Differential Equation :— A equation involving differential coefficient is known as differential equation. For example

(i) $\frac{dy}{dx} = \sin x$ (ii) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = e^x$ (iii) $\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0$

Order of a Differential Equation :— The order of the highest derivative present in the differential equation is known as order. In above differential equations (i), (ii) and the order are 1, 2 and 2 respectively.

Degree of a Differential Equation :— The power of the highest derivative present in the differential equation is know as degree. The degree is made free from radical and fractional powers. In above equations (i), (ii) and (iii) the degree are 1, 1 and 2 respectively.

Linear Differential Equation of First Order :—

$$\frac{dy}{dx} + Py = Q \quad , \text{ where P and Q are function of } x \text{ only or constant.}$$

On multiply by $e^{\int P dx}$ in above equation, we get

$$\frac{dy}{dx}.e^{\int P dx} + Py\, e^{\int P dx} = Q\, e^{\int P dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left(y e^{\int P dx}\right) = Q\, e^{\int P dx}$$

or $\quad y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$

or Solution is $\boxed{y.I.F = \int Q.I.F.dx + C}$, where I.F (Integrating Factor) $= e^{\int P dx}$

Que ① Solve the following differential equations —

(i) $\frac{dy}{dx} + \frac{3}{x}y = \frac{1}{x^4}$

(ii) $\cos^2 x \frac{dy}{dx} + y = \tan x$

(ii) $x \log x \frac{dy}{dx} + y = 2\log x$

(iv) $(1+x)\frac{dy}{dx} - y = (1+x)^2 e^x$

Sol (i) $\frac{dy}{dx} + \frac{3}{x} y = \frac{1}{x^4}$, here $P = \frac{3}{x}$ and $Q = \frac{1}{x^4}$

Now $I.F = e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = x^3$

$\therefore$ sol is $y . x^3 = \int \frac{1}{x^4} . x^3 dx + c$

or $y x^3 = \int \frac{1}{x} dx + c$ or $y x^3 = \log x + c$ Ans

(ii) $\frac{dy}{dx} + \sec^2 x . y = \tan x . \sec^2 x$, here $P = \sec^2 x$ and $Q = \tan x \sec^2 x$

Now $I.F. = e^{\int P dx} = e^{\int \sec^2 x dx} = e^{\tan x}$

$\therefore$ sol is $y . e^{\tan x} = \int \tan x \sec^2 x . e^{\tan x} dx + C$

(put $\tan x = t$ or $\sec^2 x dx = dt$)

$= \int t e^t dt + c$

$= t e^t - e^t + c$

$= \tan x \, e^{\tan x} - e^{\tan x} + C$

or $y = \tan x - 1 + c . e^{-\tan x}$ Ans

(iii) $\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}$, here $P = \frac{1}{x \log x}$ and $Q = \frac{2}{x}$

Now $I.F. = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$

$\therefore$ sol is $y . \log x = \int \frac{2}{x} \log x \, dx + c = (\log x)^2 + c$ Ans

(iv) $\frac{dy}{dx} - \frac{1}{1+x} y = (1+x) e^x$, here $P = -\frac{1}{1+x}$ and $Q = (1+x) e^x$

Now $I.F = e^{\int P dx} = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$

$\therefore$ sol is $y . \frac{1}{1+x} = \int (1+x) e^x . \frac{1}{(1+x)} dx + c = e^x + c$ Ans

Linear Differential Equation of n^{th} Order With Constant Coefficients :—

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots a_n y = Q \quad \text{--- (1)}$$

where $a_0, a_1, a_2 \cdots a_n$ are constant and Q is the function of x or may be constant.

Now write $\frac{d}{dx} \cong D$, where D is known as differential operator.

$$\therefore (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots a_n) y = Q \quad \text{--- (2)}$$

The general solution or complete solution of equation (1) or (2) is

C.S., $y = C.F. + P.I.$, where C.F. (complementary function) involved n arbitrary constants and P.I. (particular integral) does not involve any arbitrary constant.

Working Rule For Finding C.F. :—

Step 1 :- Put m for D in equation (2) and coefficient of y equal to zero, we get the auxilary equation that is

AE is $\quad a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots a_n = 0$

Step 2 :- Find the roots of the A.E.

Step 3 :- Write the CF according to roots of the AE from the following table —

S.No.	Nature of auxilary equation (AE)	Complementary function (CF)
1-	The real and distinct roots m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2-	The real and same roots m_1, m_1, m_1	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
3-	One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$
4-	Two pair of complex roots $\alpha \pm i\beta, \alpha \pm i\beta$	$e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

Que (2) :- Solve the following differential equations

(i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ (ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 30y = 0$

(iii) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$ (iv) $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

(v) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$ (vi) $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0$, given that $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $y'''(0) = 1$

Sol:- (i) $(D^2 - 3D + 2)y = 0$, the auxlary equation is

$m^2 - 3m + 2 = 0$ or $(m-1)(m-2) = 0$ or $m = 1, 2$

hence, $C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{x} + c_2 e^{2x}$

and $P.I. = 0$, hence complete solution

C.S. $y = CF + PI = c_1 e^{x} + c_2 e^{2x}$ Ans

(ii) $(D^2 + D - 30)y = 0$

Hence A.E. is $m^2 + m - 30 = 0$ or $(m-5)(m+6) = 0$ or $m = 5, -6$

$\therefore$ $C.F. = c_1 e^{5x} + c_2 e^{-6x}$ and $P.I. = 0$

$\therefore$ C.S., $y = c_1 e^{5x} + c_2 e^{-6x}$ Ans

(iii) $(D^2 - 8D + 16)y = 0$

$\therefore$ A.E. is $m^2 - 8m + 16 = 0$ or $(m-4)^2 = 0$ or $m = 4, 4$

$\therefore$ $C.F. = (c_1 + c_2 x)e^{4x}$ and $P.I. = 0$

$\therefore$ C.S. $y = C.F. + P.I. = (c_1 + c_2 x)e^{4x}$ Ans

(iv) $(D^3 - 2D^2 + 4D - 8)y = 0$

$\therefore$ A.E. is $m^3 - 2m^2 + 4m - 8 = 0$

or $(m-2)(m^2+4) = 0$ or $m = 2, \pm 2i$

$\therefore$ $C.F. = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$, and $P.I. = 0$

$\therefore$ C.S., $y = C.F. + P.I. = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$ Ans

(v) $(D^3 - 3D^2 + 4)y = 0$

$\therefore$ A.E. is $m^3 - 3m^2 + 4 = 0$

or $(m+1)(m^2 - 4m + 4) = 0$

or $(m+1)(m-2)^2 = 0$ or $m = -1, 2, 2$

$\therefore$ $C.F. = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$ and $P.I. = 0$

$\therefore$ C.S., $y = CF. + P.I. = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$ Ans

(VI) $(D^4 + D^2)\, y = 0$

$\therefore$ A.E. is $m^4 + m^2 = 0$ or $m^2(m^2+1) = 0$ or $m = 0, 0, \pm i$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{0x} + c_3 \cos x + c_4 \sin x$, and P.I. $= 0$

$\therefore$ C.S. $y =$ C.F. + P.I $= c_1 + c_2 x + c_3 \cos x + c_4 \sin x$ —— (1)

Now differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = y'(x) = c_2 - c_3 \sin x + c_4 \cos x \quad \text{—— (2)}$$

Again differentiating second and and third times, we get

—— (3)

$$\frac{d^2y}{dx^2} = y''(x) = -c_3 \cos x - c_4 \sin x$$

—— (4)

and $\dfrac{d^3y}{dx^3} = y'''(x) = c_3 \sin x - c_4 \cos x$

Now putting $x = 0$ in equations (1), (2), (3) and (4) we get

—— (5)

$y(0) = 0 = c_1 + c_3$

—— (6)

$y'(0) = 0 = c_2 + c_4$

—— (7)

$y''(0) = 0 = -c_3$

—— (8)

and $y'''(0) = 1 = -c_4$

Hence $c_4 = -1$, $c_3 = 0$, $c_2 = 1$ and $c_1 = 0$, putting these values in equation (1), we have

$y = x - \sin x$ <u>Ans</u>

Formula For Finding the Particular Integral (P.I.) :—

Let the equation $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots a_n)\, y = Q$

or $F(D)\, y = Q$

hence P.I. $= \dfrac{1}{F(D)} Q$

Case I :— If $Q = e^{ax}$, then

P.I. $= \dfrac{1}{F(D)} e^{ax} = \dfrac{e^{ax}}{F(a)}$, provided that $F(a) \neq 0$

Exceptional Cases :-

(i) If $F(a) = 0$, then P.I. $= \frac{1}{F(D)} e^{ax} = \frac{x\, e^{ax}}{F'(a)}$, provided $F'(a) \neq 0$

(ii) If $F'(a) = 0$, then P.I. $= \frac{1}{F(D)} e^{ax} = \frac{x^2 e^{ax}}{F''(a)}$, provided $F''(a) \neq 0$, and so on.

Que ③ :- Solve the following differential equations —

(i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ (ii) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$

(iii) $(D-1)^3 y = e^x$ (iv) $(D^3-1)\, y = (1+e^x)^2$

(v) $(D^2+4D+4)\, y = 2\sinh 2x$ (vi) $(D^4 - a^4)\, y = e^{ax}$

Sol :- (i) $(D^2 - 3D + 2)\, y = e^{3x}$

A.E. is $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$

Now P.I. $= \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{e^{3x}}{3^2 - 3\cdot 3 + 2} = \frac{e^{3x}}{2}$

hence C.S. $y =$ C.F + P.I. $= c_1 e^x + c_2 e^{2x} + \frac{e^{3x}}{2}$ Ans

(ii) $(D^3 + 2D^2 - D - 2)\, y = e^x$

A.E. is $m^3 + 2m^2 - m - 2 = 0$

$\Rightarrow (m-1)(m^2 + 3m + 2) = 0 \Rightarrow (m-1)(m+1)(m+2) = 0 \Rightarrow m = 1, -1, -2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$

Now P.I. $= \frac{1}{D^3 + 2D^2 - D - 2} e^x$

$= \frac{x}{3D^2 + 4D - 1} e^x$ (Since $F(a) = 0$)

$= \frac{x e^x}{6}$

$\therefore$ C.S. $y =$ C.F. + P.I. $= c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} + \frac{x e^x}{6}$ Ans

(iii) A.E. is $(m-1)^3 = 0$ or $m = 1, 1, 1$

$$\therefore \text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^x$$

$$\text{P.I.} = \frac{1}{(D-1)^3} e^x$$

$$= \frac{x}{3(D-1)^2} e^x \quad (\text{Since } F(a) = 0)$$

$$= \frac{x^2}{6(D-1)} e^x \quad (\text{Since } F'(a) = 0)$$

$$= \frac{x^3 e^x}{6}$$

$$\therefore \text{C.S. } y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x + c_3 x^2) e^x + \frac{x^3 e^x}{6} \quad \underline{\text{Ans}}$$

(iv) A.E. is $(m^3 - 1) = 0$

$$\Rightarrow (m-1)(m^2+m+1) = 0 \Rightarrow m = 1, -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}$$

$$\therefore \text{C.F.} = c_1 e^x + e^{-\frac{1}{2}x}\left(c_2 \cos \tfrac{\sqrt{3}}{2} x + c_3 \sin \tfrac{\sqrt{3}}{2} x\right)$$

$$\text{P.I.} = \frac{1}{D^3-1}(1+e^x)^2 = \frac{1}{D^3-1}\left(1 + 2e^x + e^{2x}\right)$$

$$= \frac{1}{D^3-1} e^{0x} + \frac{2}{D^3-1} e^x + \frac{1}{D^3-1} e^{2x}$$

$$= \frac{1}{0-1} + \frac{2x}{3D^2} e^x + \frac{e^{2x}}{2^3-1}$$

$$= -1 + \frac{2}{3} x e^x + \frac{1}{7} e^{2x}$$

$$\therefore \text{C.S. } y = \text{C.F} + \text{P.I} = c_1 e^x + e^{-\frac{1}{2}x}\left(c_2 \cos \tfrac{\sqrt{3}}{2} x + c_3 \sin \tfrac{\sqrt{3}}{2} x\right) - 1 + \frac{2}{3} x e^x + \frac{e^{2x}}{7}$$

(v) $(D^2 + 4D + 4)\, y = 2 \sinh 2x$

A.E. is $m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

$$\therefore \text{C.F.} = (c_1 + c_2 x)\, e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2+4D+4}\left(e^{2x} - e^{-2x}\right), \quad \text{Since } \sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$

$$= \frac{1}{D^2+4D+4} e^{2x} - \frac{1}{D^2+4D+4} e^{-2x}$$

$$= \frac{e^{2x}}{2^2 + 4 \cdot 2 + 4} - \frac{x}{2D+4} e^{-2x} = \frac{e^{2x}}{16} - \frac{x^2}{2} e^{-2x}$$

$$\therefore \text{C.S. } y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^{-2x} + \frac{e^{2x}}{16} - \frac{x^2 e^{-2x}}{2} \quad \underline{\text{Ans}}$$

(vi) $(D^4 - a^4)\, y = e^{ax}$

A.E. is $m^4 - a^4 = 0 \Rightarrow (m^2 - a^2)(m^2 + a^2) = 0 \Rightarrow m = a, -a, \pm ai$

$\therefore$ C.F. $= c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

$$\text{P.I.} = \frac{1}{D^4 - a^4} e^{ax} = \frac{x}{4D^3} e^{ax} \quad , \text{ Since } F(a) = 0$$

$$= \frac{x e^{ax}}{4a^3}$$

$\therefore$ C.S. $y = \text{C.F.} + \text{P.I.} = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax + \dfrac{x e^{ax}}{4a^3}$ **Ans**

Case II :— If $Q = \sin ax$ or $\cos ax$ and $\sin(ax+b)$ or $\cos(ax+b)$, then

$$\text{P.I.} = \frac{1}{F(D^2)} \sin ax = \frac{\sin ax}{F(-a^2)} \quad , \text{ provided } F(-a^2) \neq 0$$

and
$$\text{P.I.} = \frac{1}{F(D^2)} \cos ax = \frac{\cos ax}{F(-a^2)} \quad , \text{ provided } F(-a^2) \neq 0$$

Similarly
$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{\sin(ax+b)}{F(-a^2)} \quad , \text{ provided } F(-a^2) \neq 0$$

Exceptional Cases :— If $F(-a^2) = 0$, then

$$\text{P.I.} = \frac{1}{F(D^2)} \sin ax = \frac{x}{F'(D^2)} \sin ax \quad , \text{ provided } F'(-a^2) \neq 0$$

Similarly
$$\frac{1}{F(D^2)} \cos ax = \frac{x}{F'(D^2)} \cos ax \quad , \text{ provided } F'(-a^2) \neq 0$$

Que ④ :- Solve the following differential equations

(i) $\dfrac{d^2y}{dx^2} + 4y = \sin x + \cos 3x$ (ii) $\dfrac{d^4y}{dx^4} + a^4 y = \cos ax$

(iii) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \cos x$ (iv) $\dfrac{d^2y}{dx^2} + 4y = 2\cos^2 x$

(v) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin 2x$ (vi) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x$

Sol :- (i) $(D^2 + 4)\, y = \sin x + \cos 3x$

A.E. is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$

$$P.I. = \frac{1}{D^2+4}\sin x + \frac{1}{D^2+4}\cos 3x$$

$$= \frac{\sin x}{-1+4} + \frac{\cos 3x}{-9+4} = \frac{1}{3}\sin x - \frac{1}{5}\cos 3x$$

$\therefore$ C.S. $y = C.F. + P.I. = C_1\cos 2x + C_2\sin 2x + \frac{1}{3}\sin x - \frac{1}{5}\cos 3x$ Ans

(ii) $(D^4+a^4)y = \cos ax$

A.E. is $m^4 + a^4 = 0$

or $m^4 + a^4 + 2a^2m^2 - 2a^2m^2 = 0$

or $(m^2+a^2)^2 - (\sqrt{2}\,am)^2 = 0$

or $(m^2 - \sqrt{2}\,am + a^2)(m^2 + \sqrt{2}\,am + a^2) = 0$

or $m = \dfrac{\sqrt{2}a \pm \sqrt{2a^2-4a^2}}{2}, \dfrac{-\sqrt{2}a \pm \sqrt{2a^2-4a^2}}{2}$

$= \dfrac{a}{\sqrt{2}} \pm \dfrac{ai}{\sqrt{2}}, -\dfrac{a}{\sqrt{2}} \pm \dfrac{ai}{\sqrt{2}}$

$\therefore$ C.F. $= e^{\frac{a}{\sqrt{2}}x}\left[C_1\cos\frac{a}{\sqrt{2}}x + C_2\sin\frac{a}{\sqrt{2}}x\right] + e^{-\frac{a}{\sqrt{2}}x}\left[C_3\cos\frac{a}{\sqrt{2}}x + C_4\sin\frac{a}{\sqrt{2}}x\right]$

Now P.I $= \dfrac{1}{D^4+a^4}\cos ax = \dfrac{\cos ax}{(-a^2)(-a^2)+a^4} = \dfrac{\cos ax}{2a^4}$

Ans

$\therefore$ C.S. $y = CF + PI$.

