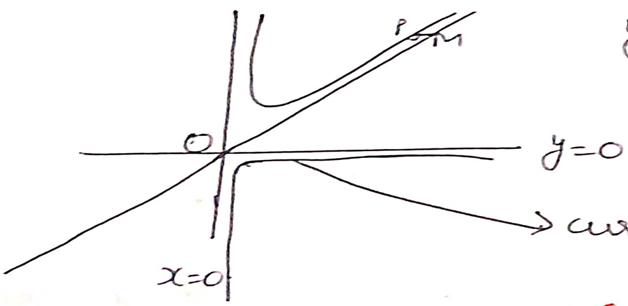
Lecture-28

Unit-I
Differential Calculus I

Contents: Asymptotes and Curve Tracing in Cartesian Form

Asymptote: An asymptote is a line or a curve that approaches a given curve arbitrary closely. An asymptote of a curve is a line such that distance between curve and the line approaches zero as they tend to infinity:-



$y=x$ is asymptote. as $PM \to 0$
$\perp$ distance from curve to line.

$x=0$ & $y=0$ are two asymptotes for curve-II

Asymptote ∥ to axes of co-ordinates

To find asymptotes ∥ to y-axis of curve $y=f(x)$.

Working Process:

1. Equate to zero the coefficient of highest power of y, present in the eqn.
2. Now factorise (if possible)

Remark:- If coefficient of highest power of y is either a constant or not resolvable into real linear factors, then there are no asymptotes ∥ to y axis

Q.1 find asymptote ∥ to y axis $x^2y^2 = a^2 \dots$

Sol. $x^2y^2 - a^2x^2 - a^2y^2 = 0$

$(x^2 - a^2)y^2 - a^2x^2 = 0.$

for asymptotes ∥ to y axis equating to zero coeff of highest power of y

$x^2 - a^2 = 0$

$\Rightarrow (x+a)(x-a) = 0$

$\boxed{\begin{array}{l} x = a \\ x = -a \end{array}}$ are asymptotes ∥ to y axis.

Q.2 find Asymptote ∥ to y axis $\frac{a^2}{x} + \frac{b^2}{y} = 1$

$a^2y + b^2x - xy = 0$

$(a^2 - x)y + b^2x = 0.$

$a^2 - x = 0$

$a^2 = x$

Hence $x = a^2$ is required asymptote ∥ to y-axis.

Method-II

Asymptote ∥ to x-axis.

working process:-

1. Equating to zero the co-efficient of highest power of x, present in the given eqn.
2. Resolve into factors (linear & real)

Remark:- If coefficient of highest power of x is either a constant or not resolvable into real factor. then there are no asymptotes ∥ to x-axis

Q.1 $x^2y^2 = a^2(x^2 + y^2)$

$x^2y^2 - a^2x^2 - a^2y^2 = 0$

$(y^2 - a^2)x^2 - a^2y^2 = 0$

$\boxed{y = \pm a}$

$y^3 - xy^2 = x^2 + 1$. Coeff of $x^2 = -1 \neq 0$

$y^3 - xy^2 - x^2 - 1 = 0$ ∴ No asymptote ∥ to x axis

$y = x(x-2)(x-3)$ Coeff of $y^3 = 1 \neq 0$

∴ No asymptote ∥ to y axis

Q. $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1.$, Q. $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1.$

Oblique Asymptote:-

Remark:- Each curve has no. of asymptotes = deg. of curve.

- for oblique asymptotes working process:-

1. find asymptotes ∥ to x & y axis
2. Let $y = mx + c$ is oblique asymptote to curve $f(x,y) = 0$.
3. put $y = m$, $x = 1$ in highest power terms of x & y. $\phi_n(m) = 0$ Let $m_1, m_2, \ldots m_n$ be the roots.
4. put $y = m$, $x = 1$ in $(n-1)^{th}$ degree term. $\phi_{n-1}(m) = 0$.

$$\Rightarrow c\phi'_n + \phi_{n-1} = 0$$

$$c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$$ for distinct values of m calculate value of c.

5. If two roots of m are equal. $y = m$, $x = 1$, next lowest degree term. $\phi_{m-2}(m) = ?$

$$\frac{c^2}{2!}\phi''_n(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi^0_{(n-2)}(m) = 0. \Rightarrow c = ?$$

6. Substitute these values in line $y = mx + c$ are required Asymptotes.

Q.1 $x^3+2x^2y-xy^2-2y^3+xy-y^2=1$ find all asym- plotes of the curve.

we have

Sol. $x^3+2x^2y-xy^2-2y^3+xy-y^2-1=0$. eqn has 3 Asymp.

|| to x-axis. No asymptote.

|| to y-axis No asymptote

For oblique Asymptote.

$$\phi_3(m) = 1+2m-m^2-2m^3=0$$
$$(1+2m)-m^2(1+2m)=0$$
$$(2m+1)(1-m^2)=0$$
$$m=-1/2,\quad m=\pm 1.$$

Now $\phi_2(m) = m-m^2$

(3-1-2 degree) 2 degree term x=1 y=m

$$c\phi_3'(m)+\phi_2(m)=0$$

$$c=\frac{-\phi_2(m)}{\phi_3'(m)} = \frac{m^2-m}{-6m^2-2m+2}.$$

At $m=1$, $c=0$

$m=-1$ $c=-1$

$m=-1/2$ $c=+1/2$

$y=x$

$y=-x-1$

$y=-\frac{1}{2}x+\frac{1}{2}$

oblique Asymptotes.

Q.2 find all asymptotes of following $x^3+3x^2y-4y^3-x+y+3=0$

Sol. No asymptotes || to x & y axis

for oblique Asymptotes.

$$\phi_3(m) = 1+3m-4m^3=0.$$

1	-4	0	3	1
	↓	-4	-4	-1
	-4	-4	-1	0

$$(m-1)(-4m^2-4m-1)=0.$$
$$m=1,\quad m=-\frac{1}{2},-\frac{1}{2}.$$

$\phi_2(m)=0$ ∵ no terms present of deg. 2.

$c\phi_3'(m)+\phi_2(m)=0$ for $m=1$ non-repeated root

$c=0$

$y=mx+c$

$\boxed{y=x}$