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DIFFERENTIAL CALCULUS-I

* Topic 1: Succesive Differentiation / n^{th} order differentiation
the process of differentiating the same function again and again is called successive differentiation.

notations: $y = f(x)$

y_1 / $\frac{dy}{dx}$ / y' / $y^{(1)}$ / Dy where $D = \frac{d}{dx}$

- Standard formula's for n^{th} derivative.

1. $y = x^m$

$y_1 = m x^{m-1}$

$y_2 = m(m-1)\, x^{m-2}$

$\vdots$

$\boxed{y_n = m \ldots \ldots (m-(n-1)) \cdot x^{m-n}}$ $m > n$

case 1: $m = n$

$\Rightarrow \boxed{y_n = n!}$ or $\boxed{y_n = m!}$

case 2: $n > m$

$\boxed{y_n = 0}$

2. $y = (ax+b)^m$

$y_1 = m\,(ax+b)^{m-1}\,(a)$

$y_2 = m\,(m-1)\,(ax+b)^{m-2}\, a^2$

$\vdots$

$\boxed{y_n = m \ldots \ldots (m-(n-1))\,(ax+b)^{m-n}\, a^n}$ $m > n$

case 1: when, $m=n$

$\boxed{y_1 = n!\, a^n}$

case 2: when, $m<n$

$\boxed{y_1 = 0}$

3) $y = e^{ax+b}$

$y_1 = e^{ax+b}\, a$

$y_2 = e^{ax+b}\, a^2$

$\vdots$

$\boxed{y_n = e^{ax+b}\, a^n}$

4) $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

$y_1 = (-1)(ax+b)^{-2} a$

$y_2 = (-1)(-2)(ax+b)^{-3} a^2$

$\vdots$

$y_n = (-1)^n n!\,(ax+b)^{-(n+1)} a^n = \boxed{\dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}} = y_n$

5) $y = \log(ax+b)$

$y_1 = \dfrac{a}{ax+b}$

$y_2 = (-1)(ax+b)^{-2} a^2$

$y_n = a^n (-1)^{n-1} (n-1)! \cdot (ax+b)^{-n}$

$\boxed{y_n = \dfrac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}}$

6) $y = a^{mx}$

$y_1 = a^{mx} \log a\; m$

$y_2 = a^{mx} \log a\; m \log a\; m$

$\vdots$

$\boxed{y_n = a^{mx} (\log^n a)(m^n)}$

7) $y = \sin(ax+b)$

$y_1 = \cos(ax+b)\, a = \sin\left(\frac{\pi}{2} + ax+b\right) a$

$y_2 = \cos\left[\frac{\pi}{2} + ax+b\right] \times a^2 = \sin\left[\frac{\pi}{2} + \frac{\pi}{2} + ax+b\right] a^2$

$\vdots$

$\boxed{y_n = \sin\left(\frac{n\pi}{2} + ax+b\right) \times a^n}$

8) $y = \cos(ax+b)$

$y_n = \cos\left(\frac{n\pi}{2} + ax+b\right) a^n$

9) $y = e^{ax} \sin(bx+c)$

$y_n = (a^2+b^2)^{n/2} e^{ax} \sin\left(bx + c + n \tan^{-1}\frac{b}{a}\right)$

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10) $y = e^{ax}\cos(bx+c)$

$y_n = (a^2+b^2)^{n/2} e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

• Power formula:-

• $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ • $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

• $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ • $\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$

numerical:

Q1. $y = \sin^2 x$ find $y_n = ?$

soln

$y = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$

Differ. 'n' times w.r.t x

~~$y_n = a^n \cos$~~

using $y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

$y_n = 0 - \frac{1\times 2^n}{2}\cos\left(2x+\frac{n\pi}{2}\right)$

$y_n = -2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$

Q2: $y = e^{3x}\cos^2 x$

soln $e^{3x}\left[\frac{1+\cos 2x}{2}\right]$

$y = \frac{e^{3x}}{2} + \frac{e^{3x}\cos 2x}{2}$

$= \frac{1}{2}3^n e^{3x} + \frac{1}{2}\left[3^2+2^2\right]^{n/2} e^{3x}\cos\left(2x+n\tan^{-1}\frac{2}{3}\right)$ ans

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Q3:) $y = \log(ax+x^2)$

$y = \log(x(a+x))$

$y = \log x + \log(a+x)$

$a=1, b=0$ $\quad a=1, b=a$

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!}{x^n} + \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}}$$

$y = \log(ax+b)$

$= \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

* Product formula:-

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\sin A\cos A = \sin 2A$

Q4) $y = \sin 3x\cos x$

$y = \frac{1}{2}(\sin 4x + \sin 2x)$

Diffn (n) times w.r.t x.

$y_n = \frac{1}{2}\left[4^n\sin\left(4x+\frac{n\pi}{2}\right) + 2^n\sin\left(2x+\frac{n\pi}{2}\right)\right]$

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Q5) $\sin^4 x$

$y = (\sin^2 x)^2$

$y = \left(\frac{1-\cos 2x}{2}\right)^2$

$= \frac{1}{4}\left(1 + \cos^2 2x - 2\cos 2x\right)$

$= \frac{1}{4}\left(1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right)$

$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\cos^4 x - 2\cos 2x\right)$

Q

* Partial fraction:- is only applied when, degree of numerator is less than degree of denominator

i) for Quadratic :

$y = \frac{1}{x^2+3x+2}$

note:

$\frac{x^2}{x^2+3x+2}$

$y = \frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} - \frac{1}{(x+2)}$

$x^2+3x+2 \overline{) x^2 } \; 1$

$x^2 + 3x + 2$

$- \quad -$

$-3x - 2$

using partial fraction.

