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"Complex variable"

(Q 65) $f(z) = z + 2\bar{z}$

$f(z) = x+iy + 2(x-iy)$

$f(z) = 3x - iy$

$u+iv = 3x - iy$

$\boxed{u = 3x}$ and $\boxed{v = -y}$

$\frac{\partial u}{\partial x} = 3$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$\hookrightarrow$ C-R eq^n not satisfied

Hence
func is not analytic anywhere.

(Q 66) $|z-2i| = 2$ under $w = \frac{1}{z}$

(Sol^n) $w = \frac{1}{z}$

$z = \frac{1}{w}$

$\left|\frac{1}{w} - 2i\right| = 2$

$|1-2iw| = 2|w|$

$|1-2i(u+iv)| = 2|u+iv|$

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$$|1+2v-2iu| = 2|u+iv|$$

$$(1+2v)^2 + (2u)^2 = 4[u^2+v^2]$$

$$\boxed{1+4v=0}$$

(Q67) $f(z) = e^{\frac{z}{(z-2)}}$

(Sol^n) $f(z) = e^{\frac{z}{(z-2)}} = e^{\frac{z-2+2}{(z-2)}} = e \cdot e^{\frac{2}{z-2}}$

$$f(z) = e\left[1 + \frac{2}{z-2} + \left(\frac{2}{z-2}\right)^2 \times \frac{1}{2!} + \dots\right]$$

It is Laurent's expansion about $z=2$

(Q68) $f(z) = u+iv$

$$u-v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$f(z) = u+iv$ —(i)

$if(z) = iu - v$ —(ii)

(i) + (ii);

$f(z)(1+i) = (u-v) + i(u+v)$

Let say $f(z)(1+i) = F(z)$

$(u-v) = U$

$(u+v) = V$

$F(z) = U + V$

$$\frac{\partial U}{\partial x} = \frac{(\cosh y - \cos x)(0+\sin x+\cos x) - (e^y - \cos x + \sin x)(0+\sin x)}{(\cosh y - \cos x)^2}$$

replace x by z and $y=0$

So, $\phi(z,0) = \dfrac{(1-\cos z)(0+\sin z+\cos z) - (1-\cos z+\sin z)(0+\sin z)}{(1-\cos z)^2}$

$\phi(z,0) = \dfrac{(1-\cos z)(0+\sin z+\cos z)-(1-\cos z+\sin z)(0+\sin z)}{(1-\cos z)^2}$

$\phi(z,0) = \dfrac{(1-\cos z)(\sin z+\cos z) - (1-\cos z+\sin z)\sin z}{(1-\cos z)^2}$

$\phi(z,0) = \dfrac{\cancel{\sin z} + \cos z - \cancel{\cos z \sin z} - \cos^2 z - \cancel{\sin z} + \cancel{\sin z \cos z} - \sin^2 z}{(1-\cos z)^2}$

$\phi(z,0) = \dfrac{\cos z - 1}{(1-\cos z)^2} = \dfrac{-1}{1-\cos z} = \dfrac{-1}{2}\operatorname{cosec}^2\dfrac{z}{2}$

$\dfrac{\partial u}{\partial y} = \dfrac{(\cosh y-\cos x)[e^y-0+0] - (e^y-\cos x+\sin x)(\sinh y-0)}{(\cosh y-\cos x)^2}$

~~$\phi(z,0) = (1-0)(1-0)[e$~~

replace x by z & y by 0

$\dfrac{\partial u}{\partial y} = \dfrac{(1-\cos z)[1]-[1-\cos z](0)}{(1-\cos z)^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{1-\cos z}{(1-\cos z)^2} = \dfrac{1}{1-\cos z}$

$\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\operatorname{cosec}^2\dfrac{z}{2}$

$$F(z)\ \cancel{f(z)} = \int [\phi(z,0) - i\psi(z,0)]\,dz + c$$

$$f(z) = \int \left[-\frac{1}{2}\operatorname{cosec}^2\frac{z}{2} - \frac{1}{2}\operatorname{cosec}^2\frac{z}{2} \right] dz + c$$

$$f(z) = -\frac{1}{2}\int \left[\operatorname{cosec}^2\frac{z}{2} + i\operatorname{cosec}^2\frac{z}{2} \right] dz + c$$

$$F(z)\ \cancel{f(z)} = -\frac{1}{2}\left[-\cot\frac{z}{2}\times 2 - i\cot\frac{z}{2}\times 2 \right] + c$$

$$F(z) = \cancel{f(z)} = -\frac{1}{2}\left[-2\cot\frac{z}{2} - i2\cot\frac{z}{2} \right] + c$$

$$F(z) = \cancel{f(z)} = \cot\frac{z}{2} + i\cot\frac{z}{2} + c$$

$$\cancel{f(z) =} \boxed{\cancel{f(z) = (1+i)\cot\frac{z}{2}}}$$

$$F(z) = (1+i)\cot\frac{z}{2} + c$$

we have $(1+i)f(z) = F(z)$

so,

$$(1+i)f(z) = (1+i)\cot\frac{z}{2} + c$$

$$f(z) = \cot\frac{z}{2} + \frac{c}{(1+i)}$$

Given

$$f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$$

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So, put $z = \frac{\pi}{2}$ in $f(z)$

