

$$f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \dots = 3,$$

Engineering Mathematics - I

MA5-103

Module - III

Differential Calculus - II

Lecture - 34

Contents:- Taylor's and Maclaurian's Theorem of one variable

Taylor's Theorem and expansion of function for single variable: Let $f(x)$ be a function of x . If the function $f(x+h)$ can be expanded in positive powers of h , where x is independent of h , then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots + \frac{h^n}{n!} f^n(x) + \dots$$

where h is a small distance.

Other forms of Taylor's Theorem

1) To expand a function about point $x=a$.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots$$

2) To expand a function in powers of $x-a$. replace h by $x-a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots + \frac{(x-a)^n}{n!} f^n(a) + \dots$$

3. To expand a function at $a=0$ or about origin also known as **Maclaurian's Theorem**.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

Remark:- To expand a function in powers of x . we use 3rd form of Maclaurian's Theorem.

$$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Working Process :- To expand a function $f(x)$ into Taylor's or Maclaurin's series.

- 1). Consider $f(x)$
- 2). Differentiate $f(x)$, number of times and find $f'(x)$, $f''(x)$, $f'''(x)$, ... and so on.
3. Use appropriate form as required.
4. For Maclaurin's series find $f(0)$, $f'(0)$, $f''(0)$
5. Substitute all the values into series.

Remark :- If not mentioned, by default we use Maclaurin's Theorem.

Q.1 Expand e^x in power of x .

To expand a function into powers of x , we use Maclaurin's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$$

$$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$$

$$f^n(x) = e^x \Rightarrow f^n(0) = e^0 = 1$$

Substitute these values in (1), we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Ans.

Q.2 Expand $\sin x$ in powers of x .

Q.3 Expand $\cos x$ in powers of x .

Q.4 Expand $\log(1+x)$ in powers of x .
Then find $\log\left(\frac{1+x}{1-x}\right)$ & hence calculate $\log\left(\frac{11}{9}\right)$

Sol:- $f(n) = \log(1+n)$ $\left| \begin{array}{l} f(0) = \log 1 = 0 \\ f'(0) = 1 \\ f''(0) = -1 \\ f'''(0) = 2 \\ f''''(0) = -6 \end{array} \right.$

$$\begin{aligned} f'(n) &= \frac{1}{1+n} \\ f''(n) &= \frac{-1}{(1+n)^2} \\ f'''(n) &= \frac{2}{(1+n)^3} \\ f''''(n) &= \frac{-6}{(1+n)^4} \end{aligned}$$

using MacLaurin's Theorem, we know

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$\log(1+n) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) \dots$$

$$\log(1+n) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6}{4!}x^4 + \dots \quad | \text{Ans}$$

changing x to $-x$, we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\begin{aligned} \log\left(\frac{1+x}{1-x}\right) &= \log(1+x) - \log(1-x) \\ &= \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \right] \end{aligned}$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[-\frac{x^2}{2} - \frac{x^3}{3} + \dots \right]$$

$$\log\left(\frac{11}{9}\right) \quad \text{put } x = \frac{1}{10}$$

$$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{300} + \dots \right] \approx 0.2067 \quad | \text{Ans.}$$

Q.5 expand $8x^3 + 7x^2 + x - 6$ in powers of $(x-2)$ & calculate $f(2.1)$

Sol. using 2nd form of

$$f(x) = 8x^3 + 7x^2 + x - 6$$

$$f'(x) = 24x^2 + 14x + 1$$

$$f''(x) = 48x + 14$$

$$f'''(x) = 48$$

$$\begin{aligned} f(2) &= 16 + 28 + 2 - 6 = 40 \\ f'(2) &= 24 + 28 + 1 = 53 \\ f''(2) &= 48 + 14 = 38 \\ f'''(2) &= 48 \end{aligned}$$

Percentage Error: $\frac{8x}{n} \times 100$ is known as percent of

we get

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2)$$

$$= 40 + 53(x-2) + \frac{38}{2!} (x-2)^2 + \frac{12}{3!} (x-2)^3$$

$$\boxed{f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 \text{ Ans.}}$$

$$f(2.1) = 40 + 53(0.1) + 19(0.1)^2 + 2(0.1)^3 = 45.49$$

Q6. Imp. expand $\log(x+h)$ in powers of x .

sol. To expand in powers of x we consider x as ~~constant~~ variable & h as variable.

We know $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$
because we have to expand in powers of x .
interchanging x & h .

$$f(h+x) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \dots$$

$$f(x+h) = \log(x+h)$$

$$f(x) = \log x$$

$$f(h) = \log h$$

$$\therefore \log(x+h) = \log h + x \frac{1}{h} + \frac{x^2}{2!} \left(-\frac{1}{h^2}\right) + \frac{x^3}{3!} \left(\frac{2}{h^3}\right) + \dots$$

Ans.

Problems for Practice:-

Q1. find expansions of $\sin^{-1}x$, $e^x \cos x$, $\log(2ex)$

Q2. obtain $\tan^{-1}x$ in powers of $(x-1)$

Q3. expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending powers of x .

Q4. Evaluate $\sqrt{95.17}$ using Taylor's Theorem.

$$\text{we get } f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2).$$

Module III
Differential Calculus II

Lecture-35

Content: - Taylor's and MacLaurin's Theorem for function of two variables.

Taylor's Theorem for function of two variables:-

Let $f(x, y)$ be a function of two variables x & y . Then $f(x+h, y+k)$ can be expanded in positive powers of h & k , where x & y are independent of h & k respectively, and h, k are small distances.

$$f(x+h, y+k) = f(x, y) + \left[h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right] + \frac{1}{2!} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \frac{1}{3!} [h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}]$$

Other forms of Taylor's Theorem of two variable

1) To expand function $f(x, y)$ about point $x=a, y=b$.

$$f(a+h, b+k) = f(a, b) + [h f_x(a, b) + k f_y(a, b)] + \frac{1}{2!} [h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)] + \dots$$

2) To expand a function in powers of $(x-a) \neq (y-b)$.

replace h by $(x-a)$ & k by $(y-b)$

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \frac{1}{2!} [(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \dots$$

3) To expand function $f(x, y)$ about $(0, 0)$ or origin or in powers of $x \neq y$. we use MacLaurin's Theorem.

$$f(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \dots$$

Working Process:- To expand a function $f(x, y)$ into Taylor's & Maclaurian's series.

- Step I Consider $f(x, y)$
- II Differentiate $f(x, y)$, $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$, $f_{yy}(x, y)$ and so on.
- III we appropriate form as required.
- IV for Maclaurian's series find $f(0, 0)$, $f_x(0, 0)$, $f_y(0, 0)$, $f_{xx}(0, 0)$, $f_{xy}(0, 0)$, $f_{yy}(0, 0)$
- V substitute all these values into series.
- Q1 Expand $e^x \sin y$ in powers of x & y upto terms of 3rd degree.

Sol:- Let $f(x, y) = e^x \sin y$
To expand $f(x, y)$ in powers of x & y , we will use Maclaurian's Theorem.

$$f(x, y) = f(0, 0) + (x) f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \dots \quad (1)$$

$f(x, y) = e^x \sin y$	$f(0, 0) = 0$
$f_x(x, y) = e^x \cos y$	$f_x(0, 0) = 0$
$f_y(x, y) = e^x \cos y$	$f_y(0, 0) = 1$
$f_{xx}(x, y) = e^x \sin y$	$f_{xx}(0, 0) = 0$
$f_{xy}(x, y) = e^x \cos y$	$f_{xy}(0, 0) = 1$
$f_{yy}(x, y) = -e^x \sin y$	$f_{yy}(0, 0) = 0$
$f_{xxx}(x, y) = e^x \sin y$	$f_{xxx}(0, 0) = 0$
$f_{xxy}(x, y) = e^x \cos y$	$f_{xxy}(0, 0) = 1$
$f_{xyy}(x, y) = -e^x \sin y$	$f_{xyy}(0, 0) = 0$
$f_{yyy}(x, y) = -e^x \cos y$	$f_{yyy}(0, 0) = -1$

substituting these values in eqn (1)

$$e^x \sin y = 0 + x(0) + y(1) + \frac{1}{1!} [x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0] + \frac{1}{3!} [2x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot 0] + y^3 \cdot (-1) + \dots$$

$$e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6}$$

[UPTU-2006, 14, 11, 12]

Q2. Expand $\tan^{-1} \frac{y}{x}$ in neighbourhood of (1, 1) upto and inclusive of second degree terms.

