Unit - III

Sequence :- The list of numbers written in a definite order is called a sequence, for example 1, 4, 7, 10, 13, 16 ..... and denoted by $\{S_n\}$.

Series :- The sum of terms of an infinite sequence is called an infinite series, for example $1+4+7+10+13+16+ \cdots \infty = \sum_{n=1}^{\infty} u_n$

Therefor sequence is an order list of numbers and series is the sum of a list of numbers i.e. $S_n = \sum_{n=1}^{n} u_n$.

Convergence and Divergence of a infinite series :-

(i) If $\lim_{n\to\infty} S_n =$ finite and unique, then series is called convergent.

(ii) If $\lim_{n\to\infty} S_n =$ infinite $(-\infty$ or $+\infty)$, " " " divergent.

(iii) If $\lim_{n\to\infty} S_n =$ not an unique (more than one values) then " " oscillatory.

Que ① – Examine the nature of the following series –

(i) $1+2+3+4+ \cdots \infty$

(ii) $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+ \cdots \infty$

(iii) $3-3+3-3+3- \cdots \infty$

Sol – (i) Now $S_n = 1+2+3+4+ \cdots \infty = \frac{n(n+1)}{2}$ $\left\{\begin{array}{l}\because \text{ in AP.} \\ \therefore S_n = \frac{n}{2}[2a+(n-1)d]\end{array}\right.$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$, i.e. the series is divergent.

(ii) Let $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+ \cdots = \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$ $\left\{\begin{array}{l}\because \text{ in GP} \\ S_n = \frac{a(1-r^n)}{1-r}, r<1 \\ = \frac{a(r^n-1)}{r-1}, r>1\end{array}\right.$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2\left(1-\frac{1}{\infty}\right) = 2$ (i.e. finite and unique) $\left\{ S_\infty = \frac{a}{1-r}\right.$

Hence series is convergent.

(iii) $S_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$

i.e. S_n is not an unique.

Hence the series is oscillatory.

Que 2 :– Show that the geometric series $\sum_{n=0}^{\infty} r^n$, $r>0$ is convergent when $r<1$ and divergent, when $r \geq 1$.

OR

Examine the nature of the series $1+r+r^2+r^3+ \cdots \infty$ according to the condition (i) $r<1$, (ii) $r>1$, (iii) $r=1$, (iv) $r=-1$

Sol :– (i) when $r<1$, then $S_n = \frac{1(1-r^n)}{(1-r)} = \frac{1}{1-r} - \frac{r^n}{1-r}$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{1-r} - \frac{r^n}{1-r}\right] = \lim_{n\to\infty}\left(\frac{1}{1-r}\right) - \lim_{n\to\infty}\left(\frac{r^n}{1-r}\right) = \frac{1}{1-r} - 0 = \frac{1}{1-r}$ (i.e. finite)

$\left(\because \lim_{n\to\infty} r^n = 0 \text{ if } r<1\right.$
$\left. = \infty \text{ if } r>1\right)$

Hence series is convergent.

(ii) when $r>1$, then $S_n = \frac{r^n-1}{r-1} = \frac{r^n}{r-1} - \frac{1}{r-1}$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{r^n}{r-1} - \frac{1}{r-1}\right] = \lim_{n\to\infty}\left(\frac{r^n}{r-1}\right) - \lim_{n\to\infty}\left(\frac{1}{r-1}\right) = \frac{\infty}{r-1} - \frac{1}{r-1} = \infty$

Hence series is divergent.

(iii) when $r=1$, then, $S_n = 1+1+1+1+ \cdots$ (n times) $= n$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$, Hence series is divergent.

(iv) when $r=-1$, then, $S_n = 1-1+1-1+ \cdots$
$= \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$

Hence series is oscillatory.

Que 3 :– Examine the nature of the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \infty$.

Sol :– The nth term of the series i.e. $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\therefore$
$u_1 = \frac{1}{1} - \cancel{\frac{1}{2}}$
$u_2 = \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$
$u_3 = \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$
$u_4 = \cancel{\frac{1}{4}} - \frac{1}{5}$
$\vdots$
$u_{n-1} = \frac{1}{n-1} - \cancel{\frac{1}{n}}$
$u_n = \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$S_n = 1 - \frac{1}{n+1}$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right] = 1 - 0 = 1$, (finite)

Hence series is convergent.

Que 4 :– Examine the nature of the following series —

(i) $1+4+9+16+ \cdots \infty$, Ans:- divergent Hint $S_n = \Sigma n^2 = \frac{n(n+1)(n+2)}{6}$

(ii) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \cdots \infty$, Ans:- Convergent, $S_n = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$

(iii) $1^3+2^3+3^3+ \cdots \infty$, Ans:- Divergent $S_n = \Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(iv) $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+ \cdots \infty$ Ans:- Convergent. $u_n = \frac{1}{n(n+2)}$
$= \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$
$\therefore S_n = \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$

Properties of Convergent and Divergent Series:—

(i) If we add a finite number of terms to a convergent, divergent or oscillatory series, the resulting series continues to be so.

(ii) If we subtract a finite number of terms to a convergent, divergent or oscillatory series the resulting series continues to be so.

