Que 1: a) $\int_0^1 \int_1^2 xy(1+x+y)\,dy\,dx$

$$\int_0^1 \int_1^2 (xy + x^2y + xy^2)\,dy\,dx$$

$$\int_0^1 \left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_1^2 dx$$

$$\int_0^1 \left[\left(\frac{4x}{2} + \frac{4x^2}{2} + \frac{8x}{3}\right) - \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x}{3}\right)\right] dx$$

$$\int_0^1 \left[\frac{3x}{2} + \frac{3x^2}{2} + \frac{7x}{3}\right] dx$$

$$\frac{3}{2}\int_0^1 \frac{x^2}{2} + \frac{3}{2}\int_0^1 \frac{x^3}{3} + \frac{7}{3}\int_0^1 \frac{x^2}{2}\,dx$$

$$\frac{3}{2}\times\frac{1}{2} + \frac{3}{2}\times\frac{1}{3} + \frac{7}{3}\times\frac{1}{2} = \frac{3}{4} + \frac{3}{6} + \frac{7}{6}$$

$$= \frac{3}{4} + \frac{\cancel{10}^{\,5}}{\cancel{6}\,3} = \frac{3}{4} + \frac{5}{3} = \frac{9+20}{12} = \frac{29}{12}$$

(b) $$\int_1^a \int_1^b \frac{dy\,dx}{xy} = \int_1^a \frac{1}{x}\left[\int_1^b \frac{dy}{y}\,dy\right] dx$$

$$\int_1^a \frac{1}{x}\left[\log y\right]_1^b dx = \log b \int_1^a \frac{1}{x}\,dx$$

$$= \log b \cdot \log a \text{ Ans}$$

(c) $$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_0^1 \left[x e^{y/x}\right]_0^{x^2} dx$$

$$\int_0^1 \left[x e^{x^2/x} - x e^{0/x}\right] dx = \int_0^1 (x e^x - x)\, dx$$

$$\int_0^1 \underset{D\ I}{x e^x}\, dx - \int_0^1 x\, dx = \left[x e^x - e^x - \frac{x^2}{2}\right]_0^1$$

$$= \left(e - e - \frac{1}{2}\right) - (0 - 1 - 0)$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$ Ans

c) $$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_0^1 \left[x e^{y/x} \right]_0^{x^2} dx$$

$$= \int_0^1 x \left[e^{x^2/x} - e^0 \right] dx = \int_0^1 x\left(e^x - 1\right) dx$$

$$= \int_0^1 x e^x\, dx - \int_0^1 x\, dx = \left[x e^x - e^x \right]_0^1 - \frac{1}{2}\left[x^2 \right]_0^1$$

$$= \left((e - e) - (0 - 1)\right) - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

2. $$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{(1+x^2+y^2)} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{(\sqrt{1+x^2})^2 + y^2}$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4}\, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

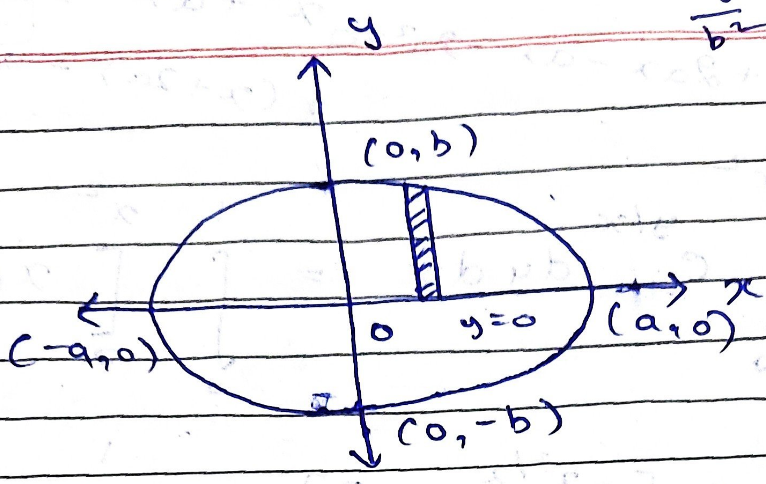
$$= \frac{\pi}{4} \left[\log\left\{x + \sqrt{1+x^2}\right\}\right]_0^1 = \frac{\pi}{4} \log(1+\sqrt{2})$$

3. Evaluate the integrals over the area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y^2 = \frac{b^2}{a^2}(a^2 - x$ $\qquad \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$

PAGE No.
DATE

3.



Limits of integration $y = 0$ to $y = \frac{b}{a}\sqrt{a^2-x^2}$

$x = 0$ to $x = a$

$$I = 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2 \, dy\, dx$$

$$= 4\int_0^a \frac{1}{3}\Big[(x+y)^3\Big]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{4}{3}\int_0^a \left(x + \frac{b}{a}\sqrt{a^2-x^2}\right)^3 dx$$

$$- \frac{4}{3}\int_0^a x^3\, dx$$

Let $I_1 = \frac{4}{3}\int_0^a \left(x + \frac{b}{a}\sqrt{a^2-x^2}\right)^3 dx$

Put $x = a\sin\theta$ then $x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \pi/2$

✓The Good Paper

and $dx = a\cos\theta \, d\theta$ then

$$I_1 = \frac{4}{3}\int_0^{\pi/2} \left(a\sin\theta + \frac{b}{\cancel{a}} \cdot \cancel{a}\cos\theta \right)^3 \cdot a\cos\theta \, d\theta$$

$$= \frac{4a}{3}\int_0^{\pi/2} (a\sin\theta + b\cos\theta)^3 \cos\theta \, d\theta$$

$$= \frac{4}{3}a\int_0^{\pi/2} \left(a^3\sin^3\theta + b^3\cos^3\theta + 3a^2b\sin^2\theta\cos\theta + 3ab^2\sin\theta\cos^2\theta\right)\cos\theta \, d\theta$$

$$= \frac{4a}{3}\int_0^{\pi/2} a^3\sin^3\theta\cos\theta \, d\theta + \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta \, d\theta$$

$$+ 4a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta \, d\theta + \frac{4}{3}a^2b^2\int_0^{\pi/2}\sin\theta\cos^3\theta \, d\theta$$

Using $\displaystyle\int_0^{\pi/2}\sin^m\theta\cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

$$= \frac{4}{3}a^4\cdot\frac{\Gamma_2\cdot\Gamma_1}{2\Gamma_3} + \frac{4}{3}ab^3\cdot\frac{\Gamma_{1/2}\cdot\Gamma_{5/2}}{2\Gamma_3} + 4a^3b\cdot\frac{\Gamma_{3/2}\cdot\Gamma_{3/2}}{2\Gamma_3}$$

$$+ 4a^2b^2\cdot\frac{\Gamma_{\cancel{3/2}}\cdot\Gamma_2}{2\Gamma_3}$$

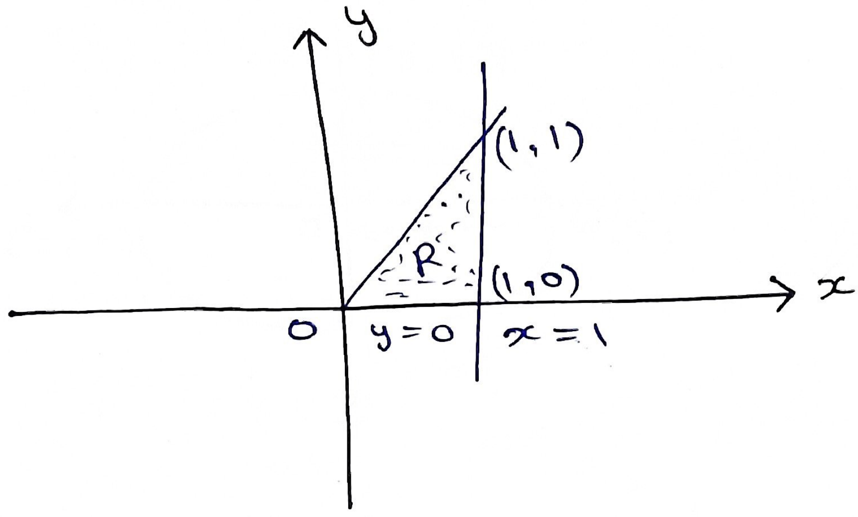
$$= \frac{a^4}{3} + \frac{ab^3}{\cancel{3}}(\sqrt{\pi})\cdot\frac{\cancel{3}}{4}(\sqrt{\pi}) + a^3b\cdot\frac{\pi}{4} + a^2b^2\cdot\cancel{\sqrt{\pi}}$$

Now $I = I_1 + I_2$

$$I = I_1 - \frac{1}{3} a^4$$

$$I_1 = \frac{a^4}{3} + ab^3\left(\frac{\pi}{4}\right) + a^3b\left(\frac{\pi}{4}\right) + a^2b^2$$

4. $y=0$, $y=x$ and $x=1$



Limits of integration: $x=0$ to $x=1$ and $y=0$ to $y=x$ then

$$I = \int_0^1 \int_0^x \sqrt{4x^2-y^2}\, dy\, dx$$

$$= \int_0^1 \left[\frac{y}{2}\sqrt{4x^2-y^2} + \frac{2x}{2}\sin^{-1}\left(\frac{y}{2x}\right) \right]_0^x dx$$

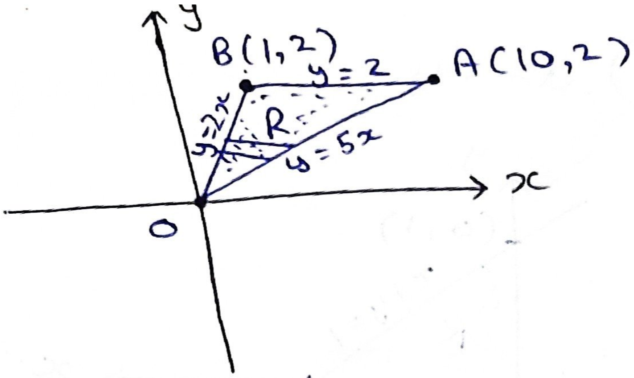
$$= \int_0^1 \left(\frac{x}{2}\sqrt{3x^2} + x \cdot \frac{\pi}{6} \right) dx$$

$$= \frac{\sqrt{3}}{2}\int_0^1 x^2\, dx + \frac{\pi}{6}\int_0^1 x\, dx$$

$$= \frac{\sqrt{3}}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{\pi}{6}\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{\sqrt{3}}{6} + \frac{\pi}{12} = \frac{1}{3}\left[\frac{\sqrt{3}}{2} + \frac{\pi}{4}\right]$$

5.



Limits of integration: $y = 0$ to $y = 2$ and

$x = \frac{y}{2}$ to $x = \frac{y}{5}$

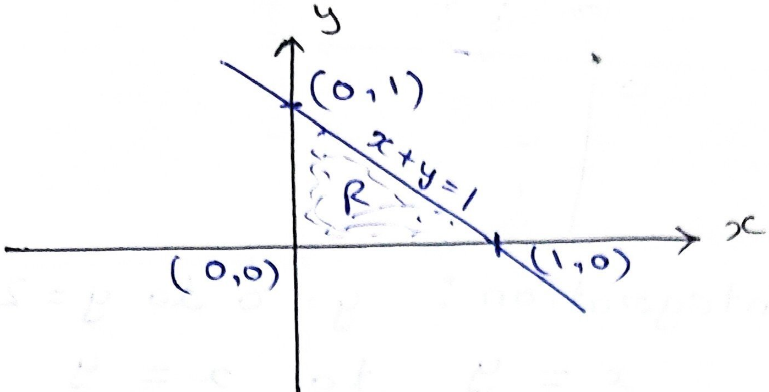
$$\int_{y=0}^{y=2} \int_{x=\frac{y}{2}}^{x=\frac{y}{5}} \sqrt{xy - y^2}\, dx\, dy$$

$$= \int_{y=0}^{y=2} \frac{1}{y} \left[\frac{2}{3} (xy - y^2)^{3/2} \right]_{\frac{y}{2}}^{y/5} dy$$

$$= \int_{y=0}^{y=2} \frac{1}{y} \left[\frac{2}{3} \left\{ \left[\frac{y^2}{5} - y^2 \right]^{3/2} - \left(\frac{y^2}{2} - y^2 \right)^{3/2} \right\} \right] dy$$

$$= \int_{y=0}^{y=2} \frac{1}{y} \left[\frac{2}{3} \left(-\frac{4y^2}{5} \right)^{3/2} - \left(\frac{-y^2}{2} \right)^{3/2} \right] dy$$

6. $\iint (x^2+y^2)\,dx\,dy$



Limits of integration : $y = 0$ to $y = 1$

and $x = 0$ to $x = 1-y$

$$= \int_0^1 \int_0^{1-y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y} dy$$