Hence compute $f(1.1, 0.9)$

Sol. Let $f(x, y) = \tan^{-1} \frac{y}{x}$

$$f_x(x, y) = \frac{1}{1+y^2} \cdot \frac{x-y}{x^2} \Rightarrow \frac{-y}{x^2+y^2}$$

$$f_y(x, y) = \frac{1}{1+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f_{xx}(x, y) = \frac{(-y)(-1) \cdot 2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$f_{yy} = \frac{(x)(-1) \cdot 2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_{xy} = \left[\frac{-1(x^2+y^2) + y(2x)}{(x^2+y^2)^2} \right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f(x, y) = f(1, 1) + (x-1)f_x(1, 1) + (y-1)f_y(1, 1) + \frac{1}{2!} [(x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1) f_{xy}(1, 1) + (y-1)^2 f_{yy}(1, 1)] + \dots$$

$$= \frac{\pi}{4} + (x-1) \left(-\frac{1}{2}\right) + (y-1) \frac{1}{2} + \frac{1}{2!} \left[(x-1)^2 \frac{1}{2} + 0 + (y-1)^2 \left(-\frac{1}{2}\right) \right]$$

$$= \frac{\pi}{4} - \frac{(x-1)}{2} + \frac{(y-1)}{2} + \frac{1}{4} \left[(x-1)^2 - (y-1)^2 \right] + \dots$$

To calculate $f(1.1, 0.9) = f(1+0.1, 1-0.1)$ Ans.

replace x by 1.1 & $y = 0.9$

we get

$$= \frac{\pi}{4} + \frac{1}{2}(0 \cdot 1) + \frac{1}{2}(-1) + \frac{1}{4}(0 \cdot 1)^2 - \frac{1}{4}(-1)^2$$

$$= 0.6857 \text{ (approx)}$$

Q.3 Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$

$$\text{Sol. } f(x, y) = x^2y + 3y - 2$$

$$fx(x, y) = 2xy$$

$$fy(x, y) = x^2 + 3$$

$$fxn(x, y) = 2y$$

$$fxny(x, y) = 2x$$

$$fny(x, y) = 0$$

$$fxnn = 0$$

$$fxny = 2$$

$$fnyy = 0$$

$$fyyy = 0$$

$$f(1, -2) = -2 - 6 - 2 = -10.$$

$$fx(1, -2) = -4$$

$$fy(1, -2) = 4$$

$$fxn(1, -2) = -4$$

$$fxny(1, -2) = 2$$

$$fnyy(1, -2) = 0$$

$$\therefore f(x, y) = -10 + (x-1)(-4) + (y+2)4 + \frac{1}{2!} [(x-1)^2(-4)]$$

$$+ 2(x-1)(y+2)(2) + (y+2)^2(0)] + \frac{1}{3!} [0 + 3(x-1)^2(y+2)x2 + 0]$$

$$f(x, y) = -10 - 4(x-1) + 4(y+2) + [-2(x-1)^2 + 2(x-1)(y+2)]$$

$$+ (x-1)^2(y+2) + \dots$$

Q. Problems for Practice

Ans.

expand following into Taylor's Series.

1) y^x in powers of $(x-1)$ & $(y-1)$

2) $e^x \tan^{-1} y$ in powers of $(x-1)$ and $(y-1)$.

3) $e^x \log(1+y)$ in powers of x & y as far as term of 3rd degree.

4) expand $e^x \cos y$ at $(1, \frac{\pi}{4})$

5) e^{xy} at $(1, 1)$

$$= \frac{\pi}{4} + \frac{1}{2} (0 \cdot 1) + \frac{1}{2} (-1) + \frac{1}{2} (0 \cdot 1)^2 - \frac{1}{2} (-1)^2$$

Engineering Mathematics - I

KAS-103

Module-III

Lecture - 3.7

Differential Calculus II

Content: Extrema of function of two variables.

Extrema of function of two variables: - A maximum or minimum value of a function is called an extreme value or extremum.

Minimum Value: - Consider a function $f(x, y)$, then function has relative minimum at the point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) .

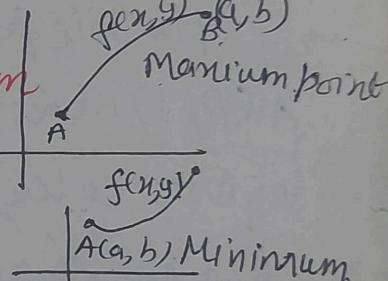
Maximum Value: - Consider a function $f(x, y)$, then function has relative maximum value at point (a, b) if $f(x, y) \leq f(a, b)$ for all point (x, y)

The point where function is either minimum or maximum is called extreme values or extremum.

Saddle point: - The point where function is neither maximum nor minimum is known as saddle point.

Working Process: - To find minimum or maximum for function $f(x, y)$

1. Consider $f(x, y)$
2. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
3. Put $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, and solve these equation to obtain points [known as stationary points]
4. for each solution compute $\lambda = f_{xx}$, $\gamma = f_{xy}$ & $\beta = f_{yy}$ and $\Delta = \lambda\beta - \gamma^2$
- 5) if $\Delta > 0$ we check further value of λ .



Q.3 Expand $x^3 + y^3 - 3xy$ in powers of $(x-1)^2 (y+2)^3$

1. If $\lambda > 0$, then point is minima.
2. If $\lambda < 0$, then point of maxima.
3. If $\lambda = -s^2 < 0$, then saddle point. Neither max nor minima.
4. If $\lambda = -s^2 = 0$, case is doubtful requires further investigation.

Q.1 find relative maximum and minimum values of

$$x^3 + y^3 - 3xy - 12x + 20.$$

Soln. 1) Consider $f(x, y) = x^3 + y^3 - 3xy - 12x + 20$.

$$\frac{\partial f}{\partial x} = 3x^2 - 12, \quad \frac{\partial f}{\partial y} = 3y^2 - 3.$$

$$2) \text{ put } 3x^2 - 12 = 0, \quad 3y^2 - 3 = 0 \Rightarrow x = \pm 2, y = \pm 1.$$

∴ possible points or stationary points are.

A(2, 1), B(2, -1), C(-2, 1), D(-2, -1)

4. Now calculating $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = 0$.

$$\lambda = -s^2 = 6x(6y) - 0 = 36xy.$$

points	x	y	λ	Extreme value.
1. (2, 1)	2	1	$72 - 0 = 72 > 0$	Minima at (2, 1)
2 (2, -1)	2	-1	$-72 < 0$	Saddle point (2, -1)
3 (-2, 1)	-2	1	$-72 < 0$	Saddle point (-2, 1)
4. (-2, -1)	-2	-1	$72 > 0$	Maxima at (-2, -1).

Maximum value $\stackrel{(-2, -1)}{=} 38$

Minimum value at $(2, 1) \quad 8 + 1 - 3 - 24 + 20 = -2$

Q.2 Determine the points where function $f(x, y) = x^3 + y^3 - 3xy$ has maximum or minimum value.