(iii) If we multiply with a non zero constant in a convergent, divergent or oscillatory series, the resulting series continues to be so.

(iv) In an infinite series if its all terms are +ive, then it is either convergent or divergent, it can not be oscillatory.

Test of Divergence :— The infinite series of +ive terms Σu_n is divergent if $\lim\limits_{n\to\infty} u_n \neq 0$.

Note :- If $\lim\limits_{n\to\infty} u_n = 0$, then test does not mean that series is convergent.

Que(5): Test the convergent of the following series —

(i) $\Sigma \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \infty$, (ii) $\sqrt[7]{\frac{1}{1+1}} + \sqrt[7]{\frac{2}{2+1}} + \sqrt[7]{\frac{3}{3+1}} + \cdots \infty$

(iii) $\sum\limits_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$, (iv) $\sum\limits_{n=1}^{\infty}\left[n \log\left(\frac{3n+2}{3n-2}\right) - 1\right]$

Sol :- (i) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim\limits_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 1 \neq 0$ i.e. divergent series

(ii) $u_n = \sqrt[7]{\frac{n}{n+1}} = \left(\frac{n}{n+1}\right)^{1/7} = \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{7}}$

$\therefore \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{7}} = 1 \neq 0$. i.e. divergent series.

(iii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\left[\cos\left(\frac{1}{n}\right)\right] = \cos 0 = 1 \neq 0$ i.e. divergent series.

(iv) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\left[n \cdot \log\left(\frac{3n+2}{3n-2}\right) - 1\right] = \lim\limits_{n\to\infty}\left[n \log \frac{\left(1+\frac{2}{3n}\right)}{\left(1-\frac{2}{3n}\right)} - 1\right]$

$= \lim\limits_{n\to\infty}\left[n\left\{\log\left(1+\frac{2}{3n}\right) - \log\left(1-\frac{2}{3n}\right)\right\} - 1\right]$

$= \lim\limits_{n\to\infty}\left[n\left\{\left(\frac{2}{3n} - \cancel{\frac{1}{2}\left(\frac{2}{3n}\right)^2} + \frac{1}{3}\left(\frac{2}{3n}\right)^3 - \cancel{\cdots}\right) - \left(-\frac{2}{3n} - \cancel{\frac{1}{2}\left(\frac{2}{3n}\right)^2} - \frac{1}{3}\left(\frac{2}{3n}\right)^3 - \cancel{\cdots}\right)\right\} - 1\right]$

$= \lim\limits_{n\to\infty}\left[n\left\{2\left(\frac{2}{3n} + \frac{1}{3}\left(\frac{2}{3n}\right)^3 + \cdots\right)\right\} - 1\right]$

$= \lim\limits_{n\to\infty}\left[2\left\{\frac{2}{3} + \frac{1 \cdot 2^3}{3 \cdot 3^3 n^2} + \cdots\right\} - 1\right]$

$= \frac{4}{3} - 1 = \frac{1}{3} \neq 0$ i.e. divergent series.

Hyperharmonic Series Test OR p - Series Test :—

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{n^p} + \cdots \infty$, is known as hyperharmonic series or p- series.

The series is converges for $p > 1$ and diverges for $p \leq 1$.

Proof :- Case-I :- when $p > 1$

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \cdots$$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots$$

$$< \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots$$

$$< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots$$

$$< 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots$$

$$< 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \cdots$$

$$< 1 + \left(\frac{1}{2}\right)^{p-1} + \left(\frac{1}{2}\right)^{2(p-1)} + \left(\frac{1}{2}\right)^{3(p-1)} + \cdots$$

$$< \frac{1}{1-\left(\frac{1}{2}\right)^{p-1}}$$ (in GP series, sum of the infinite terms $= \frac{a}{1-r}$)

$<$ finite number if $p > 1$

i.e. convergent.

Case II :- When $p = 1$, then

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$ $\left\{ \because \frac{1}{3} > \frac{1}{4},\ \frac{1}{5} > \frac{1}{8},\ \frac{1}{6} > \frac{1}{8} \text{ etc.} \right.$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$> \infty$, i.e. divergent $\left\{ \text{after the first term it is in GP Series and common ratio is 1. i.e. } \frac{a}{1-r} = \frac{a}{0} = \infty \right.$

Case III :- When $p < 1$, then

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$ $\left\{ \text{Since } \frac{1}{2^p} > \frac{1}{2} \text{ if } p<1,\ \frac{1}{3^p} > \frac{1}{3} \text{ if } p<1 \text{ etc} \right.$

$>$ divergent series.

(from above case II)

Comparison Test – If Σu_n and Σv_n be two infinite series of +tive terms, such that $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (a non-zero and finite), then Σu_n and Σv_n are eighter both convergent or both divergent.

Note – (i) To test the convergence of series, this comparison test is usefull. We compare Σu_n with an auxilary series Σv_n (p-series) whose convergence or divergence is already known.

