0^2 = \frac{1}{2}(2\pi - 2) = \pi - 1 \quad \text{—— (2)}$$

$$a_n = \frac{1}{c} \int_0^{2c} f(x) \cos \frac{n\pi x}{c} dx = \int_0^{2} \left(\frac{\pi - x}{2}\right) \cos n\pi x \, dx = \frac{1}{2}\left[(\pi - x) \frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_0^2$$

$$= \frac{1}{2n^2\pi^2}\left[-\cos 2n\pi + \cos 0\right] = 0 \quad \text{—— (3)}$$

$$b_n = \frac{1}{c} \int_0^{2c} f(x) \sin \frac{n\pi x}{c} dx = \int_0^{2} \left(\frac{\pi - x}{2}\right) \sin n\pi x \, dx = \frac{1}{2}\left[(\pi - x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^2$$

$$= \frac{-1}{2n\pi}\left[(\pi - 2) - \pi\right] = \frac{1}{n\pi} \quad \text{—— (4)}$$

Hence from equation (1), we get

$$f(x) = \frac{\pi - x}{2} = \frac{\pi - 1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x$$

Ans

Que (17) :- Obtain Fourier series for the function $f(x) = \begin{cases} \pi x & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2 \end{cases}$

Sol :- Let Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right]$ —— (1)

Here $c = 1$, hence

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \text{—— (2)}$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[x\left(\frac{\sin n\pi x}{n\pi}\right) + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\frac{\cos n\pi - 1}{n^2\pi^2}\right] + \pi\left[-\frac{\cos 2n\pi + \cos n\pi}{n^2\pi^2}\right]$$

$$= \frac{\pi}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} -\frac{4}{\pi n^2} & , \text{if } n \text{ is odd} \\ 0 & , \text{if } n \text{ is even} \end{cases} \quad \text{—— (3)}$$

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi}\right] + \pi\left[\frac{\cos n\pi}{n\pi}\right] = 0 \quad \text{—— (4)}$$

hence from equation (1), we get

$$f(x) = \frac{\pi}{2} - \frac{\pi}{4}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right] \quad \text{Ans}$$

Que (18) :- Find the Fourier series in $(0, 2)$ of the function $f(x) = 4 - x^2$.

Sol :- Here $c = 1$, hence the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$ —— (1)

where $a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx = \int_0^2 (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}$ —— (2)

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx = \int_0^2 (4-x^2)\cos n\pi x\,dx$$

$$= \left[(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2 = -\frac{4}{n^2\pi^2} \quad \text{—— (3)}$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx = \int_0^2 (4-x^2)\sin n\pi x\,dx$$

$$= \left[(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2 = \frac{4}{n\pi} \quad \text{—— (4)}$$

$\therefore$ from eqn (1) $f(x) = 4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$ Ans

Que (19) :- Find the Fourier series for $f(x) = |x|$, $-2 < x < 2$.

Sol :- By definition of modulus $f(x) = |x| = \begin{cases} -x , & -2 < x < 0 \\ x , & 0 < x < 2 \end{cases}$

Here $c = 2$, let the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$ ——— (1)

$$\therefore a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0}(-x)\,dx + \int_{0}^{2} x\,dx\right] = -\frac{1}{2}\left[\frac{x^2}{2}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_{0}^{2} = 2$$

Since $f(x)$ is even function, hence $b_n = 0$ and also we can find a_0 by the following method also ——— (2)

$$\therefore a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \int_{0}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{0}^{2} = 2$$

Similarly $a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$

$$= \left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2 = \frac{4}{\pi^2 n^2}\left[(-1)^n - 1\right] = \begin{cases} -\frac{8}{\pi^2 n^2} , & \text{if } n \text{ is odd} \\ 0 , & \text{if } n \text{ is even} \end{cases}$$ ——— (3)

Hence from equation (1), we get

$$f(x) = |x| = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right]$$ Ans

Que (20) :- Find the Fourier series for the function $f(x) = \begin{cases} 0 , & \text{when } -2 < x < -1 \\ K , & \text{when } -1 < x < 1 \\ 0 , & \text{when } 1 < x < 2 \end{cases}$

Sol :- Here $c = 2$, hence let the Fourier Series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$ — (1)

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{-1} 0\,dx + \int_{-1}^{1} K\,dx + \int_{1}^{2} 0\,dx\right]$$

$$= \frac{1}{2}\int_{-1}^{1} K\,dx = K\int_{0}^{1} dx = K$$ ——— (2)

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_{-1}^{1} K\cos\frac{n\pi x}{2}\,dx = K\int_{0}^{1}\cos\frac{n\pi x}{2}\,dx$$

$$= K\left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^1 = \frac{2K}{n\pi}\sin\frac{n\pi}{2}$$ ——— (3)

putting $n = 1, 2, 3, \cdots$ in equation (3), we get

$a_1 = \frac{2K}{\pi}$, $a_2 = 0$, $a_3 = \frac{-2K}{3\pi}$, $a_4 = 0$, $a_5 = \frac{2K}{5\pi}$ . .