(iii) $(D^2+D+1)y = \cos x$

A.E. is $m^2+m+1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\therefore$ C.F. $= e^{-\frac{1}{2}x}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right)$

P.I. $= \dfrac{1}{D^2+D+1}\cos x = \dfrac{1}{-1+D+1}\cos x$

$= \dfrac{1}{D}\cos x = \int \cos x\,dx = \sin x$

$\therefore$ C.S. $y = CF + PI = e^{-\frac{x}{2}}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right)$ Ans

(iv) $(D^2+4)\,y = 2\cos^2 x$

A.E. is $m^2+4=0 \Rightarrow m = \pm 2i$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \dfrac{1}{D^2-4}\, 2\cos^2 x = \dfrac{1}{D^2+4}(1+\cos 2x)$

$= \dfrac{1}{D^2+4} e^{0x} + \dfrac{1}{D^2+4}\cos 2x$

$= \dfrac{1}{0+4} + \dfrac{x}{2D}\cos 2x$

$= \dfrac{1}{4} + \dfrac{x}{2}\int \cos 2x\,dx = \dfrac{1}{4} + \dfrac{1}{4} x \sin 2x$

$\therefore$ C.S. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{4}(1 + x \sin 2x)$ Ans

(v) $(D^2+3D+2)\,y = \sin 2x$

A.E. is $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m = -1, -2$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \dfrac{1}{D^2+3D+2}\sin 2x = \dfrac{1}{-4+3D+2}\sin 2x$

$= \dfrac{(3D+2)}{(3D-2)(3D+2)}\sin 2x = \dfrac{(3D-2)}{9D^2-4}\sin 2x$

$= \dfrac{(3D+2)}{9(-4)-4}\sin 2x = \dfrac{6\cos 2x + 2\sin 2x}{-40}$

$\therefore$ C.S. $y =$ CF + PI $= c_1 e^{-x} + c_2 e^{-2x} - \dfrac{1}{20}(3\cos 2x + \sin 2x)$ Ans

(vi) $(D^3-3D^2+4D-2)\,y = e^x + \cos x$

A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

or $(m-1)(m^2-2m+2) = 0 \Rightarrow m = 1, 1\pm i$

$\therefore$ C.F $= c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$

P.I. $= \dfrac{1}{D^3-3D^2+4D-2} e^x + \dfrac{1}{D^3-3D^2+4D-2}\cos x$

$= \dfrac{x}{3D^2-6D+4} e^x + \dfrac{1}{-D+3+4D-2}\cos x$

$= \dfrac{x e^x}{3-6+4} + \dfrac{1}{3D+1}\cos x = \dfrac{x e^x}{1} + \dfrac{3D-1}{9D^2-1}\cos x = x e^x - \dfrac{1}{10}(-3\sin x - \cos x)$

$\therefore$ C.S. $y =$ CF + PI $= c_1 e^x + e^x(c_2 \cos x + c_3 \sin x) + x e^x + \dfrac{1}{10}(3\sin x + \cos x)$

Que (4) (VII) :- Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ and find the value of y when $x = \frac{\pi}{2}$ being given that $y = 3$, $\frac{dy}{dx} = 0$, when $x = 0$.

Sol :- $(D^2 + 2D + 10)y = -37\sin 3x$

A.E. is $m^2 + 2m + 10 = 0 \Rightarrow m = -1 \pm 3i$

$\therefore$ C.F. $= e^{-x}(c_1\cos 3x + c_2\sin 3x)$

P.I. $= \frac{1}{D^2+2D+10}(-37\sin 3x)$

$= \frac{-37}{-9+2D+10}\sin 3x = \frac{-37}{2D+1}\sin 3x$

$= \frac{-37(2D-1)}{4D^2-1}\sin 3x = \frac{-37(2D-1)}{4(-9)-1}\sin 3x$

$= 6\cos 3x - \sin 3x$

$\therefore$ C.S. $y = \text{C.F.} + \text{PI} = e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x$ —— (1)

put $x=0$ in eq^n (1), we get

$\therefore y(0) = 3 = c_1 + 6 \Rightarrow c_1 = -3$

differentiating equation (1) w.r.t. we get

$\frac{dy}{dx} = e^{-x}(-3c_1\sin 3x + 3c_2\cos 3x) - e^{-x}(c_1\cos 3x + c_2\sin 3x) - 18\sin 3x - 3\cos 3x$ —— (2)

put $x=0$ in eq^n (2), we get

$\frac{dy}{dx} = 0 = 3c_2 - c_1 - 3 \Rightarrow c_2 = 0$ (since $c_1 = -3$)

hence from equation (1), we get

$\therefore y = (6 - 3e^{-x})\cos 3x - \sin 3x$ —— (3)

Now again put $x = \frac{\pi}{2}$ in equation (3), we get

$y = -\sin\frac{3\pi}{2} = 1$

Ans

Case III :- If $Q = x^n$, where n is possitive integer, then

$$PI = \frac{1}{F(D)} x^n = \frac{1}{1 \pm \phi(D)} x^n = [1 \pm \phi(D)]^{-1} x^n$$

Again expand by Binomial theorem and operate over x^n.

Note :- (i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ (ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

Que (5) :- Solve the following differential equations

(i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^2$ (ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + 2x + 1$

(iii) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 1 + x^2$ (iv) $\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$

Sol :- (i) $(D^2 + 2D + 2) y = x^2$

A.E. is $m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$

$\therefore CF = e^{-x}(c_1 \cos x + c_2 \sin x)$

$$P.I. = \frac{1}{D^2+2D+2} x^2 = \frac{1}{2\left(1 + \frac{D^2+2D}{2}\right)} x^2 = \frac{1}{2}\left[1 + \left(\frac{D^2+2D}{2}\right)\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1 - \left(\frac{D^2+2D}{2}\right) + \left(\frac{D^2+2D}{2}\right)^2 - \cdots\right] x^2$$

$$= \frac{1}{2}\left[x^2 - \left(\frac{2+4x}{2}\right) + 2\right] = \frac{1}{2}(x^2 - 2x + 1)$$

$\therefore$ C.S. $y = C.F. + PI = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x^2 - 2x + 1)$ **Ans**

(ii) $(D^2 - 3D + 2) y = x^2 + 2x + 1$

A.E. is $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$\therefore C.F. = c_1 e^x + c_2 e^{2x}$

$$P.I. = \frac{1}{D^2 - 3D + 2}(x^2 + 2x + 1) = \frac{1}{2}\left[1 + \left(\frac{D^2-3D}{2}\right)\right]^{-1}(x^2 + 2x + 1)$$

$$= \frac{1}{2}\left[1 - \left(\frac{D^2-3D}{2}\right) + \left(\frac{D^2-3D}{2}\right)^2 - \cdots\right](x^2 + 2x + 1)$$

$$= \frac{1}{2}\left[x^2 + 2x + 1 - \left(\frac{2 - 6x - 6}{2}\right) + \frac{9}{4} \cdot 2\right]$$

$$= \frac{1}{2}\left(x^2 + 2x + 1 - 1 + 3x + 3 + \frac{9}{2}\right) = \frac{1}{2}\left(x^2 + 5x + \frac{15}{2}\right)$$

$\therefore$ C.S. $y = C.F. + PI = c_1 e^x + c_2 e^{2x} + \frac{1}{2}\left(x^2 + 5x + \frac{15}{2}\right)$ **Ans**

(iii) $(D^3+3D^2+2D)y = 1+x^2$

A.E. is $m^3+3m^2+2m=0$ or $m(m^2+3m+2)=0$

or $m(m+1)(m+2)=0 \Rightarrow m=0,-1,-2$

$\therefore$ C.F. $= c_1 + c_2 e^{-x} + c_3 e^{-2x}$

P.I. $= \dfrac{1}{D^3+3D^2+2D}(1+x^2)$

$= \dfrac{1}{2D\left(1+\frac{D^3+3D^2}{2D}\right)}(1+x^2) = \dfrac{1}{2D}\left[1+\left(\dfrac{D^2+3D}{2}\right)\right]^{-1}(1+x^2)$

$= \dfrac{1}{2D}\left[1-\left(\dfrac{D^2+3D}{2}\right)+\left(\dfrac{D^2+3D}{2}\right)^2 - \cdots\right](1+x^2)$

$= \dfrac{1}{2D}\left[1+x^2-\left(\dfrac{2+6x}{2}\right)+\dfrac{9}{4}\cdot 2\right]$

$= \dfrac{1}{2D}\left[x^2-3x+\dfrac{9}{2}\right] = \dfrac{1}{2}\int\left(x^2-3x+\dfrac{9}{2}\right)dx = \dfrac{1}{2}\left(\dfrac{x^3}{3}-\dfrac{3x^2}{2}+\dfrac{9}{2}x\right)$

$\therefore$ C.S. $y = $ C.F. $+$ P.I. $= c_1+c_2e^{-x}+c_3e^{-2x}+\dfrac{1}{2}\left(\dfrac{x^3}{3}-\dfrac{3x^2}{2}+\dfrac{9}{2}x\right)$ Ans

(iv) $(D^2+1)y = e^{2x}+\cosh 2x + x^3$

A.E. is $m^2+1=0 \Rightarrow m=\pm i$

$\therefore$ C.F. $= c_1\cos x + c_2 \sin x$

P.I. $= \dfrac{1}{D^2+1}e^{2x} + \dfrac{1}{D^2+1}\cosh 2x + \dfrac{1}{D^2+1}x^3$

$= \dfrac{e^{2x}}{4+1} + \dfrac{1}{D^2+1}\left(\dfrac{e^{2x}+e^{-2x}}{2}\right) + [1+D^2]^{-1}x^3$

$= \dfrac{e^{2x}}{5} + \dfrac{1}{5}\left(\dfrac{e^{2x}+e^{-2x}}{2}\right) + [1-D^2+D^4-\cdots]x^3$

$= \dfrac{e^{2x}}{5} + \dfrac{1}{5}\cosh 2x + (x^3-6x)$

$\therefore$ C.S. $y = $ C.F. $+$ PI $= c_1\cos x + c_2\sin x + \dfrac{e^{2x}}{5}+\dfrac{1}{5}\cosh 2x + x^3-6x$ Ans

Case IV :— If $Q = e^{ax}\phi(x)$, then $\frac{1}{F(D)} e^{ax}\phi(x) = \frac{e^{ax}}{F(D+a)} \phi(x)$

Que (6) Solve the following differential equations

(i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^x$ (ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

(iii) $\frac{d^2y}{dx^2} - 4y = x^2 e^{3x}$ (iv) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}\cos 3x$

Sol:— (i) $(D^2 - 2D + 1)y = x^2 e^x$

A.E. is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$ C.F. $= (c_1 + c_2 x)e^x$

P.I. $= \frac{1}{D^2 - 2D + 1} x^2 e^x = \frac{1}{(D-1)^2} e^x x^2$

$= \frac{e^x}{(D+1-1)^2} x^2 = \frac{e^x}{D^2} x^2 = \frac{e^x}{D} \int x^2 dx$

$= \frac{e^x}{D}\left(\frac{x^3}{3}\right) = e^x \int \frac{x^3}{3} dx = e^x \frac{x^4}{12}$

$\therefore$ C.S. $y =$ C.F. + P.I. $= (c_1 + c_2 x)e^x + \frac{e^x x^4}{12}$ Ans

(ii) $(D^2 - 2D + 1)y = x e^x \sin x$

C.F. $= (c_1 + c_2 x)e^x$

P.I. $= \frac{1}{(D-1)^2} e^x(x \sin x) = \frac{e^x}{(D+1-1)^2} x \sin x$

$= \frac{e^x}{D^2} x \sin x = \frac{e^x}{D} \int x \sin x \, dx$

$= \frac{e^x}{D}(-x\cos x + \sin x) = e^x \int (-x \cos x + \sin x) dx$

$= -e^x (x \sin x + 2\cos x)$

$\therefore$ C.S. $y =$ C.F. + P.I. $= (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x)$ Ans

(iii) $(D^2 - 4)y = x^2 e^{3x}$

A.E. is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

$\therefore$ CF $= c_1 e^{2x} + c_2 e^{-2x}$

PI $= \frac{1}{D^2 - 4} e^{3x} x^2 = \frac{e^{3x}}{(D+3)^2 - 4} x^2 = \frac{e^{3x}}{D^2 + 6D + 5} x^2$

$= \frac{e^{3x}}{5}\left[1 + \left(\frac{D^2 + 6D}{5}\right)\right]^{-1} x^2$

$$= \frac{e^{3x}}{5}\left[1 - \left(\frac{D^2+6D}{5}\right) + \left(\frac{D^2+6D}{5}\right)^2 - \cdots\cdots\right] x^2$$

$$= \frac{e^{3x}}{5}\left[x^2 - \left(\frac{2+12x}{5}\right) + \frac{36}{25}\cdot 2\right]$$

$$= \frac{e^{3x}}{5}\left(x^2 - \frac{12}{5}x + \frac{62}{25}\right)$$

$\therefore$ C.S. $y = C.F. + P.I. = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{3x}}{5}\left(x^2 - \frac{12}{5}x + \frac{62}{25}\right)$ Ans

(iv) $(D^2 - 4D + 13)y = e^{2x}\cos 3x$

A.E. is $m^2 - 4m + 13 = 0 \Rightarrow m = 2 \pm 3i$

C.F. $= e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

$$P.I. = \frac{1}{D^2-4D+13} e^{2x}\cos 3x = \frac{e^{2x}}{(D+2)^2 - 4(D+2) + 13}\cos 3x$$

$$= \frac{e^{2x}}{D^2+9}\cos 3x = \frac{x e^{2x}}{2D}\cos 3x$$

$$= \frac{x e^{2x}}{2}\int \cos 3x\, dx = \frac{x e^{2x}\sin 3x}{6}$$

$\therefore$ C.S. $y = C.F. + P.I = e^{2x}(c_1\cos 3x + c_2 \sin 3x) + \frac{1}{6} x e^{2x}\sin 3x$ Ans

Que (6) (v) :– A body executes damped forced vibrations given by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2 x = e^{-kt}\sin \omega t$$

Solve the equation for both cases, when $\omega^2 \neq b^2 - k^2$ and $\omega^2 = b^2 - k^2$.