Sol. Let $f(x, y) = x^3 + y^3 - 3xy$.

$$f_x = 3x^2 - 3ay = 0$$

$$f_y = 3y^2 - 3ax = 0$$

$\frac{\partial^2 f}{\partial x^2} = 6x$
 $\frac{\partial^2 f}{\partial y^2} = 6y$
 $\frac{\partial^2 f}{\partial x \partial y} = -3$

$\lambda = f_{xx} = 6x$
 $\lambda = f_{yy} = 6y$
 $\lambda = f_{xy} = -3$

a function $f(x,y)$ into MacLaurin's series.

$$3x^2 - 3ax = 0$$

$$x(x-a) = 0$$

$$\Rightarrow x=0 \text{ and } x=a$$

$$\lambda = f_{xx} = 6x, \quad \mu = f_{xy} = -3a, \quad \nu = f_{yy} = 6y.$$

$$\lambda - \mu^2 = 36xy - 9a^2$$

$$\text{At } (0,0) = 0 - 9a^2 = -9a^2 \text{ for all values of } a < 0. \therefore \text{Saddle point.}$$

$$\text{At } (a,a) = 36a^2 - 9a^2 = 27a^2 > 0.$$

$$\lambda = 6a \quad \begin{array}{l} \lambda > 0, \lambda = 6a > a \Rightarrow \text{point of minima.} \\ \lambda < 0, \lambda = 6a < 0 \Rightarrow \text{point of maxima.} \end{array}$$

Max value a^3
Min value $-a^3$

Q3 Discuss Maxima and minima of function.

$$f(x,y) = \cos x \cos y \cos(x+y)$$

$$\text{sol. } f_x(x,y) = \cos y [-\sin x \cos(x+y) - \cos x \sin(x+y)] \\ = -\cos y \sin(2x+y)$$

$$f_y(x,y) = \cos x [-\sin y \cos(x+y) - \sin x \cos(x+y)] \\ = -\cos x \sin(2y+x)$$

For stationary values. $f_x = 0 \quad \cos y \sin(2x+y) = 0 \quad \text{--- (1)}$

$$f_y = 0 \quad \cos x \sin(2y+x) = 0 \quad \text{--- (2)}$$

(1) \Rightarrow either $\cos y = 0$ or $\sin(2x+y) = 0$

$$y = \frac{\pi}{2}$$

$$2x+y = 0 \mid \pi \mid 2\pi$$

$$2x+y = (-1)^n n\pi$$

$$2y+x = (-1)^n n\pi$$

$$\text{from (2) } x = \frac{\pi}{2}$$

$$At y = \frac{\pi}{2}$$

$$2x+y = 0$$

$$x+2y = 0$$

$$-3y = 0$$

$$\Rightarrow x = -\frac{\pi}{2}$$

$$y = 0, \text{ if } x = 0$$

at (1,1)

we get

when $2x+y=\pi$ we get $(\pi/3, \pi/3)$
 $x+2y=\pi$

∴ stationary points are $(0,0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{2\pi}{3}, \frac{2\pi}{3})$

$$g = f_{xx} = -2 \cos y \cos(2x+y)$$

$$h = f_{yy} = -2 \cos x \cos(2y+x)$$

$$g + h^2 = 4 \cos x \cos y \cos(2x+y) \cos(2y+x) - \cos^2(2y+x)$$
$$= 4 - 1 = 3 > 0 \text{ at } (0,0)$$

$\lambda = -2 < 0 \therefore (0,0)$ is point of maxima.

at $(\frac{\pi}{2}, \frac{\pi}{2})$ $g + h^2 = 0 - 0 = 0 \therefore$ case is doubtful.

$$\text{at } (\frac{\pi}{3}, \frac{\pi}{3}) \quad g + h^2 = 4 \times \frac{1}{2} \times \frac{1}{2} \cos \pi \cos \pi + \cos^2 \frac{\pi}{3}$$
$$= 1 + \frac{1}{4} = \frac{5}{4} > 0.$$

$\lambda = -2 \times \frac{1}{2} \cos \pi = -1 > 0 \therefore$ point of minima.

$$\text{at } (\frac{2\pi}{3}, \frac{2\pi}{3}) \quad g + h^2 = 4 \times -\frac{1}{2} \times -\frac{1}{2} (1)(1) + \left(\frac{1}{4}\right)$$
$$\therefore 4 + 1 = 2 > 0.$$

$\lambda = -2 \left(-\frac{1}{2}\right)(1) = 1 > 0 \text{ pt of minima.}$

Problems for Practice:

201. 1. find minimum value of $f(x,y) = x^2 + y^2$

201. 2. find minimum value of $x^2 + y^2 + 6x + 12 = 0$.

201. 3. find stationary points of $5x^2 + 10y^2 + 12xy - 4x - 6y + 1$

201. 4. Test the function $f(x,y) = 3y^2(6-x-y)$ for maxima and minima for points not at the origin.

13Q5. Show that minimum value of $xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$

Engineering Mathematics - I

1.1.3-103.

Module - III

Lecture - 38

Differential Calculus - II

Topic

Content: Lagrange's Method of Multipliers for Maxima and Minima of function.

Lagrange's Method of Multipliers: To find extreme value of function of several variable (more than two), when variables are connected by a relation. Lagrange's Method is used.

Working Process:-

1) Consider a function $u = f(x, y, z)$ where x, y, z are connected by a relation $\phi(x, y, z) = 0$ to be maximised or minimised.

2). $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \dots \text{①}$

3). $\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \quad \dots \text{②}$

Multiplying (2) by λ (unknown multiplier) and adding (1) we get

$$du = (fx + \lambda \phi_x) dx + (fy + \lambda \phi_y) dy + (fz + \lambda \phi_z) dz$$

3) $\lambda u - du = 0$ for stationary points.

or. $\begin{cases} fx + \lambda \phi_x = 0 \\ fy + \lambda \phi_y = 0 \\ fz + \lambda \phi_z = 0 \end{cases} \quad \text{eqns give value of } \lambda, x, y, z$

4) Substitute these values. which gives extreme value of u .

Remark:-

3) $e^x \cos(y)$ in terms of 3rd degree.

$x = -a < 0 \therefore (0, 0)$ is point of minima.

Q.1 Find the minimum values of $x^2 + y^2 + z^2$ when $x+y+z=3a$.

Sol. Let $f(x, y, z) = x^2 + y^2 + z^2$ $\stackrel{(1)}{\therefore} \phi(x, y, z) = x + y + z$
 $\text{let } F = f + d\phi$
 $\text{put } dF = (fx + d\phi_x)dx + (fy + d\phi_y)dy + (fz + d\phi_z)dz$

for maximum or minimum value we put $dF = 0$.

$$\begin{aligned} fx + d\phi_x &= 0 & 2x + 1 &= 0 & \Rightarrow x = -1/2 \\ fy + d\phi_y &= 0 & 2y + 1 &= 0 & \Rightarrow y = -1/2 \\ fz + d\phi_z &= 0 & 2z + 1 &= 0 & \Rightarrow z = -1/2 \end{aligned}$$

we get $x = y = z$ substituting in eqn (2),
 $3x = 3a \Rightarrow x = a$.

\therefore Minimum value at (a, a, a) is. $3a^2$ Ans.

Q.2 Divide 24 into three parts such that continued product of first, square of second and cube of third may be maximum.