Similarly $b_n = \frac{1}{2}\int_{-1}^{1} K\sin\frac{n\pi x}{2}\,dx = 0$ ——— (4)

Hence from equation (1), we get

$$f(x) = \frac{K}{2} + \frac{2K}{\pi}\left[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \cdots\right]$$ Ans

Half Range Series :—

Time Period $T=\pi$ $(0<x<\pi)$	Time Period $T=c$ $(0<x<c)$
Half Range Cosine Series :— $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$ and $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$	**Half Range Cosine Series :—** $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{c}$ where $a_0=\frac{2}{c}\int_0^{c}f(x)\,dx$ and $a_n=\frac{2}{c}\int_0^{c}f(x)\cos\frac{n\pi x}{c}\,dx$
Half Range Sine Series :— $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$ where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$	**Half Range Sine Series :—** $f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{c}$ where $b_n=\frac{2}{c}\int_0^{c}f(x)\sin\frac{n\pi x}{c}\,dx$

Que (21) :- Obtain half range cosine and sine series for $f(x)=x$ in $0<x<\pi$, hence deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$$

Sol :- (i) Let half range cosine series $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ —— (1)

where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi$ —— (2)

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)+\frac{\cos nx}{n^2}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]=\begin{cases}\frac{-4}{\pi n^2}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases} \quad —— (3)$$

Hence from equation (1), we get

$$f(x)=x=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right) \quad —— (4)$$

Now putting $x=0$ in equation (4), we get

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right)$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$$ Ans

(ii) Let half range sine series $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$ —— (1)

where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)+\frac{\sin nx}{n^2}\right]_0^{\pi}$

$$=-\frac{2}{n}(-1)^n \quad —— (2)$$

$$\therefore f(x)=x=2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right)$$ Ans

Que (22):- Expand $\pi x - x^2$ in a half range sine series in the interval $(0,\pi)$ upto the first three terms.

Sol:- Let half range Sine Series $f(x) = \pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$ ——— (1)

where $b_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left[1-(-1)^n\right] = \begin{cases} \frac{8}{\pi n^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even}\end{cases} \quad \text{——— (2)}$$

Hence from equation (1), we get

$$f(x) = \pi x - x^2 = \frac{8}{\pi}\left(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right)$$

Ans

Que (23):- If $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$, Show that

(i) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right)$ (ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \cdots\right)$

Sol:- (i) Let half range sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ——— (1)

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{-\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2} \quad \text{——— (2)}$$

hence from equation (1) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right)$

(ii) Let half range cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ——— (1)

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{8}\right) + \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{3\pi^2}{8}\right] = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \quad \text{——— (2)}$$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{1}{n^2}\cos n\pi - \cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$a_n = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right] \quad \text{——— (3)}$$

$\therefore a_1 = 0,\ a_2 = \frac{-2}{\pi \cdot 1^2},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{-2}{\pi \cdot 3^2}$, hence from (1)

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \cdots\right]$$

Ans

Que (24) :- Find a series of cosines of multiples of x which will represent $x \sin x$ in the interval $(0, \pi)$ and show that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}$.

Sol:- Let half range cosine series $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ —— (1)

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx = 2$, Same as question no. 6.

and $a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx = \frac{2(-1)^{n+1}}{n^2-1}$ and $a_1 = -\frac{1}{2}$.

Now putting $x = \frac{\pi}{2}$, we get the result.

Que (25) :- Expand $f(x) = x$, $0 < x < 2$ as a half range (i) Since series (ii) Cosine series.

