Engineering Mathematics-I
KAS-103.
Module IV

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Multiple Integrals

Lecture-46

Content: Change of variables in double Integrals (Cartesian to Polar)

Change of variables in double Integrals:- The evaluation of double integral can be simplified by suitable change of variables.

Convert from Cartesian Co-ordinates to Polar co-ordinates

Let the variables x, y in the function $f(x,y)$.

put $x = r\cos\theta$, $y = r\sin\theta$.

such that $x^2+y^2=r^2$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

Where J denotes Jacobian

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} \Rightarrow \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\Rightarrow r.$$

$$\boxed{\text{Hence } \iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Area: $A = \iint_R dx\,dy = \iint_{R'} r\,dr\,d\theta$.

Working process:- To evaluate double integrals into Polar co-ordinates.

1) Find limits for r and θ in region R.
2) Change x & y (Cartesian co-ordinates) into polar.

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co-ordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$

3. Substitute limits for r and θ, now in form of r and θ and $dx\,dy = r\,dr\,d\theta$

4. Evaluate.

Q.1 Evaluate area of circle $x^2+y^2=4$ by changing into polar co-ordinates.

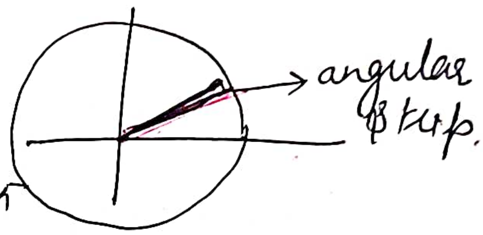
Sol. We know Area $= \iint_R dx\,dy$ — R is region $x^2+y^2=4$, $r=2$

To convert into polar co-ordinates.

$x = 2\cos\theta$

$y = 2\sin\theta$

$|J| = r$ substituting.



θ varies from 0 to 2π

r varies from

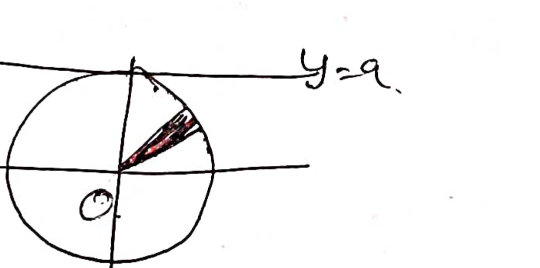
$r^2 = 4 \Rightarrow 0 \le r \le 2$.

$$\int_0^{2\pi}\int_0^{2} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 = \int_0^{2\pi} 2\,d\theta = \left[2\theta\right]_0^{2\pi} = 4\pi$$

Ans

Q2 Evaluate by changing into polar co-ordinates

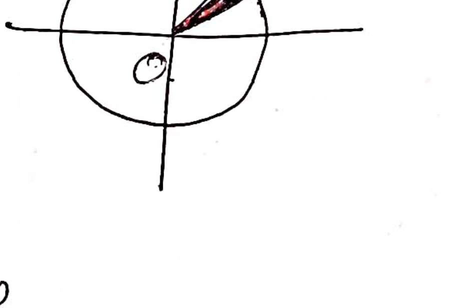
$$\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy.$$



$0 \le y \le a$

$0 \le x \le \sqrt{a^2-y^2}$

$\Rightarrow x^2+y^2=a^2$



$r^2 = a^2 \Rightarrow r = a$.

$0 \le r \le a$.

θ varies from 0 to $\pi/2$.

$$I = \int_0^{\pi/2}\int_0^{a} r^2\sin^2\theta \; r \times r \; dr\,d\theta$$

Multiple Integrals — Unit IV

function $f(x,y,z)$

... of region

$$\int r^4 \sin^2\theta \, dr \, d\theta = \int_0^{\pi/2} \left[\sin^2\theta \frac{r^5}{5}\right]_0^3 d\theta = \frac{3^5}{5}\int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$\frac{3^5}{5} \cdot \frac{\Gamma\frac{2+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{2+0+2}{2}} \Rightarrow \frac{3^5}{10} \cdot \frac{\frac{1}{2}\pi}{1\cdot 2} \Rightarrow \frac{\pi 3^5}{20}$$

Ans. using $\beta(m,n)$ relation.