So, $\frac{3-i}{2} = \cot\frac{\pi}{4} + \frac{c}{(1+i)}$

$$\frac{3-i}{2} = 1 + \frac{c}{(1+i)}$$

$$\frac{3-i-2}{2} = \frac{c}{(1+i)}$$

$$\frac{1-i}{2} = \frac{c}{(1+i)}$$

$$\frac{(1-i)(1+i)}{2} = c$$

$$\frac{1-i^2}{2} = c$$

$$\frac{1+1}{2} = c$$

$$\boxed{c = 1}$$

So,

$$\boxed{f(z) = \cot\frac{z}{2} + 1}$$

$\boxed{\text{Que - 71}}$

$$w = \frac{z}{1-z}$$

$$w(1-z) = z$$

$$\therefore \frac{(1-z)}{z} = \frac{1}{w}$$

$$\frac{1}{z} - 1 = \frac{1}{w}$$

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$y \geq 0$

$$\frac{1}{z} = \frac{1+w}{w}$$

$$\boxed{z = \frac{w}{1+w}}$$

$$x+iy = \frac{u+iv}{1+u+iv}$$

$$x+iy = \frac{u+iv}{(1+u)+iv}$$

$$x+iy = \frac{(u+iv)(u+1-iv)}{[(u+1)+iv](u+1-iv)}$$

$$x+iy = \frac{u^2+u-iuv+iuv+iv+v^2}{(u+1)^2+v^2}$$

Here, $x = \frac{u^2+u+v^2}{(u+1)^2+v^2}$

$y = \frac{v}{(u+1)^2+v^2}$

for upper of z plane

$y \geq 0$

So,

$$\frac{v}{(u+1)^2+v^2} \geq 0$$

$$v \geq 0$$

Hence, for upper half of z plane onto the upper half of w plane,

\$. we have $|z| = 1$

$$\left|\frac{w}{1+w}\right| = 1$$

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$$|w| = |1+w|$$

$$|u+iv| = |(u+1)+iv|$$

$$u^2+v^2 = (u+1)^2+v^2$$

$$\cancel{u^2}+\cancel{v^2} = \cancel{u^2}+1+2u+\cancel{v^2}$$

$$\boxed{-\frac{1}{2} = u}$$

$\boxed{\text{Que-72}}$

$z_1 = z$	$w_1 = w$
$z_2 = i$	$w_2 = 0$
$z_3 = -i$	$w_3 = 1$
$z_4 = 1$	$w_4 = \infty$

(sol) We know

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(w_3/w_4-1)}{(w_2-w_3)(1-w_1/w_4)}$$

$$\frac{(z-i)(-i-1)}{(i+i)(1-z)} = \frac{(w-0)(0-1)}{(0-1)(1-0)}$$

$$\frac{(z-i)(-i-1)}{(2i)(1-z)} = \frac{(w)(1)}{1}$$

$$w = \frac{-zi-z+i^2+i}{2(i-zi)}$$

$$w = \frac{-zi - z - 1 + i}{zi(1-z)}$$

$$w = \frac{-zi + i - z - 1}{zi(1-z)}$$

$$w = \frac{-zi + i - 1 - z}{z(i - zi)} = -\cancel{(1-i)z+}$$

$$w = \frac{-zi^2 + i^2 - i - zi}{-z(1-z)}$$

$$w = \frac{z - 1 - i - zi}{zz - z} = \frac{z(1-i) - 1(1+i)}{-(-zz+z)}$$

$$w = \frac{z(i-1) + 1(1+i)}{-zz+z}$$

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[Que-73]

$$\oint_C \frac{e^z}{z(1-z)^3}\, dz$$

(i) $C \equiv |z| = \frac{1}{2}$ (ii) $|z-1| = \frac{1}{2}$ (iii) $|z| = 2$

(Sol^n) here $f(z) = \oint \frac{e^z}{z(1-z)^3}\, dz$

for poles, put denominator $= 0$

So,,

$z(1-z)^3 = 0$

$z = 0$	$z = 1$
simple pole	order $= 3$

poles is 0, 1

$\text{Res}(z=0) = \lim_{z\to 0} (z) \times \frac{e^z}{z(1-z)^3} = \lim_{z\to 0} \frac{e^z}{(1-z)^3}$