Sol :- (i) Let half range sine series $f(x) = x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$ —— (1)

Here $\boxed{c = 2}$, Now

$$b_n = \frac{2}{c}\int_0^{c} f(x) \sin \frac{n\pi x}{c} dx = \frac{2}{2}\int_0^{2} x \sin \frac{n\pi x}{2} dx = \left[x\left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= -\frac{4}{\pi n}\cos n\pi = \frac{-4(-1)^n}{\pi n} \quad \text{—— (2)}$$

$\therefore$ from (1) $\quad f(x) = x = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}$ Ans

(ii) Let half range cosine series $f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$ —— (1)

where $a_0 = \frac{2}{c}\int_0^{c} f(x) dx = \frac{2}{2}\int_0^{2} x \, dx = \left[\frac{x^2}{2}\right]_0^2 = 2$ —— (2)

$$a_n = \frac{2}{c}\int_0^{c} f(x) \cos \frac{n\pi x}{c} dx = \frac{2}{2}\int_0^{2} x \cos \frac{n\pi x}{2} dx = \left[x\left(\frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \frac{4}{\pi^2 n^2}(\cos n\pi - 1) = \frac{4}{\pi^2 n^2}\left[(-1)^n - 1\right] \quad \text{—— (3)}$$

$\therefore \quad f(x) = x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos \frac{n\pi x}{2}$ Ans

Que (26) :- Find half range sine series for e^x in $0 < x < 1$.

Sol:- Let half range sine series $f(x) = e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x$ —— (1), here $\boxed{c = 1}$

$$\therefore b_n = \frac{2}{1}\int_0^{1} e^x \sin n\pi x \, dx = 2\left[\frac{e^x}{1+(n\pi)^2}\left(\sin n\pi x - n\pi \cos n\pi x\right)\right]_0^1$$

$$= \frac{2}{1+n^2\pi^2}\left[e^1 \cdot (\sin n\pi - n\pi \cos n\pi) - e^0(0 - n\pi)\right] = \frac{2n\pi}{1+n^2\pi^2}\left[1 - e(-1)^n\right] \quad \text{—— (2)}$$

hence from equation (1).

$$f(x) = e^x = 2\pi \sum_{n=1}^{\infty} \frac{n\left[1 - e(-1)^n\right]}{1+n^2\pi^2} \sin n\pi x$$ Ans

Que (27) :- Develop $\sin\frac{\pi x}{l}$ in half range cosine series in the range $0<x<l$.

Sol :- Here $c = l$, hence half range cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ —— (1)

where $a_0 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{l}{\pi}\cos\frac{\pi x}{l}\right]_0^l = -\frac{2}{\pi}\left[\cos\pi - \cos 0\right] = \frac{4}{\pi}$ —— (2)

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^l \sin\frac{(n+1)\pi x}{l}\,dx - \frac{1}{l}\int_0^l \sin\frac{(n-1)\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[-\frac{l}{(n+1)\pi}\cos\frac{(n+1)\pi x}{l}\right]_0^l + \frac{1}{l}\left[\frac{l}{(n-1)\pi}\cos\frac{(n-1)\pi x}{l}\right]_0^l$$

$$= \frac{1}{(n+1)\pi}\left[-(-1)^{n+1}+1\right] + \frac{1}{(n-1)\pi}\left[(-1)^{n-1}-1\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$ —— (3)

If n is odd then $a_n = \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$

If n is even then $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{4}{\pi(n-1)(n+1)}$$ —— (4)

but $n \neq 1$.

Now $a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\,dx = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,dx = \frac{1}{l}\left[\frac{-l}{2\pi}\cos\frac{2\pi x}{l}\right]_0^l$

$$= \frac{1}{2\pi}\left(-\cos 2\pi + 1\right) = 0$$ —— (5)

$\therefore$ from (1) $f(x) = \sin\frac{\pi x}{l} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1.3} + \frac{\cos\frac{4\pi x}{l}}{3.5} + \cdots\right]$ Ans

Que (28) :- Find the Fourier half range cosine series of $f(t) = \begin{cases} 2t, & 0<t<1 \\ 2(2-t), & 1<t<2 \end{cases}$

Sol :- Here $c = 2$, Let $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$ —— (1)

$$a_0 = \frac{2}{c}\int_0^c f(t)\,dt = \frac{2}{2}\int_0^2 f(t)\,dt = \int_0^1 2t\,dt + \int_1^2 2(2-t)\,dt = \left[t^2\right]_0^1 + \left[4t - t^2\right]_1^2 = 2$$ —— (2)

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{c}\,dt = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}\,dt = \int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= 2\left[t\left(\frac{\sin\frac{n\pi t}{2}}{\frac{n\pi}{2}}\right) + \frac{\cos\frac{n\pi t}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^1 + 2\left[(2-t)\frac{\sin\frac{n\pi t}{2}}{\frac{n\pi}{2}} - \frac{\cos\frac{n\pi t}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_1^2$$

$$= \frac{8}{\pi^2 n^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$ —— (3)

$\therefore$ from (1)

$$f(t) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)\cos\frac{n\pi t}{2}$$

Ans