Engineering Mathematics-I
KAS 103.
Lecture-43

Module-IV
Multivariable Calculus-I

Content: Double Integrals.

Double Integrals:- consider a function of two variables $f(x,y)$ be continuous having two variables x and y with in region R bounded by a curve c.

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx.$$

Evaluation of Double Integrals:- The method of evaluating double integral depends upon the nature of curves bounding by region R. Let R be region bounded by curves $x = x_1$, $x = x_2$ and $y = y_1$ and $y = y_2$.

Case I:- When x_1, x_2 are functions of y and y_1 & y_2 are constants.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \boxed{\int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\,dx}\,dy,$$

integration is being carried from the inner to the outer rectangle. horizontal strip.

The strip is ⊥ to the constant limit.

Case II when y_1, y_2 are functions of x and x_1 & x_2 are constants.

$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \boxed{\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy}\,dx$$

vertical strip.

Case III When x_1, x_2, y_1, y_2 are constants:

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

Remark:- integration is performed w.r.t. respect to variable limit first, then with respect to constant limit.

for constant limit, order of integra...

Q.1 Evaluate. $\int_0^3\int_0^1 (x^2+3y^2)\,dy\,dx$.

$\Rightarrow 0 \le x \le 3, \quad 0 \le y \le 1.$

first integrating with respect to y.

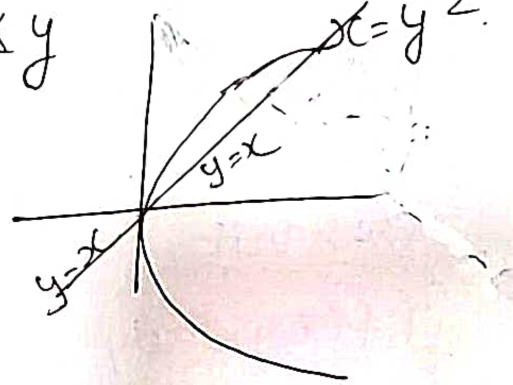
$$\int_0^3 \left[x^2y + \frac{3y^3}{3}\right]_0^1 dx = \int_0^3 \left[x^2y + y^3\right]_0^1 dx$$

$$= \int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^3 = 9+3 = 12 \text{ Ans.}$$

Q.2 Evaluate. $\int_0^1\int_{y^2}^{y} (1+xy^2)\,dx\,dy$.

Sol. - $0 \le y \le 1$ and $y^2 \le x \le y$

integrating with respect to x.



$$\int_0^1 \left[x + \frac{x^2y^2}{2}\right]_{y^2}^{y} dy$$

$$= \int_0^1 \left[y + \frac{y^4}{2} - y^2 - \frac{y^6}{2}\right] dy$$

$$= \left[\frac{y^2}{2} + \frac{y^5}{10} - \frac{y^3}{3} - \frac{y^7}{14}\right]_0^1 = \frac{1}{2} + \frac{1}{10} - \frac{1}{3} - \frac{1}{14}$$

$$= \frac{105+21-70-15}{210}$$

$$= \frac{126-85}{210} = \frac{41}{210} = \frac{41}{210}$$

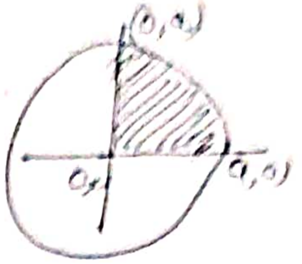
Q.3 evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\left[\sin^{-1}y\right]_0^1 dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\left[\frac{\pi}{2} - 0\right] dx$$

$\sin^{-1} y\Big]_0^1 = \frac{\pi^2}{4}$ Ans.

Q. Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of circle $x^2+y^2=a^2$

Sol. Region is $x^2+y^2=a^2$; $x\geq 0, y\geq 0$



$$\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$\int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$y = \pm\sqrt{a^2-x^2}$

over +ve quadrant

$y=\sqrt{a^2-x^2}$

$$\int_0^a x\,\frac{(a^2-x^2)}{2}\,dx$$

$0\leq y\leq \sqrt{a^2-x^2}$

$0\leq x\leq a$

$$\frac{1}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right] = \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \boxed{\frac{a^4}{8}}$$ Ans.

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Q. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$.

$$I = \int_0^1\left[\int_0^{\sqrt{1+x^2}}\frac{dy}{(1+x^2)+y^2}\right]dx = \int_0^1\left[\int_0^{\sqrt{1+x^2}}\frac{dy}{(\sqrt{1+x^2})^2+y^2}\right]dx.$$

$$= \int_0^1\frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}dx = \int_0^1\frac{1}{\sqrt{1+x^2}}\left[\frac{\pi}{4}\right]dx$$

$$= \frac{\pi}{4}\int_0^1\frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(\sqrt{2}+1)\right]$$

Problems for Practice.

1. Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$z_1 = -0.5$

Q.2 Evaluate $\iint y \, dx \, dy$ over the part of the plane bounded by line $y = x$ and parabola $y = 4x - x^2$.

Q.3 Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx dy$ over the first quadrant bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Q4. Evaluate $\iint_S xy \, dx dy$ over the positive quadrant for which $x + y \leq 1$

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Q.5 Evaluate $\iint_S xy \, dx dy$ where A is domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$

Q.6 Evaluate $\iint xy(x+y) \, dx dy$ over the area between $y = x^2$ and $y = x$.

along strip. $y = x^2$, $y = x$

$$\int_0^1 \int_{x^2}^{x} xy(x+y) \, dy \, dx$$

$\Rightarrow x^2 = x$

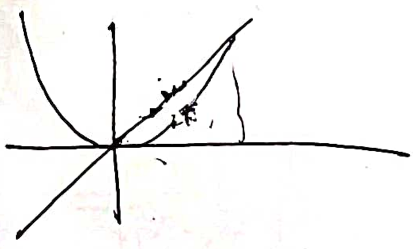
$x(x-1) = 0$

$x = 0, 1$

$0 \leq x \leq 1$

$$\int_0^1 \left[\frac{x^2 y^2}{2} + \frac{x y^3}{3}\right]_{x^2}^{x} dx$$

$$\int_0^1 \left[-\frac{x^6}{2} - \frac{x^7}{3} + \frac{x^4}{2} + \frac{x^4}{3}\right] dx$$



$$\left[-\frac{x^7}{14} - \frac{x^8}{24} + \frac{x^5}{10} + \frac{x^5}{15}\right]_0^1$$

$$\Rightarrow -\frac{1}{14} - \frac{1}{24} + \frac{1}{10} + \frac{1}{15}$$

$$= \frac{-60 - 35 + 84 + 56}{840} = \frac{45}{840} = \frac{3}{56} \text{ Ans.}$$

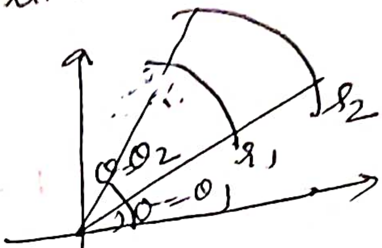
plane 14 287

$y = 4x - 4$

Evaluation of double integrals in Polar Co-ordinates

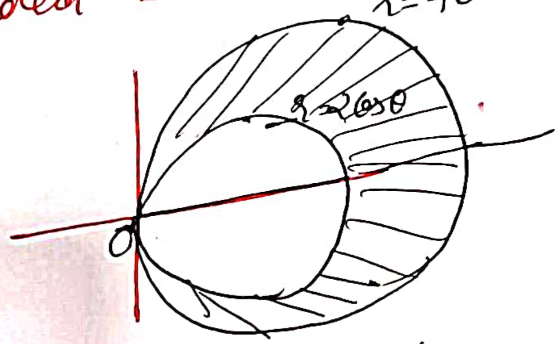
To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\, dr\, d\theta$ over the region bounded by the straight lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves r_1 and r_2, we first integrate with respect to r between the limits $r = r_1$ and r_2 (treating θ as constant). The resulting integrand is then integrated with respect to θ, between constant limits θ_1 & θ_2.

$$I = \int_{\theta_1}^{\theta_2}\int_{r=r_1=f(\theta)}^{r=r_2=g(\theta)} f(r,\theta)\, dr\, d\theta$$



Evaluate.

$\iint r^3\, dr\, d\theta$ over the area bounded between circles $r = 2\cos\theta$, $r = 4\cos\theta$



θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{256\cos^4\theta}{4} - \frac{16\cos^4\theta}{4}\right] d\theta$$

$$= \frac{240}{4}\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta$$

$\because \cos^4\theta$ is even function

$$\int_{-a}^{a} f(\theta)\,d\theta = 2\int_0^a f(\theta)$$

$$= 120\int_0^{\pi/2}\cos^4\theta\, d\theta$$

using $\int_0^{\pi/2}\cos^p\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$

$$= 120 \times \frac{3\times 1}{4\times 2}\frac{\pi}{2}$$

$$= 120\,\frac{\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{2\times 2} = \frac{45}{2}\pi$$

$$= \frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{2\,\Gamma 3}$$

Problems for practice.

1) $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

2) $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$.

3) Evaluate $\iint_R r^2\sin\theta\,dr\,d\theta$, where R is the region bounded by semi circle $r = 2a\cos\theta$ and above initial line.