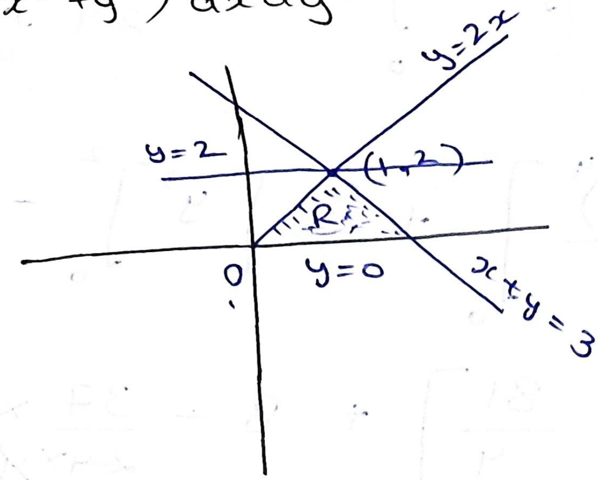
$$= -\frac{1}{3}\int_0^1 (1-y)^3\,dy + \int_0^1 y^2(1-y)\,dy$$

$$= -\frac{1}{3}\left[\frac{(1-y)^4}{4}\right]_0^1 + \left[\int_0^1 y^2\,dy - \int_0^1 y^3\,dy\right]$$

$$= \frac{1}{12} + \frac{1}{3} - \frac{1}{4} = \frac{1+4-3}{12}$$

$$= \frac{2}{12} = \frac{1}{6} \text{ Ans}$$

7. $\iint (x^2+y^2)\,dx\,dy$



Limits of integration $y=0$ to $y=2$
and $x=\frac{y}{2}$ to $x=3-y$

$$I=\int_{y=0}^{y=2}\int_{\frac{y}{2}}^{3-y}(x^2+y^2)\,dx\,dy$$

$$I=\int_{y=0}^{y=2}\left[\frac{x^3}{3}+xy^2\right]_{\frac{y}{2}}^{3-y}dy$$

$$I=\int_{y=0}^{y=2}\left[\left(\frac{(3-y)^3}{3}+y^2(3-y)\right)-\left(\frac{y^3}{24}+\frac{y^3}{2}\right)\right]$$

$$I=\int_{y=0}^{y=2}\left[\frac{(3-y)^3}{3}+3y^2-y^3-\frac{y^3}{24}-\frac{y^3}{2}\right]$$

$$I=\int_{y=0}^{y=2}\left[\frac{(3-y)^3}{3}+3y^2-\frac{37y^3}{24}\right]dy$$

$$I = \frac{1}{3}\int_{y=0}^{2} -(3-y)^3\,dy + 3\int_0^2 y^2\,dy - \frac{37}{24}\int_0^2 y^3\,dy$$

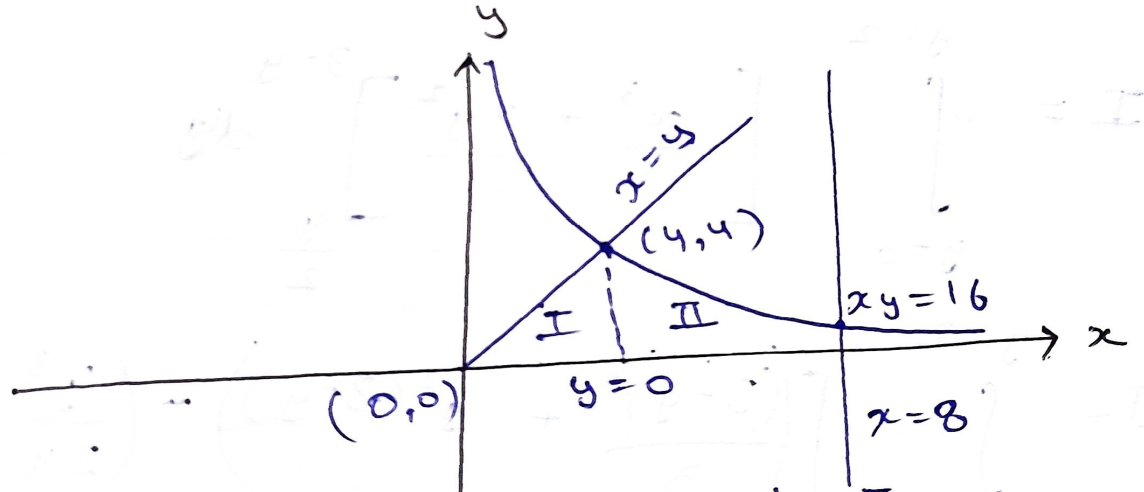
$$I = -\frac{1}{3}\left[\frac{(3-y)^4}{4}\right]_0^2 + \frac{3}{3}\left[y^3\right]_0^2 - \frac{37}{24\times 4}\left[y^4\right]_0^2$$

$$I = -\frac{1}{3}\left[\frac{1}{4} - \frac{81}{4}\right] + 8 - \frac{37\times \cancel{4}\times \cancel{4}}{\cancel{24}_{6}\times \cancel{4}}$$

$$= \frac{20}{3} + 8 - \frac{37}{6} = \frac{40+48-37}{6}$$

$$= \frac{51}{6} \text{ Ans}$$

8.



Part I

Limits of integration

$x = 0$ to $x = 4$

$y = 0$ to $y = x$

Part II

Limits of integration

$x = 4$ to $x = 8$

$y = 0$ to $y = \frac{16}{x}$

$$I_1 = \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dy\,dx = \int_{x=0}^{4} x^2 [y]_0^x\,dx$$

$$= \int_0^4 x^3\,dx = \left[\frac{x^4}{4}\right]_0^4 = 4^3 = 64 \quad \text{--- (1)}$$

$$I_2 = \int_{x=0}^{4}\int_{y=0}^{16/x} x^2\,dy\,dx = \cancel{x^2}\int_{x=0}^{4} x^2 [y]_0^{16/x}\,dx$$

$$= \int_0^4 x^2 \times \frac{16}{x}\,dx = 16\int_{x=0}^{4} x\,dx = 16\times 4^4$$

$$I_1 + I_2 = 64 \times 64 + 64$$

Hence $I = 65 \times 64 = 4160$

9. $$\int_0^{\pi/2}\int_0^{a\cos\theta} r\left(\sqrt{a^2 - r^2}\right)dr\,d\theta$$

$$-\frac{1}{2}\int_0^{\pi/2}\int_0^{a\cos\theta}\left(-2r\sqrt{a^2-r^2}\,dr\right)d\theta$$

$$= -\frac{1}{2}\times\frac{2}{3}\int^{\pi/2}\left[(a^2 - r^2)^{\frac{3}{2}}\right]^{a\cos\theta} d\theta$$

$$\cancel{= \frac{-1}{3}\int_0^{\pi/2} a^2\cos^2\theta}$$

$$= \frac{-1}{3}\int_0^{\pi/2} \left[\left(a^2 - a^2\cos^2\theta\right)^{3/2} - \left(a^2\right)^{3/2}\right] d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2} \left[\left(a^2(\sin^2\theta)\right)^{3/2} - a^3\right] d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2} \left(a^3\sin^3\theta - a^3\right) d\theta$$

$$= \frac{-a^3}{3}\int_0^{\pi/2} \left(\sin^3\theta - 1\right) d\theta$$

$$= \frac{-a^3}{3}\int_0^{\pi/2} \sin^3\theta\, d\theta + \frac{a^3}{3}\int_0^{\pi/2} d\theta$$

$$= \frac{-a^3}{3}\int_0^{\pi/2} \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right) d\theta + \frac{a^3}{3}\left(\frac{\pi}{2}\right)$$

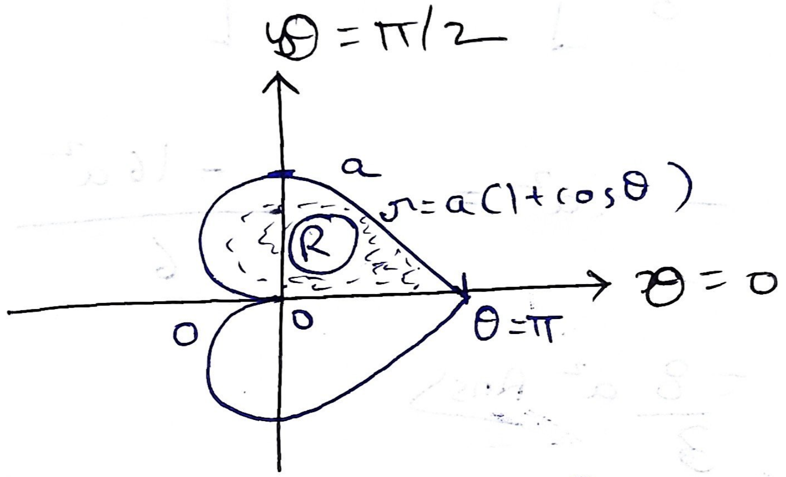
$$= \frac{-a^3}{\cancel{3}}\int_0^{\pi/2} \frac{\cancel{3}}{4}\sin\theta\, d\theta + \frac{a^3}{12}\int_0^{\pi/2} \sin 3\theta\, d\theta + \frac{\pi}{2}\left(\frac{a^3}{3}\right)$$

$$= \frac{+a^3}{4}\left[\cos\theta\right]_0^{\pi/2} - \frac{a^3}{36}\left[\cos 3\theta\right]_0^{\pi/2} + \frac{a^3\pi}{6}$$

$$= -\frac{a^3}{4} + \frac{a^3}{36} + \frac{a^3}{6}\pi = \frac{a^3}{6}\pi - \frac{8a^3}{36}$$

$$= \frac{a^3\pi}{6} - \frac{2a^3}{9} = \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) \text{ Ans}$$

10. $\iint r\sin\theta \, dr\, d\theta$



Limits of integration $\theta = 0$ to $\theta = \pi$
and $r = 2a$ to $r = a(1+\cos\theta)$

$$\int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=2a}^{a(1+\cos\theta)} r\,dr\right] d\theta = \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_{2a}^{a(1+\cos\theta)} d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\frac{a^2(1+\cos\theta)^2}{2} - \frac{4a^2}{2}\right] d\theta$$

$$= \int_{\theta=0}^{\pi} a^2 \sin\theta \left[\frac{(1+\cos\theta)^2}{2} - 2\right] d\theta$$

$$-\frac{a^2}{2}\int_0^{\pi} -\sin\theta\,(1+\cos\theta)^2\,d\theta \;-\; 2a^2\int_0^{\pi}\sin\theta\,d\theta$$

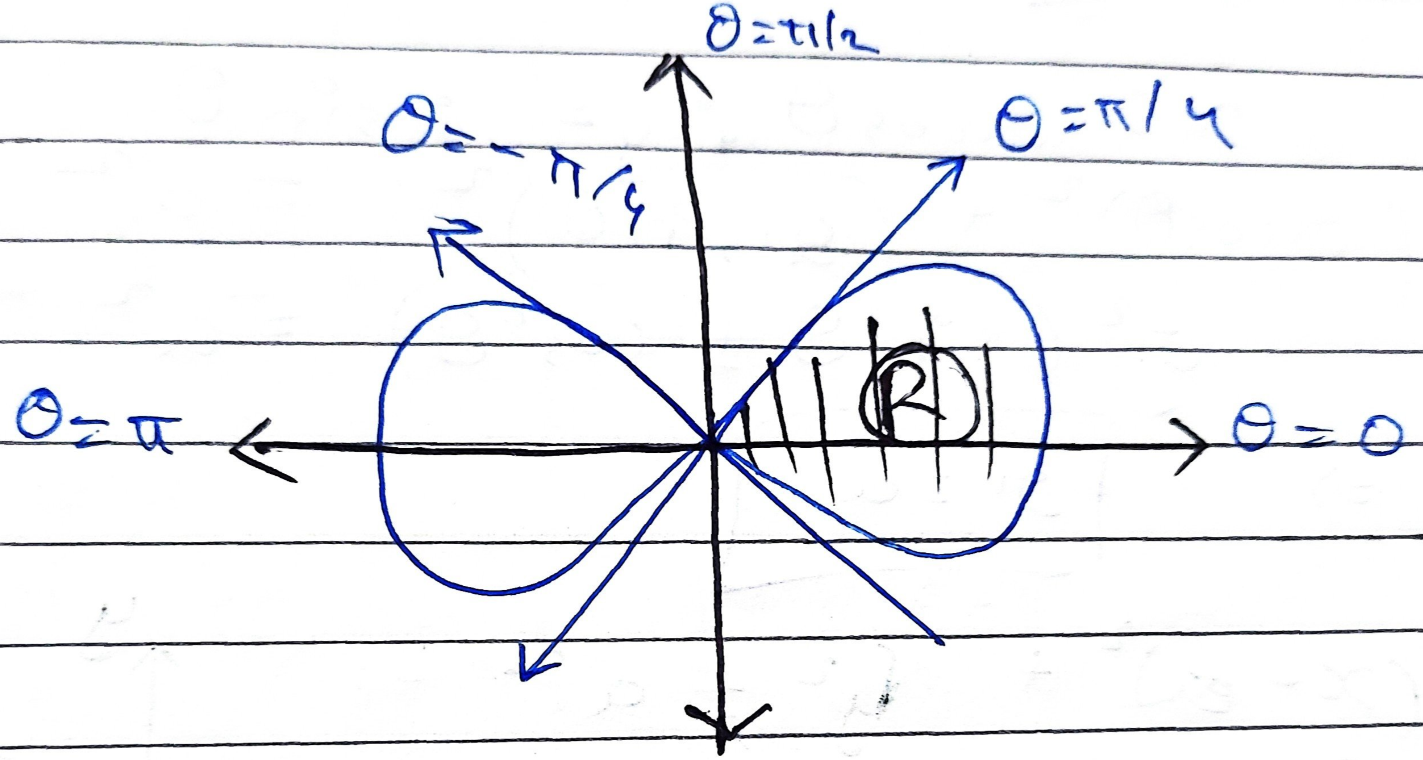
$$-\frac{a^2}{2}\left[\frac{(1+\cos\theta)^3}{3}\right]_0^{\pi} + 2a^2\left[\cos\theta\right]_0^{\pi}$$

$$-\frac{a^2}{2}\left[\frac{0}{3}-\frac{8}{3}\right] + 2a^2\left[-1-1\right]$$

$$\frac{8a^2}{6} - 4a^2 = \frac{-16a^2}{6} = \frac{-8a^3}{3}$$

$$= -\frac{8}{3}a^2 \ \underline{\text{Ans}}$$

Q1.