Sol. $x+y+z=24$ where x, y, z are three parts of

function to be maximised. $f(x, y, z) = xyz^2z^3$

Consider $F = f + d\phi$

$$F = xyz^2z^3 + \lambda[x+y+z-24] \quad \text{--- (1)}$$

for Max value $dF = 0$

$$fx + d\phi_x = 0 \quad y^2z^3 + \lambda = 0 \quad \text{--- (2)}$$

$$fy + d\phi_y = 0 \quad 2xyz^3 + \lambda = 0 \quad \text{--- (3)}$$

$$fz + d\phi_z = 0 \quad 3xyz^2 + \lambda = 0 \quad \text{--- (4)}$$

$$(2) - (3) \text{ gives } y^2z^3 - 2xyz^3 = 0 \quad yz^3[1 - 2x] = 0$$

$$(3) - (4) \text{ gives } 2xyz^3 - 3xyz^2 = 0 \quad 2yz^2[2z - 3x] = 0$$

$$(4) - (2) \text{ gives } 3xyz^2 - y^2z^3 = 0 \quad y^2z^2[3x - 3] = 0$$

$$y = 2x \quad 2z = 3y \quad \text{or } z = 3x.$$

$$(1, 1), (2, 2), (3, 3), (4, 4)$$

$$\begin{aligned} & x^2 + y^2 = 2 \\ & (x+y)^2 = 2 \\ & x^2 + y^2 + 2xy = 2 \\ & x^2 + y^2 + 2xy + 2y^2 = 2 \\ & x^2 + 3y^2 + 2xy = 2 \end{aligned}$$

$$\text{Ex. } x+2y+3z=124$$

$$x=24$$

$$y=2x=48, z=3x=48 \times 3 = 82.$$

$$x=24, y=48, z=82 \quad \text{Ans.}$$

Q.3 A wire of length b is cut into ω small parts which are bent in the form of a square and a circle. Find least value of sum of areas so found using Lagrange's Method of multipliers.

$$\text{Sol. Let } x+y=b$$

x is side of square.
 y is perimeter of circle.

$$x=4a$$

$$\omega a = \pi/4$$

$$2\pi a = y$$

$$a = \frac{y}{2\pi}$$

$$\therefore \text{Area of square} = \frac{x^2}{16}$$

$$\text{Area of circle} = \frac{y^2}{4\pi}$$

$$f = \left(\frac{x^2}{16} + \frac{y^2}{4\pi} \right) + \lambda [x+y-b]$$

$$df=0$$

$$fx + \lambda \phi_x = 0 \Rightarrow \frac{\partial x}{16} + \lambda = 0 \quad (1)$$

$$fy + \lambda \phi_y = 0 \Rightarrow \frac{\partial y}{4\pi} + \lambda = 0 \quad (2)$$

$$(1) - (2) \text{ gives}$$

$$\frac{x}{8} = \frac{y}{2\pi} \Rightarrow x = \frac{4}{\pi}y \quad (3)$$

$$\text{Substitute this value in } x+y=b$$

$$\frac{4}{\pi}y + y = b$$

$$(4+\pi)y = b\pi$$

$$y = \frac{b\pi}{4+\pi}$$

$$\lambda = \frac{4}{\pi} \left[\frac{b\pi}{4+\pi} \right]$$

$$\Rightarrow \frac{4b}{4+\pi}$$

Partial Derivative
Partial derivatives of
let $f_1(x, y, z, u, v, w)$
 $f_2(x, y, z, u, v, w)$

Content

Jacobi
useful
Cartesian
polar

Jacobian
+ u and
variables x

$\frac{\partial y}{\partial x}$

The
 $\frac{\partial (u, v)}{\partial (x, y)}$

$\frac{\partial u}{\partial x}$
 $\frac{\partial v}{\partial x}$
 $\frac{\partial u}{\partial y}$
 $\frac{\partial v}{\partial y}$

and
 $\frac{\partial u}{\partial z}$
 $\frac{\partial v}{\partial z}$

in
fun
les
feel

$$= 4 - 1 = 3 \quad \text{Ans.}$$

$$\boxed{\text{Area} = \frac{x^2}{16} + \frac{y^2}{4\pi} = \frac{b^2}{4(4+\pi)}} \quad \text{Ans.}$$

Problems for Practice:-

Q.1 Find maximum and minimum distance of pt. $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$

Q.2 Find the volume of largest rectangular parallel-pipe that can be inscribed in ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Q.3 Find the minimum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$.

Q.4 The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.

Q.5 Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$.

Content: A
Q1 Find #
Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13 Q14 Q15 Q16 Q17 Q18 Q19 Q20 Q21 Q22 Q23 Q24 Q25 Q26 Q27 Q28 Q29 Q30 Q31 Q32 Q33 Q34 Q35 Q36 Q37 Q38 Q39 Q40 Q41 Q42 Q43 Q44 Q45 Q46 Q47 Q48 Q49 Q50 Q51 Q52 Q53 Q54 Q55 Q56 Q57 Q58 Q59 Q60 Q61 Q62 Q63 Q64 Q65 Q66 Q67 Q68 Q69 Q70 Q71 Q72 Q73 Q74 Q75 Q76 Q77 Q78 Q79 Q80 Q81 Q82 Q83 Q84 Q85 Q86 Q87 Q88 Q89 Q90 Q91 Q92 Q93 Q94 Q95 Q96 Q97 Q98 Q99 Q100

$$t = f_{yy} y = -2 \cos x \cos(ay + b)$$

Engineering Mathematics - I

K AS-103

Module-III

Content: - Problems based on Lagrange's Method and
Maxima and Minima.

Lecture - 38

Differential Calculus-II

- Q1 Find the dimensions of rectangular box of maximum capacity whose surface area is given when
 (1) Box is open at the top.
 (2) Box is closed.

Sol. Let x, y, z be dimensions of box. So volume $V = xyz$.
 Total Surface area is. $S = xy + 2yz + 2zx$ — (1)

$$F = xyz + \lambda [xy + 2yz + 2zx - S]$$

$$dF = 0$$

$$yz + \lambda [y + 2z] = 0 \quad \text{--- (2)}$$

$$xz + \lambda [x + 2z] = 0 \quad \text{--- (3)}$$

$$xy + \lambda [2y + 2x] = 0 \quad \text{--- (4)}$$

$$xyz + \lambda [xy + 2yz + 2zx] = 0$$

$$xyz + \lambda [xy + 2yz] = 0$$

$$xyz + \lambda [2yz + 2zx] = 0$$

Subtract -

$$\lambda [xy + 2yz - xy - 2yz] = 0$$

$$\lambda [2yz] = 0$$

$$\lambda [2y - 2z] = 0$$

$$\lambda [y - z] = 0$$

$$\text{i.e. } x = y = z$$

$$\lambda y [2z - y] = 0$$

$$y = 2z - 2x$$

$$x = 2z - 2y$$

$$\text{let } z = k$$

$$x^2 + 2(2z)z + 2z(2z)z$$

$$4z^2 + 4z^2 + 4z^2 = S$$

$$z^2 = \frac{S}{12}$$

$$z = \frac{1}{2} \frac{S}{\sqrt{3}}, \quad y = \frac{S}{\sqrt{3}}$$

8

Substituting in (1)

for closed
 $s = 2xy + 2yz + 2zx$
 $x = y = z = \sqrt{\frac{s}{6}}$

Q.2 If $u = ax^2 + by^2 + cz^2$ and $x^2 + y^2 + z^2 = 1$, and $lx + my + nz = 0$.
 prove that stationary points satisfy eqns.

$$\frac{e^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$$

Q.3. $F = ax^2 + by^2 + cz^2 + d_1(x^2 + y^2 + z^2 - 1) + d_2(lx + my + nz)$

$$2ax + 2d_1x + ld_2 = 0 \quad (1)$$

$$2by + 2d_1y + md_2 = 0 \quad (2)$$

$$2cz + 2d_1z + nd_2 = 0 \quad (3)$$

Multiplying (1) by x , (2) by y , (3) by z .

$$2[ax^2 + by^2 + cz^2] + 2d_1[x^2 + y^2 + z^2] + d_2[lx + my + nz] = 0$$

$$2d_1 = -\frac{2u}{a} \quad \text{or } \boxed{d_1 = -\frac{u}{a}}$$

substituting in eqns (1), (2) and (3)

$$2an - 2xu + ld_2 = 0$$

$$x = -\frac{d_2 l}{2(a-u)} = \frac{d_2 l}{2(u-a)}$$

$$2by - 2yu + md_2 = 0$$

$$y = \frac{d_2 m}{2(u-b)}$$

$$2cy - 2zu + nd_2 = 0$$

$$z = \frac{d_2 n}{2(u-c)}$$

$$lx + my + nz = 0$$

$$\frac{1}{2} \left[\frac{e^2}{a(u-a)} + \frac{m^2}{b(u-b)} + \frac{n^2}{c(u-c)} \right] = 0.$$

$$\boxed{\frac{e^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0}$$

Hence proved.