Sol :– A.E. is $m^2 - 2km + b^2 = 0 \Rightarrow m = -k \pm \sqrt{k^2-b^2} = -k \pm i\sqrt{b^2-k^2}$

Case I :– when $\omega^2 \neq b^2 - k^2$

$\therefore$ C.F. $= e^{-kt}(c_1 \cos\sqrt{b^2-k^2}\, t + c_2 \sin\sqrt{b^2-k^2}\, t)$

$$P.I. = \frac{1}{D^2+2kD+b^2} e^{-kt}\sin\omega t = \frac{e^{-kt}}{(D-k)^2 + 2k(D-k) + b^2}\sin\omega t$$

$$= e^{-kt}\cdot\frac{1}{D^2+(b^2-k^2)}\sin\omega t = \frac{e^{-kt}\sin\omega t}{-\omega^2+b^2-k^2}$$

$\therefore$ C.S. $x = e^{-kt}(c_1\cos\sqrt{b^2-k^2}\, t + c_2 \sin\sqrt{b^2-k^2}\, t) + \frac{e^{-kt}\sin\omega t}{b^2-k^2-\omega^2}$

Case II :– when $\omega^2 = b^2 - k^2$, then C.F. $= e^{-kt}(c_1\cos\omega t + c_2 \sin\omega t)$

$$P.I. = \frac{1}{D^2+2kD+b^2} e^{-kt}\sin\omega t = \frac{e^{-kt}}{D^2+b^2-k^2}\sin\omega t$$

$$= \frac{e^{-kt}\, t}{2D}\sin\omega t = \frac{t e^{-kt}}{2}\left(-\frac{\cos\omega t}{\omega}\right)$$

$\therefore$ C.S. $x = C.F. + P.I = e^{-kt}(c_1\cos\omega t + c_2\sin\omega t) - \frac{t}{2\omega} e^{-kt}\cos\omega t$ Ans

Case-V :– If $Q = x^n \sin ax$ or $x^n \cos ax$, then

$$\frac{1}{F(D)} x^n \sin ax = \text{Imaginary part of } \frac{1}{F(D)} x^n e^{iax} = \text{I.p. of } e^{iax} \cdot \frac{1}{F(D+ia)} x^n$$

$$\frac{1}{F(D)} x^n \cos ax = \text{Real part of } \frac{1}{F(D)} x^n e^{iax} = \text{R.p. of } e^{iax} \cdot \frac{1}{F(D+ia)} x^n$$

Que (7) :– Solve the following differential equations

(i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x$ (ii) $\frac{d^2y}{dx^2} + y = x \cos 2x$

(iii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 e^{-x} \cos x$

Sol :– (i) $(D^2 - 2D + 1) y = x \sin x$

A.E. is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

C.F. $= (c_1 + c_2 x) e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x \sin x = \text{Imaginary part of } \frac{1}{D^2 - 2D + 1} x e^{ix}$$

$$= \text{I.p. of } e^{ix} \cdot \frac{1}{(D+i)^2 - 2(D+i) + 1} x = \text{I.p. of } e^{ix} \frac{1}{D^2 + 2(i-1)D - 2i} x$$

$$= \text{I.p. of} \cdot \frac{e^{ix}}{-2i} \left[1 - \left\{\frac{D^2 + 2(i-1)D}{2i}\right\}\right]^{-1} x$$

$$= \text{I.p. of } \frac{e^{ix}}{-2i} \left[1 + \frac{D^2 + 2(i-1)D}{2i} + \cdots\right] x$$

$$= \text{I.p. of} \cdot \frac{e^{ix}}{-2i} \left[x + \frac{\cancel{2}(i-1)}{\cancel{2}i}\right]$$

$$= \text{I.p. of } e^{ix} \left[\frac{x}{-2i} + \frac{(i-1)}{2}\right]$$

$$= \text{I.p. of } (\cos x + i \sin x) \left[\left(\frac{x+1}{2}\right) i - \frac{1}{2}\right]$$

$$= \left(\frac{x+1}{2}\right) \cos x - \frac{1}{2} \sin x$$

$\therefore$ C.S. $y = \text{C.F.} + \text{PI} = (c_1 + c_2 x) e^x + \left(\frac{x+1}{2}\right) \cos x - \frac{1}{2} \sin x$ Ans

(ii) $(D^2 + 1) y = x \cos 2x$

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

$$P.I. = \frac{1}{D^2+1} x\cos 2x = \text{Real part of } \frac{1}{D^2+1} x e^{2ix}$$

$$= R.p. \text{ of } e^{2ix} \cdot \frac{1}{(D+2i)^2+1} x = R.p. \text{ of } e^{2ix} \cdot \frac{1}{D^2+2iD-3} x$$

$$= R.p. \text{ of } \frac{e^{2ix}}{-3} \left[1 - \left(\frac{D^2+2iD}{3}\right)\right]^{-1} x$$

$$= R.p. \text{ of } \frac{e^{2ix}}{-3} \left[1 + \left(\frac{D^2+2iD}{3}\right) + \cdots\right] x$$

$$= R.p. \text{ of } \frac{e^{2ix}}{-3} \left[x + \frac{0+2i}{3}\right]$$

$$= R.p. \text{ of } \frac{(\cos 2x + i\sin 2x)}{-3} \left[x + \frac{2}{3} i\right]$$

$$= -\frac{1}{3}\left(x\cos 2x - \frac{2}{3}\sin 2x\right)$$

$$\therefore C.S.\ y = C.F. + P.I. = c_1\cos x + c_2 \sin x + \frac{2}{9}\sin 2x - \frac{1}{3} x\cos 2x$$ Ans

(iii) $(D^2+2D+1)y = x^2 e^{-x}\cos x$

A.E. is $m^2+2m+1=0 \Rightarrow m = -1, -1$

$$\therefore C.F. = (c_1 + c_2 x)e^{-x}$$

$$P.I. = \frac{1}{(D+1)^2} e^{-x} x^2 \cos x = e^{-x} \cdot \frac{1}{(D-1+1)^2} x^2\cos x$$

$$= e^{-x} \frac{1}{D^2} x^2 \cos x = e^{-x} \cdot \frac{1}{D} \int x^2 \cos x \, dx$$

$$= e^{-x} \cdot \frac{1}{D}\left[x^2 \sin x - 2x(-\cos x) + 2(-\sin x)\right]$$

$$= e^{-x} \int (x^2 \sin x + 2x\cos x - 2\sin x)\,dx$$

$$= e^{-x}\left[x^2(-\cos x) - 2x(-\sin x) + 2\cos x + 2x\sin x + 2\cos x + 2\cos x\right]$$

$$= e^{-x}(-x^2\cos x + 4x\sin x + 6\cos x)$$

$$\therefore C.S.\ y = C.F. + P.I. = (c_1 + c_2 x)e^{-x} + e^{-x}(-x^2\cos x + 4x\sin x + 6\cos x)$$ Ans

Case VI :- General Method :- If $Q = \phi(x)$ (any function of x)

$$\boxed{\frac{1}{D-a}\phi(x) = e^{ax}\int e^{-ax}\phi(x)\,dx}$$

Que 8 :- Solve the following differential equations –

(i) $\frac{d^2y}{dx^2} + 9y = \sec 3x$ (ii) $\frac{d^2y}{dx^2} + y = \tan x$

(iii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ (iv) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin e^{-x}$

(v) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = e^{-x}\sec^3 x$ (vi) $\frac{d^2y}{dx^2} + y = x - \cot x$

Sol :- (i) $(D^2+9)y = \sec 3x$

A.E. is $m^2+9=0 \Rightarrow m = \pm 3i$

$\therefore$ C.F. $= c_1\cos 3x + c_2 \sin 3x$

$$\text{P.I.} = \frac{1}{D^2+9}\sec 3x = \frac{1}{(D+3i)(D-3i)}\sec 3x$$

$$= \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right]\sec 3x = \frac{1}{6i}\left[\frac{1}{D-3i}\sec 3x - \frac{1}{D+3i}\sec 3x\right]$$

$$= \frac{1}{6i}\left[A - B\right] \quad \text{—— (1)}$$

Now $A = \frac{1}{D-3i}\sec 3x = e^{3i}\int e^{-3i}\sec 3x\,dx$

$$= e^{3i}\int(\cos 3x - i\sin 3x)\sec 3x\,dx$$

$$= e^{3i}\int(1 - i\tan 3x)\,dx$$

$$= e^{3i}\left(x + \frac{i}{3}\log\cos 3x\right)$$

Similarly $B = \frac{1}{D+3i}\sec 3x = e^{-3i}\left(x - \frac{i}{3}\log\cos 3x\right)$

Hence from eqn (1) we get

$$\text{P.I.} = \frac{1}{6i}\left[e^{3i}\left(x + \frac{i}{3}\log\cos 3x\right) - e^{-3i}\left(x - \frac{i}{3}\log\cos 3x\right)\right]$$

$$= \frac{x}{3}\left(\frac{e^{3i} - e^{-3i}}{2i}\right) + \frac{1}{9}\left(\frac{e^{3i}+e^{-3i}}{2}\right)\log\cos 3x$$

$$= \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\log\cos 3x$$

$\therefore$ C.S. $y = \text{C.F.} + \text{P.I.} = c_1\cos x + c_2\sin x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\log\cos 3x$

Ans

(ii) $(D^2+1)y = \tan x$

$C.F = c_1 \cos x + c_2 \sin x$

$P.I. = \frac{1}{D^2+1}\tan x = \frac{1}{(D-i)(D+i)}\tan x = \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\tan x$

$= \frac{1}{2i}\left[\frac{1}{D-i}\tan x - \frac{1}{D+i}\tan x\right] = \frac{1}{2i}[A-B]$ —— ①

Now $A = \frac{1}{D-i}\tan x = e^{ix}\int e^{-ix}\tan x\,dx$

$= e^{ix}\int(\cos x - i\sin x)\frac{\sin x}{\cos x}dx = e^{ix}\int\left[\sin x - i\left(\frac{1-\cos^2 x}{\cos x}\right)\right]dx$

$= e^{ix}\int[\sin x - i(\sec x - \cos x)]dx$

$= e^{ix}[-\cos x - i\log(\sec x + \tan x) + i\sin x]$

$= e^{ix}[-(\cos x - i\sin x) - i\log(\sec x + \tan x)]$

$= e^{ix}[-e^{-ix} - i\log(\sec x + \tan x)]$

$= -1 - ie^{ix}\log(\sec x + \tan x)$

Similarly $B = \frac{1}{D+i}\tan x = -1 + ie^{-ix}\log(\sec x + \tan x)$

Putting the values of A and B in equation ①, we get

$P.I. = \frac{1}{2i}\left[-\cancel{1} - ie^{ix}\log(\sec x + \tan x) + \cancel{1} - ie^{-ix}\log(\sec x + \tan x)\right]$

$= -\left(\frac{e^{ix}+e^{-ix}}{2}\right)\log(\sec x + \tan x) = -\cos x\log(\sec x + \tan x)$

$\therefore$ C.S. $y = C.F + PI = c_1\cos x + c_2\sin x - \cos x\log(\sec x + \tan x)$ Ans

(iii) $(D^2+3D+2)y = e^{e^x}$

$C.F. = c_1 e^{-x} + c_2 e^{-2x}$

$P.I. = \frac{1}{D^2+3D+2}e^{e^x} = \frac{1}{(D+1)(D+2)}e^{e^x} = \left[\frac{1}{D+1} - \frac{1}{D+2}\right]e^{e^x}$

$= \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x}$

$= e^{-x}\int e^x e^{e^x}dx - e^{-2x}\int e^{2x}e^{e^x}dx$, put $e^x = t \Rightarrow e^x dx = dt$

$= e^{-x}\int e^t\,dt - e^{-2x}\int te^t\,dt$

$= e^{-x}\cdot e^t - e^{-2x}(te^t - e^t) = e^{-x}\cdot e^{e^x} - e^{-x}\cdot e^{e^x} + e^{-2x}e^{e^x} = e^{-2x}e^{e^x}$

$\therefore$ C.S. $y = C.F. + PI = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}e^{e^x}$ Ans

(iv) $(D^2-3D+2)y = \sin e^{-x}$

$C.F. = c_1 e^{x} + c_2 e^{2x}$

$$PI = \frac{1}{D^2-3D+2}\sin e^{-x} = \frac{1}{(D-1)(D-2)}\sin e^{-x} = \frac{1}{D-}\sin e^{-x} - \frac{1}{D-}\sin e^{-x}$$

$$= e^{2x}\int e^{-2x}\sin e^{-x}\,dx - e^{x}\int e^{-x}\sin e^{-x}\,dx \quad , \text{ put } e^{-x} = t \Rightarrow -e^{-x}dx = dt$$

$$= e^{2x}\int t\sin t\,(-dt) - e^{x}\int \sin t\,(-dt)$$

$$= e^{2x}(t\cos t - \sin t) - e^{x}\cos t$$

$$= e^{2x}(\cancel{e^{-x}\cos e^{-x}} - \sin e^{-x}) - \cancel{e^{x}\cos e^{-x}} = -e^{2x}\sin e^{-x}$$

$\therefore$ C.S. $y = C.F. + P.I. = c_1 e^{x} + c_2 e^{2x} - e^{2x}\sin e^{-x}$ Ans

(v) $(D^2+2D+2)y = e^{-x}\sec^3 x$

A.E. is $m^2+2m+2=0 \Rightarrow m = -1 \pm i$

$C.F. = e^{-x}(c_1\cos x + c_2\sin x)$

$$P.I. = \frac{1}{D^2+2D+2}e^{-x}\sec^3 x = \frac{e^{-x}}{(D-1)^2+2(D-1)+2}\sec^3 x = \frac{e^{-x}}{D^2+1}\sec^3 x$$

$$= e^{-x}\frac{1}{(D-i)(D+i)}\sec^3 x = \frac{e^{-x}}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec^3 x$$

$$= \frac{e^{-x}}{2i}\left[\frac{1}{D-i}\sec^3 x - \frac{1}{D+i}\sec^3 x\right] = \frac{e^{-x}}{2i}[A-B] \quad \text{--- (1)}$$

Now $A = \frac{1}{D-i}\sec^3 x = e^{ix}\int e^{-ix}\sec^3 x\,dx = e^{ix}\int(\cos x - i\sin x)\sec^3 x\,dx$

$$= e^{ix}\int\left(\sec^2 x - i\frac{\sin x}{\cos^3 x}\right)dx \quad , \text{ put } \cos x = t \Rightarrow -\sin x\,dx = dt$$

$$= e^{ix}\left(\tan x - \frac{i}{2}\sec^2 x\right)$$

Similarly $B = \frac{1}{D+i}\sec^3 x = e^{-ix}\left(\tan x + \frac{i}{2}\sec^2 x\right)$

Hence from eqn (1)

$$P.I. = \frac{e^{-x}}{2i}\left[e^{ix}\left(\tan x - \frac{i}{2}\sec^2 x\right) - e^{-ix}\left(\tan x + \frac{i}{2}\sec^2 x\right)\right]$$

$$= e^{-x}\left[\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\tan x - \frac{1}{2}\left(\frac{e^{ix}+e^{-ix}}{2}\right)\sec^2 x\right]$$

$$= e^{-x}\left[\sin x\tan x - \frac{1}{2}\cos x\sec^2 x\right]$$

$$= e^{-x}\left[\sin x\tan x - \frac{1}{2}\sec x\right]$$

$\therefore$ C.S. $y = C.F. + P.I. = e^{-x}\left[c_1\cos x + c_2\sin x + \sin x\tan x - \frac{1}{2}\sec x\right]$ Ans

Homogeneous Linear Differential Equation or Cauchy - Euler Equation :

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots a_n y = R \quad \text{—(i)}$$

where $a_0, a_1, a_2 \cdots a_n$ are constant and R is a function of x or constant. To solve this type of differerential equation, we put

$$x = e^z \quad \text{or} \quad z = \log x \quad \text{or} \quad \frac{dz}{dx} = \frac{1}{x}$$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = Dy, \quad \left(\text{where } D \equiv \frac{d}{dz}\right)$$

angain $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D(D-1)y$$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Putting these values in the above equation (i), we get a linear differental equation with constant coefficients.