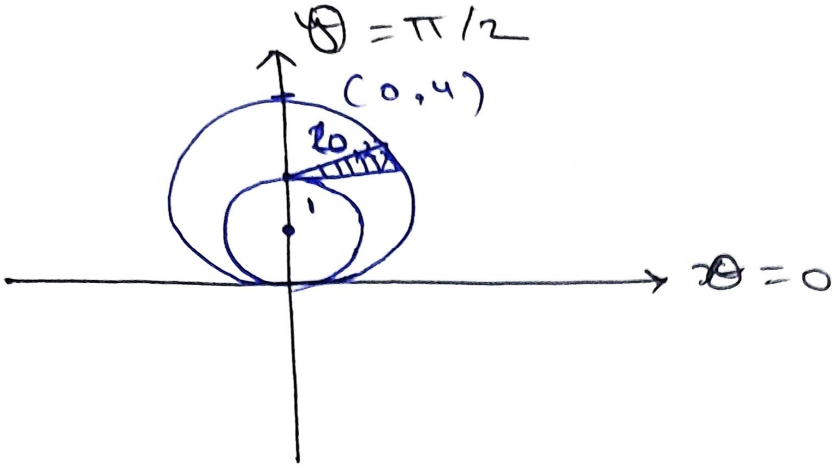


$$\theta = -\frac{\pi}{4} \text{ to } \theta = \frac{\pi}{4}$$

$$r = 0 \text{ to } r = a\sqrt{\cos 2\theta}$$

$$-\frac{1}{2} \int_{\theta = \pi/4}^{\theta = \pi/4} \left[\int_0^{a\sqrt{\cos 2\theta}} (-2r)(a^2 - r^2)^{-1/2} \, dr \right]$$

12. $\iint r^3 \, dr \, d\theta$



Limits of integration : $\theta = 0$ to $\theta = \pi$

$r = 2\sin\theta$ to $r = 4\sin\theta$

$$I = \int_0^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta \ni$$

$$I = \frac{1}{4} \int_0^{\pi} \left(256 \sin^4\theta - 16 \sin^4\theta \right) d\theta$$

$$= \frac{240}{4} \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta \quad \because \sin^4\theta \text{ is an even function}$$

$$= 120 \int_0^{\pi/2} \sin^4\theta \, d\theta = \overset{60}{\cancel{120}} \cdot \frac{\Gamma_{5/2} \cdot \Gamma_{1/2}}{\cancel{2} \, \Gamma_3}$$

$\cancel{60}^{15}$: $\dfrac{3}{\cancel{2}} \cdot \dfrac{1}{\cancel{2}} \cdot \dfrac{\Gamma_{1/2} \cdot \Gamma_{1/2}}{2}$

$\underline{\dfrac{45}{2} \pi}$ Ans

e $\log y$ e^x

13. $$I = \int_{y=1}^{e} \int_{x=1}^{\log y} \int_{z=1}^{e^x} \log z \, dz \, dx \, dy$$

$$I = \int_{y=1}^{e} \int_{x=1}^{\log y} \left[z \log z - z \right]_{1}^{e^x} dx \, dy$$

$$I = \int_{y=1}^{e} \int_{x=1}^{\log y} \left[(e^x \log e^x - e^x) - (1 \cdot \log 1 - 1) \right]$$

$$I = \int_{1}^{e} \int_{1}^{\log y} (e^x \log e^x - e^x + 1) \, dx \, dy$$

$$I = \int_{1}^{e} \int_{1}^{\log y} (x e^x - e^x + 1) \, dx \, dy$$

$$I = \int_{y=1}^{e} \int_{x=1}^{\log y} \left[x e^x - e^x - e^x + x \right] dxy$$

$$I = \int_{y=1}^{e} \left[\left(\log y \cdot e^{\log y} - 2 e^{\log y} + \log y \right) - \left(1 \cdot e^1 - 2e^1 + 1 \right) \right] dy$$

$$I = \int_{1}^{e} \left(y \log y - 2y + \log y + e - 1 \right) dy$$

PAGE No.
DATE

14. $$I=\int_{a=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)\,dz\,dy\,dx$$

$$I=\int_{x=0}^{a}\int_{y=0}^{a-x}\left[xz+yz+\frac{z^2}{2}\right]_0^{a-x-y}dy\,dx$$

$$I=\int_{x=0}^{a}\int_{y=0}^{a-x}\left[x(a-x-y)+y(a-x-y)+\frac{(a-x-y)^2}{2}\right]dy\,dx$$

$$I=\int_{0}^{a}\int_{0}^{a-x}\left(ax-x^2-xy+ay-xy-y^2+\frac{a^2+x^2+y^2-2ax+2xy-2ay}{2}\right)dy\,dx\ \cancel{dx\,dy}$$

$$I=\int_{0}^{a}\int_{0}^{a-x}\left(\cancel{ax}-\cancel{ax}-\frac{x^2}{2}-\cancel{xy}+\cancel{xy}+\cancel{ay}-\cancel{ay}-xy-\frac{y^2}{2}+\frac{a^2}{2}\right)dy\,dx\ \cancel{dx\,dy}$$

$$I=\int_{x=0}^{a}\int_{y=0}^{a-x}\left(\frac{a^2}{2}-\frac{x^2}{2}-\frac{y^2}{2}-xy\right)dy\,dx\ \cancel{dx\,dy}$$

$$I=\frac{1}{2}\int_{0}^{a}\int_{0}^{a-x}\left(a^2-x^2-y^2-2xy\right)dy\,dx$$

PAGE No.
DATE

$$I = \frac{1}{2}\int_0^a \left[a^2y - x^2y - \frac{y^3}{3} - \frac{\cancel{2}xy^2}{\cancel{2}} \right]_0^{a-x} dx$$

$$I = \frac{1}{2}\int_0^a \left[a^2(a-x) - x^2(a-x) - \frac{(a-x)^3}{3} - x(a-x)^2 \right] dx$$

$$I = \frac{1}{2}\int_0^a \left(a^3 - a^2x - ax^2 - x^3 - \frac{(a-x)^3}{3} - x(a^2+x^2-2ax) \right] dx$$

$$I = \frac{1}{2}\int_0^a \left(a^3 - 2a^2x - ax^2 - 2x^3 - \frac{(a-x)^3}{3} \cancel{-x^3} + 2ax \right) dx$$

$$I = \frac{1}{2}\int_0^a (a^3 - 2a^2x)\, dx - a\int_0^a x^2\, dx - 2\int_0^a x^3\, dx - 3\int_0^a (a-x)^3 dx + 2a\int_0^a x\, dx$$

$$I = \frac{1}{2}a^2 \int_0^a (a - 2\cancel{3}x)\, dx - a\left[\frac{x^3}{3}\right]_0^a - \frac{2}{4}\left[x^4\right]_0^a + \frac{3}{4}\left[(a-x)^4\right]_0^a + a\left[x^2\right]_0^a$$

✓The Good Paper

PAGE No.
DATE

$$I = \frac{1}{2}(a-2)\left[\frac{x^2}{2}\right]_0^a - \frac{a^4}{3} - \frac{1}{2}a^4$$

$$+ \frac{3}{4}\left[-a^4\right] + a^3$$

$$I = \frac{1}{4}(a-2)(a^2) - \frac{a^4}{3} - \frac{1}{2}a^4 - \frac{3}{4}a^4 + a^3$$

$$I = \frac{a^3}{4} - \frac{a^2}{2} - \frac{a^4}{3} - \frac{1}{2}a^4 - \frac{3a^4}{4} + a^3$$

$$I = \frac{5a^3}{4} - \frac{13a^4}{12} - \frac{a^2}{2}$$

15-(ii) $I = \int_{z=2}^{3}\int_{y=1}^{2}\int_{x=0}^{1}(x+y+z)\,dx\,dy\,dz$

$$I = \int_{z=2}^{3}\int_{y=1}^{2}\left[\frac{x^2}{2} + xy + xz\right]_0^1 dy\,dz$$

$$I = \int_{z=2}^{3}\int_{y=1}^{2}\left(\frac{1}{2} + y + z\right) dy\,dz$$

$$I = \int_{z=2}^{3}\left[\frac{1}{2}y + \frac{y^2}{2} + zy\right]_1^2 dz$$

$$I = \int_{z=2}^{3} \left[\frac{y}{2} + \frac{y^2}{2} + zy \right]_{1}^{2} dz$$

$$I = \int_{z=2}^{3} \left[\left(1 + 2 + 2z\right) - \left(\frac{1}{2} + \frac{1}{2} + z\right) \right]$$

$$I = \int_{z=2}^{3} \left(3 + 2z - 1 - z\right) dz$$

$$I = \int_{2}^{3} (z + 2)\, dz$$

$$I = \left[\frac{z^2}{2} + 2z \right]_{2}^{3}$$

$$I = \left(\frac{9}{2} + 6\right) - \left(\frac{4}{2} + 4\right)$$

$$= \frac{5}{2} + 2 = \frac{9}{2} \text{ Ans}$$

1(iii) $I = \int_{x=0}^{1} \int_{y=0}^{x^2} \int_{z=0}^{x+y} (x - 2y + z)\, dz\, dy\, dx$

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} \left[xz - 2yz + \frac{z^2}{2} \right]_0^{x+y} dy\, dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} \left(x(x+y) - 2y(x+y) + \frac{(x+y)^2}{2} \right) dy\, dx$$

$$= \int_{x=0}^{1} \left[\frac{x(x+y)^2}{2} \right]_0^{x^2} - 2 \int y(x+y)\, dy + \frac{1}{6} [(x+y)^3]$$

$$= \int_{x=0}^{1} \frac{x(x+x^2)^2}{2} - 2 \int (xy + y^2)\, dy + \frac{1}{6} \left[(x+y)^3 \right]_0^{x^2}$$

PAGE No.
DATE

16. $$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \int_{z=0}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$$

$$I = \int_0^{\pi/2} \int_0^{a\cos\theta} r \, [z]_0^{\sqrt{a^2-r^2}} \, dr \, d\theta$$

$$I = \int_0^{\pi/2} \int_0^{a\cos\theta} \left(r \cdot \sqrt{a^2-r^2} \, dr\right) d\theta$$

$$I = \int_0^{\pi/2} -\frac{1}{2} \int_0^{a\cos\theta} -2r\sqrt{a^2-r^2} \, dr \, d\theta$$

$$I = -\frac{1}{2} \int_0^{\pi/2} \frac{2}{3}\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$I = -\frac{1}{3} \int_0^{\pi/2} \left[\left(a^2(1-\cos^2\theta)\right)^{3/2} - (a^2)^{3/2}\right]$$

$$I = -\frac{1}{3} \int_0^{\pi/2} (a^3 \sin^3\theta - a^3) \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} (1 - \sin^3\theta) \, d\theta$$

PAGE No.
DATE

$$I = \frac{a^3}{3}\int_0^{\pi/2} d\theta - \frac{a^3}{3}\int_0^{\pi/2} \sin^3\theta \, d\theta$$

$$= \frac{a^3}{3}\left(\frac{\pi}{2}\right) - \frac{a^3}{3}\left(\frac{\Gamma_2 \cdot \Gamma_{1/2}}{2\,\Gamma_{5/2}}\right)$$

$$= \frac{a^3\pi}{6} - \frac{a^3}{3}\left(\frac{1}{\cancel{2}} \cdot \frac{\sqrt{\cancel{\pi}}}{\frac{3\cdot 1\cdot\sqrt{\cancel{\pi}}}{\cancel{2}\,2}}\right)$$

$$= \frac{a^3\pi}{6} - \frac{a^3}{3}\cdot\frac{2}{3} = \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) \text{ Ans}$$

17. We have $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

and $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

$$\left[-x^n \cdot e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} \cdot e^{-x} dx$$

$$\Gamma(n+1) = 0 + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

(iii) $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$

$\Gamma(n+1) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1\, \Gamma_1$

$$\boxed{\Gamma(n+1) = n!}$$

18. We have $\Gamma(n) = \int_0^\infty e^{-kx} x^{n-1}\, dx$

Put $kx = y$ then $x = \frac{y}{k}$ and $dx = \frac{dy}{k}$

then $\cancel{\Gamma(n)} = \int_0^\infty e^{-y} \cdot \left(\frac{y}{k}\right)^{n-1} \cdot \frac{dy}{k}$

$$\cancel{\Gamma(n)} = \int_0^\infty e^{-y} \cdot \frac{y^{n-1}}{k^n} \cdot dy$$

$$I = \frac{1}{k^n} \int_0^\infty e^{-y} \cdot y^{n-1}\, dy \Rightarrow \boxed{I = \frac{\Gamma n}{k^n}}$$

∞

19. $I = \int_0^\infty e^{-x^{1/n}} dx$ put

$x^{1/n} = y$ then $\frac{1}{n} \cdot (x)^{1/n-1} dx = dy$

and $\frac{1}{n} \cdot \frac{(x^{1/n})}{x} dx = dy$

$dx = \frac{nx\,dy}{x^{1/n}} \Rightarrow dx = \frac{n \cdot y^n \cdot dy}{y}$

$dx = n y^{n-1} dy$ ---- (2)

$I = \int_0^\infty e^{-y} \cdot n y^{n-1} dy = n \int_0^\infty e^{-y} \cdot y^{n-1} dy$

$= n \Gamma(n)$

✓The Good Paper

To prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

and hence evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

<u>Soln</u>: We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Now $\beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$

Put $y = \frac{1}{1+x}$: $\left.\begin{array}{l} y=0,\ x\to\infty \\ y=1,\ x=0 \end{array}\right]$