Content: Jacobian's and Properties of Jacobians.

Jacobians :- Jacobian is a functional determinant, useful in transformation of variables from Cartesian to polar, cylindrical and spherical polar co-ordinates in multiple integrals.

Jacobian for two functions of two variables :- Let u and v are functions of two independent variables $x \neq y$, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \begin{array}{l} \text{is called} \\ \text{Jacobian of} \\ u \text{ & } v \text{ with respect to } x \neq y \end{array}$$

Notation : The notation used for Jacobian is

$$J \text{ or } \frac{\partial(u, v)}{\partial(x, y)} \text{ or } J(u, v)$$

Q.1. If $u = x^2$, $v = y^2$ find $\frac{\partial(u, v)}{\partial(x, y)}$

Sol.
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy \text{ Ans.}$$

Q.2 If $u = x(1-y)$ and $v = xy$ find $\frac{\partial(u, v)}{\partial(x, y)}$

Sol.
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x(-xy) + xy = x \text{ Ans.}$$

Jacobian for three functions of three variables :-

Let u, v and w are functions of three independent variables x, y, z . Then the determinant denoted by- $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

Partial Derivatives using Jacobians: - We can find partial derivatives of implicit functions.

Let $J(u, v, w) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$ is called Jacobian of u, v and w with respect to x, y and z respectively. Then $\frac{\partial(u, v)}{\partial(x, y)}$ similarly if u, v, t and x, y, z then $\frac{\partial(u, v, t)}{\partial(x, y, z)}$

Remark - $\frac{\partial(u, v)}{\partial(u, v)} = 1$

- Q1. The formula can be extended for n independent variables.
 Q2. The Jacobian is determinant. no. of functions = no. of variables.

Q1 Calculate $J(u, v, w)$ of $u = x + 2y + 3z, v = x + 2y + 3z$ and $w = 2x + 3y + 5z$.

Sol. $J(u, v, w) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 2$.

Q2 If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$ find $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$

Sol. $J(y_1, y_2, y_3) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix} = \frac{1}{x_1 x_2 x_3} \begin{vmatrix} -x_2 x_3 & x_3 & x_2 \\ x_3 & -\frac{x_3 x_1}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1 x_2}{x_3} \end{vmatrix}$$

$$= \frac{4}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -2x_2 x_3 & x_1 x_3 & x_1 x_2 \\ x_3 x_2 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix} = \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4 \text{ Ans.}$$

Properties of Jacobians.

Chain Rule: If u and v are functions of x, y and s, t are functions of x, y .

$$\text{then } \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, s)} \times \frac{\partial(x, s)}{\partial(x, y)}$$

Similarly if u, v, w are functions of x, s, t and ℓ, s, t are functions of x, y, z .

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{\partial(u, v, w)}{\partial(x, s, t)} \times \frac{\partial(x, s, t)}{\partial(x, y, z)}$$

Q. 2 If $J_1 = \frac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \frac{\partial(x, y)}{\partial(u, v)}$
 then $J_1 \cdot J_2 = 1$. i.e. $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$.

3. If $u_1 = f(x_1)$
 $u_2 = f(x_1, x_2)$
 $u_3 = f(x_1, x_2, x_3)$
 \vdots
 $u_n = f(x_1, x_2, x_3, \dots, x_n)$.

$$\text{then } J = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \dots & 0 \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$J(u_1, u_2, \dots, u_n) = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}$$

Q. 1 v.v. Imp. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$
 find $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ and $\frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$

Sol. $J(x, y, z) = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} r \sin \theta \cos \phi & r \sin \theta \cos \phi & r \cos \theta \sin \phi \\ r \sin \theta \sin \phi & r \sin \theta \sin \phi & -r \sin \theta \\ 0 & 0 & 1 \end{vmatrix}$

find relation between them. they are functionally related, hence $w = x^2y^2z^2$

Part 2

Let $J = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$

$$= \sin\theta\cos\phi [x^2\sin^2\theta\cos\phi] + x^2\cos^2\theta\cos\phi\sin\theta\cos\phi - \sin\theta\sin\phi \\ = x^2\sin^2\theta\sin\phi - x^2\cos^2\theta\cos\phi$$

$$\frac{\partial x}{\partial u} = \\ = x^2\sin^2\theta + x^2\cos^2\theta\sin\phi \\ = x^2\sin\theta$$

then $J' = \frac{\partial(x, \theta, \phi)}{\partial(x, y, z)}$ using property II

$$\begin{cases} J' = \frac{1}{J} \\ J = \frac{1}{x^2\sin\theta} \end{cases}$$

\Rightarrow Problems for Practice

Q3 If $u = xy_3$, $v = xy + yz + zx$ and $w = x + y + z$ find.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}$$

Q3 If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{vz}$ & $u = x\sin\theta\cos\phi$, $v = x\sin\theta\sin\phi$, $w = x\cos\theta$ find $J(x, y, z)$

Q4 If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \sin x_3$. find $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3$. using property III.

Q5 If $u = xy_3$, $v = x^2 + y^2 + z^2$, $w = x + y + z$. find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

Q6 If $u = x(1-x^2)^{-1/2}$, $v = y(1-x^2)^{-1/2}$, $w = z(1-x^2)^{-1/2}$ and $x^2 = x^2 + y^2 + z^2$ then $\frac{\partial(u, v, w)}{\partial(x, y, z)} = (1-x^2)^{-5/2}$

Q7 Verify chain rule.

Q7 $x = u$, $y = u \tan v$, $z = w$

Q8 $x = u^v$, $y = \frac{u+v}{u-v}$ find $\frac{\partial(u, v)}{\partial(x, y)}$

Problems for Practice

Q1 If radius is r with r & θ as possible respectively - 1

Contents: Jacobians for Implicit functions
Functional Relationship, partial derivative
using Jacobian.

Implicit Function: A function or relation in which the dependent variable is not isolated on one side of the equation. Thus $x^2 + xy - y^2 = 1$ represents an implicit relation.

Jacobian of implicit functions: If $u_1, u_2, u_3, \dots, u_n$ are implicit functions of variables x_1, x_2, \dots, x_n then

$$f_1(u_1, u_2, u_3, \dots, u_n, x_1, x_2, \dots, x_n) = 0$$

$$f_2(u_1, u_2, u_3, \dots, u_n, x_1, x_2, \dots, x_n) = 0$$

$$f_n(u_1, u_2, u_3, \dots, u_n, x_1, x_2, \dots, x_n) = 0$$

$$\frac{\partial(u_1, u_2, u_3, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \frac{\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}}{\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(u_1, u_2, \dots, u_n)}}$$

Q. If $x+y+z=4, y+z=uv, z=uvw$ then show

$$\text{that } \frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$$

$$f_1 = x+y+z-u=0$$

$$f_2 = y+z-uv=0$$

$$f_3 = z-uvw=0$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} = -u^2 v$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Hence $J = u^2 v$
Ans.

variable $f(x+h)$ can be independent of h , where x is independent of h .