$\text{Res}(z=0) = \frac{e^0}{(1-0)^3} = \frac{1}{1} = 1$

So,, $\boxed{\text{Res}(z=0) = 1}$

$\text{Res}(z=1) = \lim_{z\to 1} \left(\frac{1}{(3-1)!} \frac{d^2}{dz^2} \left[(1-z)^3 \frac{e^z}{z(1-z)^3} \right] \right)$

$\text{Res}[\text{at } z=1] = \lim_{z\to 1} \left[\frac{1}{2} \frac{d^2}{dz^2} \left(\frac{e^z}{z} \right) \right]$

$$\lim_{z\to 1}\left[\frac{1}{2}\frac{d}{dz}\left(\frac{ze^z - e^z}{z^2}\right)\right]$$

$$\lim_{z\to 1}\left[\frac{1}{2}\left(\frac{z^2[(z\cancel{\times}e^z + e^z) - e^z] - 2z(ze^z - e^z)}{z^4}\right)\right]$$

$$\lim_{z\to 1}\left[\frac{1}{2}\left(\frac{z^2[ze^z] - 2z^2e^z - e^z \times -2z}{z^4}\right)\right]$$

$$\lim_{z\to 1}\left[\frac{1}{2}\left(\frac{z^3e^z - 2z^2e^z + 2ze^z}{z^4}\right)\right]$$

$$\frac{1}{2}\lim_{z\to 1}\left[\frac{z^3e^z - 2z^2e^z + 2ze^z}{z^4}\right]$$

$$\frac{1}{2}\left[\frac{e - 2e + 2e}{1}\right] = \frac{e}{2}$$

Here our pole of order 3 was in Non standard form i.e $(1-z)^3$

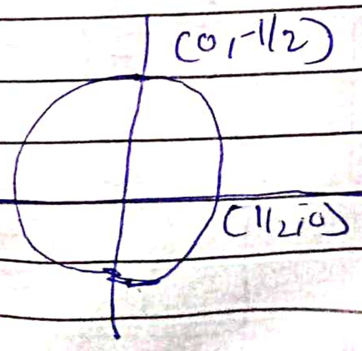
But we know standard form is $(z-1)^3$

So,

$$(1-z)^3 = (-1)^3(z-1)^3 = -(z-1)^3$$

So, $\boxed{\text{Res (at } z=1) = -\frac{e}{2}}$

(ii) for region $|z| = \frac{1}{2}$



$z=1$ lies outside

But $z=0$ is inside

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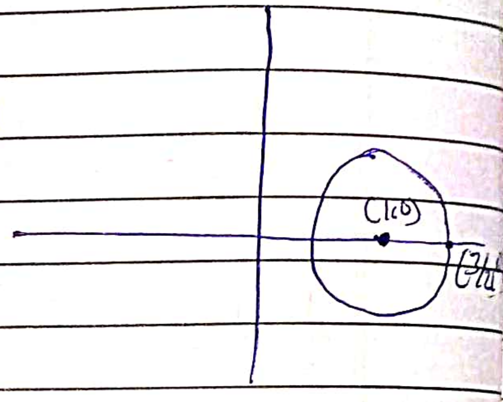
So,
$$\oint \frac{e^z}{z(1-z)^3} = 2\pi i \times R_1 = 2\pi i \times 1$$

$$\boxed{\oint \frac{e^z}{z(1-z)^3} = 2\pi i}$$

for region $|z-1| = \frac{1}{2}$

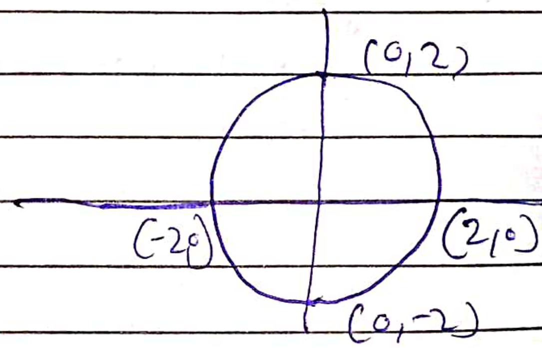
region is a circle with $(1,0)$ and radius $\frac{1}{2}$



$z=0$ lies outside the region but $z=1$ lie inside.