$dy = \frac{-1}{(1+x)^2}\,dx$; $1-y = 1-\frac{1}{1+x} = \frac{x}{x+1}$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+x}\right)^{m-1}\left(\frac{x}{1+x}\right)^{n-1}\left(\frac{-1}{(1+x)^2}\right)dx$$

$$= -\int_\infty^0 \frac{x^{n-1}}{(1+x)^{m-1+n-1+2}}\,dx$$

$$= -\int_\infty^0 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\boxed{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx} \quad \because \beta(m,n) = \beta(n,m)$$

$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}} dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

$$= \beta(9,15) - \beta(15,9)$$

$$= 0$$

RELATION BETWEEN β and γ function

→ To proove that $\beta(m,n) = \frac{\gamma_m \gamma_n}{\gamma_{m+n}}$

$\because \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\gamma_n}{k^n}$

$\Rightarrow \gamma_n = k^n \int_0^\infty e^{-kx} x^{n-1} dx$

$$\gamma_n = z^n \int_0^\infty e^{-zx} x^{n-1} dx \text{ ——— (1)}$$

multiplying (1) by $e^{-z} z^{m-1}$, we get

$$\gamma_n (e^{-z} z^{m-1}) = (z^n e^{-z})(z^{m-1}) \int_0^\infty e^{-zx} x^{n-1} dx$$

$$\gamma_n (e^{-z} z^{m-1}) = z^{(m+n)-1} \int_0^\infty e^{-(1+x)z} x^{n-1} dx \text{ ——— (2)}$$

Now integrating both sides w.r.t z from $z=0$ to $z=\infty$

$$\Rightarrow \Gamma_n \times \int_0^\infty e^{-z} z^{m-1}\,dz = \int_0^\infty \left\{ z^{(m+n)-1} \int_0^\infty e^{-(1+x)z} x^{n-1}\,dx \right\} dz$$

$$\Rightarrow \Gamma_n \times \Gamma_m = \int_{x=0}^\infty \left[\int_0^\infty e^{-(1+x)z} z^{(m+n)-1}\,dz \right] x^{n-1}\,dx$$

$$\Gamma_n \times \Gamma_m = \int_0^\infty \frac{\Gamma_{m+n}\, x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\frac{\Gamma_m \Gamma_n}{\Gamma_{m+n}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma_m \Gamma_n}{\Gamma_{m+n}}$$

Note:

Evaluate $\int_0^1 x^{1/2}(1-x)^2\,dx$

$$= \beta\left(\frac{3}{2},3\right) = \frac{\Gamma_{3/2}\,\Gamma_3}{\Gamma_{9/2}}$$

$$= \frac{\cancel{\Gamma_{3/2}} \cdot 2!}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\cancel{\Gamma_{3/2}}} = \frac{16}{105}$$

22. $\beta(l, m) \cdot \beta(l+m, n) \cdot \beta(l+m+n, p)$

$$\beta(l, m) = \frac{\Gamma_l \cdot \Gamma_m}{\Gamma_{l+m}} \quad \text{—} \quad (1)$$

$$\beta(l+m, n) = \frac{\Gamma_{l+m} \cdot \Gamma_n}{\Gamma_{l+m+n}} \quad \text{—} \quad (2)$$

$$\beta(l+m+n, p) = \frac{\Gamma_{l+m+n} \cdot \Gamma_p}{\Gamma_{l+m+n+p}} \quad \text{—} \quad (3)$$

From (1), (2) and (3)

$\beta(l, m) \cdot \beta(l+m, n) \cdot \beta(l+m+n, p)$

$$= \frac{\Gamma_l \cdot \Gamma_m}{\cancel{\Gamma_{l+m}}} \cdot \frac{\cancel{\Gamma_{l+m}} \cdot \Gamma_n}{\cancel{\Gamma_{l+m+n}}} \cdot \frac{\cancel{\Gamma_{l+m+n}} \cdot \Gamma_p}{\Gamma_{l+m+n+p}}$$

$$= \frac{\Gamma_m \cdot \Gamma_n \cdot \Gamma_l \cdot \Gamma_p}{\Gamma_{m+n+l+p}}$$

23: To proove $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$

23. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

$$\beta(m+1,n) = \frac{m\Gamma m \cdot \Gamma n}{\Gamma m+n+1}$$

$$\beta(m,n+1) = \frac{n\Gamma m \cdot \Gamma n}{\Gamma m+n+1}$$

$$\beta(m+1,n) + \beta(m,n+1) = \frac{(m+n)\,\Gamma m \cdot \Gamma n}{\Gamma m+n+1}$$

$$= \frac{\cancel{(m+n)}\,\Gamma m \cdot \Gamma n}{\cancel{(m+n)}\,\Gamma m+n}$$

$$\beta(m+1,n) + \beta(m,n+1) = \beta(m,n)$$

24. We know $\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1}\, dx$

Put $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta\, d\theta$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2} \cdot 2\sin\theta \cdot \cos\theta\, d\theta$$

$$\beta(m,n) = \frac{2}{2}\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

$$\boxed{\frac{1}{2}(\beta(m,n)) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}$$

✓The Good Paper

PAGE No.

DATE

25. $I = \int_0^{\pi/2} \sin^3 x \cdot \cos^{5/2} x \, dx$

We know $\int_0^{\pi/2} \sin^m \theta \cdot \cos^n \theta \, d\theta = \dfrac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

$$\int_0^{\pi/2} \sin^3 \theta \cdot \cos^{5/2} \theta \, d\theta = \frac{\Gamma 2 \cdot \Gamma 7/4}{2\,\Gamma 15/4}$$

$$= \frac{1 \cdot \frac{3}{4} \cdot \cancel{\Gamma 3/4}}{2 \times \frac{11}{4} \cdot \frac{7}{4} \cdot \frac{3}{4} \cdot \cancel{\Gamma 3/4}}$$

$$= \frac{\cancel{3} \times 4 \times 4 \times \cancel{4}}{2 \times \cancel{4} \times 11 \times 7 \times \cancel{3}} = \frac{\overset{8}{\cancel{16}}}{77 \times \cancel{2}} = \frac{8}{77}$$

26. i) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta \, d\theta$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta = \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta \, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right) \quad (\because \beta(m,n) = \beta(n,m))$$

$$\frac{\Gamma 3/4 \cdot \Gamma 1/4}{2 \times \Gamma 1} = \Gamma 3/4 \cdot \Gamma 1/4$$

ta

PAGE No. | DATE

27. $I = \int_0^\infty \frac{x^{n-1}}{(1+x)} dx$ Put $x = \tan^2\theta$ then

$x = 0 \Rightarrow \theta = 0$

$x = \infty \Rightarrow \theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{\tan^{2n-2}\theta \cdot 2\tan\theta \cdot \cancel{\sec^2\theta}\, d\theta}{\cancel{\sec^2\theta}}$$

$$I = 2\int_0^{\pi/2} \tan^{2n-1}\theta\, d\theta$$

$$I = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{-2n+1}\theta\, d\theta$$

$$= \cancel{2} \cdot \frac{\Gamma_{2n} \cdot \Gamma_{1-n}}{\cancel{2} \cdot \Gamma_1} = \Gamma_n \cdot \Gamma_{1-n}$$

$$= \Gamma n \cdot \Gamma(1-n)$$

(ii) $\Gamma(1/4) \cdot \Gamma(3/4) = \Gamma(1/4) \cdot \Gamma(1-1/4)$

$$= \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,\pi$$

(iii) $\Gamma(1/3) \cdot \Gamma(2/3) = \Gamma(1/3) \cdot \Gamma\left(1-\frac{1}{3}\right)$

$$= \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\sqrt{3}\pi}{3}$$

28 (ii) $\int_0^{\pi/2} \sin^n\theta \cdot \cos^{-n}\theta \, d\theta = \frac{\Gamma_{\frac{n+1}{2}} \cdot \Gamma_{\frac{1-n}{2}}}{2 \cdot \Gamma_1}$

$= \frac{\Gamma_{\frac{1+n}{2}} \cdot \Gamma_{\frac{1-n}{2}}}{2} = \frac{1}{2} \frac{\pi}{\sin\left(\frac{1+n}{2}\right)\pi}$

$= \frac{\pi}{2\cos\frac{n\pi}{2}} = \frac{\pi}{2} \sec\frac{n\pi}{2}$

(iii) $\int_0^1 x^5 (1-x^3)^{10} \, dx$

Put $x^3 = t$ then $3x^2 dx = dt$

$= \frac{1}{3} \int_0^1 x^3 \cdot 3x^2 (1-x^3)^{10} \, dx$

$\frac{1}{3} \int_0^1 t \cdot (1-t)^{10} \cdot dt = \beta(2, 11)$

$= \frac{1}{3} \cdot \frac{\Gamma_2 \cdot \Gamma_{11}}{\Gamma_{13}} = \frac{1}{3} \cdot \frac{1 \cdot 10!}{12!}$

$= \frac{1}{3} \cdot \frac{1}{12 \times 11} = \frac{1}{132 \times 3} = \frac{1}{396}$

PAGE NO.

DATE

(iii) $\int_0^2 x(8-x^3)^{1/3}\, dx$

$2\int_0^2 x\left(1-\left(\frac{x}{2}\right)^3\right)^{1/3} dx$

~~Put $\frac{x}{2} = y$ then $dx = 2dy$~~

$\cancel{4\int_0^1 2y \cdot (1-y^3)^{1/3}\, 2dy}$

Put $\left(\frac{x}{2}\right)^3 = y$ the $3\left(\frac{x}{2}\right)^2 \cdot \frac{1}{2}\, dx = dy$

$\Rightarrow dx = \frac{2}{3}\frac{dy}{\left(\frac{x}{2}\right)^2} \Rightarrow dx = \frac{2}{3}\frac{dy}{y^{2/3}}$

Also when $x=0$; $y=0$
$x=2$; $y=1$

$2\int_0^1 2y^{1/3} \cdot (1-y)^{1/3} \cdot \frac{2}{3} y^{-2/3} \cdot dy$

$\frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}\, dy$

$= \frac{1}{4}\left(\frac{\quad}{\sin\frac{\pi}{4}} \right) = \quad 4 \qquad 2\sqrt{2}$

(30.) Duplication formula

To proove that

$$\Gamma_m \Gamma_{m+\frac{1}{2}} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma_{2m} : \text{ where } m \text{ is a +ve integer.}$$

we have

$$\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \, dx = \frac{\Gamma_m \Gamma_n}{2\Gamma_{m+n}}$$

Put $n = \frac{1}{2}$ i.e. $2n-1 = 0$ in (1), we get

$$\int_0^{\pi/2} \sin^{2m-1} x (\cos x)^0 \, dx = \frac{\Gamma_m \Gamma_{1/2}}{2\Gamma_{m+\frac{1}{2}}}$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} x \, dx = \frac{\sqrt{\pi}.\Gamma_m}{2\Gamma_{m+\frac{1}{2}}}$$

$\cancel{\Rightarrow \int_0^{\pi/2} \sin^{2m-1} x \, dx =}$ Again put $n = m$ in (1)

we get

$$\int_0^{\pi/2} \sin^{2m-1} x \cos^{2m-1} x \, dx = \frac{\Gamma_m \Gamma_m}{2\Gamma_{2m}}$$

$$\Rightarrow \int_0^{\pi/2} \left(\frac{2\sin x \cos x}{2} \right)^{2m-1} dx = \frac{(\Gamma_m)^2}{2\Gamma_{2m}}$$

$$\Rightarrow \frac{1}{2^{2m-1}} \int_0^{\pi/2} (\sin 2x)^{2m-1} dx = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

Put $2x = y \Rightarrow dx = \frac{1}{2} dy$

when $x = 0, \quad y = 0$

$x = \frac{\pi}{2}, \quad y = \pi$

$$\Rightarrow \frac{1}{2} \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin y)^{2m-1} dy = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

$$\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \quad : \text{If } f(2a-x) = f(x)$$

$[\because \sin\{\pi - x\} = \sin x]$

$$\Rightarrow \frac{1}{2} \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin y)^{2m-1} dy = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

Using (2), we get

$$\Rightarrow \frac{1}{2^{2m-1}} \times \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma m}{\Gamma m+\frac{1}{2}} = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

$$\Rightarrow \Gamma m \, \Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m \quad \text{Ans}$$

$$2^{1-2m}\,\beta\left(m,\tfrac{1}{2}\right) = \frac{1}{2^{2m-1}}\,\frac{\Gamma m\cdot\Gamma 1/2}{\Gamma m+\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\frac{\Gamma m}{\Gamma m+\frac{1}{2}} = \frac{\sqrt{\pi}}{2^{2m-1}}\times\frac{\Gamma m\cdot\Gamma m}{\Gamma m\;\Gamma m+\frac{1}{2}}$$

$$\because\ \Gamma m\;\Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\Gamma 2m$$

$$= \frac{\sqrt{\pi}}{2^{2m-1}}\times\frac{2^{2m-1}}{\sqrt{\pi}}\;\frac{\Gamma m\;\Gamma m}{\Gamma 2m}$$

$$= \frac{\Gamma m\times\Gamma m}{\Gamma m+n} = \beta(m,m)$$

32. (i) $\int_0^\infty e^{-\sqrt{x}} . x^{1/4} dx$

Put $\sqrt{x} = y$ then $x = y^2$ and

$dx = 2y dy$ --- (1)

$$\int_0^\infty e^{-y} \cdot y^{1/2} . 2y dy = 2\int_0^\infty e^{-y} . y^{3/2} dy$$

$$= 2 . \Gamma(5/2) = \cancel{2} . \frac{3}{\cancel{2}} . \frac{1}{2} . \sqrt{\pi}$$

$$= \frac{3\sqrt{\pi}}{2}$$

(ii) $\int_0^1 \left(\frac{x^3}{(1-x^3)}\right)^{1/3} dx = \int_0^1 \left(x^3 . (1-x^3)^{-1}\right)^{1/3} dx$

$$\int_0^1 x . \left(\frac{1}{1-x^3}\right)^{1/3} dx$$

Put $x^3 = y$ then $x = y^{1/3}$

~~and $3x^2 dx = dy$~~

$$dx = \frac{dy}{3x^2} = \frac{dy}{3y^{2/3}}$$

DATE

$$\frac{1}{3}\int_0 y^{1/3}\cdot(1-y)^{-1/3}\cdot y^{-2/3}\,dy$$

$$\frac{1}{3}\int_0^1 y^{-1/3}(1-y)^{-1/3}\,dy = \frac{1}{3}\beta\left(\frac{2}{3},\frac{2}{3}\right)$$

$$= \frac{\frac{1}{3}\cdot\left(\Gamma\frac{5}{3}\right)^2}{\left(\Gamma 4/3\right)}$$