(ii) How we find the Σv_n ie auxilary series :– Write the n^{th} term of the given series and retain only the highest power of n in the numerator and denominator both, the resulting term is v_n. for example

Let (i) $u_n = \frac{\sqrt{n}}{n^3+1}$, then $v_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$

(ii) If $u_n = \sin\frac{1}{n} = \frac{1}{n} - \frac{1}{\lfloor 3 \cdot n^3} + \frac{1}{\lfloor 5 \cdot n^5} - \cdots$, then $v_n = \frac{1}{n}$

(iii) If $u_n = \frac{\sqrt{2n-1}}{(2n+2)(2n+4)}$, then $v_n = \frac{\sqrt{n}}{n \cdot n} = \frac{1}{n^{3/2}}$.

Que (6) :– Test whether the following series is convergent or divergent.

(i) $\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{1}{n}$

(ii) $\sum_{n=1}^{\infty} \frac{n(n+4)}{(n+2)(n+3)(n+5)}$

(iii) $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

(iv) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)$

(v) $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

(vi) $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1} - \sqrt{n-1}}{n}\right)$

(vii) $\frac{1}{1\cdot 2} + \frac{2}{3\cdot 4} + \frac{3}{5\cdot 6} + \cdots \infty$

(viii) $\frac{1}{4\cdot 6} + \frac{\sqrt{3}}{6\cdot 8} + \frac{\sqrt{5}}{8\cdot 10} + \cdots \infty$

Sol – (i) $u_n = \frac{1}{n}\sin\frac{1}{n} = \frac{1}{n}\left(\frac{1}{n} - \frac{1}{\lfloor 3 \cdot n^3} + \frac{1}{\lfloor 5 \cdot n^5} - \cdots\right) = \frac{1}{n^2} - \frac{1}{\lfloor 3 \cdot n^4} + \frac{1}{\lfloor 5 \cdot n^6} - \cdots$

Let $v_n = \frac{1}{n^2}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[1 - \frac{1}{\lfloor 3 \cdot n^2} + \frac{1}{\lfloor 5 \cdot n^4} - \cdots\right] = 1$ (ie. finite and no zero)

Hence comparison test is applied, hence Σu_n & Σv_n are both convergent or both divergent.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent. Since in p-series $p = 2 > 1$, the series is convergent.

$\therefore \Sigma u_n$ is convergent.

(ii) $u_n = \dfrac{n(n+4)}{(n+2)(n+3)(n+5)}$, Let $v_n = \dfrac{n \cdot n}{n \cdot n \cdot n} = \dfrac{1}{n}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{n^2(n+4)}{(n+2)(n+3)(n+5)} = \lim\limits_{n\to\infty} \dfrac{\left(1+\frac{4}{n}\right)}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{5}{n}\right)} = 1$ (finite & non zero)

Hence comparison test is applied i.e. both Σu_n & Σv_n are convergent or divergent.

Now $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent.

$\therefore$ Σu_n is divergent.

(iii) $u_n = \sqrt{(n^3+1)} - \sqrt{n^3} = n^{3/2}\left(1+\frac{1}{n^3}\right)^{1/2} - n^{3/2} = n^{3/2}\left[\left(1+\frac{1}{n^3}\right)^{1/2} - 1\right]$

$= n^{3/2}\left[1 + \frac{1}{2}\cdot\frac{1}{n^3} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{\lfloor 2}\cdot\frac{1}{n^6} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{\lfloor 3}\cdot\frac{1}{n^9} + \cdots - 1\right]$

$= n^{3/2}\left[\cancel{1} + \frac{1}{2n^3} - \frac{1}{8\cdot n^6} + \frac{1}{16 n^9} - \cdots - \cancel{1}\right]$

$= \dfrac{1}{2n^{3/2}} - \dfrac{1}{8\cdot n^{9/2}} + \cdots$

$\therefore$ Let $v_n = \dfrac{1}{n^{3/2}}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\left[\frac{1}{2} - \frac{1}{8\cdot n^3} + \cdots\right] = \frac{1}{2}$ (finite and non zero)

Hence comparison test is applied i.e. both Σu_n & Σv_n are convergent or divergent.

Now $\Sigma v_n = \Sigma \dfrac{1}{n^{3/2}} = \Sigma \dfrac{1}{n^p}$, $p = \frac{3}{2} > 1$ i.e. convergent

Hence Σu_n is convergent.

(iv) Same as above, $u_n = \left[\sqrt{n^2+1} - n\right] = \frac{1}{2n} - \frac{1}{8\cdot n^3} + \cdots$ and let $v_n = \frac{1}{n}$ i.e. divergent.

(v) $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$= n^2\left(1+\frac{1}{n^4}\right)^{1/2} - n^2\left(1-\frac{1}{n^4}\right)^{1/2}$

$= n^2\left[\cancel{1} + \frac{1}{2n^4} - \cancel{\frac{1}{8n^8}} + \frac{1}{16n^{12}} - \cancel{\cdots} - \left(\cancel{1} - \frac{1}{2n^4} - \cancel{\frac{1}{8n^8}} - \frac{1}{16n^{12}} - \cancel{\cdots}\right)\right]$

$= n^2\left[\frac{1}{n^4} + \frac{1}{8n^{12}} + \cdots\right]$

$= \dfrac{1}{n^2} + \dfrac{1}{8n^{10}} + \cdots$

Let $v_n = \dfrac{1}{n^2}$

Now $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\left[1 + \frac{1}{8n^8} + \cdots\right] = 1$ (finite and non zero)

Hence comparison test can be applied i.e. both Σu_n & Σv_n are conv. or div.