Shortcut

$\frac{\frac{1}{(x+1)}}{(x+2)} + \frac{\frac{1}{(x+2)}}{(x+1)}$ $= 1 + \frac{3x-2}{(x-2)(x-1)}$

now $\to = 0$

$\therefore x = -2$

$\frac{1}{\frac{-1}{x+2}} + \frac{1}{\frac{1}{x+1}}$

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now, for y_n

$y = \frac{1}{(x+1)} - \frac{1}{(x+2)}$

$a=1, b=1$ $\quad a=1, b=2$

$y_n = \frac{(-1)^n n!\, 1^n}{(x+1)^{n+1}} - \frac{(-1)^n n!\, 1^n}{(x+2)^{n+1}}$

Q2 $y = \frac{x}{x^2-5x+6}$

$y = \frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$

using partial fraction,

$\frac{\frac{x}{(x-2)}}{(x-3)} + \frac{\frac{x}{(x-3)}}{(x-2)}$

$\frac{\frac{3}{1}}{(x-3)} + \frac{\frac{2}{-1}}{(x-2)}$

$y = \frac{3}{(x-3)} - \frac{2}{(x-2)}$

Diff 'n' times wrt (x)

$a=1, b=-3 \quad a=1, b=-2$

$y_n = \frac{1}{ax+b} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$= \frac{3(-1)^n n!\, 1^n}{(x-3)^{n+1}} - \frac{2(-1)^n n!\, 1^n}{(x-2)^{n+1}}$

Q3) If $y = \frac{x^2}{x^2-3x+2}$ find $y_n = ?$

$$y = 1 + \frac{3x-2}{(x-1)(x-2)}$$

$$y = 1 + \frac{\frac{3x-2}{x-1}}{x-2} + \frac{\frac{3x-2}{x-2}}{x-1}$$

$$y = 1 + \frac{4}{x-2} - \frac{1}{x-1}$$

Diff n times wrt to x.

$$y_n = 0 + \frac{4(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}$$

Q4) $y = \frac{x-3}{x^3-3x^2+2x}$

$$y = \frac{x-3}{x(x-1)(x-2)}$$

$$y = \frac{\frac{x-3}{x}}{(x-1)(x-2)} + \frac{\frac{x-3}{x-1}}{x(x-2)}$$

$$y = \frac{\frac{x-3}{(x-1)(x-2)}}{x} + \frac{\frac{(x-3)}{x(x-2)}}{(x-1)} + \frac{\frac{(x-3)}{x(x-1)}}{(x-2)}$$

$$y = \frac{-3}{2x} + \frac{2}{x-1} - \frac{1}{2(x-2)}$$

* Typical Question:-

Q1) find n^{th} derivative $y = \frac{1}{x^2+4}$?

Sol^n $y = \frac{1}{x^2+4} = \frac{1}{(x+2i)(x-2i)}$

using partial fraction.

$$\frac{\frac{1}{x+2i}}{x-2i} + \frac{\frac{1}{x-2i}}{x+2i}$$

$$y = \frac{1}{4i(x-2i)} - \frac{1}{4i(x+2i)}$$

$a=1 \quad b=-2i \qquad a=1 \quad b=2i$

$$\frac{1}{4i}\frac{(-1)^n n!}{(x-2i)^{n+1}} - \frac{1}{4i}\frac{(-1)^n n!}{(x+2i)^{n+1}}$$

*** $y = \frac{x^n-1}{x-1}$ find y_n

$$y = \frac{(x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \dots 1)}{(x-1)}$$

$\boxed{y_n = 0}$

$$* \boxed{a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b)}$$

Q3) $y = (x^2-1)^m$ find y_{2m}

$$y = {}^mC_0 (x^2)^m (-1)^0 + {}^mC_1 (x^2)^{m-1}(-1)^1 + \dots \dots {}^mC_m (x^2)^0 (-1)^m$$

$= 1 \times x^{2m} \times 1 + \dots\dots$

differentiating wrt x '2m' times.

$= (2m)! + 0+0+0+0 \dots$

$y_{2m} = (2m)!$ Ans

Q) $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ ~~find~~ prove $y_n = 2(-1)^{n-1}(n-1)!\,\sin n\theta \sin^n\theta$

put $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$\boxed{y = 2\tan^{-1}x}$$

Diff w.r.t to x.

$$y_1 = \frac{2}{1+x^2}$$

$$y_1 = \frac{2}{(x+i)(x-i)}$$

$$= 2\left[\frac{\frac{\cancel{2}1}{x-i}}{x+1} + \frac{\frac{1}{x+i}}{x-i}\right]$$

$$= 2\left[\frac{\frac{1}{-2i}}{x+i} + \frac{\frac{1}{2i}}{x-i}\right]$$

$$= \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

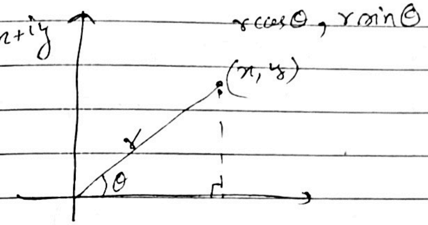
Differe. 'n' times w.r.t to x.

$$y_{n+1} = \frac{1}{i}\left[\frac{(-1)^n n!}{(x-i)^{n+1}} - \frac{(-1)^n n!}{(x+i)^{n+1}}\right]$$

replace n by $n-1$

$$y_n = \frac{1}{i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

$z = x+iy$



for $x = r\cos\theta$
$1 = r\sin\theta$.

$$y_n = \frac{\cancel{2}(-1)^{n-1}(n-1)!}{i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n i}\left[\frac{1}{(\cos\theta - i\sin\theta)^n} - \frac{1}{(\cos\theta + i\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n i}\left[\frac{1}{\cos n\theta - i\sin n\theta} - \frac{1}{\cos n\theta + i\sin n\theta}\right]$$

$$\left(\frac{\cancel{\cos n\theta} + i\sin n\theta - \cancel{\cos n\theta} + i\sin n\theta}{\cos^2 n\theta - (i\sin n\theta)^2}\right)$$

$$= \frac{2(-1)^{n-1}(n-1)\,i\,\sin n\theta}{r^n}$$

$x = r\cos\theta$
$1 = r\sin\theta$
$r = \dfrac{1}{\sin\theta}$

$$= 2(-1)^{n-1}(n-1)\,i\sin n\theta\,\sin^n\theta$$

ans

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Topic 2 **Leibnitz Theorem:-** It is used to differentiate product of the two function n times.