33.(i) $$\frac{\Gamma 1/3\cdot\Gamma 5/6}{\Gamma 2/3} = \frac{\Gamma 1/3\cdot\Gamma_{\frac{1}{3}+\frac{1}{2}}}{\Gamma_{2/3}}$$

$$= \frac{\frac{\sqrt{\pi}}{2^{\frac{2}{3}-1}}\cdot\cancel{\Gamma(2/3)}}{\cancel{\Gamma(2/3)}}$$

$$= \frac{\sqrt{\pi}}{2^{-1/3}} = 2^{1/3}\cdot\sqrt{\pi}$$

(iii) 34. $$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$$ Put $x = \tan^2\theta$ then $dx = 2\tan\theta\sec^2\theta\cdot d\theta$

and $x=0 \Rightarrow \theta = 0$

$x=\infty \Rightarrow \theta = \pi/2$

$$\int_0^{\pi/2}\frac{\tan^{2p-2}\theta}{\cancel{\sec^2\theta}}\cdot 2\tan\theta\,\cancel{\sec^2\theta}\,d\theta$$

$$= 2\int_0^{\pi/2}\tan^{2p-1}\theta\cdot d\theta$$

DATE

$$2\int_0^{\pi/2} \sin^{2p-1}\theta \cdot \cos^{-(2p-1)}\theta \, d\theta$$

$$= \frac{\cancel{2}\, \Gamma_p \cdot \Gamma_{1-p}}{\cancel{2} \cdot \Gamma_1} = \Gamma_p \cdot \Gamma_{1-p} = \frac{\pi}{\sin p\pi}$$

35. $$\int_0^1 \left(\frac{1}{1+x^4}\right) dx \times \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

Put $x^4 = y$ then $x = y^{1/4}$

and $4x^3 dx = dy \Rightarrow dx = \frac{1}{4}\frac{dx}{y^{3/4}}$

$$I_1 = \int_0^1 \left(\frac{1}{1+x^4}\right) dx = \frac{1}{4}\int_0^1 \frac{y^{-3/4}}{(1+y)} \cdot dy$$

$n = 1/4$ then

$$I_1 = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\,\pi$$

$$I_2 = \int_0^{\pi} \cos^{1/2}\theta \cdot \sin^{-1/2}\theta \, d\theta = \frac{\Gamma_{1/4} \cdot \Gamma_{3/4}}{2 \cdot \Gamma_1}$$

$$= \Gamma_{1/4} \cdot \Gamma_{(1-1/4)}$$

$$= \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \Gamma(2m)$$

$$= \frac{\sqrt{\pi}}{2^{-1/2}} \cdot \Gamma_{1/2}$$

$$I_2 = = \sqrt{2} \cdot \pi$$

$$I = I_1 \cdot I_2 = \sqrt{2}\pi \cdot \sqrt{2}\pi = 2\pi^2$$

27. Given $I = \iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\,dy\,dz$

Put $\left(\frac{x}{a}\right)^p = u$ then $\frac{x}{a} = u^{1/p}$

$x = au^{1/p} \Rightarrow \cancel{dx = \frac{a}{p} \cdot u^{1-1/p}\, du}$

$$dx = \frac{a}{p} \cdot u^{1/p-1}\, du$$

Similarly $dy = \frac{b}{q} \cdot v^{1/q-1}\, dv$

$$dz = \frac{c}{r} \cdot w^{1/r-1}\, dw$$

$$\iiint (au^{1/p})^{l-1} \cdot (bv^{1/q})^{m-1} \cdot (cw^{1/r})^{n-1} \cdot \frac{a}{p} \cdot \frac{b}{q} \cdot \frac{c}{r} \cdot u^{1/p-1}\, v^{1/q-1} \cdot w^{1/r-1}\, du\,dv\,dr$$

$$= \frac{abc}{pqr} \iiint a^{l-1} \cdot b^{m-1} \cdot c^{n-1} \cdot u^{\frac{l}{p}-1}\, v^{\frac{m}{q}-1} \cdot w^{\frac{n}{r}-1} \cdot du\,dv\,dr$$

$$= \frac{a^l b^m \cdot c^n}{pqr} \iiint u^{l/p-1}\, v^{m/q-1} \cdot w^{n/r-1}\, dv\,du\,dr$$

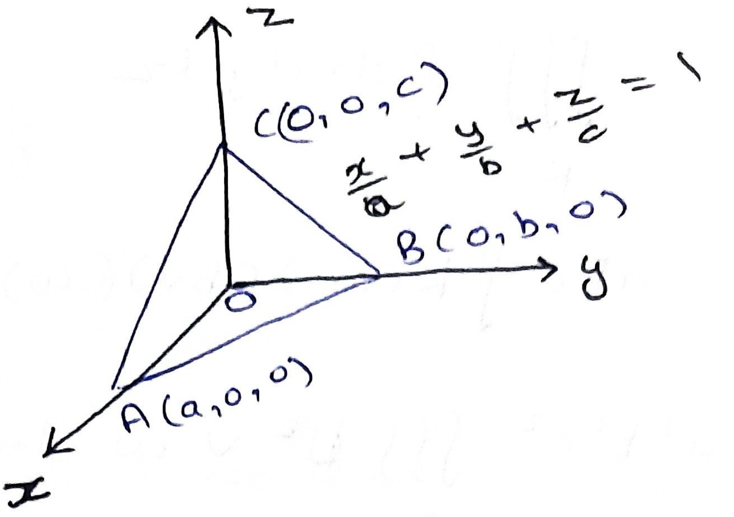
$$= \frac{a^l b^m c^n}{p q r} \cdot \frac{\Gamma_{l/p} \cdot \Gamma_{m/q} \cdot \Gamma_{n/r}}{\Gamma_{\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1}} \quad \text{Ans}$$

PAGE No. DATE

$$I = \frac{27}{8} abc \cdot \frac{\frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$I = \frac{\cancel{27}^{\,9}}{8} abc \times \frac{1\pi\cancel{\sqrt{\pi}}}{\cancel{8}\,2} \times \frac{\cancel{4}}{\cancel{3}\cancel{\sqrt{\pi}}}$$

Que 38:



$$V = \iiint_R dx\,dy\,dz$$

Subject to condition $x \geq 0,\ y \geq 0,\ z \geq 0$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leq 1$$

Put $\frac{x}{a} = u \Rightarrow x = au \Rightarrow dx = a\,du$

$\frac{y}{b} = v \Rightarrow y = bv \Rightarrow dy \Rightarrow b\,dv$

$\frac{z}{c} = w \Rightarrow z = cw \Rightarrow dz = c\,dw$

$$J = \iiint_R a\,du \cdot b\,dv \cdot c\,dw$$

$$= abc \iiint_R du\,dv\,dw$$

$$= abc \cdot \frac{\Gamma_1 \Gamma_1 \Gamma_1}{\Gamma_{1+1+1+1}} = abc \cdot \frac{1}{3!}$$

$$= \frac{abc}{6} \text{ Ans}$$

$$\text{Mass} = \iiint_R \rho \, dx\, dy\, dz$$

$$= abc \int k(au)(bv)(cw)\, du\, dv\, dw$$

$$= k a^2 b^2 c^2 \iiint_R u\, v\, w\, du\, dv\, dw$$

$$= k a^2 b^2 c^2 \cdot \frac{2 \times 2 \times 2}{6!}$$

$$= k a^2 b^2 c^2 \cdot \frac{\cancel{8}\ \cancel{2}}{1 \times \cancel{2} \times 3 \times \cancel{4} \times 5 \times 6}$$

$$= \frac{k a^2 b^2 c^2}{90}$$

VVI

39. $V = \iiint dx\,dy\,dx$ ---- (1)

Subject to condition $x \geq 0,\ y \geq 0,\ z \geq 0$
and $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$

Put $\left(\frac{x}{a}\right)^{2/3} = u \Rightarrow \frac{x}{a} = u^{3/2}$

or $x = au^{3/2} \Rightarrow dx = \frac{3}{2} a u^{1/2} du$

Similarly $dy = \frac{3}{2} b v^{1/2} dv$

$dz = \frac{3}{2} c w^{1/2} dw$

$I = \iiint \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot abc \cdot u^{1/2} \cdot v^{1/2} \cdot w^{1/2}\, du\,dv\,dw$

---- (2)

Subject to conditions $u \geq 0,\ v \geq 0,\ w \geq 0$
and $u + v + w \leq 1$

Apply drichlet's theorem in (2) we get

$I = \frac{27}{8} abc \cdot \frac{\Gamma_{3/2} \cdot \Gamma_{3/2} \cdot \Gamma_{3/2}}{\Gamma_{5/2}}$

✓The Good Paper

$$I = \frac{27}{8} abc \cdot \frac{\frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$I = \frac{\overset{9}{\cancel{27}}}{8} abc \times \frac{1\pi\sqrt{\cancel{\pi}}}{\cancel{8}\,2} \times \frac{\cancel{4}}{\cancel{3}\sqrt{\cancel{\pi}}}$$

$$\boxed{I = \frac{9\pi\, abc}{16}}$$

$1 \quad e^2$

Que 41 $I = \iiint\limits_R x^2 y z \, dx\, dy\, dz$

Subject to condition $x=0, y=0, z=0$
$x+y+z=1$

Apply Dichlet's then

$l=3, m=2, n=2$

$$= \frac{\Gamma_3 \Gamma_2 \Gamma_2}{\Gamma_8} = \frac{2! \times 1! \times 1!}{7!}$$

Que 40 $\frac{x}{a} = u \Rightarrow x = au \Rightarrow dx = a\,du$

$\frac{y}{b} = v \Rightarrow y = bv \Rightarrow dy = b\,dv$

$\frac{z}{c} = w \Rightarrow z = cw \Rightarrow dz = c\,dw$

$$I = \iiint\limits_R a^2u^2 \cdot bv \cdot cw \cdot abc\, du\, dv\, dw$$

$$I = a^3b^2c^2 \iiint\limits_R u^2 v w \, du\, dv\, dw$$

$$I = a^3b^2c^2 \cdot \frac{\cancel{1}2}{7!} \Rightarrow I = \frac{a^3b^2c^2}{2520}$$

$$I = \frac{a^3b^2c^2}{2520}$$

Que 42 $I = \iiint_R e^{-(x+y+z)}\,dx\,dy\,dz$

Subject to conditions $x>0,\ y>0,\ z>0$

& $x+y+z<a$ $(a>0)$

OR $I = \iiint_R x^{1-1}\,y^{1-1}\,z^{1-1} \cdot e^{-(x+y+z)}\,dx\,dy\,dz$

Subject to condition : $x>0,\ y>0,\ z>0$

$\cancel{\&}\ 0 < x+y+z < a$

Apply Liouville's extension

$l=1,\ m=1,\ n=1,\ h_1=0,\ h_2=0,\ t=x+y+z$

$\therefore \iiint x^{1-1}\,y^{1-1}\,z^{1-1} \cdot e^{-(x+y+z)}\,dx\,dy\,dz$

$= \dfrac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 3} \int_0^a e^{-t} \cdot t^{1+1+1-1}\,dt$

$= \dfrac{1}{2} \int_0^a \underset{D}{t^2} \cdot \underset{I}{e^{-t}}\,dt$

$\dfrac{1}{2}\left[t^2(-e^{-t}) - 2t(e^{-t}) + 2(-e^{-t}) \right]_0^a$

$= \dfrac{1}{2}\left[-a^2e^{-a} - 2ae^{-a} - 2e^{-a} - (-2) \right]$

Que 43: $I = \iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}} = \frac{\pi^2}{8} a^2$

Let $I = \iiint_R \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$

Subject to condition $x \geq 0$; $y \geq 0$; $z \geq 0$

and $a^2 - x^2 - y^2 - z^2 > 0$

or $0 < x^2+y^2+z^2 < a^2$ or

$0 < \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} < 1$

Hence integral becomes

$I = \frac{1}{a} \iiint_R \frac{dx\,dy\,dz}{\sqrt{1-\left(\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2+\left(\frac{z}{a}\right)^2\right)}}$

Put $\frac{x^2}{a^2} = u \Rightarrow adx = \frac{a}{2} u^{-1/2} du$

$dy = \frac{a}{2} v^{-1/2} dv$; $dz = \frac{a}{2} w^{-1/2} dw$

$I = \frac{1}{a} \iiint u^{-1/2} v^{-1/2} w^{-1/2} \frac{1}{\sqrt{1-(u+v+w)}} du\,dv\,dw$

$$I = \frac{1}{a} \times \frac{a^3}{8} \iiint_R u^{-1/2} v^{-1/2} w^{-1/2} \frac{1 \, du\,dv\,dw}{\sqrt{1-(u+v+w)}}$$