Que (9) :- Solve the following differential equations:—

(a) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (\log x)\sin(\log x)$

(b) $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2\log x$

(c) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$

(d) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 24x^2$

(e) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$

(f) $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x\log x$

(g) $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = xe^x$

Sol (a) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (\log x)\sin(\log x)$ ——— (1)

put $x = e^z \Rightarrow z = \log x \quad \therefore \frac{dz}{dx} = \frac{1}{x}$

and $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$, where $D \equiv \frac{d}{dz}$

$\therefore \quad [D(D-1) + D + 1]y = z\sin z$

or $(D^2+1)y = z\sin z$ ——— (2)

Auxilary equation is $m^2+1=0 \Rightarrow m = \pm i$

$C.F. = c_1\cos z + c_2\sin z = c_1\cos(\log x) + c_2\sin(\log x)$

$P.I = \frac{1}{D^2+1} z\sin z$

$=$ Imaginary part of $\frac{1}{D^2+1} z e^{iz}$

$=$ " " $e^{iz}\frac{1}{(D+i)^2+1} z$

$=$ " " $e^{iz}\frac{1}{D^2+2iD} z$

$=$ " " $e^{iz}\frac{1}{2iD}\left[1+\frac{D}{2i}\right]^{-1} z$

$=$ " " $e^{iz}\frac{1}{2iD}\left[1-\frac{D}{2i}+\left(\frac{D}{2i}\right)^2 - \cdots\right] z$

$=$ " " $\frac{e^{iz}}{\cancel{2i}} \cdot \frac{1}{2iD}\left[z - \frac{1}{2i}\right]$

$=$ " " $\frac{e^{iz}}{2}\left[\frac{z^2}{2i} + \frac{1}{2}z\right]$

$=$ " " $(\cos z + i\sin z)\cdot\frac{1}{4}\cdot\left[-z^2 i + z\right]$

$= \frac{1}{4}\left(z\sin z - z^2\cos z\right)$

$= \frac{1}{4}\left[\log x \sin(\log x) - (\log x)^2\cos(\log x)\right]$

$\therefore$ C.S. $y = c_1\cos(\log x) + c_2\sin(\log x) + \frac{1}{4}\left[\log \sin(\log x) - (\log x)^2\cos(\log x)\right]$

(b) give equation can be written as

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3 \log x \quad \text{——— (1)}$$

put $x = e^z \Rightarrow z = \log x$ and $D \equiv \frac{d}{dz}$, then equation (1) reduces to

$$[D(D-1)(D-2) + 3D(D-1) + D] = z\, e^{3z}$$

or $\quad D^3 y = z e^{3z} \quad$ ——— (2)

Auxilary equation is $\quad m^3 = 0 \quad \Rightarrow m = 0, 0, 0$

$\therefore$ C.F. $= c_1 + c_2 z + c_3 z^2 = c_1 + c_2 \log x + c_3 (\log x)^2$

$$P.I. = \frac{1}{D^3} z e^{3z} = e^{3z} \cdot \frac{1}{(D+3)^3} z = e^{3z} \cdot \frac{1}{27}\left[1 + \frac{D}{3}\right]^{-3} z$$

$$= e^{3z} \cdot \frac{1}{27}\left[1 - \frac{3D}{3} + \left(\frac{D}{3}\right)^2 \cdots\cdots\right] z$$

$$= \frac{e^{3z}}{27}(z - 1) = \frac{x^3}{27}(\log x - 1)$$

$\therefore$ complete sol is $\quad y = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{x^3}{27}(\log x - 1)$

(c) $$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x \quad \text{——— (1)}$$

put $x = e^z$ and $D \equiv \frac{d}{dz}$, we get

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\, y = e^z + z$$

$\Rightarrow (D^3 + 1)\, y = e^z + z \quad$ ——— (2)

Auxilary equation is $\quad m^3 + 1 = 0 \quad \Rightarrow (m+1)(m^2 - m + 1) = 0$

$\quad \Rightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$

$$\therefore \text{C.F.} = c_1 e^{-z} + e^{\frac{z}{2}}\left(c_2 \cos\frac{\sqrt{3}}{2} z + c_3 \sin\frac{\sqrt{3}}{2} z\right)$$

$$= \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$$

$$P.I. = \frac{1}{D^3+1}(e^z + z) = \frac{1}{D^3+1} e^z + \frac{1}{D^3+1} z$$

$$= \frac{e^z}{1+1} + [1 + D^3]^{-1} z = \frac{e^z}{2} + (1 - D^3 + \cdots) z = \frac{e^z}{2} + z$$

$$= \frac{x}{2} + \log x$$

$\therefore$ complete solution is $\quad y =$ C.F. + P.I.

(d) $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 24x^2$ ——— ①

put $x = e^z \Rightarrow z = \log x$ and $D \equiv \frac{d}{dz}$ then

$[D(D-1)(D-2) + 3D(D-1) + D]\, y = 24\, e^{2z}$

$\Rightarrow \quad D^3 y = 24\, e^{2z}$ ——— ②

A.E. is $\quad m^3 = 0 \Rightarrow m = 0, 0, 0$

C.F. $= c_1 + c_2 z + c_3 z^2 = c_1 + c_2 \log x + c_3 (\log x)^2$

P.I. $= \frac{1}{D^3} 24\, e^{2z} = \frac{24\, e^{2z}}{8} = 3e^{2z} = 3x^2$

complete sol. $y = c_1 + c_2 \log x + c_3 (\log x)^2 + 3x^2$

(e) $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$ ——— ①

put $x = e^z \Rightarrow z = \log x$ and let $D \equiv \frac{d}{dz}$ then

$[D(D-1) - 3D + 5]\, y = e^{2z} \sin z$

$\Rightarrow (D^2 - 4D + 5)\, y = e^{2z} \sin z$ ——— ②

A.E. is $\quad m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i$

C.F. $= e^{2z}(c_1 \cos z + c_2 \sin z) = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)]$

P.I. $= \frac{1}{D^2 - 4D + 5} e^{2z} \sin z$

$= e^{2z} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 5} \sin z$

$= e^{2z} \cdot \frac{1}{D^2 + 1} \sin z$

$= e^{2z} \cdot \frac{z}{2D} \sin z$

$= e^{2z} \cdot \frac{z}{2} \int \sin z\, dz$

$= \frac{z e^{2z}}{2}(-\cos z) = -\frac{x^2 (\log x) \cos(\log x)}{2}$

$\therefore$ Complete sol is $y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{x^2}{2}(\log x)\cos(\log x)$

(f) $x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x \log x$ ——— (1)

Let $x = e^z \Rightarrow z = \log x$ and let $D \equiv \frac{d}{dz}$ then

$[D(D-1) + 5D + 4]\, y = z e^z$

$\Rightarrow (D^2 + 4D + 4)\, y = z e^z$ ——— (2)

A.E. is $m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

$\therefore$ C.F. is $= (c_1 + c_2 z)\, e^{-2z} = \frac{1}{x^2}(c_1 + c_2 \log x)$

$$P.I. = \frac{1}{D^2+4D+4} z e^z = \frac{1}{(D+2)^2} z e^z$$

$$= e^z \cdot \frac{1}{(D+3)^2} z = \frac{e^z}{9}\left[1 + \frac{D}{3}\right]^{-2} z$$

$$= \frac{e^z}{9}\left[1 - 2\frac{D}{3} + \cdots\right] z$$

$$= \frac{e^z}{9}\left(z - \frac{2}{3}\right)$$

$$= \frac{x}{9}\left(\log x - \frac{2}{3}\right)$$

$\therefore$ Complete sol. is $y = \frac{1}{x^2}(c_1 + c_2 \log x) + \frac{x}{9}\left(\log x - \frac{2}{3}\right)$

(g) $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = x e^x$ ——— (1)

Let $x = e^z \Rightarrow z = \log x$, and let $D \equiv \frac{d}{dz}$ we get

$[D(D-1) + 2D]\, y = e^z e^{e^z}$

$\Rightarrow (D^2 + D)\, y = e^z e^{e^z}$ ——— (2)

A.E. is $m^2 + m = 0 \Rightarrow m = 0, -1$

C.F. $= c_1 + c_2 e^{-z} = c_1 + c_2 x^{-1}$

$$P.I. = \frac{1}{D^2+D} e^z e^{e^z} = \frac{1}{D(D+1)} e^z \cdot e^{e^z} = \left[\frac{1}{D} - \frac{1}{D+1}\right] e^z e^{e^z}$$

$$= \frac{1}{D} e^z e^{e^z} - \frac{1}{D+1} e^z e^{e^z} = \int e^z e^{e^z} dz - e^{-z}\int e^z (e^z e^{e^z})\, dz$$

(Since $\frac{1}{D+a}\phi(x) = e^{-ax}\int e^{ax}\phi(x)\,dx$)

$$= \int e^z e^{e^z} dz - e^{-z}\int e^{2z} e^{e^z} dz$$

$$= e^{e^z} - e^{-z}(e^z - 1)\, e^{e^z} = e^{-z} \cdot e^{e^z} = x^{-1} e^x$$

$\therefore$ complete solution is $y = c_1 + c_2 x^{-1} + x^{-1} e^x$

Reducible Case — An equation of the form

$$a_0 (a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 (a+bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots a_n y = R$$

can be reduced to linear differential equations with constant coefficients, by the substitution

$$a+bx = e^z \quad \text{i.e.} \quad z = \log(a+bx) \Rightarrow \frac{dz}{dx} = \frac{b}{a+bx}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a+bx} \cdot \frac{dy}{dz}$$

$$\Rightarrow (a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = bDy \qquad \left(\text{Since } D \equiv \frac{d}{dz}\right)$$

Similarly $(a+bx)^2 \frac{d^2 y}{dx^2} = b^2 D(D-1) y$,

$$(a+bx)^3 \frac{d^3 y}{dx^3} = b^3 D(D-1)(D-2) y \quad \text{and so on.}$$

Que (10) :— Solve the following differential equations —

(a) $(1+x)^2 \dfrac{d^2 y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\{\log(1+x)\}$

(b) $(1+2x)^2 \dfrac{d^2 y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$

Sol (a) :— Put $(1+x) = e^z \Rightarrow z = \log(1+x) \Rightarrow \dfrac{dz}{dx} = \dfrac{1}{1+x}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{1+x}\frac{dy}{dz} \Rightarrow (1+x)\frac{dy}{dx} = \frac{dy}{dz} = Dy.$$

Similarly $(1+x)^2 \dfrac{d^2 y}{dx^2} = D(D-1) y$

$$\therefore [D(D-1) + D + 1] y = 4\cos z.$$

$$\Rightarrow (D^2+1) y = 4\cos z$$

A.E. is $m^2+1 = 0 \Rightarrow m = \pm i$

C.F. $= c_1 \cos z + c_2 \sin z = c_1 \cos\log(1+x) + c_2 \sin\log(1+x)$

$$\text{P.I.} = \frac{1}{D^2+1} 4\cos z = \frac{4z}{2D}\cos z = 2z\int \cos z\, dz = 2z\sin z$$

$$= 2\log(1+x)\sin\{\log(1+x)\}$$

C.S. $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x)\sin\{\log(1+x)\}$

(b) $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$

Put $(1+2x) = e^z \Rightarrow z = \log(1+2x)$, and let $\dfrac{d}{dz} \equiv D$, then

$(1+2x)\dfrac{dy}{dx} = 2Dy$ and $(1+2x)^2 \dfrac{d^2y}{dx^2} = 4D(D-1)y$.

$\therefore \quad [4D(D-1) - 12D + 16]\, y = 8e^{2z}$

$\Rightarrow \quad (D^2 - 4D + 4)\, y = 2e^{2z}$

A.E. is $\quad m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

C.F. $= (c_1 + c_2 z)\, e^{2z} = [c_1 + c_2 \log(1+2x)](1+2x)^2$

$$P.I. = \frac{1}{D^2 - 4D + 4}\, 2e^{2z}$$

$$= 2 \cdot \frac{z}{2D - 4}\, e^{2z}$$

$$= 2 \cdot \frac{z^2}{2} e^{2z} = z^2 e^{2z} = \{\log(1+2x)\}^2 (1+2x)^2$$

C.S. $\quad y = [c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2](1+2x)^2$

Simultaneous Differential Equations :—

If two or more dependent variables are functions of a single independent variable, then the equations involving their derivatives are called simultaneous equation. For example if $x = f(t)$ and $y = \phi(t)$ then the derivatives

$$\frac{dx}{dt} + ay = f(t) \quad \text{and} \quad \frac{dy}{dt} + bx = \phi(t)$$

Que (11) :— Solve the following simultaneous equations —

(a) $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

and $\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

(b) $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t$ and $\frac{dx}{dt} + y - x = \cos t$

(c) $\frac{dx}{dt} + 4x + 3y = t$ and $\frac{dy}{dt} + 2x + 5y = e^t$

(d) $\frac{dx}{dt} + y = \sin t$ and $\frac{dy}{dt} + x = \cos t$

given that $x = 2$ and $y = 0$ when $t = 0$.

(e) $t\frac{dx}{dt} + y = 0$ and $t\frac{dy}{dt} + x = 0$

given that $x(1) = 1$ and $y(-1) = 0$

(f) $\frac{dx}{dt} + 5x - 2y = t$ and $\frac{dy}{dt} + 2x + y = 0$

being given $x = y = 0$ when $t = 0$

(g) $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = 2z$, $\frac{dz}{dt} = 2x$

Sol (a) :- The given differential equation can be written as

$$Dx + (D-2)y = 2\cos t - 7\sin t \quad \text{——— } ①$$

$$(D+2)x - Dy = 4\cos t - 3\sin t \quad \text{——— } ②$$

on operating D and $(D-2)$ in equation ① and ② respectively

$$D^2x + D(D-2)y = -2\sin t - 7\cos t \quad \text{——— } ③$$

$$(D^2-4)x - D(D-2)y = 4(D-2)\cos t - 3(D-2)\sin t$$

or $(D^2-4)x - D(D-2)y = -4\sin t - 8\cos t - 3\cos t + 6\sin t$

$$= 2\sin t - 11\cos t \quad \text{——— } ④$$

on adding equation ③ & ④ we get

$$(D^2 + D^2 - 4)x = -18\cos t$$

or $(D^2-2)x = -9\cos t$ ——— ⑤

A.E. is $m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$

$$C.F. = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$$

$$P.I. = \frac{1}{D^2-2}(-9\cos t) = \frac{-9\cos t}{-1-2} = 3\cos t$$

$$\therefore x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 3\cos t \quad \text{——— } ⑥$$

Putting this value in equation ②, we get

$$\frac{dy}{dt} = \frac{dx}{dt} + 2x + 4\cos t - 3\sin t$$

$$= c_1\sqrt{2}e^{\sqrt{2}t} - c_2\sqrt{2}e^{-\sqrt{2}t} - 3\sin t + 2\left(c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 3\cos t\right)$$

$$- 4\cos t + 3\sin t$$

$$\frac{dy}{dt} = (\sqrt{2}+2)c_1 e^{\sqrt{2}t} + (2-\sqrt{2})c_2 e^{-\sqrt{2}t} + 2\cos t \quad \text{——— } ⑦$$

on integration we get

$$y = (1+\sqrt{2})c_1 e^{\sqrt{2}t} - (\sqrt{2}-1)c_2 e^{-\sqrt{2}t} + 2\sin t + c_3 \quad \text{——— } ⑧$$

equation ⑥ and ⑧ is complete solution.