$\boxed{y = u.v}$

$$\frac{d^n}{dx^n}[u.v]$$

$$= {}^nC_0 U_n v + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 - - - - {}^nC_n U V_n.$$

where u, v are function of x and subscripts denotes derivatives.

remark, v is a weak function (polynomial)

Q:1) $y = e^{2x} x^2$ find y_n
using lebnitz then. write this here.

let $v = x^2$ $u = e^{2x}$

$v_1 = 2x$	$U_n = e^{2x} 2^n$	note:
$v_2 = 2$	$U_{n-1} = e^{2x} 2^{n-1}$	${}^nC_0 = 1$ ${}^nC_1 = n$
$v_3 = 0$	$U_{n-2} = e^{2x} 2^{n-2}$	${}^nC_2 = \frac{n(n-1)}{2}$
		${}^nC_3 = \frac{n(n-1)(n-2)}{6}$

$$= \boxed{1 \times e^{2x} 2^n x^2 + n e^{2x} 2^{n-1} (2x) + \frac{n(n-1)}{2} e^{2x} 2^{n-2} (2) + 0}$$

Working process:-

1. consider y in terms of x $y = f(x)$ - (i)
2. Differentiate (i) wrt to 'x'.

$$y_1 = \frac{d}{dx} f(x) - \text{(ii)}$$

(3). 1. take LCM if possible
2. if sq. root, squaring both side.

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3. cross multiply
4. Try to convert RHS interms of y.

3. Differentiate again wrt to 'x'
4. Shift all terms on LHS and apply leebnitz to differentiate 'n' times.

$${}^nC_0 U_n v + {}^nC_1 U_{n-1} V_1 + - - - - - {}^nC_n U V_n$$

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Q: if $y = \cancel{\tan^{-1}x}$, show that $\cancel{(1+x^2)y_{n+2}}$ $+ 2nxy_n + n(n-1)y_n$

Q: if $y = \sin^{-1}x$ show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

solⁿ Step 1: $y = \sin^{-1}x$ — (i)

Step 2: Diff. (i) w.r.t. to x

$$y_1 = \frac{1}{\sqrt{1-x^2}} \quad \text{— (ii)}$$

$$y_1^2 = \frac{1}{(1-x^2)}$$

$$y_1^2 (1-x^2) = 1 \quad \text{— (iii)}$$

Step 3: Diff (iii) w.r.t. to x

$$(1-x^2)\, 2y_1 \frac{dy_1}{dx} + y_1^2(-2x) = 0$$

$$-2y_1\left(xy_1 - (1-x^2)y_2\right) = 0$$

$$(1-x^2)y_2 - xy_1 = 0 \quad \text{— (iv)}$$

now, differentiating 'n' times w.r.t. to 'x'

$$\left[{}^nC_0\, y_{n+2}(1+x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)\right]$$

$$-\left[{}^nC_0\, y_{n+1}\, x + {}^nC_1\, y_n(1)\right] = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} + \frac{n(n-1)}{2} y_n(-2) - xy_{n+1} - ny_n = 0$$

$$(1-x^2)y_{n+2} - xy_{n+1}(2n+1) - ny_n(n-\cancel{1}+1) \quad \text{— (v)}$$

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if $\cos\left[\frac{\log y}{m}\right] = x$ show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n$

Q: $\cos(\log y) = x$

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1}$

$- (n^2+m^2)y_n = 0$

$$y = e^{m\cos^{-1}x} \quad \text{— (i)}$$

$$y_1 = e^{m\cos^{-1}x} \times \frac{-m}{\sqrt{1-x^2}} \quad \text{— (ii)}$$

$$y_1^2(1-x^2) = m^2 (e^{m\cos^{-1}x})^2$$

$$y_1^2(1-x^2) = m^2y^2 \quad \text{— (iii)}$$

$$(1-x^2)\,2y_1y_2 + y_1^2(-2x) = m^2\, 2yy_1$$

$$(1-x^2)y_2 - xy_1 - m^2y = 0 \quad \text{— (iv)}$$

$${}^nC_0(1-x^2)y_{n+2} + {}^nC_1(0-2x)y_{n+1} + {}^nC_2(-2)y_n - \left[{}^nC_0\, xy_{n+1} + {}^nC_1\, y_n\right] - m^2y_n = 0$$

$$= (1-x^2)y_{n+2} + 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$= (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \quad \text{— (v)}$$

Step-5 put $x=0$ in eqⁿ (1) $y = f(x)$

$\Rightarrow y(0) = ?$

Step-6 put $x=0$ in eqⁿ (2).

$y_1(0) = 1$

Step-7 put $x=0$ in eq (4)

Step-8 put $x=0$ in eq (5)

→ let the result — (6)

Step-9: case 1 → n is odd in eq 6

→ n is even in eq 6

continue

put $x=0$ in eq ①

$y(0) = \sin^{-1} 0 = 0$

$\boxed{y(0)=0}$

put $x=0$ in eqn ②

$y_1(0) = \frac{1}{\sqrt{1-0}}$

$\boxed{y_1(0)=1}$

put $x=0$ in eqn ④

$\boxed{y_2(0)=0}$

put $x=0$ in eqn ⑤

$y_{n+2}(0) = n^2 y_n(0)$ ---- ⑥

case I: when n is even,

$n = 2, 4, 6, \ldots$

$n=2$ in eqn 6

$y_4(0) = 2^2 y_2(0) = 2^2 \times 0 = 0$

$\boxed{y_4(0)=0}$

$\Rightarrow \boxed{y_n(0)=0}$ when n is even.

case II: when n is odd $n = 1, 3, 5, \ldots$

$n=1$ in 6

$y_3(0) = 1^2 y_1(0) = 1^2 \times 1$

$n=3$

$y_5(0) = 3^2 y_3(0) = 3^2 \times 1^2 \times 1$

hence, $\boxed{y_n(0) = 1 \times 1^2 \times 3^2 \times \cdots (n-2)^2}$, n is odd.

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Q. $y = \sin(m \sin^{-1} x)$ find $y_n(0)$.

$\Rightarrow$ Diff wrt to 'x'

$y_1 = \cos(m \sin^{-1} x) \frac{d}{dx}(m \sin^{-1} x)$

$y_1 = \frac{m \cos(m \sin^{-1} x)}{\sqrt{1-x^2}}$ --- (ii)

squaring both side.