$u > 0, v > 0, w > 0, \quad 0 < u + v + w < 1$

$l = 1/2, \; m = 1/2, \; n = 1/2, \; h_1 = 0, \; h_2 = 1$

$t = (u+v+w)$

Apply Liouville's extension of Dirichlet formula:-

$$I = \frac{a^2}{8} \times \frac{\Gamma 1/2 \cdot \Gamma 1/2 \cdot \Gamma 1/2}{\Gamma 3/2} \int_0^1 \frac{1}{\sqrt{1-t}} t^{\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{1}} \, dt$$

$$I = \frac{a^2}{8} \cdot \frac{\pi \cdot \cancel{\sqrt{\pi}}}{\frac{1}{2} \cdot \cancel{\sqrt{\pi}}} \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} \, dt$$

$$= \frac{a^2 \pi}{4} \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} \, dt = \frac{a^2 \pi}{4} \beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$\frac{\pi}{4} a^2 \cdot \frac{\frac{1}{2} \cdot \sqrt{\pi} \cdot \sqrt{\pi}}{\cancel{1}} = \frac{\pi a^2 \cdot \pi}{8}$$

$$= \frac{\pi^2 a^2}{8} \text{ Ans}$$

Que 44 : Let $I = \iiint_R \sqrt{\frac{1-(x^2+y^2+z^2)}{1+(x^2+y^2+z^2)}}\, dx\,dy\,dz$

Subject to condition $x>0,\ y>0,\ z>0$
and $0 < x^2+y^2+z^2 < 1$

Put $x^2 = u \Rightarrow 2x\,dx = du \Rightarrow dx = \frac{1}{2}u^{-1/2}du$

$dy = \frac{1}{2}v^{-1/2}dv$; $dz = \frac{1}{2}w^{-1/2}dw$

$l = \frac{1}{2} = m = n = \frac{1}{2}$

$= \frac{1}{8}\iiint_R \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\cdot du\,dv\cdot dw.$

$\frac{1}{8}\cdot\frac{\Gamma_{1/2}\cdot\Gamma_{1/2}\cdot\Gamma_{1/2}}{\Gamma_{3/2}}\cdot\int_0^1\sqrt{\frac{1-t}{1+t}}\,dt$

$\frac{1}{8}\cdot\frac{\sqrt{\pi}\cdot\sqrt{\pi}\cdot\cancel{\sqrt{\pi}}}{\frac{1}{2}\cdot\cancel{\sqrt{\pi}}}$

$\frac{\pi}{4}\int_0^1\sqrt{\frac{1-t}{1+t}}\cdot t^{1/2+\cancel{1/2}+\cancel{1/2}-\cancel{1}}\,dt$

$\frac{\pi}{4}\int_0^1\sqrt{\frac{1-t}{1+t}}\,t^{1/2}\,dt$

$$\frac{\pi}{4}\int_0^1 t^{1/2}\cdot\frac{1-t}{\sqrt{1-t^2}}\,dt = \frac{\pi}{4}\int_0^1 t^{1/2}\cdot(1-t^2)^{-1/2}\,dt$$

$$-\frac{\pi}{4}\int_0^1 t^{3/2}\cdot(1-t^2)^{-1/2}\,dt$$

Now $I = \frac{\pi}{4}\left[I_1 - I_2\right]$

~~Put $1-t^2 = v^2$ then~~

Now put $t^2 = z$ then $2t\,dt = dz$

$$\Rightarrow dt = \frac{1}{2}z^{-1/2}\,dz$$

$$I_2 = \int_0^1 (z^{1/2})^{3/2}\,(1-z)^{-1/2}\left(\frac{1}{2}z^{-1/2}\right)dz$$

$$= \frac{1}{2}\int_0^1 z^{1/4}\cdot(1-z)^{-1/2}\,dz = \frac{1}{2}\,\beta\left(\frac{5}{4},\frac{1}{2}\right)$$

$$I_1 = \frac{1}{2}\int_0^1 t^{1/2}\,(1-t^2)^{-1/2}\,dt$$

$$= \frac{1}{2}\int_0^1 (z^{1/2})^{1/2}\,(1-z)^{-1/2}\cdot z^{-1/2}\,dz$$

$$= \frac{1}{2}\int_0^1 z^{-1/4}\cdot(1-z)^{-1/2}\,dz = \beta\left(\frac{3}{4},\frac{1}{2}\right)$$

$$= \frac{1}{2}\,\frac{\gamma 3/4\cdot\gamma 1/2}{\gamma 5/4}$$

$= 2 \sin \theta$ *and* $r = 4 \sin \theta$

TRIPLE INTEGRAL

$$I = \frac{\pi}{4} \times \frac{1}{2} \left[\frac{\Gamma_{3/4} \cdot \Gamma_{1/2}}{\Gamma_{5/4}} - \frac{\Gamma_{5/4} \cdot \Gamma_{1/2}}{\Gamma_{7/4}} \right]$$

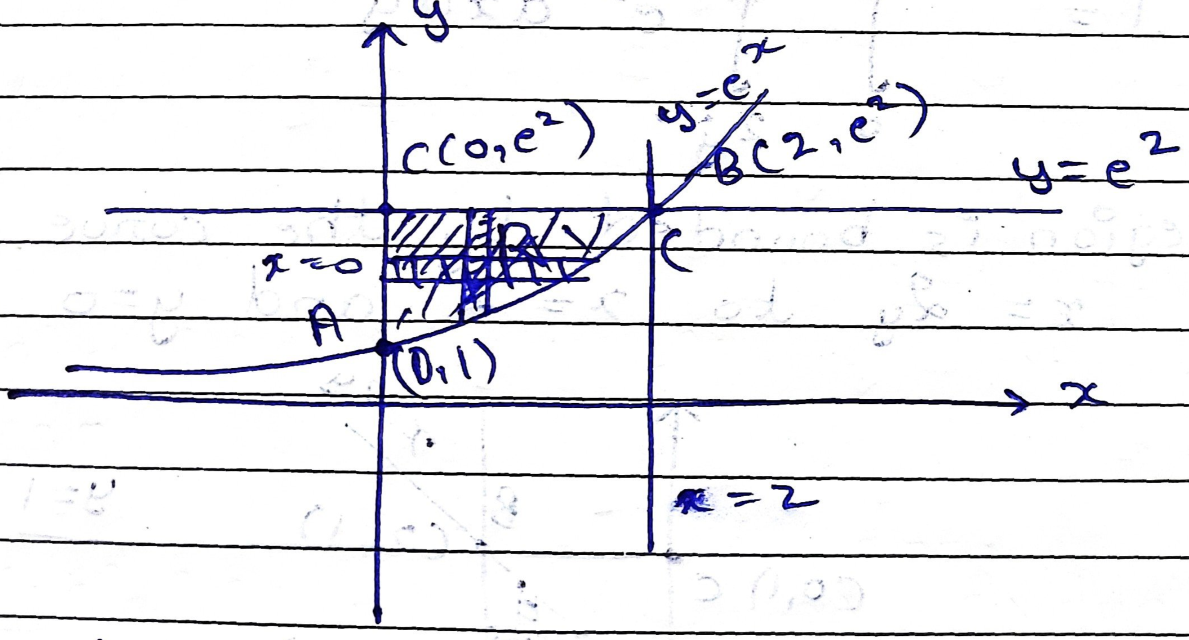
$$= \frac{\pi}{8} \left[\frac{\Gamma_{3/4} \cdot \Gamma_{1/2}}{\frac{1}{4} \cdot \Gamma_{1/4}} - \frac{\Gamma_{5/4} \cdot \Gamma_{1/2}}{\frac{3}{4} \cdot \Gamma_{3/4}} \right]$$

$$= \frac{\pi}{2} \left[\frac{\Gamma_{3/4} \cdot \Gamma_{1/2}}{\Gamma_{1/4}} - \frac{\Gamma_{5/4} \cdot \Gamma_{1/2}}{3 \cdot \Gamma_{3/4}} \right]$$

45. $I = \int_0^1 \int_{e^x}^{e^2} \frac{dy\,dx}{\log y}$

The region is bounded by the curves
$y = e^x$ to $y = e^2$ and $x = 0$ to $x = 1$

Region of integration



New limits of integration.
$x = 0$ to $x = \log y$
$y = 1$ to $y = e^2$

✓The Good Paper

PAGE No.
DATE

$$I = \int_0^1 \int_{e^x}^{e^2} \frac{dy\,dx}{\log y} = \int_1^{e^2} \int_0^{\log y} \frac{dx\,dy}{\log y}$$

$$I = \int_1^{e^2} \frac{1}{\log y} \left[x \right]_0^{\log y} dy$$

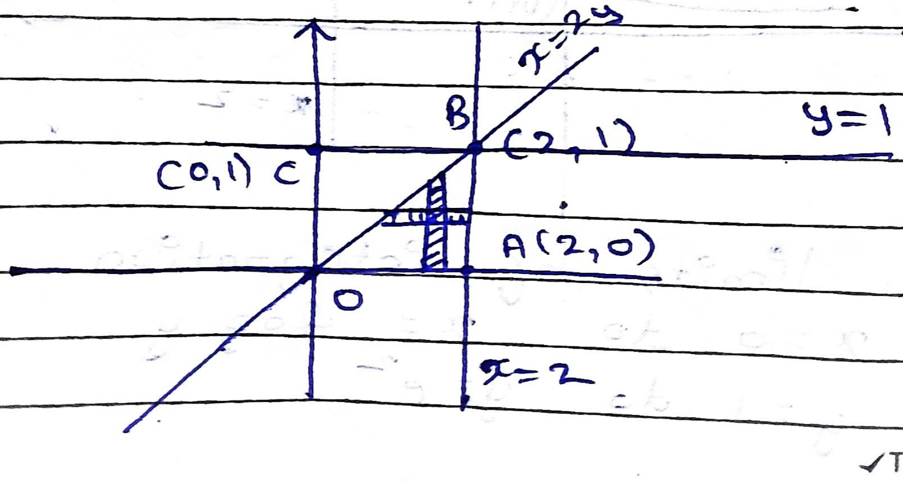
$$I = \int_{y=1}^{e^2} \frac{1}{\cancel{\log y}} \cdot \cancel{\log y}\, dy$$

$$I = \int_1^{e^2} dy \Rightarrow I = \left[y \right]_1^{e^2}$$

$$\boxed{I = e^2 - 1} \quad \text{Ans}$$

46. $$I = \int_0^1 \int_{2y}^{2} e^{x^2}\, dx\, dy$$

Region is bounded by the curve $x = 2y$ to $x = 2$ and $y = 0$ to $y = 1$



✓The Good Paper

Limits of integration
$y = 0$ to $y = \frac{x}{2}$ and $x = 0$ to $x = 2$

$$I = \int_0^1 \int_{2y}^{2} e^{x^2}\, dx\, dy = \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx$$

$$I = \int_0^2 e^{x^2} \cdot \frac{x}{2}\, dx = \frac{1}{2} \int_0^2 x \cdot e^{x^2}\, dx$$

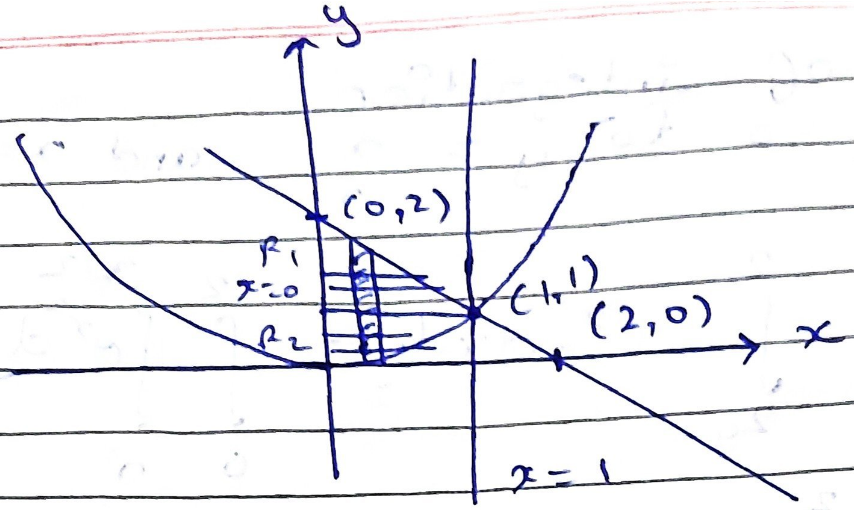
$$I = \frac{1}{4} \int_0^2 2x \cdot e^{x^2}\, dx$$ Put $x^2 = t$
then $2x\,dx = dt$

$$I = \frac{1}{4} \int_0^2 e^t\, dt \Rightarrow I = \frac{1}{4} \left[e^t \right]_0^2$$

$$I = \frac{1}{4}(e^2 - 1)$$ Ans

47. $$I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$$

Region is bounded by curves
$y = x^2$ to $y = 2 - x$ and
$x = 0$ to $x = 1$.