(b) The given equation can be written as

$$(D+3)x + Dy = \sin t \quad \text{—— (1)}$$

and $$(D-1)x + y = \cos t \quad \text{—— (2)}$$

operating D in equation (2) we get

$$D(D-1)x + Dy = -\sin t \quad \text{—— (3)}$$

Subtracting (3) from (1), we get

$$[(D+3) - D(D-1)]x = 2\sin t$$

or $$(D^2-2D-3)x = -2\sin t \quad \text{—— (4)}$$

A.E is $m^2-2m-3=0 \Rightarrow m=-1, 3$

$$C.F = c_1 e^{-t} + c_2 e^{3t}$$

$$P.I. = \frac{1}{D^2-2D-3}(-2\sin t) = \frac{-2}{-1-2D-3}\sin t = \frac{1}{D+2}\sin t$$

$$= \frac{D-2}{D^2-4}\sin t = \frac{(D-2)\sin t}{-1-4} = -\frac{1}{5}(\cos t - 2\sin t)$$

$$\therefore x = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{5}(\cos t - 2\sin t) \quad \text{—— (5)}$$

Putting the value of x in equation (2), we get

$$y = \cos t + x - \frac{dx}{dt}$$

$$= \cos t + c_1 e^{-t} + c_2 e^{3t} - \frac{1}{5}(\cos t - 2\sin t) - \left[-c_1 e^{-t} + 3c_2 e^{3t} + \frac{1}{5}\sin t + \frac{2}{5}\cos t\right]$$

$$= 2c_1 e^{-t} - 2c_2 e^{3t} + \frac{2}{5}\cos t + \frac{1}{5}\sin t \quad \text{—— (6)}$$

equation (5) & (6) gives the complete solution.

(c) The given equation can be written as

$$(D+4)x + 3y = t \quad \text{———} \ (1)$$

and $$2x + (D+5)y = e^t \quad \text{———} \ (2)$$

operating $(D+5)$ in equation (1) we get

$$(D+5)(D+4)x + 3(D+5)y = (D+5)t$$

or $$(D^2+9D+20)x + 3(D+5)y = 1+5t \quad \text{———} \ (3)$$

on multiply equation (2) by 3 and subtracting from (3), we get

$$(D^2+9D+14)x = 1+5t-3e^t \quad \text{———} \ (4)$$

A.E. is $m^2+9m+14=0 \Rightarrow m=-2,-7$

$$C.F. = c_1 e^{-2t} + c_2 e^{-7t}$$

$$P.I. = \frac{1}{D^2+9D+14}(1+5t-3e^t)$$

$$= \frac{1}{D^2+9D+14} e^{0 \cdot t} + \frac{5}{D^2+9D+14} t - \frac{3}{D^2+9D+14} e^t$$

$$= \frac{1}{0+9(0)+14} + \frac{5}{14}\left[1+\frac{D^2+9D}{14}\right]^{-1} t - \frac{3e^t}{1^2+9(1)+14}$$

$$= \frac{1}{14} + \frac{5}{14}\left(t - \frac{9}{14}\right) - \frac{e^t}{8}$$

$$= \frac{5}{14}t - \frac{e^t}{8} - \frac{31}{196}$$

$$\therefore x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{14}t - \frac{e^t}{8} - \frac{31}{196} \quad \text{———} \ (5)$$

from equation (1) & (5) we have

$$3y = t - \frac{dx}{dt} - 4x$$

$$= t - \left(-2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{5}{14} - \frac{e^t}{8}\right) - 4\left(c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{14}t - \frac{e^t}{8} - \frac{31}{196}\right)$$

$$= -2c_1 e^{-2t} + 3c_2 e^{-7t} - \frac{3}{7}t + \frac{5}{8}e^t + \frac{27}{98}$$

$$\therefore y = \frac{1}{3}\left[-2c_1 e^{-2t} + 3c_2 e^{-7t} - \frac{3}{7}t + \frac{5}{8}e^t + \frac{27}{98}\right] \quad \text{———} \ (6)$$

equation (5) & (6) gives complete solution.

(d) given equation can be written as

$$Dx + y = \sin t \quad \text{——— (1)}$$

and $x + Dy = \cos t$ ——— (2)

On operating D in equation (1) and subtracting equation (2) we get

$$(D^2 - 1)x = 0 \quad \text{——— (3)}$$

A.E. $m^2 - 1 = 0 \Rightarrow m = \pm 1$

C.F. $x = c_1 e^{t} + c_2 e^{-t}$ ——— (4)

from equation (1) and (4) we get

$$y = \sin t - \frac{dx}{dt} = \sin t - c_1 e^{t} + c_2 e^{-t} \quad \text{— (5)}$$

At $t = 0$, $x = 2$ and $y = 0$ putting in (4) and (5) we get

$2 = c_1 + c_2$ and $0 = -c_1 + c_2$

$\therefore c_1 = c_2 = 1$

$\therefore x = e^{t} + e^{-t}$ and $y = \sin t - e^{t} + e^{-t}$.

(e) $t\dfrac{dx}{dt} + y = 0$ — (1) and $t\dfrac{dy}{dt} + x = 0$ ——— (2)

Let $t = e^{z} \Rightarrow z = \log t$ and let $\dfrac{d}{dz} \equiv D$ then

$\dfrac{dx}{dt} = \dfrac{dx}{dz} \cdot \dfrac{dz}{dt} = \dfrac{dx}{dz} \cdot \dfrac{1}{t} \Rightarrow t\dfrac{dx}{dt} = \dfrac{dx}{dz} \equiv Dx$ Similarly $t\dfrac{dy}{dt} \equiv Dy$

$\therefore Dx + y = 0$ — (3) and $Dy + x = 0$ ——— (4)

On eliminating y in above equation, we get

$$(D^2 - 1)x = 0 \quad \text{——— (5)}$$

A.E. $m^2 - 1 = 0 \Rightarrow m = \pm 1$

Sol. is $x = c_1 e^{z} + c_2 e^{-z} = c_1 t + c_2 t^{-1}$ ——— (6)

from equation (1) and (6) we get

$$y = -t\frac{dx}{dt} = -t(c_1 - c_2 t^{-2}) = -c_1 t + c_2 t^{-1} \quad \text{— (7)}$$

putting at $t = 1$, $x = 1$, in equation (6) we get

$$1 = c_1 + c_2 \quad \text{——— (8)}$$

again at $t = -1$, $y = 0$ putting in (7), we get

$$0 = -c_1 + c_2 \quad \text{——— (9)}$$

from (8) and (9) we get $c_1 = c_2 = \frac{1}{2}$ hence

$x = \frac{1}{2}(t + t^{-1})$

and $y = \frac{1}{2}(t^{-1} - t)$

(f) given equation can be written as

$$(D+5)\,x - 2y = t \quad \text{——— (1)}$$

$$2x + (D+1)\,y = 0 \quad \text{——— (2)}$$

On eliminating x i.e. multiply (1) by 2 and operating $(D+5)$ in (2) and then subtracting, we get

$$[-4 - (D+5)(D+1)]\,y = 2t \quad \text{or} \quad (D^2+6D+9)\,y = -2t \quad \text{——— (3)}$$

A.E. $\quad m^2+6m+9 = 0 \Rightarrow (m+3)^3 = 0 \Rightarrow m = -3, -3$

C.F. $= (c_1 + c_2 t)\,e^{-3t}$

$$\text{P.I.} = \frac{1}{(D+3)^2}(-2t)$$

$$= -\frac{2}{9}\left[1+\frac{D}{3}\right]^{-2} t$$

$$= -\frac{2}{9}\left[1 - \frac{2D}{3} + \cdots\right] t = -\frac{2t}{9} + \frac{4}{27}$$

$$\therefore \quad y = (c_1 + c_2 t)\,e^{-3t} - \frac{2}{9}t + \frac{4}{27} \quad \text{——— (4)}$$

from equation (2) and (4) we get

$$2x = -\frac{dy}{dt} - y$$

$$= -\left[(-3)(c_1+c_2 t)\,e^{-3t} + c_2 e^{-3t} - \frac{2}{9}\right] - \left[(c_1+c_2 t)\,e^{-3t} - \frac{2}{9}t + \frac{4}{27}\right]$$

$$\therefore \quad x = \left[\left(c_1 - \frac{c_2}{2}\right) + c_2 t\right] e^{-3t} + \frac{t}{9} + \frac{1}{27} \quad \text{——— (5)}$$

putting $x = y = 0$ at $t = 0$ in equation (4) and (5) we get

$$0 = c_1 + \frac{4}{27} \quad \text{and} \quad c_1 - \frac{c_2}{2} + \frac{1}{27} = 0$$

$$\Rightarrow \quad c_1 = -\frac{4}{27} \quad \text{and} \quad c_2 = -\frac{2}{9}$$

$$\therefore \quad x = -\frac{1}{27}(1+6t)\,e^{-3t} + \frac{1}{27}(1+3t)$$

$$\text{and } y = -\frac{2}{27}(2+3t)\,e^{-3t} + \frac{2}{27}(2-3t)$$

(g) $\frac{dx}{dt} = 2y$ —— (1) , $\frac{dy}{dt} = 2z$ —— (2) , $\frac{dz}{dt} = 2x$ —— (3)

differentating (1), we get

$$\frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 4z$$

again differentating $\frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 8x$

or $(D^3 - 8)x = 0$ —— (4)

A.E is $m^3 - 8 = 0 \Rightarrow (m-2)(m^2+2m+4) = 0$

$\Rightarrow m = 2, \ -1 \pm i\sqrt{3}$

$\therefore \ x = c_1 e^{2t} + e^{-t}(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t)$ —— (5)

from (1) and (5) we get

$$y = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}\left[2c_1 e^{2t} - e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t) + \sqrt{3}\,e^{-t}(-c_2\sin\sqrt{3}t + c_3\cos\sqrt{3}t)\right]$$

$$= c_1 e^{2t} + \frac{e^{-t}}{2}\left[(\sqrt{3}c_2 + c_3)\sin\sqrt{3}t + (c_2 - \sqrt{3}c_3)\cos\sqrt{3}t\right]$$ —— (6)

again from (2) and (6) we get

$$z = \frac{1}{2}\frac{dy}{dt}$$

$$= \frac{1}{2}\cdot 2c_1 e^{2t} + \frac{1}{2}\cdot\frac{e^{-t}}{2}\left[(\sqrt{3}c_2 + c_3)\sin\sqrt{3}t + (c_2 - \sqrt{3}c_3)\cos\sqrt{3}t\right]$$

$$-\frac{\sqrt{3}e^{-t}}{4}\left[(\sqrt{3}c_2 + c_3)\cos\sqrt{3}t - (c_2 - \sqrt{3}c_3)\sin\sqrt{3}t\right]$$

$$= c_1 e^{2t} + \frac{e^{-t}}{4}\left[\{(\sqrt{3}c_2 + c_3) + \sqrt{3}(c_2 - \sqrt{3}c_3)\}\sin\sqrt{3}t + \{(c_2 - \sqrt{3}c_2) - \sqrt{3}(\sqrt{3}c_2 + c_3)\}\cos\sqrt{3}t\right]$$

$$= c_1 e^{2t} + \frac{e^{-t}}{2}\left[(\sqrt{3}c_2 - c_3)\sin\sqrt{3}t - (c_2 + \sqrt{3}c_3)\cos\sqrt{3}t\right]$$

Linear Differential Equation of Second Order :—

An equation of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ is known as linear differential equation of second order; where P, Q and R are functions of x or constant.

A linear differential equation of second order with variable coefficient can be solved by changing dependent and independent variables.

(I) **By changing dependent Variable** :— Using this method we can solved the differential equation by two method one is when one solution of complementary function is know and other is normal form.

(i) **When one part of complementary function is known** :—

Let the equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{——— (1)}$$

Let $y = u$ be a given solution of the complementary function. Since $y=u$ be the solution of one part of C.F. hence $y=u$ must be satisfy the auxilary equation. i.e.

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \quad \text{——— (2)}$$

Let $y = uv$ be the complete solution of (1)

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

or $$v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + u\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right] + 2\frac{du}{dx}\frac{dv}{dx} = R$$

or $$v \cdot 0 + u\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right] + 2\frac{du}{dx}\cdot\frac{dv}{dx} = R \quad \text{(using equation (2))}$$

$$\Rightarrow \frac{d^2v}{dx^2} + P\frac{dv}{dx} + \frac{2}{u}\frac{du}{dx}\frac{dv}{dx} = \frac{R}{u}$$

or $$\boxed{\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u}} \quad \text{——— (3)}$$

Let $\frac{dv}{dx} = z$ then $\frac{d^2v}{dx^2} = \frac{dz}{dx}$ hence from (3)

$$\boxed{\frac{dz}{dx} + \left[P + \frac{2}{u}\frac{du}{dx}\right] z = \frac{R}{u}} \quad \text{———— (4)}$$

This is a linear differential equation of first order, and ~~the~~ hence it can be solved easely. Again on integration we find value of v.

<u>Note</u> :- If the one part of C.F. is not given, then we can try to satisfy the following conditions —

S.N.	condition	one part of C.F.
1	$a^2 + aP + Q = 0$	e^{ax}
(i)	put $a = 1$, $1 + P + Q = 0$	e^{x}
(ii)	put $a = -1$, $1 - P + Q = 0$	e^{-x}
2	$m(m-1) + mxP + Qx^2 = 0$	x^m
(i)	put $m = 1$, $P + Qx = 0$	x
(ii)	put $m = 2$, $2 + 2Px + Qx^2 = 0$	x^2

Proof — Let $y = e^{ax}$ be a part of C.F. of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, then

$$a^2 e^{ax} + Pa\,e^{ax} + Q\,e^{ax} = 0 \quad \Rightarrow \quad a^2 + aP + Q = 0 .$$

Que (12) :- Solve the following differential equations

(a) $y'' - 4xy' + (4x^2 - 2)y = 0$, given that $y = e^{x^2}$ is a solution.

(b) $x^2 y'' - (x^2 + 2x)y' + (x+2)y = x^3 e^x$, given that $y = x$ is a solution.

(c) $xy'' - y' + (1-x)y = x^2 e^{-x}$, given that $y = e^x$ is a solution of C.F.

(d) $xy'' - (2x-1)y' + (x-1)y = 0$

(e) $y'' - \cot x\, y' - (1 - \cot x)y = e^x \sin x$

(a) $y'' - 4xy' + (4x^2 - 2)y = 0$ ——— ①

Let $y = u = e^{x^2}$ is a solution of one part of C.F.

Let $y = uv$ be the complete solution. Hence we know that

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \qquad \text{——— ②}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-4x + \frac{2}{e^{x^2}}\left(2x\,e^{x^2}\right)\right]\frac{dv}{dx} = 0 \qquad (\text{Since } R = 0)$$

$$\Rightarrow \frac{d^2v}{dx^2} + 0 \cdot \frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \frac{dv}{dx} = c_1$$

$$\Rightarrow v = c_1 x + c_2$$

$\therefore$ Complete sol. is $y = uv = e^{x^2}(c_1 x + c_2)$

(b) given equation can be written as

$$y'' - \left(\frac{x+2}{x}\right)y' + \left(\frac{x+2}{x}\right)y = xe^x \qquad \text{——— ①}$$

given that $y = u = x$ is a sol.