Again $\Sigma v_n = \Sigma \frac{1}{n^2} = \Sigma \frac{1}{n^p}$, $p = 2 > 1$, i.e. convergent.

Hence Σu_n is convergent.

Let Σu_n be an infinite series of +ive terms, then

Ratio Test or D'Alembert's Ratio test :— $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \begin{cases} <1, & \text{then } \Sigma u_n \text{ is divergent} \\ >1, & \text{then } \Sigma u_n \text{ is convergent} \\ =1, & \text{then ratio test is fail.} \end{cases}$

Raabe's Test :— $\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \begin{cases} <1, & \text{then } \Sigma u_n \text{ is divergent} \\ >1, & \text{then } \Sigma u_n \text{ is convergent} \\ =1, & \text{then Raabe's test is fail.} \end{cases}$

Logarithmic Test :— $\lim_{n\to\infty}\left[n \log \frac{u_n}{u_{n+1}}\right] = \begin{cases} <1, & \text{then } \Sigma u_n \text{ is divergent} \\ >1, & \text{then } \Sigma u_n \text{ is convergent} \\ =1, & \text{then logarithmic test is fail.} \end{cases}$

Cauchy's Root Test :— $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \begin{cases} <1, & \text{then } \Sigma u_n \text{ is convergent} \\ >1, & \text{then } \Sigma u_n \text{ is divergent} \\ =1, & \text{then Cauchy's test is fail.} \end{cases}$
or
(Cauchy's n^{th} root test)

Questions Based on Ratio Test —

Que 7 :— Test for the convergence of the following series —

(i) $1 + \frac{\lfloor 2}{2^2} + \frac{\lfloor 3}{3^3} + \frac{\lfloor 4}{4^4} + \cdots \infty$ (ii) $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$

(iii) $\sum_{n=1}^{\infty} \frac{3\cdot6\cdot9\cdots3n}{4\cdot7\cdot10\cdots(3n+1)}\cdot\frac{2^n}{(3n+2)}$ (iv) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

Sol :— (i) Here $u_n = \frac{\lfloor n}{n^n}$ $\therefore$ $u_{n+1} = \frac{\lfloor n+1}{(n+1)^{n+1}}$ (Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$)

Now $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\lfloor n}{\lfloor n+1}\cdot\frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$

Hence by ratio test Σu_n is convergent.

(ii) Here $u_n = \frac{n}{1+2^n}$, then $u_{n+1} = \frac{n+1}{1+2^{n+1}}$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1+2^{n+1}}{1+2^n} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\cdot\frac{\cancel{2^n}\left(2+\frac{1}{2^n}\right)}{\cancel{2^n}\left(1+\frac{1}{2^n}\right)} = 2 > 1$

Hence by ratio test Σu_n is convergent.

(iii) $u_n = \frac{3\cdot6\cdot9\cdots3n}{4\cdot7\cdot10\cdots(3n+1)}\cdot\frac{2^n}{(3n+2)}$, hence $u_{n+1} = \frac{3\cdot6\cdot9\cdots3n\,(3n+3)}{4\cdot7\cdot10\cdots(3n+1)(3n+4)}\cdot\frac{2^{n+1}}{(3n+5)}$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{\cancel{3\cdot6\cdot9\cdots3n}}{\cancel{3\cdot6\cdot9\cdots3n}(3n+3)}\cdot\frac{\cancel{4\cdot7\cdot10\cdots(3n+1)}(3n+4)}{\cancel{4\cdot7\cdot10\cdots(3n+1)}}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{(3n+5)}{(3n+2)}\right]$

$= \lim_{n\to\infty}\frac{(3n+4)}{(3n+3)}\cdot\frac{1}{2}\cdot\frac{(3n+5)}{(3n+2)}$

$= \lim_{n\to\infty}\frac{\left(1+\frac{4}{3n}\right)}{\left(1+\frac{3}{3n}\right)}\cdot\frac{1}{2}\cdot\frac{\left(1+\frac{5}{3n}\right)}{\left(1+\frac{2}{3n}\right)} = \frac{1}{2} < 1$

Hence by ratio test Σu_n is divergent.

(iv) $u_n = \frac{n^2}{3^n}$ and $u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$, Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\cdot\frac{3^{n+1}}{3^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^2}\cdot 3 = 3 > 1$

Hence series is convergent.

(vi) $u_n = \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n} = \dfrac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{n(\sqrt{n+1}+\sqrt{n-1})} = \dfrac{(\cancel{n}+1)-(\cancel{n}-1)}{n(\sqrt{n+1}+\sqrt{n-1})} = \dfrac{2}{n(\sqrt{n+1}+\sqrt{n-1})}$

Let $v_n = \dfrac{2}{n(\sqrt{n}+\sqrt{n})} = \dfrac{1}{n^{3/2}}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{\frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}}{\frac{1}{n^{3/2}}} = \lim\limits_{n\to\infty} \dfrac{2}{\frac{1}{n^{1/2}}(\sqrt{n+1}+\sqrt{n-1})} = \lim\limits_{n\to\infty} \dfrac{2}{\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$

$= \dfrac{2}{\sqrt{1}+\sqrt{1}} = \dfrac{2}{2} = 1$ (finite & nonzero)

Hence Σu_n & Σv_n both are convergent and divergent.

