3 - Laplace Transform

3·1 Introduction :- Many times a mathematical equation can not be solved as it is but, if transformed into another form by an standard rule, then it becomes easier, to find the solution of the given mathematical equation. One of such ruler known as Laplace Transform.

Using Laplace transform we can solve boundary differential equations without the necessity of first finding the general solution.

French mathematician Pierre Simon Marquis De Laplace (1749 - 1827) was a professor in Paris, he developed this theory.

3·2 Definition — Let $f(t)$ be a function of t for all t; then Laplace transform of $f(t)$ denoted by $L\{f(t)\}$ is defined as

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt ,$$

provided that the integral exists.

Where s is a parameter may be real or complex.

3.3 Formulae :—

(2)

(1) $L\{1\} = \frac{1}{s}$, $s>0$

Proof :– $L\{1\} = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}(0-1) = \frac{1}{s}$

(2) $L\{t^n\} = \frac{\lfloor n}{s^{n+1}}$ if n is a positive integer.

$= \frac{\Gamma(n+1)}{s^{n+1}}$ if n is fraction.

Proof :– $L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n dt$

put $st = x$ then $t = \frac{x}{s}$ or $dt = \frac{dx}{s}$

$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx = \frac{\Gamma(n+1)}{s^{n+1}}$

$\left(\text{Since } \Gamma(n+1) = \int_0^\infty e^{-x} x^n dx\right)$

(3) $L\{e^{at}\} = \frac{1}{s-a}$, $s>a$

Proof :– $L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$

$= \frac{1}{-(s-a)}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty = \frac{1}{-(s-a)}[0-1] = \frac{1}{s-a}$

(4) $L\{\sin at\} = \frac{a}{s^2+a^2}$, $s>0$

Proof. $L\{\sin at\} = L\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\} = \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$

$= \frac{1}{2i}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2i} \frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$

(5) Similarly $L\{\cos at\} = \frac{s}{s^2+a^2}$, $s>0$

(6) $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s^2>a^2$

Proof :– $L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2} \cdot \frac{2a}{s^2-a^2} = \frac{a}{s^2-a^2}$

(7) Similarly $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $s^2>a^2$

Questions based on the formula (3.3)

Que ① Find the Laplace transform of

(a) $t^{-1/2}$ (b) $\cos^2 2t$ (c) $\sin^3 2t$ (d) $\sinh^2 2t$ (e) $\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3$

Sol — (a) $L\{t^{-1/2}\} = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} = \dfrac{\Gamma\frac{1}{2}}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$

(b) $L\{\cos^2 2t\} = L\left\{\dfrac{1+\cos 4t}{2}\right\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 4t\}$

$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\dfrac{s}{s^2+16}$

(c) Since $\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\therefore \sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$\therefore L\{\sin^3 2t\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$

$= \frac{3}{4}\cdot\dfrac{2}{s^2+4} - \frac{1}{4}\cdot\dfrac{6}{s^2+36} = \dfrac{48}{(s^2+4)(s^2+36)}$

(d) $\sinh^2 2t = \left[\dfrac{e^{2t}-e^{-2t}}{2}\right]^2 = \frac{1}{4}\left[e^{4t}+e^{-4t}-2\right]$

$\therefore L\{\sinh^2 2t\} = \frac{1}{4}\left[L\{e^{4t}\} + L\{e^{-4t}\} - 2L\{1\}\right]$

$= \frac{1}{4}\left[\dfrac{1}{s-4} + \dfrac{1}{s+4} - \dfrac{2}{s}\right]$

(e) $\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3 = (\sqrt{t})^3 + \left(\frac{1}{\sqrt{t}}\right)^3 + 3\sqrt{t}\,\frac{1}{\sqrt{t}}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right) = t^{3/2} + t^{-3/2} + 3t^{1/2} + 3t^{-1/2}$

$\therefore L\left\{\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right\} = L\{t^{3/2}\} + L\{t^{-3/2}\} + 3L\{t^{1/2}\} + 3L\{t^{-1/2}\}$

$= \dfrac{\Gamma\frac{5}{2}}{s^{5/2}} + \dfrac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} + 3\cdot\dfrac{\Gamma\frac{3}{2}}{s^{3/2}} + 3\cdot\dfrac{\Gamma\frac{1}{2}}{s^{1/2}}$

$= \dfrac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}} + \dfrac{(-2\sqrt{\pi})}{s^{-1/2}} + 3\cdot\dfrac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + 3\,\dfrac{\sqrt{\pi}}{s^{1/2}}$

(Since $\Gamma\frac{1}{2} = \sqrt{\pi}$ and $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
and $\Gamma(n+1) = n\Gamma n = \lfloor n$

$= \sqrt{\pi}\left[\dfrac{3}{4s^{5/2}} - 2s^{1/2} + \dfrac{3}{2s^{3/2}} + \dfrac{3}{s^{1/2}}\right]$

3.4 Properties of Laplace transforms :—

(a) First shifting property :— If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at} f(t)\} = \bar{f}(s-a)$$

④

Proof :— $L\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = \bar{f}(s-a)$$

Note :— Applying this property the formulae 3.3 can be written as

(1) $L\{e^{at} t^n\} = \dfrac{\lfloor n}{(s-a)^{n+1}}$

(2) $L\{e^{at} \sin bt\} = \dfrac{b}{(s-a)^2+b^2}$, (3) $L\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

(5) $L\{e^{at} \sinh bt\} = \dfrac{b}{(s-a)^2-b^2}$, (6) $L\{e^{at} \cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}$