Let $y = uv$ be C.S., hence, we know that

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u} \qquad \text{——— ②}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\frac{(x+2)}{x} + \frac{2}{x}(1)\right]\frac{dv}{dx} = \frac{xe^x}{x}$$

$$\Rightarrow \frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \qquad \text{——— ③}$$

Let $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$, hence from ③ we get

$$\frac{dz}{dx} - z = e^x \qquad \text{——— ④}$$

$$\text{I.F.} = e^{\int(-1)dx} = e^{-x}$$

$$\therefore z\,e^{-x} = \int e^x e^{-x}\,dx + c_1 = x + c_1$$

$$\Rightarrow z = xe^x + c_1 e^x$$

$$\Rightarrow \frac{dv}{dx} = xe^x + c_1 e^x$$

$$\Rightarrow v = \int(xe^x + c_1 e^x)\,dx + c_2 = (x-1)e^x + c_1 e^x + c_2$$

$\therefore$ Complete sol. $y = uv = x\left[(x-1)e^x + c_1 e^x + c_2\right]$

(c) The given equation can be written as

$$y'' - \frac{1}{x}y' + \left(\frac{1-x}{x}\right)y = x e^{-x} \quad \text{--- (1)}$$

given that $y = u = e^{x}$ is a solution of C.F.

Let $y = uv$ be complete sol. of (1) hence

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u} \quad \text{--- (2)}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\frac{1}{x} + \frac{2}{e^{x}}(e^{x})\right]\frac{dv}{dx} = \frac{x e^{-x}}{e^{x}}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(2 - \frac{1}{x}\right)\frac{dv}{dx} = x e^{-2x} \quad \text{--- (3)}$$

Let $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$ hence

$$\frac{dz}{dx} + \left(2 - \frac{1}{x}\right)z = x e^{-2x} \quad \text{--- (4)}$$

$$\text{I.F.} = e^{\int(2-\frac{1}{x})dx} = e^{2x - \log x} = \frac{e^{2x}}{x}$$

$$\therefore z \cdot \frac{e^{2x}}{x} = \int x e^{-2x} \cdot \frac{e^{2x}}{x}\,dx + c_1$$

$$\Rightarrow z\,\frac{e^{2x}}{x} = \int 1\,dx + c_1 = x + c_1$$

$$\Rightarrow z = (x^2 + c_1 x)\,e^{-2x}$$

$$\Rightarrow \frac{dv}{dx} = x^2 e^{-2x} + c_1 x\, e^{-2x}$$

$$\Rightarrow v = \int x^2 e^{-2x}dx + c_1 \int x e^{-2x}\,dx + c_2$$

$$= x^2\left(\frac{-e^{-2x}}{2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right) + c_1\left[x\left(-\frac{e^{-2x}}{2}\right) - 1\left(\frac{e^{-2x}}{4}\right)\right] + c_2$$

$$= -\frac{1}{4}\left(2x^2 + 2x + 1\right)e^{-2x} - \frac{c_1}{4}(2x+1)\,e^{-2x} + c_2$$

$$\therefore \text{C.S. } y = uv$$

$$= e^{x}\left[c_2 - \frac{1}{4}(2x^2+2x+1)e^{-2x} - \frac{c_1}{4}(2x+1)e^{-2x}\right]$$

$$= c_2 e^{x} - \frac{e^{-x}}{4}\left[(2x^2+2x+1) + c_1(2x+1)\right]$$

(d) The given equation can be written as

$$y'' - \left(\frac{2x-1}{x}\right) y' + \left(\frac{x-1}{x}\right) y = 0 \quad \text{—— (1)}$$

here $P = \frac{1-2x}{x}$, $Q = \frac{x-1}{x}$ and $R = 0$

Since $1 + P + Q = 1 + \frac{1-2x}{x} + \frac{x-1}{x} = 0$

hence $y = u = e^x$ is one solution of C.F.

Let $y = uv$ be complete solution, hence

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{1-2x}{x} + \frac{2}{e^x}\cdot e^x\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0 \quad \text{—— (2)}$$

Let $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\therefore \frac{dz}{dx} + \frac{1}{x}\cdot z = 0$$

$$\Rightarrow \frac{dz}{z} = -\frac{dx}{x}$$

$$\Rightarrow \log z = -\log x + \log c_1$$

$$\Rightarrow z = \frac{c_1}{x}$$

$$\Rightarrow \frac{dv}{dx} = \frac{c_1}{x}$$

$$\Rightarrow v = c_1 \log x + c_2$$

$\therefore$ Complete sol. $y = uv$

$$= e^x(c_1 \log x + c_2)$$

(e) $y'' - \cot x \; y' - (1-\cot x)\, y = e^x \sin x$ —— (1)

here $P = -\cot x$, $Q = \cot x - 1$, $R = e^x \sin x$

Now $1 + P + Q = 1 - \cot x + \cot x - 1 = 0$, hence

$y = u = e^x$ is one solution of C.F.

Let $y = uv$ be complete solution, hence

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\cot x + \frac{2}{e^x}\cdot e^x\right]\frac{du}{dx} = \frac{e^x \sin x}{e^x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x \quad \text{—— (2)}$$

put $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\therefore \frac{dz}{dx} + (2 - \cot x)\, z = \sin x \quad \text{—— (3)}$$

$$\text{I.F.} = e^{\int(2-\cot x)dx} = e^{2x - \log \sin x} = \frac{e^{2x}}{\sin x}$$

$$\therefore z\,\frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x}\, dx + c_1$$

$$\Rightarrow z \cdot \frac{e^{2x}}{\sin x} = \int e^{2x} dx + c_1 = \frac{e^{2x}}{2} + c_1$$

$$\Rightarrow z = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

$$\Rightarrow v = -\frac{1}{2}\cos x + c_1 \cdot \frac{e^{-2x}}{1^2+2^2}\left[(-2)\sin x - \cos x\right] + c_2$$

$$\left(\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)\right)$$

$$= -\frac{1}{2}\cos x - \frac{c_1}{5}e^{-2x}(2\sin x + \cos x) + c_2$$

hence complete sol. is

$$y = uv$$
$$= e^x\left[-\frac{1}{2}\cos - \frac{c_1}{5}e^{-2x}\, 2\sin x + \cos x + c_2\right]$$

(ii) Normal form or Removel of first derivative :— Let the differential eqn

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \text{———— (1)}$$

Let $y = uv$ be complete solution of (1), hence (2)

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

putting above values in equation (1), we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \frac{v}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = \frac{R}{u} \qquad \text{—— (3)}$$

Let the coefficient of $\frac{dv}{dx}$ is zero, i.e

$$P + \frac{2}{u}\frac{du}{dx} = 0 \Rightarrow \frac{du}{dx} = -\frac{Pu}{2} \qquad \text{—— (4)}$$

$$\text{or} \quad \frac{du}{u} = -\frac{P}{2}dx$$

on integration $\log u = -\int \frac{P}{2}dx$

$$\Rightarrow u = e^{-\frac{1}{2}\int P dx} \qquad \text{———— (5)}$$

Now diff. equation (4) we get

$$\frac{d^2u}{dx^2} = -\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}u\frac{dP}{dx}$$

$$= \frac{P^2u}{4} - \frac{1}{2}u\frac{dP}{dx} \qquad \text{———— (6)}$$

putting these values in equation (3) we get

$$\frac{d^2v}{dx^2} + 0.\frac{dv}{dx} + \frac{v}{u}\left[\frac{P^2u}{4} - \frac{1}{2}u\frac{dP}{dx} + P\left(-\frac{Pu}{2}\right) + Qu\right] = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}\right]v = \frac{R}{u}$$

$$\Rightarrow \boxed{\frac{d^2v}{dx^2} + Q_1 v = R_1} \qquad \text{———— (7)}$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ and $R_1 = \frac{R}{u} = R \cdot e^{\frac{1}{2}\int P dx}$

Equation (7) easily solved for v and hence complete solution is $y = uv$.

Que (13) :- Solve the following differential by changing of dependent variable.

(a) $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$

(b) $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right) + \cos^2 x \cdot y = 0$

(c) $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)y = e^{x^2}$

(d) $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + n^2 y = 0$

Sol (a) :- $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$ —— (1)

$\therefore P = -4x \;,\; Q = 4x^2-1$ and $R = -3e^{x^2}\sin 2x$ —— (2)

Let $y = uv$ be complete sol. hence we know that

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \quad \text{—— (3)}$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ and $R_1 = \frac{R}{u}$ where $u = e^{-\frac{1}{2}\int P dx}$

$$\therefore Q_1 = 4x^2-1 - \frac{1}{2}(-4) - \frac{16x^2}{4} = 1$$

Now $u = e^{-\frac{1}{2}\int(-4x)dx} = e^{\int 2x dx} = e^{x^2}$

$\therefore R_1 = \frac{R}{u} = -3\sin 2x$

Putting these values in equation (3) we get

$$\frac{d^2v}{dx^2} + v = -3\sin 2x \quad \text{—— (4)}$$

The solution of equation (4) is

A.E. is $m^2+1 = 0 \Rightarrow m = \pm i$

C.F. is $= c_1\cos x + c_2\sin x$

P.I. $= \frac{1}{D^2+1}(-3\sin 2x) = \frac{-3\sin 2x}{-4+1} = \sin 2x$

$\therefore v = c_1\cos x + c_2\sin x + \sin 2x$

hence complete solution $y = uv$

$$= e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x)$$

(b) $$\frac{d}{dx}\left(\cos^2 x \frac{dy}{dx}\right) + \cos^2 x \cdot y = 0$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} - 2\cos x \sin x \frac{dy}{dx} + \cos^2 x \cdot y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + y = 0 \quad \text{——— (1)}$$

here $P = -2\tan x$, $Q = 1$ and $R = 0$

Let $y = uv$ be complete solution, we know that

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \quad \text{——— (2)}$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ and $R_1 = \frac{R}{u}$, where $u = e^{-\frac{1}{2}\int P dx}$

$$\therefore Q_1 = 1 - \frac{1}{2}(-2\sec^2 x) - \frac{4\tan^2 x}{4} = 1 + \sec^2 x - \tan^2 x = 2$$

Now $u = e^{-\frac{1}{2}\int(-2\tan x)dx} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$

$\therefore R_1 = 0$

Putting these values in equation (2), we get

$$\frac{d^2v}{dx^2} + 2v = 0 \quad \text{—— (3)}$$

or $(D^2+2)v = 0$

A.E. :— $m^2 + 2 = 0 \Rightarrow m = \pm\sqrt{2}\, i$

C.F. $v = c_1 \cos\sqrt{2}\, x + c_2 \sin\sqrt{2}\, x$

hence Complete solution is $y = uv$
$= \sec x \left(c_1 \cos\sqrt{2}\, x + c_2 \sin\sqrt{2}\, x\right)$

(c) $$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)y = e^{x^2} \quad \text{——— (1)}$$

here $P = -4x$, $Q = 4x^2 - 3$ and $R = e^{x^2}$

Let $y = uv$ be complete sol. we know that

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \quad \text{——— (2)}$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ and $R_1 = \frac{R}{u}$, where $u = e^{-\frac{1}{2}\int P dx}$

$$\therefore Q_1 = 4x^2 - 3 - \frac{1}{2}(-4) - \frac{16x^2}{4} = 4x^2 - 3 + 2 - 4x^2 = -1$$

and $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int(-4x)dx} = e^{\int 2x\, dx} = e^{x^2}$

$$R_1 = \frac{R}{u} = \frac{e^{x^2}}{e^{x^2}} = 1$$

Putting above values in equation (2) we get

$$\frac{d^2v}{dx^2} - v = 1 \longrightarrow (3)$$

$$\Rightarrow (D^2-1)v = 1$$

A.E. $m^2-1=0 \Rightarrow m = \pm 1$

C.F. $= c_1 e^x + c_2 e^{-x}$

P.I. $\frac{1}{D^2-1} 1 = \frac{1}{D^2-1} e^{0\cdot x} = \frac{1}{0-1} = -1$

$\therefore v = c_1 e^x + c_2 e^{-x} - 1$

hence complete solution $y = uv = e^{x^2}(c_1 e^x + c_2 e^{-x} - 1)$

(d) $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + n^2 y = 0 \longrightarrow (1)$

here $P = \frac{2}{x}$, $Q = n^2$ and $R = 0$

Let $y = uv$ be complete solution, hence

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \longrightarrow (2)$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$, $R_1 = \frac{R}{u}$ and $u = e^{-\frac{1}{2}\int P dx}$

$\therefore Q_1 = n^2 - \frac{1}{2}\left(-\frac{2}{x^2}\right) - \frac{1}{4}\left(\frac{4}{x^2}\right) = n^2 + \frac{1}{x^2} - \frac{1}{x^2} = n^2$

and $u = e^{-\frac{1}{2}\int \frac{2}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

and $R_1 = 0$

Putting above values in equation (2) we get

$$\frac{d^2v}{dx^2} + n^2 v = 0 \longrightarrow (3)$$

$\Rightarrow (D^2+n^2)v = 0$

A.E. is $m^2+n^2=0 \Rightarrow m = \pm in$

C.F. $v = c_1 \cos nx + c_2 \sin nx$

hence complete solution is $y = uv = \frac{1}{x}(c_1 \cos nx + c_2 \sin nx)$

(II) By Changing of independent Variable :-

Let $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

Let us consider $z = f(x)$ --- (2)

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d^2y}{dz^2}\cdot\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} \quad \text{--- (3)}$$

putting the values of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in equation (1) we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + P\frac{dy}{dz}\cdot\frac{dz}{dx} + Qy = R$$

$$\Rightarrow \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

$$\Rightarrow \frac{d^2y}{dz^2} + \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{--- (4)}$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \quad \text{and} \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Now choose $Q_1 = \text{Constant} = a^2$ (say) $\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2 \Rightarrow \frac{dz}{dx} = \frac{1}{a}\sqrt{Q}$

or $dz = \frac{1}{a}\sqrt{Q}\,dx \Rightarrow z = \frac{1}{a}\int Q\,dx$

Que (14) :- Solve the following equations by changing the independent variable.