Now $\Sigma v_n = \Sigma \dfrac{1}{n^{3/2}} = \Sigma \dfrac{1}{n^p}$, $n = \frac{3}{2} > 1$ i.e. convergent.

Hence Σu_n is convergent.

(vii) $\dfrac{1}{1\cdot 2} + \dfrac{2}{3\cdot 4} + \dfrac{3}{5\cdot 6} + \cdots \dfrac{n}{(2n-1)2n} + \cdots \infty$

Let $u_n = \dfrac{n}{(2n-1)2n} = \dfrac{1}{2(2n-1)}$, Let $v_n = \dfrac{1}{n}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{1}{2(2-\frac{1}{n})} = \dfrac{1}{4}$ (finite and non zero)

Hence comparison test can be applied, $\therefore$ Σu_n & Σv_n both are convergent or divergent.

Since $\Sigma v_n = \Sigma \dfrac{1}{n} = \Sigma \dfrac{1}{n^p}$, $p=1$ i.e. divergent

Hence Σu_n is divergent.

(viii) $\dfrac{1}{4\cdot 6} + \dfrac{\sqrt{3}}{6\cdot 8} + \dfrac{\sqrt{5}}{8\cdot 10} + \cdots \dfrac{\sqrt{2n-1}}{(2n+2)(2n+4)} + \cdots \infty$

$\therefore$ $u_n = \dfrac{\sqrt{2n-1}}{(2n+2)(2n+4)}$, Let $v_n = \dfrac{\sqrt{n}}{n\cdot n} = \dfrac{1}{n^{3/2}}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\left[\dfrac{(\sqrt{2n-1})\cdot n^{3/2}}{(2n+2)(2n+4)}\right] = \lim\limits_{n\to\infty}\left[\sqrt{\dfrac{2n-1}{n}}\left(\dfrac{n}{2n+2}\right)\left(\dfrac{n}{(2n+4)}\right)\right]$

$= \lim\limits_{n\to\infty}\left[\left(\sqrt{2-\frac{1}{n}}\right)\left(\dfrac{1}{2+\frac{2}{n}}\right)\left(\dfrac{1}{2+\frac{4}{n}}\right)\right] = \sqrt{2}\cdot\dfrac{1}{2}\cdot\dfrac{1}{2} = \dfrac{1}{2\sqrt{2}}$ (finite and non zero)

Hence comparison test can be applied $\therefore$ Σu_n & Σv_n both are convergent and divergent.

Since $\Sigma v_n = \Sigma \dfrac{1}{n^{3/2}} = \Sigma \dfrac{1}{n^p}$, $p = \frac{3}{2} > 1$ i.e. convergent.

Hence Σu_n is convergent.

Que 8 :- Test for the convergence of the series $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\frac{x^4}{4\cdot 5}+\cdots\infty$

Sol :- Here $u_n=\frac{x^n}{n(n+1)}$, then $u_{n+1}=\frac{x^{n+1}}{(n+1)(n+2)}$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{x^n}{x^{n+1}}\cdot\frac{(n+1)(n+2)}{n(n+1)}=\lim_{n\to\infty}\left[\frac{1}{x}\left(\frac{n+2}{n}\right)\right]=\lim_{n\to\infty}\left[\frac{1}{x}\left(1+\frac{2}{n}\right)\right]=\frac{1}{x}$

By Ratio test $\frac{1}{x}<1$ or $x>1$ then series in divergent and $\frac{1}{x}>1$ or $x<1$, then series is convergent. But for $x=1$, test is fail.

Now for $x=1$. $u_n=\frac{1}{n(n+1)}$ let $v_n=\frac{1}{n\cdot n}=\frac{1}{n^2}$

Now $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[\frac{n^2}{n(n+1)}\right]=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)=1$ (finit and nozero)

Hence comparison test can be applied, hence Σu_n & Σv_n both are convergent or divergent.

Since $\Sigma v_n=\Sigma\frac{1}{n^2}=\Sigma\frac{1}{n^p}$, $p=2>1$ i.e. convergent.

Hence the given series is convergent if $x\le 1$ and divergent if $x>1$.

Que 9 :- Test for convergence of the series $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots\infty$

Sol :- Here $u_n=\frac{x^n}{n^2+1}$, $u_{n+1}=\frac{x^{n+1}}{(n+1)^2+1}$, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(n^2+2n+2)}{n^2+1}\cdot\frac{1}{x}=\frac{1}{x}$ let $v_n=\frac{1}{n^2}$

Series is convergent if $x\le 1$ and divergent if $x>1$

Que 10 :- Examine the convergence or divergence of the series $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\cdots\infty$

Sol :- Here $u_n=\frac{x^{2n-2}}{(n+1)\sqrt{n}}$, hence $u_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{x^{2n-2}}{x^{2n}}\cdot\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}=\lim_{n\to\infty}\left[\frac{1}{x^2}\frac{(1+\frac{2}{n})}{(1+\frac{1}{n})}\sqrt{\left(1+\frac{1}{n}\right)}\right]=\frac{1}{x^2}$

By ratio test the given series is divergent if $\frac{1}{x^2}<1$ i.e. $x^2>1$

and convergent if $\frac{1}{x^2}>1$ i.e. $x^2<1$. but $x^2=1$ its fail.