Q.2 If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^2 + y^2 + z^2$, then show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{3}{u^2 + v^2 + w^2}$.

$$\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Q.3 If α, μ, ν are roots of eqⁿ $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$, prove that $\frac{\partial(x, y, z)}{\partial(\alpha, \mu, \nu)} = \frac{-(v-\alpha)(\alpha-\mu)(\mu-\nu)}{(a-b)(b-c)(c-a)}$.

Sol. Consider eqⁿ.

$$\begin{aligned} & x(a+k)(c+k) + y(a+k)(b+k) + z(a+k)(b+k) = (a+k)(b+k)(c+k) \\ & = x[b(c+k^2) + (b+c)k] + y[a(c+k^2) + (a+c)k] + z[a(b+k^2) + (a+b)k] \\ & = ab(c+k)ab + (a+b)c k + (a+b)k^2 + k^2c + k^3 \\ & = k^3 + [a+b+c - x - y - z]k^2 + k[a b + b c + c a - (b+c)x - (a+c)y \\ & \quad - (a+b)z] + abc - xbc - acy - aby = 0. \end{aligned}$$

$$\text{sum of roots} = -\frac{b}{a}.$$

$$\text{product of roots} = \frac{c}{a}.$$

$$\text{product of roots taken three at a time} = -\frac{d}{a}.$$

of eqⁿ $ax^3 + bx^2 + cx + d = 0$

$$\therefore \alpha + \mu + \nu = x + y + z - a - b - c.$$

$$\alpha\mu + \mu\nu + \nu\alpha = ab + bc + ca - (b+c)x - (a+c)y - (a+b)z.$$

$$\alpha\mu\nu = abc + acy + abz - abc.$$

$$\frac{\partial(x, y, z)}{\partial(\alpha, \mu, \nu)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(\alpha, \mu, \nu)} \left| \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \right|$$

$$\frac{\partial(F_1, F_2, F_3)}{\partial(\alpha, \mu, \nu)} = (\alpha - \mu)(\alpha - \nu)(\mu - \nu)$$

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = -(a-b)(b-c)(c-a)$$

Problems for
Q.1, Q.2, Q.3
in the form

Q.1

Q.2

Q.3

$$\frac{\partial^3 f}{\partial x^3} + \frac{\partial^3 f}{\partial x^2 \partial y} + \frac{\partial^3 f}{\partial x \partial y^2} + \frac{\partial^3 f}{\partial y^3} = -\frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} (x-a)(y-b)(z-c)$$

Q.2 If $x^2 + y^2 + u^2 + v^2 = 0$ and $uv + xy = 0$ then prove that $\frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2 y^2}{u^2 v^2}$

Q.3 If $u^3 + v^3 = x+y$, $u^2 + v^2 = x^3 + y^3$ then show that $\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 x^2}{u v (u-v)}$

Functional Relationship: Let u_1, u_2, \dots, u_n be functions of x_1, x_2, \dots, x_n . Then necessary condition for the existence of relation of form $F(u_1, u_2, \dots, u_n) = 0$ is that Jacobian $\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = 0$ or vanish identically.

i.e. variables are related to each other and we can find relation between them by hit and trial.

Remark: - If $J=0$, functions are linearly dependent else linearly independent.

Q.1 If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$. Verify that they are functionally dependent and if so find relation between them.

$$\text{Sol. } \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} ux & vy \\ vx & vy \end{vmatrix} = \begin{vmatrix} \frac{1-xy+y(x+y)}{(1-xy)^2} & \frac{(1-xy)+x^2}{(1-xy)^2} \\ \frac{1}{1+xy} & \frac{1}{1+y^2} \end{vmatrix} = 0. \quad \therefore u \text{ & } v \text{ are functional dependent.}$$

ERJ MAJ
ture - 34 Theorem of one
for single
function.

$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$ or $u = \tan v$

Q2 Determine functional dependence & find relation.

$$u = \frac{x+y}{x-y}, v = \frac{xy}{(x-y)^2}$$

Sol. $\frac{\partial u}{\partial x} = \frac{x-y - (x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$$\frac{\partial u}{\partial y} = \frac{\partial x}{(x-y)^2}, v_x = -\frac{y(x+y)}{(x-y)^3}, v_y = \frac{x(x+y)}{(x-y)^3}$$

$$J = \begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix} = \begin{vmatrix} -\frac{2y}{(x-y)^2} & \frac{\partial x}{(x-y)^2} \\ -\frac{y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix} = \frac{1}{(x-y)^5} \begin{vmatrix} -y & x \\ -y & x \end{vmatrix}$$

∴ Functional relationship exists between u and v .

changing

$$u = \frac{x+y}{x-y}$$

leg (1) $u+1 = \frac{x+y+x-y}{x-y} = \frac{2x}{x-y}$

leg f $u-1 = \frac{2x+y-x+y}{x-y} = \frac{2y}{x-y}$

$$\frac{x}{y} = \frac{u+1}{u-1}$$

$$v = \frac{xy}{(x-y)^2} \Rightarrow \frac{y^2 \left(\frac{2x}{y}\right)}{y^2 \left[\frac{x}{y} - 1\right]^2} \Rightarrow \frac{\frac{u+1}{u-1}}{\left[\frac{u+1}{u-1} - 1\right]^2} = \frac{u+1}{u-1} \cdot \frac{4}{(u-1)^2}$$

$$\Rightarrow \frac{u+1}{u-1} \cdot \frac{(u-1)^2}{4}$$

Q5 $v = (u+1)(u-1)$

Sol. at $u^2 - 4v^2 - 1 = 0$ Ans.

Problems for practice!

Q1 Show that $u = y+z, v = x+yz^2, w = x-4yz - 2y^2$ are not independent.

Q2 Show $u = x^2 + y^2 + z^2, v = x+yz+z^3, w = yz + z^2 + xy$ are not independent. If not find relation between them.

$u = x+y+z, v = x^2+y^2+z^2 - 2xy - 2yz - 2zx, \text{ & } w = x^3+y^3+z^3 - 3xyz$
 show that they are functionally related, hence find relation between them.

Partial Derivatives Using Jacobians! - we can find partial derivatives of implicit functions using Jacobian.

$$\text{Let } f_1(x, y, z, u, v, w) = 0$$

$$f_2(x, y, z, u, v, w) = 0$$

$$f_3(x, y, z, u, v, w) = 0$$

$$\frac{\partial x}{\partial u} = \frac{f_1(x, y, z, u, v, w) \cdot \frac{\partial f_1}{\partial u}}{f_1(x, y, z, u, v, w) \cdot \frac{\partial f_2}{\partial u} + f_2(x, y, z, u, v, w) \cdot \frac{\partial f_3}{\partial u}}$$

$$= \frac{(-1) \cdot \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(u, v, w)}}{\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x, y, z)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)}}$$

Remark! - first we write Jacobian in the denominator then we write Jacobian in numerator by replacing x, y, z by u, v, w .

$$\frac{\partial u}{\partial x} = \frac{(-1) \cdot \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(u, v, w)}}{\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x, y, z)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)}}$$

Q: Given $x = u+v+w, y = u^2+v^2+w^2, z = u^3+v^3+w^3$ find $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{(-1) \cdot \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)}}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2v & 2w \\ 0 & 3v^2 & 3w^2 \end{vmatrix} = -6vw \begin{vmatrix} 1 & 1 \\ v & w \end{vmatrix} = 6vw(v-w)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ 2u & 2v & 2w \\ 3u^2 & 3v^2 & 3w^2 \end{vmatrix} = 6(v-u)(w-u)(w-v)$$

$$\left[\therefore \frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)} \right] \text{ Ans.}$$

$$2) \text{ To expand } f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

Q1 find $\frac{\partial u}{\partial x}$ if $u^2 + x v^2 - xy = 0$

$$\text{Sol. } f_1 = u^2 + x v^2 - xy, \quad f_2 = u^2 + xy v + v^2 = 0$$

$$\frac{\partial u}{\partial x} = - \frac{\partial (f_1, f_2)}{\partial (x, v)} / \frac{\partial (f_1, f_2)}{\partial (u, v)}$$

$$\frac{\partial (f_1, f_2)}{\partial (x, v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = 2u v y + 4u v - 4u v$$

$$\frac{\partial (f_1, f_2)}{\partial (x, v)} = \begin{vmatrix} v^2 - y & \frac{\partial v}{\partial v} \\ y v & x y + 2 v \end{vmatrix} = -v y v^2 + v^3 - v y^2 - 2 y v$$

$$\left(\frac{\partial u}{\partial x} \right)_v = \frac{\partial (y v^2 - v^3 + x y^2 + 2 y v)}{\partial u x y + 4 u v^2 - 4 u v} / \text{treating } v \text{ as constant.}$$

Problems for Practice: -

Q1 If $u = x + y^2, v = y + z^2, w = z + x^2$

find $\frac{\partial u}{\partial u}$.