$y_1^2(1-x^2) = m^2 \cos^2(m \sin^{-1} x)$

$\because \sin^2\theta + \cos^2\theta = 1$

So $\cos^2\theta = 1 - \sin^2\theta$.

$\Rightarrow y_1^2(1-x^2) = m^2(1-y^2)$ --- (iii)

differentiating wrt to x.

$2y_1 y_2 (1-x^2) + y_1^2(-2x) = m^2(-2y y_1)$

$y_2(1-x^2) - xy_1 = -m^2 y$

$\boxed{y_2(1-x^2) - xy_1 + m^2 y = 0}$ --- (iv)

Differentiating 'n' times wrt to x using leibnitz

$\Rightarrow {}^nC_0 (1-x^2) y_{n+2} + {}^nC_1 (-2x) y_{n+1} + {}^nC_2 (-2) y_n$

$- [{}^nC_0 \, x y_{n+1} + {}^nC_1 (1) y_n] + m^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} - n(n-1) y_n - x y_{n+1} - n y_n + m^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 - m^2) y_n = 0$ --- (v) put $x=0$ in y (i)

$\Rightarrow y(0) = \sin(m \sin^{-1} 0)$

$\boxed{y(0)=0}$

put $x=0$ in eq^n (ii)

$\boxed{y_1(0)=m}$

put $x=0$ in eq^n (iv)

$\boxed{y_2(0) = -m^2 y(0) = 0}$ $\because y(0)=0$

put $x=0$ in eq^n (v)

$y_{n+2} = (n^2-m^2)\, y_n(0)$ -- (vi)

because, $\boxed{y_2(0)=0 \Rightarrow y_n(0)=0 \text{ where, } n \text{ is even}}$

when, n is odd, $n=1, 3, 5$

$\therefore$ for $n=1$ in eq (vi)

$y_3(0) = (1^2-m^2)\, y_1(0) = m(1^2-m^2)$

$y_5(0) = (3^2-m^2)\, y_3(0) = m(1^2-m^2)(3^2-m^2)$

$\Rightarrow y_n(0) = m(1^2-m^2)(3^2-m^2) \ldots (n-2)^2-m^2$

$y_n(0) = 0$; n is even

$y_n(0) = m(1^2-m^2)(3^2-m^2) \ldots (n-2)^2-m^2$; n is odd.

* Partial Derivatives:–

consider a function $z = f(x,y)$, x, y are independent variable and z is a dependent variable. Then partial derivative w.r.t (x) is obtained by keeping y as constant and differentiating w.r.t (x).

representation.

$z_x \,\Big|\, \frac{\partial z}{\partial x} \,\Big|\, f_x \,\Big|\, \frac{\partial f}{\partial x}$

and partial derivative of z w.r.t to y is obtained by keeping x as constant.

note: $\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\right]$

Q: $z = \log(x^2+y^2)$ find $\frac{\partial z}{\partial x}$?

sol^n $\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \frac{\partial}{\partial x}[x^2+y^2]$

$= \frac{1}{x^2+y^2}(2x+0)$

$\boxed{\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}}$ sol^n

note:– $\sinh^{-1} x = \log(x+\sqrt{1+x^2})$

$\cosh^{-1} x = \log[x - \sqrt{1+x^2}]$

Q: $y = [\sinh^{-1} x]^2$ find $y_n(0)$!

$y = (\log(x - \sqrt{1+x^2}))^2$

diff. w.r.t to x.

$y_1 = 2\log(x+\sqrt{1+x^2}) \frac{d}{dx}(\log(x+\sqrt{1+x^2}))$

$y_1 = 2\log(x+\sqrt{1+x^2}) \times \frac{1}{x+\sqrt{1+x^2}} \frac{d}{dx}(x+\sqrt{1+x^2})$

$y_1 = 2\log(x+\sqrt{1+x^2}) \times \frac{1}{x+\sqrt{1+x^2}} \left[1 + \frac{2x}{2\sqrt{1+x^2}}\right]$

$y_1 = \frac{2\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$

squaring both side.

$\Rightarrow (1+x^2)^2 y_1 = 4y.$

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Q) $z = \log(x^2+y^2)$ find, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}.$

Sol $\frac{\partial z}{\partial x} = \frac{2x}{(x^2+y^2)}$, $\frac{\partial z}{\partial y} = \frac{2y}{(x^2+y^2)}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$

(2) $z = \tan^{-1}\left(\frac{y}{x}\right)$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}.$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x^2}{x^2+y^2}$

- Note:- symmetric means, if we interchange x and y and equation remain the same. then $\frac{\partial^2 z}{\partial y^2} = \frac{d^2 z}{dx^2}$

$\frac{1}{2}-1$

Q5) if $u=\sqrt{x^2+y^2+z^2}$ i) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}$

ii) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$

$\frac{\partial u}{\partial x}=\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{u}$

$\frac{\partial u}{\partial y}=\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}=\frac{y}{u}$

$\frac{\partial u}{\partial z}=\frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{u}$

$\therefore$ ii) $\frac{x^2+y^2+z^2}{u^2}=\frac{u^2}{u^2}=1$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2+z^2}}=\sqrt{x^2+y^2+z^2}-\frac{2x^2}{2\sqrt{x^2+y^2+z^2}}$

$=$ or $\frac{u^2-x^2}{u^3}$

$\Rightarrow \frac{\partial^2 u}{\partial y^2}=\frac{u^2-y^2}{u^3}$

$\therefore$ i) $\frac{u^2-x^2+u^2-y^3+u^2-z^2}{u^3}$

$\Rightarrow \frac{\partial^2 u}{\partial z^2}=\frac{u^2-z^2}{u^3}$

$3u^2-(u^2)$

$\frac{2u^2}{u^3}=\frac{2}{u}$

Q. $z(x+y)=x^2+y^2$ show $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]$

$\Rightarrow z=\frac{x^2+y^2}{(x+y)}$

$\frac{\partial z}{\partial x}=\frac{2(x+y)x-(x^2+y^2)}{(x+y)^2}$

$\frac{\partial z}{\partial x}=\frac{2x}{(x+y)}-\frac{z}{(x+y)}$

$\frac{\partial z}{\partial x}=\frac{2x-z}{(x+y)}$

$\frac{\partial z}{\partial y}=\frac{2y-z}{(x+y)}$

LHS $\frac{2x-\cancel{z}-2y+\cancel{z}}{(x+y)}$

LHS $\left(\frac{2(x-y)}{(x+y)}\right)^2$

$\frac{4(x-y)^2}{(x+y)^2}$

RHS.