Now For $x^2=1$, then $u_n=\frac{1}{(n+1)\sqrt{n}}$, again let $v_n=\frac{1}{n\sqrt{n}}=\frac{1}{n^{3/2}}$

Again $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n^{3/2}}{(n+1)\sqrt{n}}=\lim_{n\to\infty}\frac{n^{3/2}}{n(1+\frac{1}{n})\sqrt{n}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1$ (finite & nonzero)

Hence comparison test can be applied i.e. both Σu_n & Σv_n are convergent or divergent.

Since $\Sigma v_n=\Sigma\frac{1}{n^{3/2}}=\Sigma\frac{1}{n^p}$, $p=\frac{3}{2}>1$ i.e. convergent.

Hence the given series is convergent if $x^2\le 1$ and divergent if $x^2>1$.

Questions Based on Raabe's Test :—

Que (11) :- Test for convergence of the series $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{1}{n}$.

Sol :- Here $u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{1}{n}$, then $u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots2n\,(2n+2)}\cdot\frac{1}{n+1}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}\cdot\frac{2\cdot4\cdot6\cdots2n(2n+2)}{2\cdot4\cdot6\cdots2n}\cdot\frac{n+1}{n}$$

$$= \lim_{n\to\infty}\frac{(2n+2)}{(2n+1)}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\frac{\left(1+\frac{2}{2n}\right)}{\left(1+\frac{1}{2n}\right)}\left(1+\frac{1}{n}\right) = 1 \quad \text{(i.e. Ratio test fail.)}$$

Now we apply Raabe's test. hence

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(n+1)}{(2n+1)\,n}-1\right] = \lim_{n\to\infty}\cancel{n}\left[\frac{\cancel{2n^2}+4n+2-\cancel{2n^2}-n}{(2n+1)\cancel{n}}\right]$$

$$= \lim_{n\to\infty}\left(\frac{3n+2}{2n+1}\right) = \lim_{n\to\infty}\left[\frac{3\cancel{n}\left(1+\frac{2}{3n}\right)}{2\cancel{n}\left(1+\frac{1}{2n}\right)}\right] = \frac{3}{2} > 1,$$

Hence by Raabe's test the given series is convergent.

Que (12) :- In question no. 8, for $x = 1$, the Ratio test is fail then we apply Raabe's test

$$\text{Now}\quad \frac{u_n}{u_{n+1}} - 1 = \frac{n+2}{n} - 1 = \frac{\cancel{n}+2-\cancel{n}}{n} = \frac{2}{n}$$

By Raabe's test,

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\cdot\frac{2}{n}\right] = 2 > 1 \text{ i.e. convergent}$$

Que (13) :- In question no. 9, for $x=1$, the Ratio test is fail, then we apply Raabe's test

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{(n^2+2n+2)}{n^2+1}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{\cancel{n^2}+2n+2-\cancel{n^2}-1}{n^2+1}\right)\right] = \lim_{n\to\infty}\frac{2n^{\cancel{2}}\left(1+\frac{1}{2n}\right)}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} = 2 > 1$$

i.e. convergent.

Que (14) :- Test the convergence of the series $1 + \frac{(\lfloor 1)^2}{\lfloor 2}x^2 + \frac{(\lfloor 2)^2}{\lfloor 4}x^4 + \cdots\infty$.

Sol :- Here $u_n = \frac{(\lfloor n)^2 x^{2n}}{\lfloor 2n}$ hence $u_{n+1} = \frac{(\lfloor (n+1))^2\, x^{2(n+1)}}{\lfloor 2(n+1)}$

$$\text{Now}\quad \frac{u_n}{u_{n+1}} = \frac{(\lfloor n)^2}{(\lfloor n+1)^2}\cdot\frac{x^{2n}}{x^{2n+2}}\cdot\frac{\lfloor 2n+2}{\lfloor 2n} = \frac{\cancel{(\lfloor n)^2}}{(n+1)^2\cancel{(\lfloor n)^2}}\cdot\frac{1}{x^2}\cdot\frac{(2n+2)(2n+1)\cancel{\lfloor 2n}}{\cancel{\lfloor 2n}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{2(n+1)(2n+1)}{(n+1)^2}\cdot\frac{1}{x^2} = \lim_{n\to\infty}\frac{2\cdot(2n+1)}{(n+1)}\cdot\frac{1}{x^2} = \lim_{n\to\infty}\frac{2\cdot2\cancel{n}\left(1+\frac{1}{2n}\right)}{\cancel{n}\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x^2}$$

$$= \frac{4}{x^2}$$

By Ratio Test

(i) If $\frac{4}{x^2} < 1$ or $x^2 > 4$, then series is divergent

(ii) If $\frac{4}{x^2} > 1$ or $x^2 < 4$, then series is convergent.