New limits of integration

for R_1	for R_2
$x=0$ to $x=2-y$	$x=0$ to $x=\sqrt{y}$
$y=1$ to $y=2$	$y=0$ to $y=1$

$I = I_1 + I_2$ where

$$I_1 = \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$I_2 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy$$

$$I_1 = \frac{1}{2} \int_1^2 y \left[x^2 \right]_0^{(2-y)} dy$$

$$= \frac{1}{2} \int_1^2 y(2-y)^2 \, dy = \frac{1}{2} \int_1^2 y(y^2+4-4y) d$$

✓ The Good Paper

$x^2+2x-x-$

$x(x+2)-1(x+2)=0$

$$I_1 = \frac{1}{2}\int_1^2 y^3+4y-4y^2\,dy$$

$$I_1 = \frac{1}{2}\left[\frac{y^4}{4}+2y^2-\frac{4}{3}y^3\right]_1^2$$

$$I_1 = \frac{1}{2}\left[\left(4+8-\frac{4}{3}\cdot 8\right)-\left(\frac{1}{4}+2-\frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[12-\frac{32}{3}-\frac{1}{4}-2+\frac{4}{3}\right]$$

$$= \frac{1}{2}\left[10-\frac{1}{4}-\frac{28}{3}\right] = \frac{1}{2}\left[\frac{120-3-112}{12}\right]$$

$$I_1 = \frac{5}{24}$$

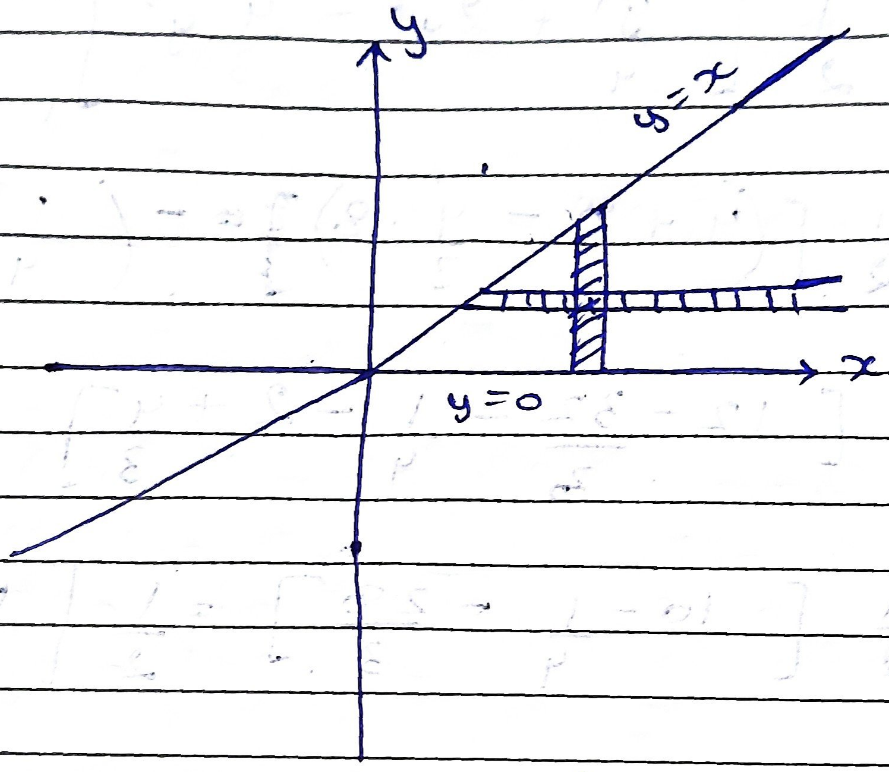
$$I_2 = \frac{1}{2}\int_0^1 y\left[x^2\right]_0^{\sqrt{y}}dy = \frac{1}{2}\int_0^1 y\cdot y\,dy$$

$$= \frac{1}{6}\left[y^3\right]_0^1 = \frac{1}{6}$$

$$I = I_1+I_2 = \frac{1}{6}+\frac{5}{24} = \frac{9}{24} = \frac{3}{8}\text{ Ans}$$

48. The region is bounded by $y=0$ to $y=x$
and $x=0$ to ∞

Region of integration



New limits of integration :

$x = y$ to $x=\infty$
$y=0$ to $y=\infty$

$$J = \int_0^{\infty}\int_0^{x} x e^{-x^2/y}\,dy\,dx = \int_0^{\infty}\int_y^{\infty} x \cdot e^{-x^2/y}\,dx\,dy \quad \text{---} (1)$$

✓The Good Paper

Put $\frac{-x^2}{y} = t$ then $\frac{-2x}{y} \cdot dx = dt$

$\Rightarrow \quad 2x\,dx = -y\,dt$ and

$x = y$ then $t = -y$

$x = \infty$ then $t = -\infty$

$\Rightarrow \cancel{I = \int_0^\infty -y \int_y^\infty e^{t}\, dt}$

$$I = \int_0^\infty \frac{-y}{2}\left[\int_{-y}^{-\infty} e^{t}\, dt\right] dy$$

$$I = \frac{-1}{2}\int_0^\infty y \left[e^{t}\right]_{-y}^{-\infty} dy$$

$$I = \frac{-1}{2}\int_0^\infty y\left(-e^{-y}\right) dy$$

$$I = \frac{1}{2}\int_0^\infty y e^{-y}\, dy = \frac{1}{2}\left[-ye^{-y} + e^{-y}\right]_0^\infty$$

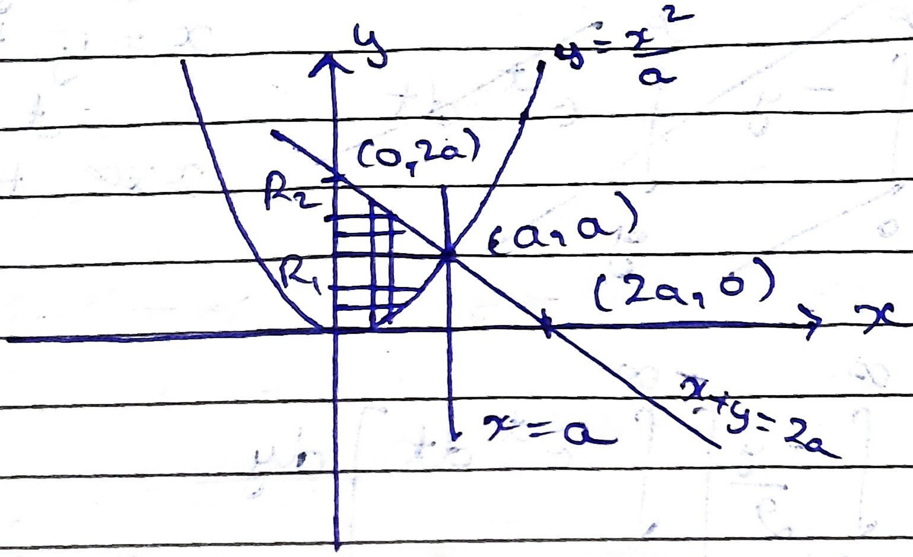
$$I = \frac{1}{2}\left[\left(-\infty \cdot e^{-\infty} + e^{-\infty}\right) - \left(-0 \cdot e^{-0} + \left(e^{-0}\right)\right)\right]$$

$$\Rightarrow I = \frac{1}{2}\left[0 - (0+1)\right] \Rightarrow \boxed{I = \frac{-1}{2}}$$

49. The region is bounded by the curves

$y = \frac{x^2}{a}$ to $y = 2a - x$

$x = 0$ to $x = a$



New limits of integration

for R_1	for R_2
$x=0$ to $x=\sqrt{ay}$	$x=0$ to $x=2a-y$
$y=0$ to $y=a$	$y=a$ to $y=2a$

$$I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\,dy\,dx =$$

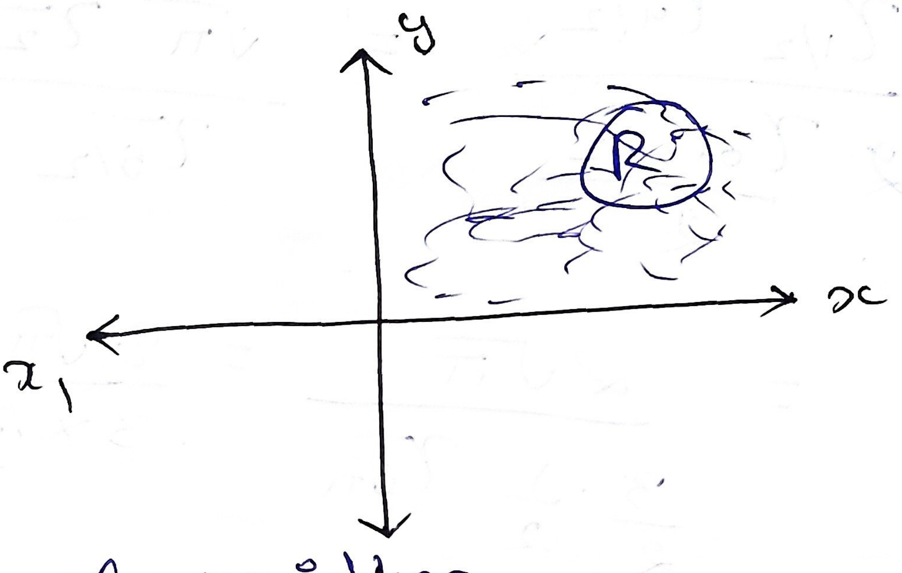
$$\int_0^a \int_0^{\sqrt{ay}} f(x,y)\,dy\,dx + \int_a^{2a} \int_0^{2a-y} f(x,y)\,dy\,dx$$

Que 51: # Let $I = \int_0^{\infty} e^{-x^2} dx$ ---- (1)

$$I = \int_0^{\infty} e^{-y^2} dy \quad \text{----} \quad (2)$$

$$\Rightarrow I^2 = \int_{y=0}^{\infty} \int_{x=0}^{\infty} e^{-(x^2+y^2)} dx\,dy \quad \text{--} \quad (3)$$

Region of integration $\quad x=0$ to $x=\infty$
$y=0$ to $y=\infty$



Change of variables
Put $x = r\cos\theta$; $y = r\sin\theta$

$$x^2+y^2 = r^2$$

$$dx\,dy = |J|\,dr\,d\theta : \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$dx\,dy = r\,dr\,d\theta$$

$$I^2 = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$I^2 = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{\infty} 2r\cdot e^{-r^2}\,dr\right]d\theta$$

$$I^2 = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[-e^{-r^2}\right]_{r=0}^{\infty} d\theta$$

$$I^2 = \frac{1}{2}\int_{0}^{\pi/2} 1\,d\theta = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$$

Now $I = \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Que 52: $I = \iint \frac{\sqrt{1-x^2-y^2}}{\sqrt{1+x^2+y^2}}\,dx\,dy$

R: over the +ve quadrant of a circle $x^2+y^2=1$

$\Rightarrow$ Change of variables:
$x = r\cos\theta$; $y = r\sin\theta$

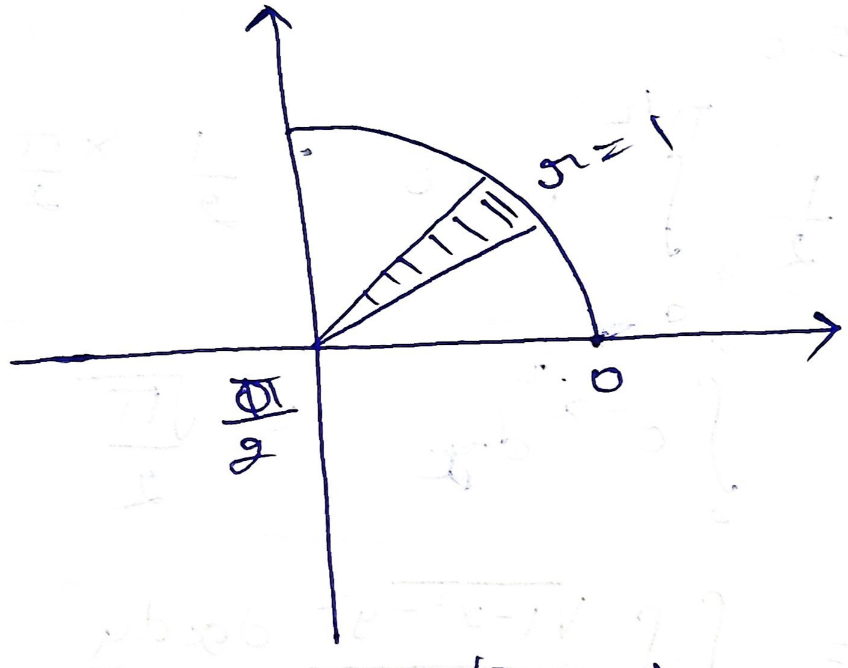
Now, $dx\,dy = |J|\,dr\,d\theta$

$dx\,dy = r\,dr\,d\theta$

$$I = \iint \frac{\sqrt{1-r^2}}{\sqrt{1+r^2}} \cdot r\,dr\,d\theta$$

$$I = \iint_R \sqrt{\frac{1-r^2}{1+r^2}} \cdot r\,dr\,d\theta$$

Region of integration



Hence $$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}} \cdot r\,dr\,d\theta$$

~~Put r = cos2θ~~

$$I = \int_0^{\pi/2} \left[\int_{r=0}^{1} \frac{r(1-r^2)}{\sqrt{1-r^4}}\,dr \right] d\theta$$

$$I = \int_0^{\pi/2} \left[\left[\int_0^1 \frac{r\,dr}{\sqrt{1-r^4}}\right] - \int_{r=0}^{1} \frac{r^3}{\sqrt{1-r^4}}\,dr \right] d\theta$$

Let $I_1 = \frac{1}{2} \int_{r=0}^{1} \frac{2r\,dr}{\sqrt{1-r^4}}$: Put $r^2 = t$
$2r\,dr = dt$

$$= \frac{1}{2} \int_{t=0}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{2} \left[\sin^{-1}(t)\right]_0^1 = \frac{\pi}{4}$$