$\frac{x+y-2x+z-2y+z}{x+y}=\frac{-x-y+2z}{(x+y)}$

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V. Imp. Composite function.

Q2) $u = f(r)$; $r^2 = x^2 + y^2$ Show that $f''(r) + \frac{1}{r} f'(r)$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]$$

$u = f(r)$

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} \qquad \because r^2 = x^2 + y^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$= f'(r)\frac{\partial r}{\partial x} \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\boxed{\frac{\partial u}{\partial x} = f'(r)\frac{x}{r}} \qquad \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\boxed{\frac{\partial u}{\partial y} = f'(r)\frac{y}{r}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\, x \times r^{-1}\right]$$

$$= \left[f''(r)\, x \frac{\partial r}{\partial x}\frac{x}{r} + \frac{f'(r)\times 1}{r} + f'(r)\, x\, (-1) r^{-2}\frac{\partial r}{\partial x}\right]$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} + \frac{x^2 f'(r)(-1)}{r^3}$$

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} + \frac{y^2 f'(r)(-1)}{r^3}$$

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Q3) $x^2 = au + bv$ — (i) show,

$y^2 = au - bv$ — (ii) $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$

adding (i) and (ii) $\qquad$ subtract (i) and (ii)

$$\frac{x^2+y^2}{2a} = u \qquad \frac{x^2-y^2}{2a} = v$$

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Q: $y = \log(x+\sqrt{1+x^2})^2$ — (i)

$(1+x^2)y_1^2 = 4y$ — (iii)

$(1+x^2)y_2 + 2xy_1 - 2 = 0$

$y_1 = \frac{2}{\sqrt{1+x^2}} \log(x+\sqrt{1+x^2})$ — (ii)

$(1+x^2)2y_1y_2 + y_1^2(2x) = 4y_1$

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(topic-4) Euler's theorem:-

• Homogeneous function: A function $f(x,y)$ is said to be homogeneous function of degree 'n' if sum of indicis (powers) of variables, x,y are same.

$f(x,y) = x^3+y^3+3x^2y+3xy^2$ ✓

$f(x,y) = x^3+y^3+3x^2y+3xy^2+2xy$ ✗

working process:-

$x = tx, \quad y = ty$

$\Rightarrow \boxed{f(tx,ty) = t^n f(x,y)}$ $f(x,y)$ is homogeneous, n is degree.

$\Rightarrow (tx)^3 + (ty)^3 + 3(tx)^2(ty) + 3(tx)(ty)^2$

$\Rightarrow t^3(x^3+y^3+3x^2y+3xy^2)$

$\Rightarrow t^3 f(x,y)$.

Q1: $f(x,y) = \frac{x^4+y^4}{(x-y)}$

$x = tx, \quad y = tx.$

$f(tx,ty) = \frac{(tx)^4+(ty)^4}{(tx-ty)}$

$= \frac{t^4(x^4+y^4)}{t(x-y)}$

$= \frac{t^3(x^4+y^4)}{(x-y)}$

$= t^3 f(x,y)$

$\therefore$ function is homogeneous and degree is 3.

Remark:-
degree of homogeneous function can be 0, -ve or in fraction.

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Statement: If 'u' is a homogeneous function in x, y of degree n, then.

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Q1) If $u = \dfrac{(x^3+y^3)}{\sqrt{x}+\sqrt{y}}$ find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$?

Sol^n Step 1: ~~of~~ Statement by Euler's theorem if u is a homo. function of deg. n then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.} \quad (i)$$

Step 2: Check u is homo. $x \to tx$, $y \to ty$.

$$u = \frac{t^3x^3 + t^3y^3}{\sqrt{tx}+\sqrt{ty}} = \frac{t^3(x^3+y^3)}{t^{1/2}(\sqrt{x}+\sqrt{y})}$$

$$= t^{3-\frac{1}{2}}u$$

$$= t^{5/2}u$$

$\boxed{\Rightarrow u \text{ is homo. function of deg. } 5/2}$

Step 3: Substitution in (i)

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}u}$$

Remark) i) Euler's theorem for 1st order is extendable.

$u(x,y,z)$; $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = nu$

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ii) If u is a homo. function of degree n, then by Euler's theorem.

(Result -2) $\boxed{x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u}$

** This result is not extendable.

Q2 ~~(i)~~) If $u = \dfrac{(\sqrt{x}+\sqrt{y})^{1/4}}{(x^2+y^2)^{1/3}}$ find $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$

if u is homogen. function then

$$u(xt, yt) = \frac{(\sqrt{xt}+\sqrt{yt})^{1/4}}{((xt)^2+(yt)^2)^{1/3}}$$

$$= \frac{t^{\frac{1}{8}}(\sqrt{x}+\sqrt{y})^{1/4}}{t^{2/3}(x^2+y^2)^{1/3}}$$

$$= \frac{-13}{24}(u)$$

$\therefore$ $= \cancel{0.8} = \cancel{\frac{43}{50}}$

0.83

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \cancel{\frac{43}{50}u} \text{ or } \frac{481}{576}u.$$

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Q3) Using Euler's theorem $\left[u = \dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}} \right]$ -- (i)

Sol^n Step 1: if u is a homo. function of deg. n in x, y
then, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$ -- (ii)

Step 2: Check u is homo. $x \to tx$, $y \to ty$.

$$t^{\frac{1}{2}-\frac{1}{3}} u$$
$$= t^{1/6} u$$

$\Rightarrow$ u is homo. function of deg $n = \dfrac{1}{6}$

Step 3: writing in (i)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{6} u \quad \text{-- (iii)}$$

Step 4: to verify.
taking log on both side on (i)

$$\log u = \log (x^{1/2} + y^{1/2}) - \log(x^{1/3} + y^{1/3})$$

Step 5:
partially diff. wrt x.