(iii) If $\frac{4}{x^2} = 1$ or $x^2 = 4$, then ratio test is fail.

Now we apply the Raabe's test, put $x^2=4$, then

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{\cancel{2}(2n+1)}{(n+1)}\cdot\frac{1}{\cancel{4}_2}-1\right] = \lim_{n\to\infty} n\left[\frac{\cancel{2n}+1-\cancel{2n}-2}{2(n+1)}\right] = \lim_{n\to\infty}\left[\frac{-n}{2(n+1)}\right]$$

$$= \lim_{n\to\infty} \frac{-\cancel{n}}{2\cancel{n}(1+\frac{1}{n})} = -\frac{1}{2} < 1 \quad \text{i.e. divergent.}$$

Hence the given series is convergent for $x^2<4$, and divergent for $x^2\geq 4$.

Que (15):- Test for convergence of the series $\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\cdots\infty$

Sol:- Here $u_n = \frac{3\cdot 6\cdot 9\cdots 3n}{7\cdot 10\cdot 13\cdots(3n+4)}\cdot x^n$, hence $u_{n+1}=\frac{3\cdot 6\cdot 9\cdots 3n(3n+3)}{7\cdot 10\cdot 13\cdots(3n+4)(3n+7)}\cdot x^{n+1}$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{3n+7}{3n+3}\cdot\frac{1}{x} = \lim_{n\to\infty}\frac{(1+\frac{7}{3n})}{(1+\frac{3}{3n})}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By Ratio test, Σu_n is convergent if $\frac{1}{x}>1$ or $x<1$ and divergent if $\frac{1}{x}<1$ or $x>1$. For $x=1$, ratio test is fail, we apply Raabe's test.

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3}-1\right) = \lim_{n\to\infty}\frac{4n}{3n+3} = \lim_{n\to\infty}\frac{4\cancel{n}}{3\cancel{n}(1+\frac{3}{3n})} = \frac{4}{3}>1$$

i.e. convergent.

Hence the given series (i) convergent if $x\leq 1$ and divergent if $x>1$.

Que (16) — Test the convergence $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\cdots\infty$

Sol:- Here $u_n = \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}\cdot\frac{x^{2n-1}}{2n-1}$, hence $u_{n+1}=\frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n}\cdot\frac{x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{2n}{2n-1}\cdot\frac{2n+1}{2n-1}\cdot\frac{x^{2n-1}}{x^{2n+1}} = \lim_{n\to\infty}\frac{(1+\frac{1}{2n})}{(1-\frac{1}{2n})^2}\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

By Ratio test (i) when $\frac{1}{x^2}>1$ or $x^2<1$, then Σu_n is convergent

(ii) when $\frac{1}{x^2}<1$ or $x^2>1$, then Σu_n is divergent

(iii) when $\frac{1}{x^2}=1$ or $(x^2=1)$ then Ratio test is fail.

Now we apply Raabe's test for $x^2=1$, then

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{2n(2n+1)}{(2n-1)(2n-1)}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{\cancel{4n^2}+2n-\cancel{4n^2}+4n-1}{(2n-1)^2}\right)\right]$$

$$= \lim_{n\to\infty}\left[n\,\frac{(6n-1)}{(2n-1)^2}\right] = \lim_{n\to\infty}\left[\frac{\cancel{n}\cdot 6\cancel{n}(1-\frac{1}{6n})}{4\cancel{n^2}(1-\frac{1}{2n})^2}\right] = \frac{3}{2}>1 \quad \text{i.e. Convergent}$$

Hence series is convergent if $x^2\leq 1$ and divergent if $x^2>1$.

Question Based on Logarithmic Test :—

Que (17): Test the convergence of the series $1 + \frac{1}{2}\cdot x + \frac{\lfloor 2}{3^2}x^2 + \frac{\lfloor 3}{4^3}x^3 + \cdots \infty$

Sol – Here $u_n = \frac{\lfloor n}{(n+1)^n}x^n$, hence $u_{n+1} = \frac{\lfloor n+1}{(n+2)^{n+1}}x^{n+1}$

Now
$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\lfloor n}{\lfloor n+1}\cdot\frac{(n+2)^{n+1}}{(n+1)^n}\cdot\frac{x^n}{x^{n+1}} = \lim_{n\to\infty}\frac{(n+2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{1}{x}$$
$$= \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{n+1}\cdot\frac{1}{x} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{1}{x} = \frac{e}{x}$$
$$\left(\text{Since } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$$

$\therefore$ By Ratio test (i) if $\frac{e}{x} > 1$ or $x < e$, then Σu_n is convergent.

(ii) if $\frac{e}{x} < 1$ or $x > e$, then Σu_n is divergent

(iii) if $\frac{e}{x} = 1$ or $x = e$, the ratio test is fail.