So,
$$\oint \frac{e^z}{z(1-z)^3} dz = 2\pi i (\text{Res at } z=1)$$
$$\boxed{= 2\pi i \times -\frac{e}{2} = -\pi e i}$$

for region
$$|z| = 2$$



$z=0,1$ both lies inside the C

So,
$$\oint_C \frac{e^z}{z(1-z)^3} dz = 2\pi i \left[\text{Res}(\text{at } z=0) + \text{Res}(\text{at } z=1)\right]$$
$$= 2\pi i \left[1 - \frac{e}{2}\right] = \pi i (2-e)$$

$$\boxed{\oint_C \frac{e^z}{z(1-z)^3} dz = \pi i (2-e)}$$

[Que-74]

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

(Soln) $f(z) = \dfrac{z^2-1}{z^2+5z+6}$

$$z^2+5z+6 \overline{\big)\, z^2-1 \,}\big(1$$
$$\underline{z^2+5z+6}$$
$$-5z-7$$

So,,

$$f(z) = \frac{(z^2+5z+6)\times 1 - 5z - 7}{z^2+5z+6}$$

$$f(z) = 1 - \frac{(5z+7)}{z^2+5z+6}$$

let say $\dfrac{5z+7}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3}$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$$

So,

$$\boxed{f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}}$$

(i) $|z| < 2$

$$f(z) = 1 + \frac{3}{2}\left(\frac{z}{2}+1\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4}\times 1 - \frac{z^3}{8} + \cdots\right]$$

$$-\frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

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$f(z)$ is a expansion of taylor's series.

(ii) $2 < |z| < 3$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$f(z) = 1 + \left[\frac{3}{z} - \frac{2\times3}{z^2} + \frac{4\times3}{z^3} - \frac{8\times3}{z^3} + \cdots\right] - \left[\frac{8}{3} - \frac{8z}{9} + \frac{8z^2}{27} - \cdots\right]$$

This is a laurent series expansion.

(iii) $|z| > 3$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right]$$

$$f(z) = 1 + \left[\frac{3}{z} - \frac{6}{z^2} + \frac{12}{z^3} - \frac{24}{z^4} + \cdots\right] - \left[\frac{8}{z} - \frac{24}{z^2} + \frac{72}{z^3} - \cdots\right]$$

This is laurent series expansion

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[Q-75]

Harmonic func :- A func is said to be Harmonic if its satisfy given laplace eqn.

i,e $\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} = 0\right]$ where $u(x,y) \Rightarrow$ func

[Q-76]

$$w = \frac{2z+3}{z+2}$$

for invariant func put $w = z$

$$z = \frac{2z+3}{z+2}$$

$$z^2 + 2z = 2z + 3$$

$$z^2 = 3$$

$$\boxed{z = \pm\sqrt{3}}$$

[Que-77]

Cauchy integral theorem
if $f(z)$ is analytic and $f'(z)$ is continous at each point within and on a simple closed curve C, then

$$\oint_C f(z)\,dz = 0$$

[Q-78]

$$f(z) = \sin\left(\frac{1}{z-a}\right)$$

We Know:,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - -$$

So,

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - - -$$

Here $z=a$ is a singular point
And Here are infinite no' of terms in principal part
So, It is essential singularity

[Que-79]

$$f(z) = \begin{cases} \frac{x^2y^5(x+iy)}{x^4+y^{10}} & , z \neq 0 \\ 0 & , z=0 \end{cases}$$

(soln)

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}} = \frac{x^3y^5 + ix^2y^6}{x^4+y^{10}}$$

$$f(z) = \frac{x^3y^5}{x^4+y^{10}} + i\frac{x^2y^6}{x^4+y^{10}}$$

Here $f(z) = u + iv$

$$u = \frac{x^3y^5}{x^4+y^{10}} \quad ; \quad v = \frac{x^2y^6}{x^4+y^{10}}$$

We know

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

So, $$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Here $\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$

↳ Hence C-R eqn satisfied

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{x^2y^5(x+iy)-0}{(x^4+y^{10})(x+iy)}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^5}{x^4+y^{10}}$$

Along $y = mx$

$$\lim_{x\to 0} \frac{x^2m^5x^5}{x^4+m^{10}x^{10}} = \frac{x^7m^5}{x^4(1+m^{10}x^6)} = 0$$

Along $y^5 = x^2$

$$\lim_{x\to 0} \frac{x^2 x^2}{x^4 + x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

$f'(0)$ doesn't unique, so $f'(0)$ does not exist

Hence, $f(z)$ is not analytic at origin.

[Que-80]

~~$f(z) = \frac{1}{2\pi i} \oint \frac{z^2 - z + 1}{x^4 + y^{10}}$~~

$$I = f(z) = \frac{1}{2\pi i} \oint_C \frac{z^2 - z + 1}{z - a}\, dz, \quad C \equiv |z-1| = \frac{1}{2}$$

(Soln)

Here, $z - a = 0$

$z = a$ is a poles.