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{2}\frac{x^{\frac{1}{2}-1}}{x^{1/2}+y^{1/2}} - \frac{x^{\frac{1}{3}-1}}{(x^{1/3}+y^{1/3})}\frac{1}{3}$$

Step 6 multiplying by x.

$$\frac{x}{u}\frac{\partial u}{\partial x} = \frac{1}{2}\frac{x^{1/2}}{(x^{1/2}+y^{1/2})} - \frac{1}{3}\frac{x^{1/3}}{(x^{1/3}+y^{1/3})} \quad \text{-- (iv)}$$

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Step 7 similarly diff wrt y.

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{1}{2}\frac{y^{(\frac{1}{2}-1)}}{(x^{1/2}+y^{1/2})} - \frac{1}{3}\frac{y^{\frac{1}{3}-1}}{(x^{1/3}+y^{1/3})}$$

Step 8 multiplying by y.

$$\frac{y}{u}\frac{\partial u}{\partial y} = \frac{1}{2}\frac{y^{1/2}}{(x^{1/2}+y^{1/2})} - \frac{1}{3}\frac{y^{1/3}}{(x^{1/3}+y^{1/3})} \quad \text{-- (v)}$$

Step 9 adding (iv) and (v)

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\frac{\cancel{(x^{1/2}+y^{1/2})}}{\cancel{(x^{1/2}+y^{1/2})}} - \frac{1}{3}\left(\frac{\cancel{x^{1/3}+y^{1/3}}}{\cancel{x^{1/3}+y^{1/3}}}\right)$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{6} u}$$

Hence, ~~proved~~ verified.

Result 3): If $f(u)$ is a homogeneous function in x, y of degree n,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}}$$

$$Q1)\ u = \sin^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^2+y^2}\right)$$

Soln

Step1: $\Rightarrow \sin u = \dfrac{x^{1/2}+y^{1/2}}{x^2+y^2}$

$f(u)$

Step2: - Statement

Step3. $y \to ty \quad x \to xt$

$$\sin u = \frac{t^{1/2}(x^{1/2}+y^{1/2})}{t^2(x^2+y^2)} = t^{-\frac{3}{2}}f(u)$$

$n = -3/2$

Step4: $\phi' = \dfrac{1}{\sqrt{1-u^2}}$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{3}{2}\left(\frac{\phi}{\cdot} \times \sqrt{1-u^2}\right)$$

$$= -\frac{3}{2} u\sqrt{1-u^2}$$

$$= -\frac{3}{2}\frac{\sin u}{\cos u}$$

$$= -\frac{3}{2}\tan u$$

Result 4:- If $f(u)$ is homo. function of degree in x, y the

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\cdot(g'(u)-1)$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$

Q. If $u = \cos^{-1}\left[\dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}\right]$ find $xu_x + yu_y$

Hence, find $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$.

Soln Step1: $\cos u = \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$ — (1)

Step2: If $f(u)$ is Homo. function of degree n

$$xu_x + yu_y = n\frac{f(u)}{f'(u)} \quad \text{— (2)}$$

Step3: $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)(g'(u)-1);$$

$$g(u) = n\frac{f(u)}{f'(u)} \quad \text{— (ii)}$$

$f(u)$ is Homo $x, tx, y \to ty$.

$$\cos u = \frac{\sqrt{tx}+\sqrt{ty}}{t^2x^2+t^2y^2} = t^{\frac{1}{2}-2}\left[\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2}\right]$$

$$= t^{-3/2}f(u)$$

$$\boxed{f(u) = \cos u \text{ is homo. function of } n = \frac{-3}{2}}$$

Step4: Sub in eqn (ii) $xu_x + yu_y = \dfrac{-3}{2}\dfrac{\cos u}{-\sin u}$

$$= \frac{3}{2}\cot u = g(u)$$

Substitution in eqution (iii)

$= \frac{3\cot u}{2}\left[\frac{-3\operatorname{cosec}^2 u - 1}{2}\right].$

Q2) Verify Euler's theorem $u = e^{(x^2+y^2)} = \exp(x^2+y^2)$

sol^n $f(u) = \log u = x^2+y^2$ -- (i)

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

$\boxed{n=2}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\log u}{1/u} = 2u\log u$ -- (iii)

Step 5: pd (i) wrt to (x)

$\frac{1}{u}\frac{\partial u}{\partial x} = 2x + 0$

x by x $\frac{x}{u}\frac{\partial u}{\partial x} = 2x$ -- (4)

$\therefore \log u$ is symmetric

$\Rightarrow \frac{y}{u}\frac{\partial u}{\partial y} = 2y^2$ -- (5)

4 + 5

$= \frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2(x^2+y^2)$

Q3) $z = x\,\phi\left(\frac{y}{x}\right) + \varphi\left(\frac{x}{y}\right)$ find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

and $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2}$

sol^n let $z = u + v$

$u = x\,\phi\left(\frac{y}{x}\right)$, $v = \varphi\left(\frac{x}{y}\right)$

Check u is homo.

$x \to tx$, $y = ty$

$u = tx\,\phi\left(\frac{ty}{tx}\right)$

$u = t^1(u)$

$\therefore$ degree $n = 1$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u$ -- (ii)

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 0$ -- (3)

v is check homo.

$x \to tx$, $y \to ty$

$v = \varphi\left(\frac{tx}{ty}\right) = t^0\varphi\left(\frac{x}{y}\right) = t^0 v \therefore v$ is homo of

deg $n = 0$

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0$ -- (iv)

$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0$ -- (v)

(ii) + (iv) ; (iii) + (v):

$$x\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\right]+y\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right]=U+G$$

$$\boxed{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\,\phi\left(\frac{y}{x}\right)}$$

* Total Differentiation:-

In total differentiation all the independent variables vary simultaneously.

$$\underset{\text{dependent}}{z}=f\underset{\text{Independent}}{(x,y)}$$

Composite function of one-variable.