Q2 If $u = x^2 + y^2 + z^2, v = xyz$ find $\frac{\partial u}{\partial u}$.

Q3 If $u = x y z, v = x^2 + y^2 + z^2, w = x + y + z$. find $\frac{\partial u}{\partial u}$.

Content: A

Approximate

Working

1. Cons

2. Re

3. Cl

4.

5

Content: Approximation.

Approximation: - using error approximation the approximate values of expressions can be determined.

Working Process :- To approximate value of expressions.

1. Consider the given expression
2. Round off each term to nearest integer. e.g. $(27.1)^{1/3} = (27)^{1/3}$
3. Consider the given expression as function $y = f(x) = x^{1/3}$.
4. Substitute these approximate numbers as variable. take $x = 27$.
5. The difference in original value and estimated value is termed as error $\delta x = 27.1 - 27 = 0.1$.
6. $\delta y = \frac{df}{dx} \delta x$
7. $f(x+\delta x) = y + \delta y$

Q1. find approximate value of $(27.2)^{1/3}$.

Sol. 1) Round off to nearest $(27)^{1/3}$

2) let $y = f(x) = x^{1/3}$ where $x = 27$.

$$3) \delta x = 27.2 - 27 = 0.2$$

$$4) \delta y = \frac{df}{dx} \delta x = \frac{1}{3} x^{-2/3} \cdot 0.2$$

$$= \frac{1}{3} (27)^{-2/3} \times 0.2$$

$$5) \frac{1}{3} \times 0.2$$

$$\delta y = 0.007$$

$$5) \text{ Hence } f(x+\delta x) = y + \delta y$$

$$= 3 + 0.007$$

$$f(27.2) = 3.007$$

$$\begin{aligned}
 f(a+h) &= f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) \\
 f(x) &= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\
 3. \text{ To estimate} & \quad [f(3.82) + f(2.1)]^{1/5}
 \end{aligned}$$

Q2 Find approximate value of $[x^2 + 2y^3]^{1/5}$

Sol Let $f(x,y) = [x^2 + 2y^3]^{1/5}$

where $x=2, y=1$

$$\begin{aligned}
 \delta x &= 4 + 3.82, \quad \delta y = 2.1 - 2, = 0.1 \\
 &= -0.18
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= \frac{1}{5} [x^2 + 2y^3]^{-4/5} \times 2x \\
 &= \frac{1}{5} [16 + 16]^{-4/5} \times 8 = \frac{1}{5} \times \frac{8}{2^4} = \frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f}{\partial y} &= \frac{1}{5} (x^2 + 2y^3)^{-4/5} \times 6y^2 \\
 &= \frac{1}{5} [16 + 16]^{-4/5} \times 6 \times 2^2 = \frac{1}{5} \times \frac{6 \times 3}{2^4} = \frac{3}{10}
 \end{aligned}$$

By total differentiation.

$$\begin{aligned}
 df &= \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y \\
 &= \frac{1}{10} (0.18) + \frac{3}{10} (0.1) = 0.012
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x+\delta x, y+\delta y) &= f(x, y) + \delta f \\
 f(3.82, 2.1) &= f(4, 2) + 0.012 = 0.012
 \end{aligned}$$

Q3 Compute $(1.99)^2 \times (1.02)^{1/10}$.

Sol: $f(x,y) = x^2 y^{1/10}$ where $x=2, \delta x=-0.01$
 $y=1, \delta y=0.02$

$$f_x(x,y) = 2x y^{1/10} = 4$$

$$f_y(x,y) = \frac{1}{10} x^2 y^{-9/10} \cdot \frac{4}{10} = \frac{4}{100} x^2 y^{-9/10}$$

$$\delta f = f_x \delta x + f_y \delta y$$

$$4x - 0.01 + \frac{4}{10} x 0.02$$

$$= -0.04 + 0.008$$

$$= -0.392$$

$$\begin{aligned}
 f(x+\delta x, y+\delta y) &= f(x,y) + \frac{\delta x}{x} f_x(x,y) + \frac{\delta y}{y} f_y(x,y) \\
 f(1.99, 3.01) &= f(2, 1) \\
 &= 2^2 \times 1^{1/10} \\
 &= 4
 \end{aligned}$$

Q4 find approximate value of $f(1.99, 3.01)$

Sol: $x=1, y=1$
 $\delta x=0.02$

$$f(x,y) =$$

$$f_x =$$

$$f(x+\delta x, y+\delta y) = f(x, y) + \delta f$$

$$\begin{aligned}f(1.99, 3.01) &= f(2, 3) + \delta f \\&= 2^2 \times 1^{1/10} + (-0.392) \\&= 4 - 0.392\end{aligned}$$

$$f(1.99, 3.01) = 3.608 \quad \text{Ans.}$$

Q.4 find approximate values of $\sqrt{(0.98)^2 + (0.01)^2 + (1.94)^2}$

$$x=1, y=2, z=2.$$

$$\delta x = -0.02, \delta y = 0.01, \delta z = -0.06.$$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

$$fx = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$fy = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad fz = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\begin{aligned}\delta f &= fx\delta x + fy\delta y + fz\delta z \\&= \frac{1}{3}(-0.02) + \frac{2}{3}(0.01) + \frac{2}{3}(-0.06)\end{aligned}$$

$$= \frac{1}{3}[-0.02 + 0.02 - 0.12] = -0.04$$

$$f(x+\delta x, y+\delta y, z+\delta z) = f(x, y, z) + \delta f$$

$$f(0.98, 2.01, 1.94) = 3 - 0.04 = 2.96 \quad \text{Ans.}$$

Problems for Practice: value. when $x=1.99, y=3.01, z=0.98$

1) find approximate value. for $f(x, y, z) = x^2 y^2 z^{1/10}$

$$f(x, y, z) = x^2 y^2 z^{1/10}$$

2) find approximate value of $(2.4)^{3.1}$

3) find approximate value of $(x)^y$ where $x=4.01$ & $y=2.99$.

4) find approximate value of $[(3.98)^2 + (2.01)^2 + 3(1.94)^2]^{1/5}$

2009-10

$= a$
 $\rightarrow + \frac{h^n}{n!} f^{(n)}(a)$
replace h by $x - a$
 $\frac{(x-a)^n}{n!}$

Ramjet year

Engineering Mathematics I

KAS-103

Lecture-42

Module-III
Differential Calculus I

Content: Errors

Errors: - When one quantity or variable is related to several others by a functional relationship it is possible to estimate the percentage change in it caused by given percentage change in the other variable.

When input variables are measured and measurement are in error, due to limits on precision of measurement, then we can estimate the effect these errors have on the output.