↳ simple pole

$$\text{Res}(at\ z = a) = \lim_{z\to a} \frac{(z-a)}{2\pi i} \times \left(\frac{z^2 - z + 1}{z - a}\right)$$

$$= \lim_{z\to a} \frac{z^2 - z + 1}{2\pi i} = \frac{a^2 - a + 1}{2\pi i}$$

$$\boxed{\text{Res}(at\ z = a) = \frac{a^2 - a + 1}{2\pi i}}$$

Case 1:-

when a lies inside the region

then

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$I = 2\pi i \times$ (Residue at $z=a$)

$I = 2\pi i \times \frac{(a^2-a+1)}{2\pi i}$

$I = \frac{\cancel{2\pi i}}{\cancel{2\pi i}}(a^2-a+1) = a^2-a+1$

So, $\boxed{I = a^2-a+1}$

Case 2nd; if a lies outside the region

$I = 2\pi i \times$ Residue of all point lie inside $\hookrightarrow 0$

$I = 2\pi i \times 0$

$\boxed{I = 0}$

[Que-81]

$f(z) = u+iv$

$u-v = e^x(\cos y - \sin y)$

Analytic func :- A func is said to be analytic at a point z_0, if it is a single value and differentiable not only at z_0 but also in neighbourhood of z_0.

Here, $f(z) = u+iv$ —①

$if(z) = iu-v$ —②

① + ②;

$(1+i)f(z) = (u-v)+i(u+v)$

Let say $F(z) = U+iV$

$U = u-v$

So, $U = e^x\cos y - e^x\sin y$

$\frac{dU}{dx} = e^x\cos y - e^x\sin y = e^x(\cos y - \sin y)$

let assume

$$\frac{\partial u}{\partial x} = \phi(x,y)$$

so, $\phi(x,y) = e^x (\cos y - \sin y)$

replace $\underline{x \text{ by } z}$ and $\underline{y \text{ by } 0}$

so,

$$\boxed{\phi(z,0) = e^z(1-0) = e^z}$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y - \cos y)$$

$$\psi(x,y) = -e^x(\sin y + \cos y)$$

$$\psi(z,0) = -e^z(1) = -e^z$$

we know,

$$F(z) = \int [\phi(z,0) - i\psi(z,0)]\, dz + c$$

$$F(z) = \int (e^z + ie^z)\, dz = (1+i)\int e^z\, dz + c$$

$$F(z) = (1+i)e^z + c$$

we have. $(1+i) f(z) = F(z)$

so,

$$(1+i) f(z) = (1+i) e^z + c$$

$$\boxed{f(z) = e^z + \frac{c}{1+i}}$$

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[Que-82]

$|z-1| = 1$ under the mapping $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

So, $|z-1| = 1$

$$\left|\frac{1}{w} - 1\right| = 1$$

$$|1-w| = |w|$$

$$|1-u-iv| = |u+iv|$$

$$(1-u)^2 + v^2 = u^2 + v^2$$

$$1 + \cancel{u^2} - 2u + \cancel{v^2} = \cancel{u^2} + \cancel{v^2}$$

$$1 - 2u = 0$$

$$1 = 2u$$

$$\boxed{u = \frac{1}{2}}$$

[Que-83]

$f(z) = \frac{1 - \cos z}{z^3}$ about the point $z = 0$

$$f(z) = \frac{1}{z^3}(1 - \cos z)$$

$$f(z) = \frac{1}{z^3}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$$

$$f(z) \; \frac{1}{z^3}\left[\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right]$$

$$f(z) = \frac{1}{2!\,z} - \frac{z}{4!} + \frac{z^3}{6!} - \cdots$$

'This is a Laurent series expansion pole $\boxed{z = 0}$

[Que-84]

$$I = \oint \frac{4+3z}{(z-2)(z-3)}\,dz \qquad c: |z|=1$$

(Soln)

Here

$$(z-2)(z-3)=0$$

$z-2=0$	$z-3=0$
$z=2$	$z=3$
↳ simple pole	↳ simple pole

$$\text{Res}(\text{at } z=2) = \lim_{z\to 2} \cancel{(z-2)} \times \frac{4+3z}{\cancel{(z-2)}(z-3)}$$

$$= \lim_{z\to 2} \cdot \frac{4+3z}{z-3}$$

$$= \frac{4+6}{2-3} = -10$$

$$\text{Res}(\text{at } z=3) = \lim_{z\to 3} \cancel{(z-3)} \times \frac{4+3z}{(z-2)\cancel{(z-3)}}$$