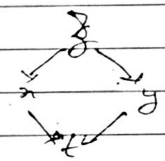
$z=f(x,y)$

$x=g_1(t)$

$y=g_2(t)$

$$\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial x}\times\frac{dx}{dt}+\frac{\partial z}{\partial y}\times\frac{dy}{dt}}$$

Q: $z=x^2+y^2$, $x=2t$, $y=4t^2$



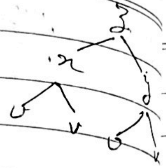
$$\therefore\ \frac{dz}{dt}=\frac{\partial z}{\partial x}\times\frac{dx}{dt}+\frac{\partial z}{\partial y}\times\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x}=2x,\quad \frac{\partial z}{\partial y}=2y$$

$$\frac{dx}{dt}=2,\quad \frac{dy}{dt}=8t$$

$$\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial x}\times\frac{dx}{} = 8t+6y+3}$$

* Composite functions of two variables:-



$$z = f(x,y) \; ; \; x = g_1(u,v) \; ; \; y = g_2(u,v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u} \qquad \text{prove}$$

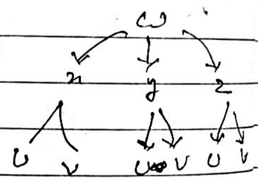
$$= \frac{u}{\sqrt{1+v^2}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

Q.1) $w = \sqrt{x^2+y^2+z^2}$ where $x = u\cos v$, $y = u\sin v$, $z = uv$. find $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v}$

Sol

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial u}}$$



$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \times \frac{1}{\cancel{2}} \times \cancel{2}x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial x}{\partial u} = \cos v \; , \; \frac{\partial y}{\partial u} = \sin v \; , \; \frac{\partial z}{\partial u} = v.$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \; , \; \frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \frac{\partial w}{\partial u} = \frac{x\cos v + y\sin v + zv}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{u\cos^2 v + u\sin^2 v + uv^2}{w}$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{1}{w} u(1+v^2) \quad \text{--- (ii)}$$

$$\boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial v}}$$

$$\frac{\partial w}{\partial v} = \frac{x}{\sqrt{x^2+y^2+z^2}} \times -u\sin v + \frac{y}{\sqrt{x^2+y^2+z^2}} \times u\cos v + \frac{zu}{\sqrt{x^2+y^2+z^2}}$$

$$= \cancel{\frac{-u^2\cos v \cdot \sin v}{w}} + \cancel{\frac{u^2\cos v \cdot \sin v}{w}} + \frac{u^2 v}{w}$$

$$= \frac{u^2}{w}\left[\cancel{\cos v \cdot \sin v}\right]$$

$$= \frac{u^2 v}{w} \quad \text{--- (iii)}$$

ans :-

$$\frac{u^2(1+v^2)}{w} - \frac{u^2 v^2}{w}$$

$$\frac{u^2(1+v^2-v^2)}{w} = \frac{u^2}{w} \quad \text{ans}$$

$$= \frac{u^2}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{u^2}{\sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2 v^2}}$$

$$= \frac{u^{\cancel{2}}}{\sqrt{\cancel{u^2}(1+v^2)}}$$

$$= \frac{u}{\sqrt{(1+v^2)}}$$

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Q1) $z = x^2 + y^2$ $\quad x = u^2 - v^2$ $\quad y = uv$ $\quad$ find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\times\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\times\frac{\partial y}{\partial u}$$

$$= 2x \times 2u + 2y v$$

$$\frac{\partial z}{\partial u} = 4xu + 2yv$$

$$= 4(u^2 - v^2)u + 2uv \times v$$

$$= 4u^3 - 2uv^2$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\times\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\times\frac{\partial y}{\partial v}$$

$$= 2x \times -2v + 2y \times u$$

$$= -4(u^2 - v^2)v + 2uv \times u$$

$$= -4u^2v + 4v^3 + 2u^2v$$

$$= 4v^3 - 2u^2v$$

Q2) $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ find $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$

Sol: $u = f(r, s, t)$ $\quad r = \frac{x}{y}$, $\quad s = \frac{y}{z}$, $\quad t = \frac{z}{x}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\times\frac{\partial r}{\partial x} + \frac{\partial f}{\partial t}\times\frac{\partial t}{\partial x}$$

$$= \frac{\partial f}{\partial r}\times\frac{1}{y} + \frac{\partial f}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\times\frac{\partial s}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\times\left(\frac{-x}{y^2}\right) + \frac{\partial f}{\partial s}\times\left(\frac{1}{z}\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial s}\times\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\times\frac{\partial t}{\partial z}$$

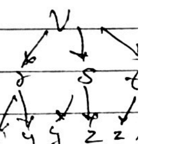
$$= \frac{\partial f}{\partial s}\times\left(\frac{-y}{z^2}\right) + \frac{\partial f}{\partial t}\times\left(\frac{1}{x}\right)$$

$$\Rightarrow \cancel{\frac{x}{y}\left(\frac{\partial f}{\partial r}\right)} - \cancel{\frac{z}{x}\frac{\partial f}{\partial t}} - \cancel{\frac{x}{y}\frac{\partial f}{\partial r}} + \cancel{\frac{y}{z}\left(\frac{\partial f}{\partial s}\right)}$$

$$- \cancel{\frac{y}{z}\left(\frac{\partial f}{\partial s}\right)} + \cancel{\frac{z}{x}\frac{\partial f}{\partial t}}$$

$= 0$ Ans

Q3) if $v = f(2x - 3y, 3y - 4z, 4z - 2x)$ find $6v_x + 4v_y + 3v_z$



let $r = 2x - 3y$ $\quad s = 3y - 4z$ $\quad t = 4z - 2x$

$\frac{\partial r}{\partial x} = 2$ $\quad \frac{\partial s}{\partial y} = 3$ $\quad \frac{\partial t}{\partial z} = 4$

$\frac{\partial r}{\partial y} = -3$ $\quad \frac{\partial s}{\partial z} = -4$ $\quad \frac{\partial t}{\partial x} = -2$