Error determination: - Consider a function $z = f(x, y)$

If δx and δy are small increments in variables x & y , then an approximate change or error in z denoted by δz can be measured as.

$$\delta z = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$$

i.e. let $z = f(x, y) \quad (1)$

$$z + \delta z = f(x + \delta x, y + \delta y) \quad (2)$$

(2) - (1) gives -

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\text{or } \delta z = f(x, y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} - f(x, y)$$

$$\delta z = \delta x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \delta y$$

$$\text{or } dz = f_x dx + f_y dy$$

Absolute error: δx is known as absolute error in x .

Relative Error: $\delta x/x$ is known as relative error in x !

Percentage Error: $\frac{\delta x}{x} \times 100$ is known as percentage error in x .

function above, $f(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$ in powers of $a-a$. replace h by $(x-a)$

$f(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a)$

Q1 Find percentage error in area of ellipse when error of 1 percent is made in measuring major and minor axis.

Q2. Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $2a$ be major and $2b$ be minor axis.

Area of ellipse = πab

$$A = \pi ab$$

$$\text{Taking, } \log A = \log \pi + \log a + \log b.$$

Total differentiating both sides.

$$\frac{1}{A} \Delta A = 0 + \frac{1}{a} \Delta a + \frac{1}{b} \Delta b$$

$$\frac{\Delta A}{A} \times 100 = \frac{\Delta a}{a} \times 100 + \frac{\Delta b}{b} \times 100$$

$$\frac{\Delta A}{A} \times 100 = 1 + 1.$$

\Rightarrow percentage error in Area = 2% Ans.

Q2 Find percentage error in area of rectangle when an error of 1 percent is made in measuring the sides.

Q3 The period T of simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$ find maximum percentage error in T due to possible error of 2% in l and 5% in g .

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Taking log both sides

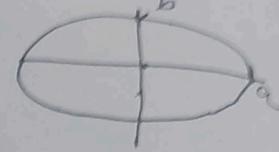
$$\log T = \log 2\pi + \frac{1}{2} [\log l - \log g]$$

Differentiating

$$\frac{1}{T} \Delta T = 0 + \frac{1}{2} \left[\frac{1}{l} \Delta l - \frac{1}{g} \Delta g \right]$$

$$\frac{1}{T} \Delta T \times 100 = \frac{1}{2} \left[\frac{1}{l} \Delta l \times 100 - \frac{1}{g} \Delta g \times 100 \right]$$

$$= \frac{1}{2} [2 \pm 5]$$



The power of l at which is constant decreased by 1.

$P = \frac{1}{2} u^3 l^2$

$\log P = \frac{1}{2} \log l^2$

$\frac{1}{P} \Delta P = \frac{1}{2} \frac{1}{l^2} \Delta l$

$\Delta P = \frac{1}{2} \frac{1}{l^2} l \Delta l$

$\Delta P = \frac{1}{2} \frac{1}{l} \Delta l$

measuring
true value of x
problem

The power 'P' required to propel a steamer of length 'l' at a speed 'u' is given by $P = \alpha u^3 l^3$ where α is constant. If u is increased by 3% & l is decreased by 1%. Find Increase in P

Sol. $P = \alpha u^3 l^3$

$$\log P = \log \alpha + 3 \log u + 3 \log l$$

$$\frac{1}{P} \delta P = 0 + \frac{3}{u} \delta u + \frac{3}{l} \delta l$$

$$\frac{\delta P}{P} \times 100 = 3 \times 3 - 3(1)$$

$$\frac{\delta P}{P} \times 100 = 6\% \quad \boxed{\text{Ans.}}$$

decreased
 $\frac{\delta l}{l} \times 100 = -1$

ANSWER PAPER	ANSWER	ANSWER	ANSWER	ANSWER
10	10	10	10	10
11	10	10	10	10
12	10	10	10	10
13	10	10	10	10
14	10	10	10	10
15	10	10	10	10

Q5 In estimating the number of bricks in a pile measured (15m x 10m x 5m), count of bricks is 100 bricks per m^3 . Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is $\text{Rs. } 200$ per 1000 bricks.

Sol. - volume of pile = $l \times b \times h$

$$\log V = \log l + \log b + \log h$$

$$\frac{\delta V \times 100}{V} = \frac{\delta l \times 100}{l} + \frac{\delta b \times 100}{b} + \frac{\delta h \times 100}{h}$$

$$\frac{\delta V \times 100}{V} = 2 + 2 + 2$$

$$\delta V = \frac{6V}{100}$$

$$8V = 6 \times \frac{(5 \times 10 \times 5)}{100} = 15 \text{ m}^3$$

No. of bricks in $8V = 15 \times 100 = 1500$.

Cost of 1 brick = $\frac{\text{Rs. } 200}{1000} = \text{Rs. } 0.2$.

Error in cost = $\text{Rs. } 0.2 \times 1500 = \text{Rs. } 3000 \quad \boxed{\text{Ans.}}$



Q6 The diameter and height of right circular cylinder are found by measurement to be 8.0 cm and 12.5 cm respectively with possible error 0.05% in each. Find maximum approximate error in volume.

$$\log(1+x)$$

changing
 $\log(1-x)$

$$\log\left(\frac{1+x}{1-x}\right)$$

$$\log\left(\frac{1+x}{1-x}\right)$$

$$\log\left(\frac{1+x}{1-x}\right)$$

$$\log\left(\frac{1+x}{1-x}\right)$$

5 Expand

using 2nd

$$f(x) = x^3$$

$$f'(x) = 6x^2$$

$$f''(x) = 12x$$

$$f'''(x) = 18$$

where x is a small
other forms of Taylor's Mean
1) To expand a function about point $x=a$
 $f(x+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots$
2) To expand a function in powers of $x-a$. replace h by $x-a$
 $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$

$$V = \pi r^2 h = \frac{1}{3} \pi r^2 h$$

$$\frac{\delta V}{V} = \frac{2}{3} \frac{\delta D}{D} + \frac{8}{3} \frac{\delta h}{h}$$

$$\frac{\delta V}{V} = \frac{2}{3} \times 0.05 + \frac{1}{12.5} \times 0.05$$

$$\delta V = 0.05 \left[\frac{1}{4.0} + \frac{1}{12.5} \right] \times V$$

$$= 0.05 [12.5 + 4.0] \times \pi \times 4 \times 4 \times 12.5$$

$$3.3 \pi \text{ cm}^3 \text{ Ans.}$$

Q.7 A balloon is in the form of right circular cylinder of
2006 $r=1.5$ and length $= 4$ m and surmounted by hemisphere
ends. If radius is increased by 0.01 m & length is
increased by 0.05 find percentage change in volume of balloon.

$$r=1.5, h=4 \text{ m}$$

$$\text{Sol. } V = \pi r^2 h + \frac{2}{3} \pi r^3 + \frac{2}{3} \pi r^3$$

$$V = \pi r^2 h + \frac{4}{3} \pi r^3$$

$$\delta V = 0 + 2\pi r^2 h + \pi r^2 8h + \frac{4}{3} \times 3 \pi r^2 8r$$

$$\delta V = 2 \times 1.5 \times 4 \times 8r + \pi (1.5)^2 \times 8h + 4\pi (1.5)^2 \times 8r$$

$$= 8 \times 1.5 \times 0.01 + \pi (1.5)^2 [0.05 + 4 \times 0.01]$$

$$= 1.5 \times 0.08 + \pi (2.25) (0.09)$$

$$\frac{\delta V}{V} = \frac{1.5 \times 0.08 + \pi (2.25) (0.09)}{\pi r^2 h + \frac{4}{3} \pi r^3}$$

$$\Rightarrow 0.389\%$$



Problems for Practice.

1) If radius & height of cone are 4 cm and 8 cm with a possible error of 0.04 and 0.08 inches respectively - find percentage error in volume of cone.

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