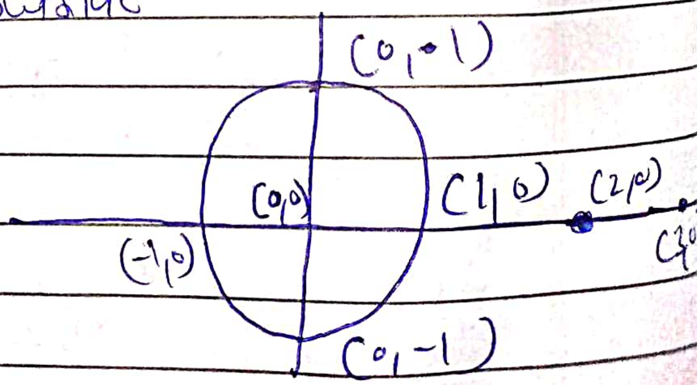
$$= \lim_{z\to 3} \frac{4+3z}{(z-2)} = \frac{13}{1} = 13$$

for region $|z|=1$

↳ both point lies outside

So,

$$\boxed{I=0}$$



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[Que-85]

$f(z) = \sqrt{|xy|}$

$f(z) = \sqrt{|xy|} + 0i = \sqrt{|xy|} + i0$

$\boxed{u = \sqrt{|xy|}}$ & $\boxed{v = 0}$

$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \frac{0-0}{x} = 0$

$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \frac{0-0}{y} = 0$

$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \frac{0-0}{y} = 0$

$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \frac{0-0}{x} = 0$

Here, $\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right]$

→ Hence C-R satisfied

$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x+iy}$

$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{|xy|}}{x+iy}$

Along $y = mx$

~~$f'(0) =$~~

$f'(0) = \lim_{x \to 0} \frac{\sqrt{|xmx|}}{x + imx}$

$f'(0) = \lim_{x \to 0} \frac{x\sqrt{|m|}}{x(1+im)} = \frac{\sqrt{|m|}}{1+im}$

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Here, $f'(0)$ depend on m, doesn't unique
So, $f'(0)$ doesn't exist
Hence $f(z)$ / func is not analytic at origin.

[Q86] [86]

$f(z) = z^3$

$f(z) = z^3 = (x+iy)^3 = x^3 + i^3y^3 + 3ixy(x+iy)$

$f(z) = x^3 - iy^3 + 3ix^2y + 3i^2xy^2$

$f(z) = x^3 - iy^3 + 3ix^2y - 3xy^2$

$f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$f(z) = u + iv$, $v \to 3x^2y - y^3$, $u \to x^3 - 3xy^2$

$u = x^3 - 3xy^2$

$\boxed{\frac{\partial u}{\partial x} = 3x^2 - 3y^2}$ $\boxed{\frac{\partial u}{\partial y} = 0 - 6xy = -6xy}$

$v = 3x^2y - y^3$

$\boxed{\frac{\partial v}{\partial x} = 6xy}$ $\boxed{\frac{\partial v}{\partial y} = 3x^2 - 3y^2}$

~~Here, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$~~

Here, $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Hence C-R satisfied

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$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are polynomials

So, they are all continous.

Hence, $f(z)$ is analytic func everywhere.

[Que - 87]

Conformal Mapping:- if the angle of intersection of the curves at the point p in Z plane is same as the angle of intersection of the curve at point p' in w plane is same in magnitude as well in rotation then the mapping is said to be conformal Mapping.

[Que - 88]

$$\int_0^{1+i} (x^2 - iy)\, dz \quad \text{along } y = x$$

(Soln) We know; $z = x + iy$

So, $dz = dx + i\,dy$

Here, $y = x$

So, $dy = dx$

So, $dz = dx + i\,dx$

$dz = (1+i)\,dx$

$$\int_0^1 (x^2 - ix)(1+i)\,dx$$

$$(1+i)\int_0^1 (x^2 - ix)\,dx = (1+i)\left[\frac{x^3}{3} - \frac{ix^2}{2}\right]_0^1$$

$$\int_0^{1+i} (x^2 - iy)\,dz = (1+i)\left[\frac{1}{3} - \frac{i}{2}\right] = (1+i)\frac{(2-3i)}{6}$$

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$$\frac{2-3i+2i-3i^2}{6} = \frac{2-3i+2i+3}{6}$$

$$= \frac{5-i}{6} = \frac{5}{6} - \frac{i}{6} \quad \text{Ans}$$

[Que - 89]

$$f(z) = \frac{\cos z}{z(z+5)} \quad \text{at } z=0$$

(Sol^n)

$$f(z) = \frac{\cos z}{z(z+5)}$$

Here $z(z+5) = 0$

$z=0$ simple pole	$z=-5$ simple pole.