$I_2 = -\frac{1}{4} \int \frac{-4r^3}{\sqrt{1-r^4}}\,dr$: $1-r^4 = t^2$
$r=0, t=0$
$r=1, t=0$

$$= -\frac{1}{4} \int_{t=1}^{0} 2\,dt = \frac{2}{4}\left[t^2\right]_0^1$$

$$= \frac{1}{2}$$

$$I = \int_{\theta=0}^{\pi/2} \left(\frac{\pi}{4} - \frac{1}{2}\right) d\theta$$

$$I = \left(\frac{\pi}{4} - \frac{1}{2}\right) \times \frac{\pi}{2} = \frac{\pi^2}{8} - \frac{\pi}{4}$$

$$I = \iint_R (x+y)^2 \, dx\,dy$$

$u = x + y$ and $v = x - 2y$

$u - v = 3y \Rightarrow y = \frac{1}{3}(u-v)$

$2u + v = 3x \Rightarrow x = \frac{1}{3}(2u+v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{3}\times 2 & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

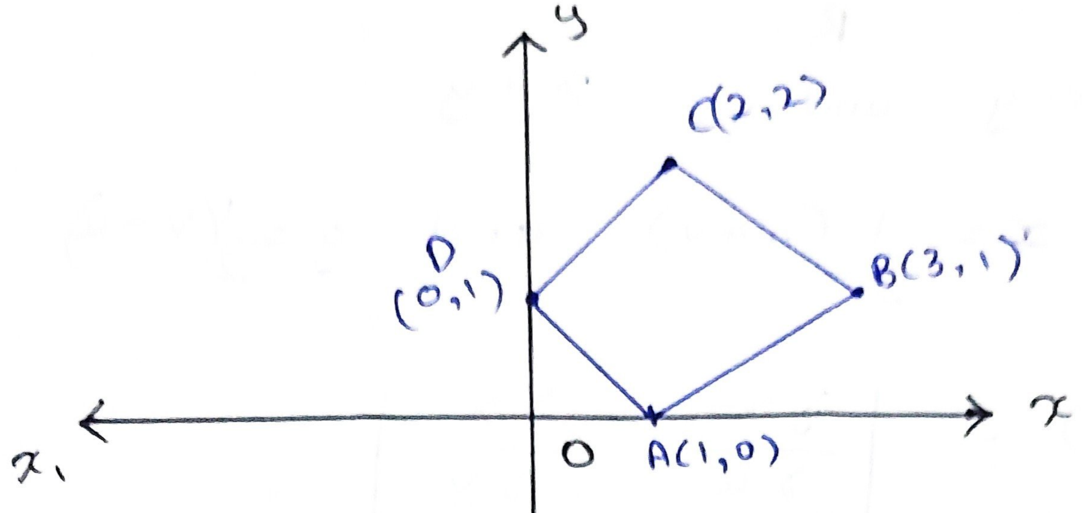
$$= \frac{1}{3} \times \frac{1}{3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \frac{1}{9}(-2-1) = \left(-\frac{1}{3}\right)$$

$$\iint_R (x+y)^2 \, dx\,dy = \iint_{R'} u^2 |J| \, du\,dv : \; |J| = \left|-\frac{1}{3}\right| = \frac{1}{3}$$

$$= \frac{1}{3} \iint_{R'} u^2 \, du\,dv$$

Region of integration



xy plane

$A(1,0) \Rightarrow u = x+y \Rightarrow u = 1$
$v = x - 2y \Rightarrow v = 1$

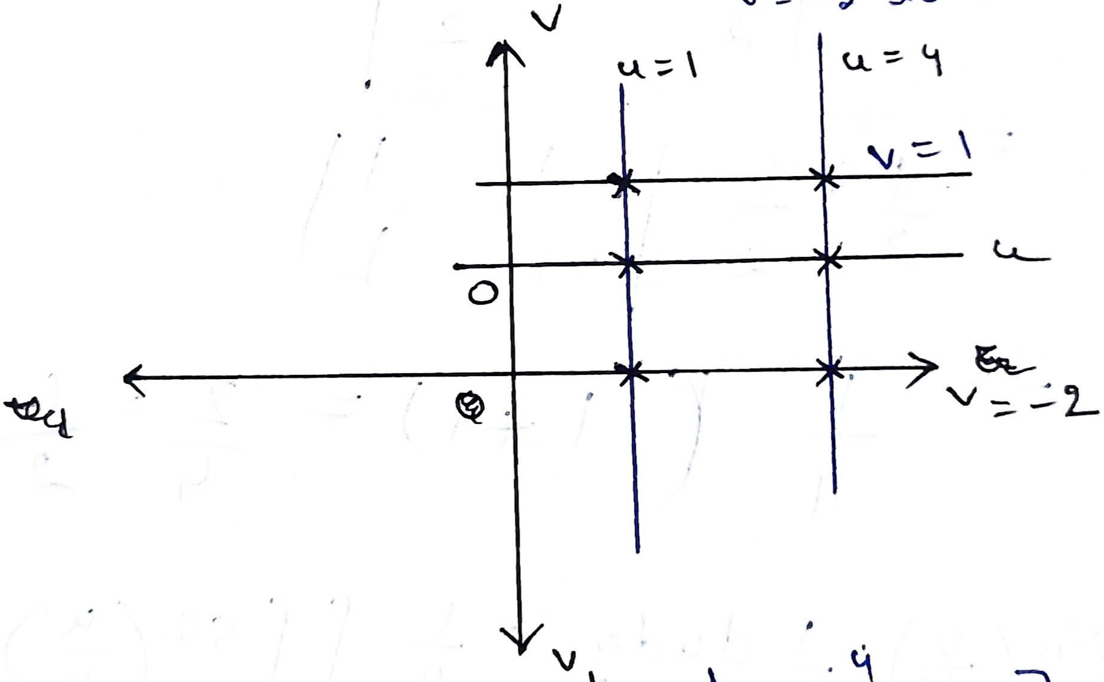
$B(3,1) \Rightarrow u = 4,\ v = 1$

$C(2,2)\ u = 4,\ v = -2$

$D(0,1)\ u = 1;\ v = -2$

$\therefore u = 1$ to 4

$v = -2$ to 1



$$I = \iint_R (x+y)^2\,dx\,dy = \frac{1}{3}\int_{v=-2}^{1}\left[\int_1^4 u^2\,du\right]dv$$

$$= \frac{1}{3}\int_{v=-2}^{1}\left[\frac{u^3}{3}\right]_1^4 dv$$

$$= \frac{1}{9}\int_{-2}^{1}(64-1)\,dv = \frac{63}{9}\int_{-2}^{1} dv$$

$$I = \iint_R \sin\left(\frac{x-y}{x+y}\right) dx\,dy$$

$u = x - y$ and $v = x + y$

Now $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

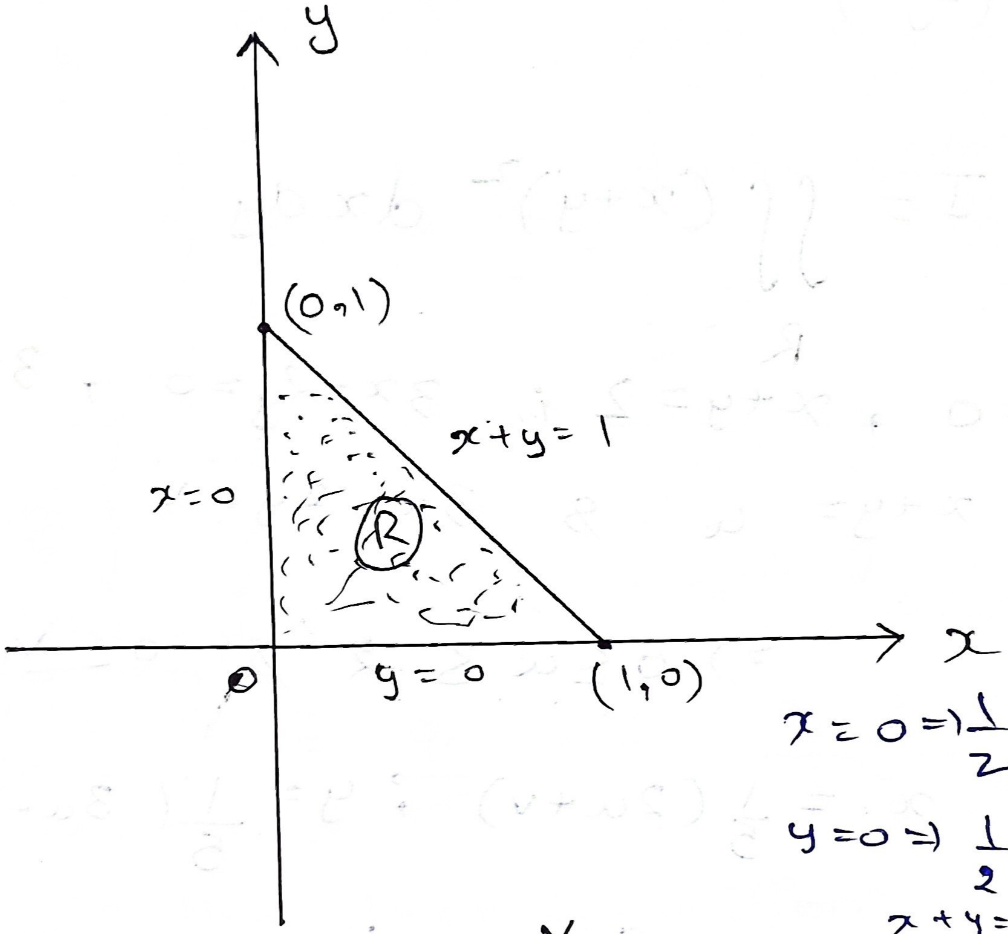
$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2} \times \frac{1}{2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

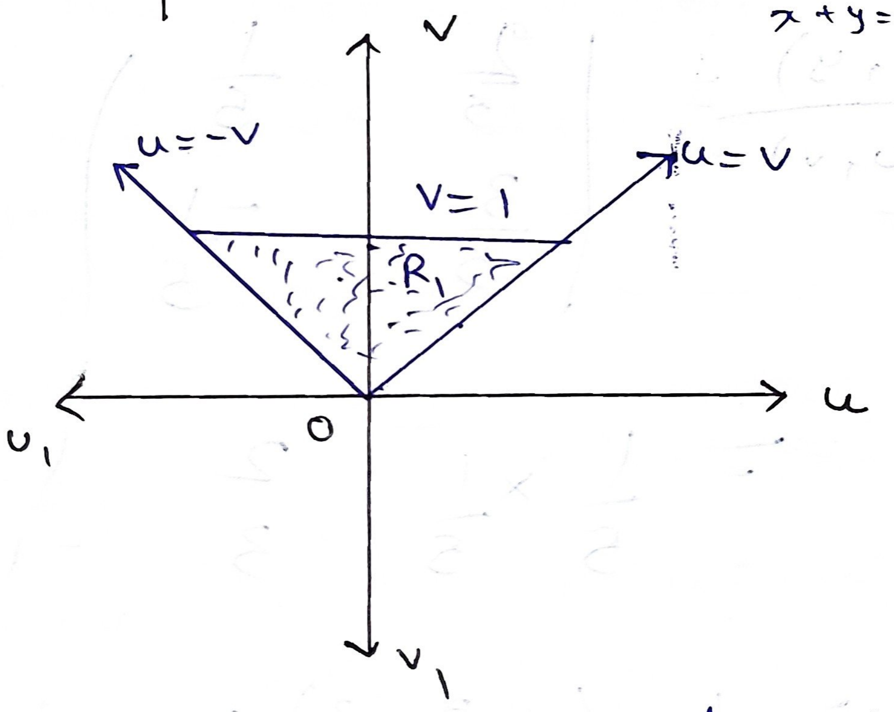
$$= \frac{1}{4}(1+1) = \frac{2}{4} = \frac{1}{2}$$

$$\iint_R \sin\left(\frac{u}{v}\right) \cdot \frac{1}{2}\, du\,dy = \frac{1}{2} \iint_R \sin\left(\frac{u}{v}\right) du\,dv$$

Region of integration



$x=0 \Rightarrow \frac{1}{2}(u+v)=0$
$u=-v$
$y=0 \Rightarrow \frac{1}{2}(v-u)=0$
$u=v$
$x+y=1 \Rightarrow v=1$



Limits of integration $u=-v$ to $u=v$
$v=0$ to $v=1$

$$I = \int\!\!\int\left[\int_{u=-v}^{v}\sin\left(\frac{u}{v}\right)du\right]dv$$

Since $\sin\left(\frac{u}{v}\right)$ is an odd function w.r.t u so

$$\int_{-v}^{v} \sin\left(\frac{u}{v}\right) du \quad I = 0$$

Que 56: $I = \iint_R (x+y)^2 \, dx\, dy$

$x+y=0$, $x+y=2$; $3x-2y=0$, $3x-2y=3$

Put $x+y = u$ & $3x-2y = v$

$\Rightarrow 0 \le u \le 2$ & $0 \le v \le 3$

Now $x = \frac{1}{5}(2u+v)$; $y = \frac{1}{5}(3u-v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{vmatrix}$$

$$= \frac{1}{5} \times \frac{1}{5} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= \frac{1}{25}(-2-3) = -\frac{1}{5}$$

$$I = \iint_R u^2 \, |J| \, du\, dv$$

$$= \iint_R u^2 \cdot \frac{1}{5} \cdot du\, dv = \frac{1}{5} \iint_R u^2 \, du\, dv$$

$$= \frac{1}{5} \int_{v=0}^{3} \int_{u=0}^{2} u^2 \, du\, dv = \frac{1}{15} \int_0^3 8 \, dv$$

$$= \frac{8}{15} \times 3 = \frac{8}{5}$$