Now we apply the logarithmic test at $x = e$.

$$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left[\log\left\{\left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{1}{e}\right\}\right] = \lim_{n\to\infty} n\left[(n+1)\log\left(1+\frac{1}{n+1}\right) - \log e\right]$$
$$= \lim_{n\to\infty} n\left[(n+1)\left\{\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} - \cdots\right\} - 1\right]$$
$$= \lim_{n\to\infty} n\left[\cancel{1} - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} - \cdots - \cancel{1}\right]$$
$$= \lim_{n\to\infty}\left[-\frac{1}{2}\cdot\frac{n}{n(1+\frac{1}{n})} + \frac{n}{3n^2(1+\frac{1}{n^2})} - \cdots\right]$$
$$= \lim_{n\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{(1+\frac{1}{n})} + \frac{1}{3}\frac{(1/n)}{(1+\frac{1}{n^2})} - \cdots\right]$$
$$= -\frac{1}{2} < 1 \quad \text{i.e. divergent.}$$

Hence the given series is divergent if $x \geq e$ and convergent if $x < e$.

Que (18):– Test the convergence $x + \frac{2^2x^2}{\lfloor 2} + \frac{3^3x^3}{\lfloor 3} + \frac{4^4x^4}{\lfloor 4} + \frac{5^5x^5}{\lfloor 5} + \cdots \infty$

Sol :– Here $u_n = \frac{n^n x^n}{\lfloor n}$ $\therefore$ $u_{n+1} = \frac{(n+1)^{n+1}}{\lfloor n+1}x^{n+1}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n^n}{(n+1)^{n+1}}\cdot\frac{\lfloor n+1}{\lfloor n}\cdot\frac{x^n}{x^{n+1}} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}\cdot\frac{1}{x} = \lim_{n\to\infty}\left\{\frac{1}{(1+\frac{1}{n})^n}\right\}\cdot\frac{1}{x}$$
$$= \frac{1}{e\cdot x}$$

By Ratio test (i) $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ i.e. convergent.

(ii) $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$ i.e. divergent

(iii) $\frac{1}{ex} = 1$ or $ex = 1$, ratio test is fail; Apply Log. Test

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[n\log\left(1+\frac{1}{n}\right)^{-n}\cdot e\right] = \lim_{n\to\infty} n\left\{-n\log\left(1+\frac{1}{n}\right) + \log e\right\}$$
$$= \lim_{n\to\infty} n\left\{-n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) - 1\right\} = \frac{1}{2} < 1$$
i.e. divergent

Que (19) :- Test for convergence of the series whose n^{th} terms are

(i) $\dfrac{n^{n^2}}{(n+1)^{n^2}}$ or $\dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$ (ii) $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Sol :- (i) Now $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}} = \left(\dfrac{n}{n+1}\right)^{n^2} \Rightarrow (u_n)^{\frac{1}{n}} = \left(\dfrac{n}{n+1}\right)^n = \left(\dfrac{1}{1+\frac{1}{n}}\right)^n$

again $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\dfrac{1}{1+\frac{1}{n}}\right)^n = \lim\limits_{n\to\infty}\dfrac{1}{\left(1+\frac{1}{n}\right)^n} = \dfrac{1}{e} < 1$

Hence by Cauchy's Root Test, the given series is convergent.

(ii) $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \dfrac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}} \Rightarrow (u_n)^{\frac{1}{n}} = \dfrac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$

Now $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \dfrac{1}{e} < 1$ i.e. convergent.

Que (20) :- Test the convergence of the series $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}\cdots\frac{1}{n^n}+\cdots\infty$

Sol :- Here $u_n = \dfrac{1}{n^n}$ or $(u_n)^{\frac{1}{n}} = \dfrac{1}{n}$

Now $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{1}{n} = 0 < 1$, hence by CR test it is convergent.

Que (21) :- Examine then convergence of $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots\infty$

Sol :- Here $u_n = \left[\left(\dfrac{n+1}{n}\right)^{n+1}-\left(\dfrac{n+1}{n}\right)\right]^{-n} \Rightarrow (u_n)^{\frac{1}{n}} = \left[\left(\dfrac{n+1}{n}\right)^{n+1}-\left(\dfrac{n+1}{n}\right)\right]^{-1}$

Now $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)\right]^{-1} = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^n-1\right\}\right]^{-1} = [e-1]^{-1}$

$= \dfrac{1}{e-1} < 1$

Hence by Cauchy's Root Test the series is convergent.

Que (22) :- Test the series for convergence and divergence $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\cdots$

Sol :- Omitting the first term of the series, then

$u_n = \left(\dfrac{n+1}{n+2}\right)^n x^n \Rightarrow (u_n)^{\frac{1}{n}} = \left(\dfrac{n+1}{n+2}\right)x$

Now $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)}x = x$

∴ By Cauchy's Root Test, the series is convergent if $x<1$ and divergent if $x>1$. The test is fail for $x=1$. Now for $x=1$ we have $u_n = \left(\dfrac{n+1}{n+2}\right)^n$

Now $\lim\limits_{n\to\infty}u_n = \lim\limits_{n\to\infty}\left(\dfrac{n+1}{n+2}\right)^n = \lim\limits_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \lim\limits_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{1}{n/2}\right)^{n/2}\right]^2} = \dfrac{e}{e^2} = \dfrac{1}{e} \neq 0$

i.e. Σu_n is divergent

Hence the given series is convergent if $x<1$ and divergent if $x\geq 1$.