Residue at $z=0$

$$\text{Res}(\text{at } z=0) = \lim_{z\to 0} (z) \frac{\cos z}{z(z+5)}$$

$$= \lim_{z\to 0} \frac{\cos z}{z+5} = \frac{1}{5}$$

$$\left[\text{Res}(\text{at } z=0) = \frac{1}{5}\right] \quad \text{Ans}$$

[Que - 90]

$$u = x^4 - 6x^2y^2 + y^4$$

(Sol^n)

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

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$$\frac{\partial u}{\partial y} = 0 - 12x^2y + 4y^3$$

$$\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{12x^2} - \cancel{12y^2} - \cancel{12x^2} + \cancel{12y^2} = 0$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Laplace eqn satisfied Hence it is ∴ a harmonic func.

$$u \text{ (real part)} = x^4 - 6x^2y^2 + y^4$$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\& \quad \frac{\partial u}{\partial y} = -12x^2y + 4y^3$$

Let say $\frac{\partial u}{\partial x} = \phi(x,y)$ and $\frac{\partial u}{\partial y} = \psi(x,y)$

and replacing $x \to z$ & $y \to 0$

So, $\phi(z,0) = 4z^3$ & $\psi(z,0) = 0$

$$f(z) = \int [\phi(z,0) - i\psi(z,0)]\, dz + c$$

$$f(z) = \int (4z^3 - i\,0)\, dz + c$$

$$\boxed{f(z) = z^4 + c}$$ Ans,,

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[Que-91]

$$f(z) = \frac{1}{(z-1)(z-2)}$$

(sol)

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

put $z = 2$

so,

$$1 = 0 + B(1)$$

$$\boxed{B = 1}$$

put $z = 1$

so,

$$1 = A(-1)$$

$$\boxed{A = -1}$$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2} = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $1 < |z| < 2$

(soln)

$$f(z) = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$f(z) = \frac{-1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right) - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$f(z) = \left(\frac{-1}{2} + \frac{z}{4} - \frac{z^2}{8} + \frac{z^3}{16} - \cdots\right) - \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\right)$$

This is laurent series expansion.

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(iii) $2<|z|$
(soln)

$$f(z) = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$f(z) = \frac{1}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right) - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)$$

$$f(z) = \left(\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right) - \left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right)$$

This is also laurent series expansion

[Que-92]

Que-79 & 92 are same

[Que-93]

$z = 0, i, 2i$ $\qquad$ $w = 5i, \infty, \frac{-i}{3}$

$z_1 = z$ $\qquad$ $w_1 = w$

$z_2 = 0$ $\qquad$ $w_2 = 5i$

$z_3 = -i$ $\qquad$ $w_3 = \infty$

$z_4 = 2i$ $\qquad$ $w_4 = \frac{-i}{3}$

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(1-w_4/w_3)}{(w_2/w_3-1)(w_4-w_1)}$$

$$\frac{(z-0)(-i-2i)}{(0+i)(2i-z)} = \frac{(w-5i)(1-0)}{(0-1)(-i/3-w)}$$

$$\frac{z(-3i)}{i(2i-z)} = \frac{3(w-5i)}{(i+3w)}$$

$$\frac{-\cancel{3}z}{(2i-z)} = \frac{\cancel{3}(w-5i)}{(i+3w)}$$

$$\frac{-z}{(2i-z)} = \frac{(w-5i)}{(i+3w)}$$

$$z(i+3w) = (z-2i)(w-5i)$$

$$zi + 3zw = zw - 5zi - 2wi + 10i^2$$

$$zi + 3zw = zw - 5zi - 2wi - 10$$

~~$zi + 3zw + 5zi$~~

$$3zw - zw + 2wi = -zi - 5zi - 10$$

$$w(3z - \cancel{z} + 2i) = -zi - 5zi - 10$$

$$w = \frac{-zi - 5zi - 10}{3z - z + 2i}$$

$$w = \frac{-6zi - 10}{2z + 2i} = \frac{\cancel{2}(-3zi - 5)}{\cancel{2}(z+i)}$$

$$w = \frac{-3zi - 5}{z + i}$$

multiply by and divide by i

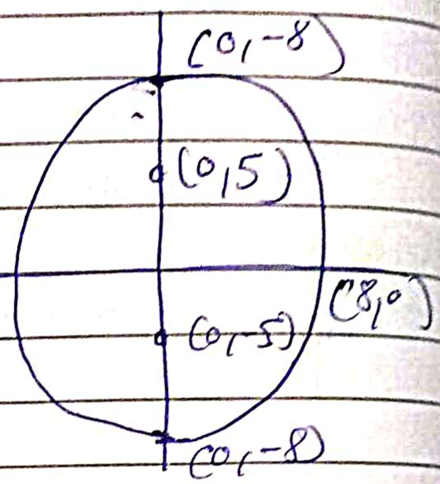
So,

$$w = \frac{-3zi^2 - 5i}{zi + i^2} = \frac{+3z - 5i}{zi - 1}$$

$$\boxed{w = \frac{3z - 5i}{zi - 1}}$$

[Que-95]