(a) $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$

(b) $\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\,\mathrm{cosec}^2 x = 0$

(c) $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$

(d) $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$

(e) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$

(a) $$\frac{d^2y}{dx^2} - \frac{1}{a}\frac{dy}{dx} + 4x^2 y = x^4$$

$$P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4$$

Let $Q_1 = 1$ (constant) ——— (1)

$$\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \Rightarrow \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = 1 \Rightarrow \left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\text{or } \frac{dz}{dx} = 2x \Rightarrow dz = 2x\,dx \Rightarrow z = x^2 \text{ ——— (2)}$$

Now $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(-\frac{1}{x}\right)(2x)}{4x^2} = 0 \text{ ——— (3)}$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4} \text{ ——— (4)}$$

hence $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\Rightarrow \frac{d^2y}{dz^2} + 0 + y = \frac{z}{4}$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = \frac{z}{4} \text{ ——— (5)}$$

hence C.F. $= c_1 \cos z + c_2 \sin z$

and $$\text{P.I.} = \frac{1}{D^2+1}\left(\frac{z}{4}\right) = \left[1+D^2\right]^{-1}\left(\frac{z}{4}\right) = \left[1 - D^2 + \cdots\right]\left(\frac{z}{4}\right) = \frac{z}{4}$$

$\therefore$ C.S. of eq^n (5) is

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4}$$

$$= c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}$$

(b) $$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4 \operatorname{cosec}^2 x \cdot y = 0 \quad \text{——— (1)}$$

here $P = \cot x$, $Q = 4 \operatorname{cosec}^2 x$ and $R = 0$

Let $Q_1 = 1$ (constant) $\Rightarrow \dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = \dfrac{4 \operatorname{cosec}^2 x}{\left(\frac{dz}{dx}\right)^2} = 1$ ——— (2)

or $\dfrac{dz}{dx} = 2 \operatorname{cosec} x$ ——— (3)

$\Rightarrow z = 2 \log(\operatorname{cosec} x - \cot x)$ or $z = 2 \log \tan \frac{x}{2}$ ——— (4)

hence $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-2 \operatorname{cosec} x \cot x + 2 \operatorname{cosec} x \cot x}{4 \operatorname{cosec}^2 x} = 0 \quad \text{——— (5)}$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0 \quad \text{——— (6)}$$

We know that, $\dfrac{d^2y}{dz^2} + P_1\dfrac{dy}{dz} + Q_1 y = R_1$ ——— (7)

$\therefore \dfrac{d^2y}{dz^2} + y = 0$ ——— (8)

$\therefore y = c_1 \cos z + c_2 \sin z$

$= c_1 \cos\left(2 \log \tan \frac{x}{2}\right) + c_2 \sin\left(2 \log \tan \frac{x}{2}\right)$

(c) $$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2 \sin x^2 \quad \text{——— (1)}$$

here $P = -\frac{1}{x}$, $Q = -4x^2$ and $R = 8x^2 \sin x^2$

Let, $Q_1 = -1$ (constant) ——— (2)

$$\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -1 \quad \text{or} \quad \frac{-4x^2}{\left(\frac{dz}{dx}\right)^2} = -1$$

$$\text{or} \quad \frac{dz}{dx} = 2x \quad \Rightarrow z = x^2 \quad \text{——— (3)}$$

$$\text{Now } P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}(2x)}{4x^2} = 0 \quad \text{——— (4)}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{8x^2 \sin x^2}{4x^2} = 2\sin x^2 = 2\sin z \quad \text{——— (5)}$$

We know that $\frac{d^2y}{dz^2} + P_1\frac{dy}{dx} + Q_1 y = R_1$

$$\text{hence} \quad \frac{d^2y}{dz^2} - y = 2\sin z \quad \text{——— (6)}$$

$$\therefore C.F = c_1 e^z + c_2 e^{-z}$$

$$P.I. = \frac{1}{D^2-1} 2\sin z$$

$$= \frac{2\sin z}{-1-1} = -\sin z$$

$\therefore$ C.S. of eqn (6) is

$$y = c_1 e^z + c_2 e^{-z} - \sin z$$

$$= c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$$

(d) $$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2\cos^2 x\, y = 2\cos^4 x \quad \text{——— (1)}$$

here $P = \tan x$, $Q = -2\cos^2 x$ and $R = 2\cos^4 x$

Let $Q_1 = -2$ (Constant) ——— (2)

$$\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -2 \quad \Rightarrow \quad \frac{-2\cos^2 x}{\left(\frac{dz}{dx}\right)^2} = -2$$

$$\text{or} \quad \frac{dz}{dx} = \cos x \quad \Rightarrow z = \sin x \quad \text{——— (3)}$$

$$\text{Now} \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sin x + \tan x \cos x}{\cos^2 x} = 0 \quad \text{——— (4)}$$

$$\text{and} \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x = 2(1-\sin^2 x) = 2(1-z^2)$$

——— (5)

We know that $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$

$$\text{hence} \quad \frac{d^2y}{dz^2} - 2y = 2(1-z^2) \quad \text{——— (6)}$$

$$\therefore C.F = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$$

$$\text{and} \quad P.I = \frac{1}{D^2-2} 2(1-z^2)$$

$$= \frac{2}{-2}\left[1 - \frac{D^2}{2}\right]^{-1}(1-z^2)$$

$$= -\left[1 - \frac{D^2}{2} + \cdots\right](1-z^2)$$

$$= -\left[1 - z^2 - \frac{2}{2}\right] = z^2$$

hence c.s. of eqⁿ (6)

$$y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2$$

$$= c_1 e^{\sqrt{2}\sin x} + c_2 e^{\sqrt{2}\sin x} + \sin^2 x$$

(e) $$\frac{d^2y}{dx^2} + \frac{1}{1+x}\frac{dy}{dx} + \frac{1}{(1+x)^2}y = \frac{4\cos\log(1+x)}{(1+x)^2} \quad \text{——— (1)}$$

here $P = \frac{1}{1+x}$, $Q = \frac{1}{(1+x)^2}$ and $R = \frac{4\cos\log(1+x)}{(1+x)^2}$

Let $Q_1 = 1$ (constant) ——— (2)

$$\Rightarrow \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \Rightarrow \left(\frac{dz}{dx}\right)^2 = Q = \frac{1}{(1+x)^2}$$

or $$\frac{dz}{dx} = \frac{1}{1+x} \Rightarrow z = \log(1+x) \quad \text{——— (3)}$$

Now $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\frac{1}{(1+x)^2} + \frac{1}{(1+x)^2}}{\left(\frac{dz}{dx}\right)^2} = 0 \quad \text{——— (4)}$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 4\cos\log(1+x) = 4\cos z \quad \text{——— (5)}$$

We know that $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

hence $$\frac{d^2y}{dz^2} + y = 4\cos z \quad \text{——— (6)}$$

$$\therefore CF = c_1\cos z + c_2\sin z \quad \text{——— (7)}$$

and $$P.I. = \frac{1}{D^2+1}(4\cos z) = \frac{4z}{2D}\cos z = 2z\int\cos z\,dz$$

$$= 2z\sin z \quad \text{——— (8)}$$

hence c.s. of eqn (6) is

$$y = c_1\cos z + c_2\sin z + 2z\sin z$$

$$= c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$$

Method of variation of Parameter :—

Let the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ——— (1)

Let $y = Au + Bv$ ——— (2) be the solution of C.F., hence it must be satisfy the auxilardy equation. where A and B are constance and u and v are the functions of x.

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad \text{——— (3)}$$

and $$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0 \qquad \text{——— (4)}$$

Again let $y = Au + Bv$ ——— (5) be the complete solution of (1), hence it must be satisfy the equation (1), where A and B are function of x not constant.

Now differentaling equation (5) we get

$$\frac{dy}{dx} = u\frac{dA}{dx} + A\frac{du}{dx} + B\frac{dv}{dx} + v\frac{dB}{dx} \qquad \text{——— (6)}$$

or $$\frac{dy}{dx} = \left(u\frac{dA}{dx} + v\frac{dB}{dx}\right) + \left(A\frac{du}{dx} + B\frac{dv}{dx}\right)$$

Let us choose $u\frac{dA}{dx} + v\frac{dB}{dx} = 0$ ——— (7), hence above equation (6) can be written as

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx} \qquad \text{——— (8)}$$

again differentating, we get

$$\frac{d^2y}{dx^2} = A\frac{d^2u}{dx^2} + \frac{dA}{dx}\frac{du}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\frac{dv}{dx} \qquad \text{——— (9)}$$

Putting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1) we get

$$A\frac{d^2u}{dx^2} + \frac{dA}{dx}\frac{du}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\cdot\frac{dv}{dx} + P\left(A\frac{du}{dx} + B\frac{dv}{dx}\right) + Q(Au+Bv) = R$$

$$\Rightarrow A\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + B\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right) + \frac{dA}{dx}\cdot\frac{du}{dx} + \frac{dB}{dx}\cdot\frac{dv}{dx} = R \qquad \text{——— (10)}$$

from eqn (3), (4) and (10), we get

$$\frac{dA}{dx}\cdot\frac{du}{dx} + \frac{dB}{dx}\cdot\frac{dv}{dx} = R \qquad \text{——— (11)}$$

Solving (7) and (11) for $\frac{dA}{dx}$ and $\frac{dB}{dx}$, we get

$$\frac{\frac{dA}{dx}}{\begin{vmatrix} 0 & v \\ R & \frac{dv}{dx} \end{vmatrix}} = \frac{\frac{dB}{dx}}{\begin{vmatrix} u & 0 \\ \frac{du}{dx} & R \end{vmatrix}} = \frac{1}{\begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix}}$$

$$\Rightarrow \frac{\frac{dA}{dx}}{-vR} = \frac{\frac{dB}{dx}}{uR} = \frac{1}{u\frac{dv}{dx} - v\frac{du}{dx}}$$

$$\therefore \frac{dA}{dx} = \frac{-vR}{u\frac{dv}{dx} - v\frac{du}{dx}} \Rightarrow A = \int \frac{-vR}{uv_1 - vu_1}\,dx + C_1 \quad \text{——— (12)}$$

$$\text{and} \quad \frac{dB}{dx} = \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}} \Rightarrow B = \int \frac{uR}{uv_1 - vu_1}\,dx + C_2 \quad \text{——— (13)}$$

Putting the value of A and B in equation (5) we get complete solution.

Que (15) :— Using the variation of parameter method, solve the following equations

(a) $\frac{d^2y}{dx^2} + y = \sec x$

(b) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$

(d) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$

(e) $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - \cot x \cdot y = \sin^2 x$

Sol (a) :— $\frac{d^2y}{dx^2} + y = \sec x$ ——— (1)

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F. $y = A\cos x + B\sin x$ ——— (2)

where A and B are constant.

Let $y = Au + Bv$ ——— (3) be the complete solution, when A and B are the functions of x and $u = \cos x$, and $v = \sin x$.

A and B are determine by the formula

$$A = \int \frac{-Rv}{uv_1 - vu_1}\,dx + c_1$$

$$= \int \frac{-\sec x \cdot \sin x}{\cos x \cos x - \sin x(-\sin x)}\,dx + c_1$$

$$= -\int \tan x\,dx + c_1 = \log \cos x + c_1$$

and $$B = \int \frac{Ru}{uv_1 - vu_1}\,dx + c_2$$

$$= \int \frac{\sec x \cdot \cos x}{\cos x \cdot \cos x - \sin x(-\sin x)}\,dx + c_2$$

$$= \int 1.\,dx + c_2 \quad = x + c_2$$

Hence c.s. is $y = Au + Bv$

$$= (\log \cos x + c_1)\cos x + (x + c_2)\sin x$$

(b) $$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \qquad \text{———} \ (1)$$

C.F. is $y = Ae^x + Be^{-x}$

Let $y = Au + Bv$ be the complete solution, where $u = e^x$ and $v = e^{-x}$

and A & B are functions of x to be determine by the formula

$$A = \int \frac{-Rv}{uv_1 - vu_1}\,dx + c_1$$

$$= \int \frac{-\frac{2}{1+e^x} \cdot e^{-x}}{e^x(-e^{-x}) - e^{-x}(e^x)}\,dx + c_1 = \int \frac{e^{-x}}{1+e^x}\,dx + c_1$$

$$= \int \frac{1}{e^x(1+e^x)}\,dx + c_1 = \int\left[\frac{1}{e^x} - \frac{1}{1+e^x}\right]dx + c_1$$

$$= \int\left[e^{-x} - \frac{e^{-x}}{1+e^{-x}}\right]dx + c_1$$

$$= -e^{-x} + \log(1+e^{-x}) + c_1$$

$$= \int \frac{e^{-2x}}{e^{-x}+1}\,dx + c_1 = -e^{-x} + \log(1+e^{-x}) + c_1$$

$$B = \int \frac{Ru}{uv_1 - vu_1}\,dx + c_2$$

$$= \int \frac{-\frac{2}{1+e^x} \cdot e^x}{-2}\,dx + c_2$$

$$= \int \frac{e^x}{1+e^x}\,dx + c_2 = \log(1+e^x) + c_2$$

hence complete sol. is

$$y = Au + Bv$$

$$= \left[\log(1+e^{-x}) + c_1\right]e^x + \left[\log(1+e^x) + c_2\right]e^{-x}$$

(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$

A.E. is $m^2 - 2m = 0 \Rightarrow m = 0, 2$

C.F. is $y = A1 + B e^{2x}$

Let $y = Au + Bv$ be C.S. where $u = 1$ and $v = e^{2x}$, and A & B are the functions of x to be determined by the equation

$$A = \int \frac{-vR}{uv_1 - vu_1} dx + c_1$$

$$= \int \frac{-e^{2x} \cdot e^x \sin x}{1(2e^{2x}) - e^{2x}(0)} dx + c_1$$

$$= -\frac{1}{2} \int e^x \sin x \, dx + c_1$$

$$= -\frac{1}{2}\left[\frac{e^x}{1+1}(\sin x - \cos x)\right] + c_1$$

$$= -\frac{e^x}{4}(\sin x - \cos x) + c_1$$

and $B = \int \frac{uR}{uv_1 - vu_1} dx + c_2$

$$= \int \frac{1 \cdot e^x \sin x}{2e^{2x}} dx + c_2$$

$$= \frac{1}{2} \int e^{-x} \sin x \, dx + c_2$$

$$= \frac{1}{2}\left[\frac{e^{-x}}{1+1}\{(-1)\sin x - \cos x\}\right] + c_2$$

$$= -\frac{1}{4}e^{-x}(\sin x + \cos x) + c_2$$

hence C.S. is $y = Au + Bv$

$$= \left[-\frac{e^x}{4}(\sin x - \cos x) + c_1\right] + \left[-\frac{1}{4}e^{-x}(\sin x + \cos x) + c_2\right]e^{2x}$$

$$= -\frac{e^x}{4}(\sin x - \cos x) + c_1 - \frac{1}{4}e^x(\sin x + \cos x) + c_2 e^{2x}$$

$$= c_1 + c_2 e^{2x} - \frac{e^x}{2}\sin x$$

(d) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2e^x$

$\Rightarrow \dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x^2}y = e^x$ ——— (i)

here $P = \frac{1}{x}$, $Q = -\frac{1}{x^2}$, $R = e^x$

Now $P + Qx = \frac{1}{x} - \frac{1}{x^2}\cdot x = 0$ hence $y = x = u$ (say) one of the solution of C.F.

Let $y = uv$ be the C.F. of eqn (1), Now using method of dependent variable, we have

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{1}{x} + \frac{2}{x}\cdot 1\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{3}{x}\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$ then $\frac{d^2v}{dx^2} = \frac{dz}{dx}$, hence

$\dfrac{dz}{dx} + \dfrac{3}{x}z = 0 \Rightarrow \dfrac{dz}{z} = -\dfrac{3}{x}dx \Rightarrow \log z = -3\log x + \log c_1$

or $z = \dfrac{c_1}{x^3} \Rightarrow \dfrac{dv}{dx} = \dfrac{c_1}{x^3} \Rightarrow v = -\dfrac{c_1}{2x^2} + c_2$

hence C.F. of equation (1) is

$$y = uv = x\left(-\frac{c_1}{2x^2} + c_2\right) = \left(-\frac{c_1}{2}\right)\cdot\frac{1}{x} + c_2 x = A\cdot\frac{1}{x} + B\cdot x$$

Let $y = Au + Bv$ be complete solution of (1), Hence by method of variation of parameter we get ($R = e^x$, $u = \frac{1}{x}$ & $v = x$)

$$A = \int \frac{-vR}{uv_1 - vu_1}dx + c_1 = \int \frac{-x\,e^x}{\frac{1}{x}\cdot 1 - x\cdot\left(-\frac{1}{x^2}\right)}dx + c_1 = \int \frac{-x e^x}{-\frac{2}{x}}dx + c_1$$

$$= \frac{1}{2}\int x^2 e^x dx + c_1$$

$$= \frac{1}{2}\left[x^2e^x - (2x)e^x + 2e^x + c_1\right] = \frac{1}{2}x^2e^x - xe^x + e^x + c_1'$$

again $B = \displaystyle\int \frac{uR}{uv_1 - vu_1}dx + c_2 = \int \frac{\frac{1}{x}\cdot e^x}{-\frac{2}{x}}dx + c_2 = -\frac{1}{2}\int e^x dx + c_2 = -\frac{1}{2}e^x + c_2$

∴ solution is $y = Au + Bv$

$$= \left[\frac{1}{2}x^2e^x - xe^x + e^x + c_1'\right]\frac{1}{x} + \left[-\frac{1}{2}e^x + c_2\right]x$$

(e) $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - \cot x \cdot y = \sin^2 x$

here $P = 1-\cot x$, $Q = -\cot x$

Now $1-P+Q = 1 - (1-\cot x) - \cot x = 0$, hence $y = e^{-x} = u$ is one part of C.F.