$\cancel{\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}(2) + \frac{\partial v}{\partial}}$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial v}{\partial t} \times \frac{\partial t}{\partial x}$$

$$V_x = \frac{\partial v}{\partial r} \times 2 + \frac{\partial v}{\partial t} \times (-2) \quad \text{— (i)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial v}{\partial s} \times \frac{\partial s}{\partial y}$$

$$V_y = \frac{\partial v}{\partial r} \times (-3) + \frac{\partial v}{\partial s} \times 3 \quad \text{— (ii)}$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial s} \times \frac{\partial s}{\partial z} + \frac{\partial v}{\partial t} \times \frac{\partial t}{\partial z}$$

$$V_z = \frac{\partial v}{\partial s} \times (-4) + \frac{\partial v}{\partial t} \times (4) \quad \text{— (iii)}$$

$$6V_x + 4V_y + 3V_z$$

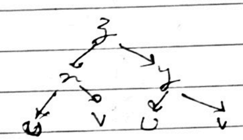
$$= (12-12)\frac{\partial v}{\partial r} + (12-12)\frac{\partial v}{\partial t} + (12-12)\frac{\partial v}{\partial s}$$

$$= 0$$

Q4) If $z = f(x,y)$ $\quad x = e^u \cos v \quad y = e^u \sin v$

Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

Sol^n



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v) \quad \text{— (i)}$$

squaring both side.

$$\left(\frac{\partial z}{\partial u}\right)^2 = \left(\frac{\partial z}{\partial x}(e^u \cos v)\right)^2 + \left(\frac{\partial z}{\partial y}(e^u \cos v)\right)^2 + 2\frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y}(e^u \cos v)(e^u \sin v) \quad \text{— (i)}$$

$$\left(\frac{\partial z}{\partial v}\right)^2 = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

$$\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \times e^{2u}\sin^2 v + \frac{\partial z}{\partial y} e^{2u}\cos^2 v - 2\frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} e^{2u} \sin v \cos v \quad \text{— (ii)}$$

adding (i) and (ii)

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \frac{\partial z}{\partial x} e^{2u} + \frac{\partial z}{\partial y} e^{2u}$$

$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

Hence proved

part-2.

Also Show that:
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2u}\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right]$$

from (i)

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x}(e^u \cos v) + \frac{\partial}{\partial y}(e^u \sin v).$$

from (ii)
$$\frac{\partial}{\partial v} = \frac{\partial}{\partial x}(-e^u \sin v) + \frac{\partial}{\partial y}(e^u \cos v)$$

$$\therefore \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$
$$= \left[\frac{\partial}{\partial x}(e^u \cos v) + \frac{\partial}{\partial y}(e^u \sin v)\right]\left[\frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)\right]$$
$$= \frac{\partial^2 z}{\partial x^2}(e^{2u}\cos^2 v) + \frac{\partial}{\partial x} e^{2u}\cos v \cdot \sin v \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} \times \frac{\partial}{\partial y} e^{2u}\sin^2 v + \frac{\partial}{\partial y} \times \frac{\partial z}{\partial x} e^{2u} \sin v \cdot \cos v \quad \text{(iii)}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$
$$= \left[\frac{\partial}{\partial x}(-e^u \sin v) + \frac{\partial}{\partial y}(e^u \cos v)\right]\left[-e^u \sin v \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} e^u \cos v\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}(e^{2u}\sin^2 v) - \frac{\partial}{\partial x} \times \frac{\partial z}{\partial y} e^{2u} \sin v \cdot \cos v$$
$$- \frac{\partial}{\partial y} \times \frac{\partial z}{\partial x}(e^{2u}\sin v \cdot \cos v) + \frac{\partial^2 z}{\partial y^2} e^{2u}\cos^2 v \quad \text{(iv)}$$

adding (iii) and (iv)

$$\frac{\partial^2 z}{\partial x^2} e^{2u} + \frac{\partial^2 z}{\partial y^2} e^{2u}$$

$$\Rightarrow \boxed{e^{-2u}\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right] = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}}$$

ans

* Total Differential Coefficient:— If $z = f(x,y)$ then dz is called total differential coefficient.

$$\boxed{dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

(Q1) if $u = e^x y z^2$ find $du = ?$

soln
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$
$$\boxed{du = e^x y z^2 dx + e^x z^2 dy + 2z e^x y\, dz}$$ ans

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* Implicit function:- A function is said to be implicit if one variable of a function cannot be expressed explicitly in terms of another.

$$\boxed{f(x,y)=c}$$

$$\boxed{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0}$$

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

Q1: $x^3+y^3+3axy = 2=f$ find $\frac{dy}{dx}$?

soln

$$\frac{\partial f}{\partial x} = 3x^2+3ay$$

$$\frac{\partial f}{\partial y} = 3y^2+3ax$$

$$\therefore \frac{dx}{dy} = \frac{-(3x^2+3ay)}{(3y^2+3ax)}$$ ans

Q2: $c = x^y + y^x$ find $\frac{\partial x}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial x}$

Variable.

Base	$x^5 = 5x^{5-1}$
power	$5^x = 5^x \log 5$

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial f}{\partial y} = x^y \log x + x y^{x-1}$$

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$$\Rightarrow \frac{dx}{dy} = \frac{-(yx^{y-1}+y^x\log y)}{(x^y\log x + xy^{x-1})}$$

Q3: If $v = x.\log xy$ find $\frac{dv}{dx}$ where, $x^3+y^3+3axy=0$.
this shows that there will be one variable i.e x.

soln

$$\frac{dv}{dx} = \frac{\partial v}{\partial x}\frac{dx}{dx} + \frac{\partial v}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\times\frac{dy}{dx}$$

$$= \left(\cancel{\log xy} + \cancel{\frac{x.y}{xy}}\right) +$$

$$= \cancel{\log xy} + \frac{x}{xy}\times x \times \frac{dy}{dx}$$

$$= \log xy + \frac{x}{y}\frac{dy}{dx}.$$

$$\frac{\partial f}{\partial x} = 3x^2+3ay$$

$$\frac{\partial f}{\partial y} = 3y^2+3ax.$$

$$\frac{dv}{dx} = \left(\frac{x}{x} + \log x + \log y\right) + \left[0 + \frac{x}{y}\left(\frac{-3x^2+3ay}{3y^2+3ax}\right)\right].$$

If $v = x.\log xy$ find $\frac{dv}{dx}$, where, $x^3+y^3+3axy=0$