2019 Q. Evaluate $\iint_R \sqrt{x^2+y^2}\,dx\,dy$ along the region bounded by $x^2+y^2=4$, $x^2+y^2=9$.

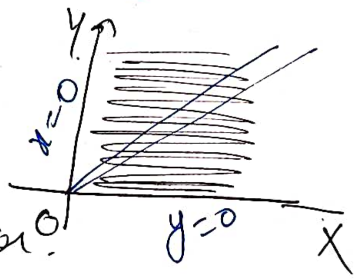
2014. Q. Evaluate $\int_0^{9}\int_0^{\sqrt{29x-x^2}} (x^2+y^2)\,dx\,dy$ by changing into polar co-ordinates.

2010, 03. Q. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing into polar co-ordinates. Hence show that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Sol. given region is entire XY plane.

$0 \le x \le \infty$

$0 \le y \le \infty$



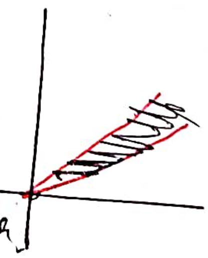
converting into polar co-or.

$0 \le x^2+y^2 \le \infty$

$0 \le r^2 \le \infty \Rightarrow 0 \le r \le \infty$

angular strip varies from 0 to ∞

& θ varies from 0 to $\pi/2$ [I quadrant]



Hence

$$I = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta =$$

$r^2 = t$

$r = \sqrt{t}$

$dr = \frac{1}{2\sqrt{t}}dt$

Integrals: Consider a function which is continuous ...

$$\int_0^{\pi/2}\int_0^{\infty} \frac{e^{-t}}{2\sqrt{t}} \sqrt{t}\, dt\, d\theta = \int_0^{\pi/2} \frac{1}{2}\, d\theta =$$

Let $I_1 = \int_0^{\infty} e^{-x^2} dx$ —(1)

$I_1 = \int_0^{\infty} e^{-y^2} dy$ —(2) Multiplying.

$$I_1^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy$$

$$I_1^2 = \frac{\pi}{4}$$

$$\boxed{I_1 = \frac{\sqrt{\pi}}{2}}$$ Hence proved.

Change of variables from cartesian to another variables.

Let $\iint_R f(x,y)\, dx\, dy$ be given integral. Let integral to be changed to u, v by relation $x = \phi(u,v)$
$y = \psi(u,v)$

then $dx\,dy$ will be replaced by $|J|\, du\, dv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}.$$

* $|J|$ = Modulus of Jacob

Hence $\iint_R f(x,y)\, dx\, dy = \iint_{R'} g(u,v)\,|J|\, du\, dv$.

2007, 14.

Q. Evaluate $\iint_R (x+y)^2\, dx\, dy$, R is region bounded by parell-o-gram $x+y=0$, $x+y=2$, $3x-2y=0$, $3x-2y=3$.

Let $x+y=u$.
$3x-2y=v$.

$$\therefore \iint_R (x+y)^2\,dx\,dy = \iint_{R'} f(u,v)\,|J|\,du\,dv$$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}.$$

$x+y=u$

$3x-2y=v.$

$$\begin{array}{r} 3x+3y=3u \\ -3x \mp 2y = -v \\ \hline y=\frac{1}{5}(3u-v) \end{array} \qquad \begin{array}{r} -2x-2y=-2u \\ +3x-2y=-v \\ \hline x=\frac{2u+v}{5} \end{array}$$

$J = -\frac{1}{5}$ $\quad |J| = \frac{1}{5}$

Regions is $u=0$, $u=2$, $v=0$, $v=3$

Hence $I = \int_0^3 \int_0^2 u^2 \frac{1}{5}\,du\,dv$

$= \frac{8}{5}$ Ans.

Problems for practice.

1) Evaluate. $\iint_R (x-y)^4 e^{x+y}\,dx\,dy$ over region R bounded by square with vertices at $(1,0)$, $(2,1)$ $(1,2)$ and $(0,1)$

2) Prove that area in positive quadrant bounded by $y^2=4ax$; $y^2=4bx$, $xy=c^2$, $xy=d^2$ is $\frac{1}{3}(d^2-c^2)\log\left(\frac{b}{a}\right)$ $d>c$, $b>a$.

2008 3). evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing into polar co-ordinates

2006. 4) Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the positive quadrant of circle $x^2+y^2=1$