$$\int \frac{\sin z}{(z^2+25)^2}\, dz \quad ; \quad C \equiv |z| = 8$$



singularities

$$z^2+25=0$$

$$\boxed{z = \pm 5i}$$

Both are inside

$$\int \frac{\sin z}{(z+5i)^2(z-5i)^2}\, dz$$

$$I = \int_{C_1} \frac{\frac{\sin z}{(z+5i)^2}}{(z-5i)^2}\, dz + \int_{C_2} \frac{\frac{\sin z}{(z-5i)^2}}{(z+5i)^2}\, dz$$

$$f_1(z) = \frac{\sin z}{(z+5i)^2} \qquad f_2(z) = \frac{\sin z}{(z-5i)^2}$$

$$I = \frac{1}{(2-1)!} 2\pi i f_1'(a) + \frac{1}{(2-1)!} 2\pi i f_2'(a)$$

$$f_1'(z) = \frac{(z+5i)^2 \cos z - \sin z \times 2(z+5i)}{(z+5i)^4}$$

$$f_1'(5i) = \frac{(10i)^2 \cos 5i - \sin 5i \times 2 \times 10i}{(10i)^4}$$

$$f_1'(5i) = \frac{10i \cos 5i - 2 \sin 5i}{(10i)^3} = \frac{10i \cos 5i - 2\sin 5i}{1000(-i)}$$

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$$f_2'(z) = \frac{(z-5i)^2 \cos z - \sin z \times 2(z-5i)}{(z-5i)^4}$$

$$f_2'(z) = \frac{(z-5i)\cos z - 2\sin z}{(z-5i)^3}$$

$$f_2'(-5i) = \frac{(-10i)\cos 5i + 2\sin 5i}{(-10i)^3}$$

$$f_2'(-5i) = \frac{-10i\cos 5i + 2\sin 5i}{-1000 i^3} = \frac{-10i\cos 5i + 2\sin 5i}{1000 i}$$

$$I = 2\pi i f_1'(5i) + 2\pi i f_2'(-5i)$$

$$I = 2\pi i \times \frac{10i\cos 5i - 2\sin 5i}{1000(-i)} + 2\pi i \times \frac{-10i\cos 5i + 2\sin 5i}{1000 i}$$

$$I = \pi \frac{[10i\cos 5i - 2\sin 5i]}{-500} + \pi \frac{[-10i\cos 5i + 2\sin 5i]}{500 i}$$

$$I = \pi \frac{[-10i\cos 5i + 2\sin 5i - 10i\cos 5i + 2\sin 5i]}{500}$$

$$I = \pi \frac{[-20i\cos 5i + 4\sin 5i]}{500}$$

$$\boxed{I = \pi \frac{[\sin 5i - 5i\cos 5i]}{125}}$$

(Que-95)

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$$

It is the form of $\int_{-\infty}^{\infty} \frac{y(x)}{F(x)}$

$$\frac{f(z)}{F(z)} = \frac{z^2}{(z^2+1)(z^2+4)}\,dz$$

Here, $F(z)=0$
for poles

$(z^2+1)(z^2+4)=0$

$z=\pm i$	$z=\pm 2i$
simple pole	simple pole

In upper half plane poles are $+i, +2i$

Pole	order
i	1
$2i$	1

$$\text{Res}(at\ z=i) = \lim_{z\to i} (z-i) \times \frac{z^2}{(z-i)(z+i)(z+2i)(z-2i)}$$

$$= \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)}$$

$$= \frac{i^2}{(2i)(i^2+4)} = \frac{-1}{(2i)(3)} = \frac{-1}{6i}$$

$$= \frac{i}{6}$$

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$$\text{Res (at } z = 2i) = \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)(z-2i)} \times (z-2i)$$

$$= \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)}$$

$$= \frac{4i^2}{(4i^2+1)(4i)} = \frac{-i}{3}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \cancel{2\pi i\left(\dots\right)} \; 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right)$$

$$= 2\pi i\left(\frac{-i}{6}\right) = \frac{-\pi i^2}{3} = \frac{\pi}{3}.$$