Let $y = uv$ be solution of given equation. hence we know that

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[1-\cot x + \frac{2}{e^{-x}}(-e^{-x})\right]\frac{dv}{dx} = 0$$

or $$\frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} = 0$$

$$\frac{dz}{dx} - (1+\cot x)z = 0 \quad \left(\text{put } z = \frac{dv}{dx}\right)$$

$IF = e^{-\int(1+\cot x)dx}$
$= e^{-x-\log \sin x}$
$= e^{-x} e^{\log \operatorname{cosec} x}$
$= \operatorname{cosec} x\, e^{-x}$.

$\therefore z \cdot IF = \int 0 \cdot dx + c_1$

$z \operatorname{cosec} x\, e^{-x} = c_1$

$~~z~~ = c_1 e^{x} \sin x$

$$\Rightarrow z = c_1 e^{x} \sin x$$

$$\Rightarrow \frac{dv}{dx} = c_1 e^{x} \sin x$$

$$\Rightarrow v = \frac{c_1 e^{x}}{2}(\sin x - \cos x) + c_2$$

hence C.F. of given equation is

$$y = uv = e^{-x}\left(\frac{c_1}{2}e^{x}(\sin x - \cos x) + c_2\right) = A(\sin x - \cos x) + Be^{-x}$$

Let $y = Au + Bv$ be C.S. where $u = \sin x - \cos x$ & $v = e^{-x}$ and

$$A = \int \frac{-vR}{uv_1 - vu_1}dx + c_1 = \int \frac{-e^{-x}\sin^2 x}{(\sin x - \cos x)(-e^{-x}) - (e^{-x})(\cos x + \sin x)}dx + c_1$$

$$= \frac{1}{2}\int \sin x\, dx + c_1 = \frac{-\cos x}{2} + c_1$$

and $$B = \int \frac{uR}{uv_1 - vu_1}dx + c_2 = \int \frac{(\sin x - \cos x)\sin^{\cancel{2}} x}{-2e^{-x}\cancel{\sin x}}dx + c_2$$

$$= \frac{1}{2}\int e^{+x}(\cos x - \sin x)\sin x\, dx + c_2$$

$$= \frac{e^{x}}{20}(3\sin 2x - \cos 2x - 5) + c_2$$

Hence complete solution is

$$y = Au + Bv$$
$$= \left[-\frac{\cos x}{2} + c_1\right](\sin x - \cos x) + \left[\frac{e^{x}}{20}(3\sin 2x - \cos 2x - 5) + c_2\right]e^{-x}$$

$-\frac{df}{0}$

$-p$

Frobenius Method – If $x=0$ is a regular singular point, the we take

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{k=0}^{\infty} a_k x^{m+k} \quad a_0 \neq 0$$

be the sol. of given diff. eqⁿ.

Now we find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and put in given eqⁿ. And then we equate the coeff. of various powe of x equal to zero.

Case I :- When the roots of indicial eqⁿ are different and not differing by an integer then — i.e. $m_1 - m_2 =$ Not integer

$$y_1 = (y)_{m=m_1} \quad \& \quad y_2 = (y)_{m=m_2}$$

C.S. $y = c_1 y_1 + c_2 y_2$

Case II :- When the roots of indicial eqⁿ are same i.e. (0,0), (1,1) (−1,−1) etc.

$$y_1 = (y)_{m=m_1} \quad \& \quad y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=m_1}$$

$\therefore$ C.S. $y = c_1 y_1 + c_2 y_2$

Case III :- When the roots of indicial eqⁿ are different and differing by an integer then ($m_1 = \frac{3}{2}$, $m_2 = \frac{1}{2}$

(a) On putting the value of m_2 then the coeff. of x be-comes ∞, hence we remove this difficulty by putting $a_0 = b_0(m - m_2)$

$$\therefore y_1 = (y)_{m=m_1} \quad \& \quad y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=m_2}$$

$\therefore$ C.S. $y = c_1 y_1 + c_2 y_2$

(b) On putting the value of m_1 or m_2 the y series not becomes infinite, then this case convert into case I i.e.

$$y_1 = (y)_{m=m_1} \quad \& \quad y_2 = (y)_{m=m_2}$$

$\therefore$ C.S. $y = c_1 y_1 + c_2 y_2$

Que ① Solve $3x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ ——— ①

Sol :- Let $y = x^m[a_0 + a_1 x + a_2 x^2 + \cdots] = \sum_{k=0}^{\infty} a_k x^{m+k}$, $a_0 \neq 0$ ——— ②

be the sol. of eqⁿ ① hence

$\therefore \frac{dy}{dx} = \sum a_k (m+k) x^{m+k-1}$ & $\frac{d^2y}{dx^2} = \sum a_k (m+k)(m+k-1) x^{m+k-2}$

putting above values in eqⁿ ①, we get

$3x \sum a_k (m+k)(m+k-1) x^{m+k-2} + 2\sum a_k (m+k) x^{m+k-1} + \sum a_k x^{m+k} = 0$

$\Rightarrow \sum a_k (m+k)[3m+3k-3+2] x^{m+k-1} + \sum a_k x^{m+k} = 0$

$\Rightarrow \sum a_k (m+k)(3m+3k-1) x^{m+k-1} + \sum a_k x^{m+k} = 0$ ——— ③

Comparing the coeff of various power x is equal to zero.

Coff x^{m-1} — $a_0 m(3m-1) = 0$

$\Rightarrow \boxed{m(3m-1) = 0}$ → indicial eqⁿ $\Rightarrow \boxed{m = 0, \frac{1}{3}}$ Since $a_0 \neq 0$

Coff of x^m — $a_1(m+1)(3m+2) + a_0 = 0 \Rightarrow a_1 = \frac{-a_0}{(m+1)(3m+2)}$

Coff of x^{m+k} — $a_{k+1}(m+k+1)(3m+3k+2) + a_k = 0$

$\Rightarrow \boxed{a_{k+1} = \frac{-a_k}{(m+k+1)(3m+3k+2)}}$.

putting $k = 0, 1, 2, 3, \ldots$ we get

$a_1 = \frac{-a_0}{(m+1)(3m+2)}$

$a_2 = \frac{-a_1}{(m+2)(3m+5)} = \frac{a_0}{(m+1)(m+2)(3m+2)(3m+5)}$

Similarly a_3, a_4 ---- etc. can be write. Hence from ② we get

$y = a_0 x^m \left[1 - \frac{x}{(m+1)(3m+2)} + \frac{x^2}{(m+1)(m+2)(3m+2)(3m+5)} - \cdots\right]$ ——— ④

Now $y_1 = (y)_{m=0} = a_0\left[1 - \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 2\cdot 5} - \cdots\right]$

and $y_2 = (y)_{m=\frac{1}{3}} = a_0 x^{\frac{1}{3}}\left[1 - \frac{x}{(\frac{1}{3}+1)(1+2)} + \frac{x^2}{(\frac{1}{3}+1)(\frac{1}{3}+2)(1+2)(1+5)} - \cdots\right]$

$= a_0 x^{\frac{1}{3}}\left[1 - \frac{\cancel{3}x}{4\cdot\cancel{3}} + \frac{\cancel{3}\cdot\cancel{3}x^2}{4\cdot\cancel{7}\cdot\cancel{3}\cdot\cancel{6}_2} - \cdots\right]$

$= a_0 x^{\frac{1}{3}}\left[1 - \frac{1}{4}x + \frac{x^2}{56} - \cdots\right]$

$\therefore$ C.S. $y = C_1 y_1 + C_2 y_2$ Ans

Que ② :- Solve $x y'' + y' + xy = 0$ —— ①

Sol :- Let $y = x^m [a_0 + a_1 x + a_2 x^2 + \cdots] = \sum a_k x^{m+k}$, $a_0 \neq 0$ be sol. of ① —— ②

$$x \sum a_k (m+k)(m+k-1) x^{m+k-2} + \sum a_k (m+k) x^{m+k-1} + x \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum a_k (m+k)^2 x^{m+k-1} + \sum a_k x^{m+k+1} = 0 \quad \text{—— ③}$$

Comparing the coeff. of various power of x is equal to zero. i.e.

Coff. of x^{m-1} $\rightarrow$ $a_0 m^2 = 0 \Rightarrow \boxed{m = 0, 0}$ Since $a_0 \neq 0$

Coff. of x^m $\rightarrow$ $a_1 (m+1)^2 = 0 \Rightarrow a_1 = 0$ Sin $(m+1)^2 \neq 0$ because $m = 0$

Coff. of x^{m+k+1} $\rightarrow$ $a_{k+2}(m+k+2)^2 + a_k = 0 \Rightarrow \boxed{a_{k+2} = \dfrac{-a_k}{(m+k+2)^2}}$ —— ④

putting $k = 0, 1, 2, 3, 4 \cdots$ we get

$$a_2 = \frac{-a_0}{(m+2)^2}$$

$$a_3 = \frac{-a_1}{(m+3)^2} = 0 = a_5 = a_7 = a_9 \cdots$$

$$a_4 = \frac{-a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2 (m+4)^2}$$

Similarly $a_4 = \dfrac{-a_4}{(m+6)^2} = \dfrac{-a_0}{(m+2)^2 (m+4)^2 (m+6)^2}$

putting above values in eqn ② we get

$$y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2 (m+4)^2} - \frac{x^6}{(m+2)^2 (m+4)^2 (m+6)^2} + \cdots\right] \quad \text{—— ⑤}$$

Now $y_1 = (y)_{m=0} = a_0 \left[1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} - \dfrac{x^6}{2^2 . 4^2 . 6^2} + \cdots\right]$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2 (m+4)^2} - \cdots\right] + a_0 x^m \left[0 + \frac{2x^2}{(m+2)^3} - \cdots\right]$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = y_1 \log x + a_0 \left[\frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\right]$$

Hence C.S. is

$y = C_1 y_1 + C_2 y_2$ Ans

Que ③ :- Solve $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-4)\,y = 0$ ——— (1)

Let $y = x^m(a_0 + a_1x + a_2x^2 + \cdots) = \sum_{k=0}^{\infty} a_k x^{m+k}$, $a_0 \neq 0$ be sol of eqⁿ ① ——— (2)

$\therefore\ x^2 \sum a_k (m+k)(m+k-1)\, x^{m+k-2} + x \sum a_k (m+k)\, x^{m+k-1} + (x^2-4)\sum a_k x^{m+k} = 0$

$\Rightarrow\ \sum a_k \left[(m+k)\{m+k-1+1\} - 4\right] x^{m+k} + \sum a_k x^{m+k+2} = 0$

$\Rightarrow\ \sum a_k\left[(m+k)^2 - 4\right] x^{m+k} + \sum a_k x^{m+k+2} = 0$

or $\sum a_k (m+k+2)(m+k-2)\, x^{m+k} + \sum a_k x^{m+k+2} = 0$ ——— (3)

equating the coeff. of different power of x is equalto zero we get

$a_0 (m+2)(m-2) = 0 \Rightarrow \boxed{m = 2, -2}$, $a_0 \neq 0$

$a_1 (m+3)(m-1) = 0 \Rightarrow a_1 = 0$ Since $(m+3)(m-1) \neq 0$

eq Coeff of x^{m+k+2} $\rightarrow$ $a_{k+2}(m+k+4)(m+k) + a_k = 0 \Rightarrow \boxed{a_{k+2} = \dfrac{-a_k}{(m+k)(m+k+4)}}$ ——— (4)

putting $k = 0, 1, 2, 3, \cdots$ we get $a_3 = a_1 = a_5 = a_7 \cdots = 0$

again $a_2 = \dfrac{-a_0}{m(m+4)}$, $a_4 = \dfrac{-a_2}{(m+2)(m+6)} = \dfrac{a_0}{m(m+2)(m+4)(m+6)}$

Hence from ② we get

$$y = a_0 x^m\left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \cdots\right] \quad \text{——— (5)}$$

$$y_1 = (y)_{m_1=2} = a_0 x^2\left[1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} - \cdots\right]$$

On putting $m_2 = -2$ the y-series becomes infinite, hence we remove this difficulty we put $a_0 = b_0(m - m_2) = b_0(m+2)$ in eqⁿ ⑤ we get

$$y = b_0 x^m\left[(m+2) - \frac{(m+2)}{m(m+4)}x^2 + \frac{x^4}{m(m+4)(m+6)} - \cdots\right]$$

$$\frac{\partial y}{\partial m} = b_0 x^m \log x\left[(m+2) - \frac{(m+2)}{m(m+4)}x^2 + \frac{x^4}{m(m+4)(m+6)} - \cdots\right]$$
$$+\ b_0 x^m\left[1 - \frac{(m+2)}{m(m+4)}\left[\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right]x^2 + \cdots\right]$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-2} = b_0 x^{-2}\log x\left[0 - 0 + \frac{x^4}{-2\cdot 2\cdot 4} - \cdots\right] + b_0 x^{-2}\left[1 + \frac{x^2}{2^2} \cdots\right]$$

Hence C.S. $y = c_1 y_1 + c_2 y_2$ Ans

Solve the following diff eqⁿ by power series

Que (1)

(a) $4xy'' + 2y' + y = 0$ Ans. $y = C_1 \cos\sqrt{x} + C_2 \sin\sqrt{x}$

(b) $2x^2y'' - xy' + (1-x^2)y = 0$

$m = 1, \frac{1}{2}$

Ans $y = C_1\left[1 + 3x^2 + \frac{3}{5}x^4 - \cdots\right] + C_2 x^{3/2}\left[1 + \frac{3}{8}x^2 - \frac{3\cdot1}{8\cdot16}\right.$

(c) $2x(1-x)y'' + (1-x)y' + 3y = 0$

Ans. $y = C_1\left[1 - 3x + \frac{3x^2}{1\cdot2} + \frac{3x^3}{3\cdot5} + \cdots\right] + C_2 x^{\frac{1}{2}}(1-x)$

Que (2) (a) $xy'' + y' - y = 0$

Ans - $y = C_1y_1 + C_2y_2$ where $y_1 = (y)_{m=0} = a_0\left[1 + x + \frac{x^2}{(\lfloor 2)^2} + \frac{x^3}{(\lfloor 3)^2} + \cdots\right]$

and $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = y_1 \log x - 2a_0\left[x + \frac{1}{(\lfloor 2)^2}\left(1+\frac{1}{2}\right)x^2 + \frac{1}{(\lfloor 3)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)\right.$

(b) $x(x-1)y'' + (3x-1)y' + y = 0$

$y_1 = a_0(1 + x + x^2 + x^3 - \cdots)$

$y_2 =$

(c) $xy'' + (1+x)y' + 2y = 0$

Ans - $y = C_1\left[1 - 2x + \frac{3}{\lfloor 2}x^2 - \frac{4}{\lfloor 3}x^3 - \cdots\right]$

$+ C_2\left[y_1 \log x + a_0\left(3x - \frac{13}{4}x^2 - \cdots\right)\right]$

Q. (3) (a) $x^2y'' + 5xy' + x^2y = 0$

Ans - $y = A\left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \cdots\right) + Bx^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4}\right.$

$- B\log x\left(\frac{1}{16} + \frac{x^2}{16} + \cdots\right)$

(b) $x^2y'' + xy' + (x^2-1)y = 0$

Ans - $y = Ax\left(1 - \frac{x^2}{2\cdot4} + \frac{x^4}{2\cdot4^2\cdot6} - \cdots\right) + Bx^{-1}\log x\left(-\frac{x^2}{2} + \right.$

$+ Bx^{-1}\left(1 + \frac{x^2}{2^2} - \frac{3}{2^2 2^3}x^4 + \cdots\right)$