(b) Second shifting property — If $L\{f(t)\} = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a), & t > \\ 0 & t < \end{cases}$

then $L\{g(t)\} = e^{-as} \bar{f}(s)$

Proof :— $L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0 \; dt + \int_a^\infty e^{-st} \cdot f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

put $t - a = x$
$dt = dx$

$$= \int_0^\infty e^{-s(a+x)} f(x)\,dx$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)\,dx$$

$$= e^{-as} \bar{f}(s)$$

(c) Change of scale property :— If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{f(at)\} = \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right)$$

Proof :— $L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$

put $at = x$ $\therefore dt = \frac{dx}{a}$

$$= \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) \cdot \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\,dx$$

$$= \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right)$$

(d) Linear Properties :— If a and b are any constant and f and g are functions of t, then

$$L\{a f(t) + b g(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$$

Proof :— $L\{a f(t) + b g(t)\} = \int_0^\infty e^{-st}[a f(t) + b\, g(t)]\,dt$

$$= a \int_0^\infty e^{-st} f(t)\,dt + b \int_0^\infty e^{-st} g(t)\,dt$$

$$= a\,L\{f(t)\} + b\,L\{g(t)\}$$

Questions based on the basic definition of Laplace transform

Que (2) Find the Laplace transform of

(a) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ (b) $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$ (c) $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & 2 \le t < \infty \end{cases}$

Sol:- (a) By the definition of Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st}\cdot \cos t\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$$

$$= \left[\frac{e^{-st}}{s^2+1}\left(-s\cos t + \sin t\right)\right]_0^\pi$$

$\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

and $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$= \frac{e^{-s\pi}}{s^2+1}(-s)(-1) - \frac{e^0}{s^2+1}(-s)$$

$$= \frac{s\left(1 + e^{-s\pi}\right)}{s^2+1}$$

(b) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^{\frac{2\pi}{3}} e^{-st}\cdot 0\,dt + \int_{\frac{2\pi}{3}}^\infty e^{-st}\cos\left(t - \frac{2\pi}{3}\right)dt$$

put $t - \frac{2\pi}{3} = x$

$dx = dt$

$$= 0 + \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)}\cos x\,dx$$

$$= e^{-\frac{2s\pi}{3}}\int_0^\infty e^{-sx}\cos x\,dx$$

$$= e^{-\frac{2}{3}s\pi}\int_0^\infty e^{-st}\cos t\,dt$$

(By the property of definite integ

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$$= e^{-\frac{2s\pi}{3}}\, L\{\cos t\}$$

$$= e^{-\frac{2\pi s}{3}}\cdot\frac{s}{s^2+1}$$

(c) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot t^2\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1 + \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2 + \left[t^2 \frac{e^{-st}}{-s}\right]_2^\infty - 2\int_2^\infty t \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \frac{(1-e^{-s})}{s} + \left(\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right) + \frac{4}{s} e^{-2s} + \frac{2}{s}\int_2^\infty t\, e^{-st}\, dt$$

$$= \frac{1}{s} + \frac{2}{s} e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s}\left[\left\{t \frac{e^{-st}}{-s}\right\}_2^\infty - \int_2^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt\right]$$

$$= \frac{1}{s} + \frac{2}{s} e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s}\left[\frac{2}{s} e^{-2s} + \frac{1}{s}\left\{\frac{e^{-st}}{-s}\right\}_2^\infty\right]$$

$$= \frac{1}{s} + \frac{2}{s} \cdot e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{4}{s^2} e^{-2s} + \frac{2}{s^3} e^{-2s}$$

Que (3) Find the Laplace transform of

(a) $t^3 e^{-2t}$ (b) $t^{-1/2} e^t$ (c) $\cos at \sinh at$ (d) $\cos t \cos 2t$ (e) $t \cos t$

Sol :— (a) We know that $L\{t^3\} = \frac{\lfloor 3}{s^4}$

$\therefore L\{e^{-2t} t^3\} = \frac{\lfloor 3}{(s+2)^4}$ (By using I shifting property)

(b) Since $L\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$

$\therefore L\{e^t t^{-1/2}\} = \sqrt{\frac{\pi}{(s-1)}}$

(c) $\cos at \sinh at = \cos at\left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{1}{2}\left[e^{at}\cos at - e^{-at}\cos at\right]$

$\because L\{\cos at\} = \frac{s}{s^2+a^2}$

$\therefore L\{\cos at \sinh at\} = \frac{1}{2}\left[L\{e^{at}\cos at\} - L\{e^{-at}\cos at\}\right]$

$$= \frac{1}{2}\left[\frac{s}{(s-a)^2+a^2} - \frac{s}{(s+a)^2+a^2}\right]$$

(d) $\cos t \; \cos 2t = \frac{1}{2} \cdot 2\cos t \; \cos 2t$

$= \frac{1}{2}\left[\cos 3t + \cos t\right]$

$\therefore L\{\cos t \cos 2t\} = \frac{1}{2}\left[L\{\cos 3t\} + L\{\cos t\}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right] = \frac{s(s^2+5)}{(s^2+1)(s^2+9)}$

(e) $t \cos t = t\left[\frac{e^{it} + e^{-it}}{2}\right]$

Since $L\{t\} = \frac{1}{s^2}$

$\therefore L\{t \cos t\} = \frac{1}{2}\left[L\{t\, e^{it}\} + L\{t\, e^{-it}\}\right]$

$= \frac{1}{2}\left[\frac{1}{(s-i)^2} + \frac{1}{(s+i)^2}\right]$

$= \frac{1}{2}\left[\frac{(s+i)^2 + (s-i)^2}{(s-i)^2(s+i)^2}\right]$

$= \frac{1}{2}\left[\frac{s^2 - 1 + 2is + s^2 - 1 - 2is}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^2}$

3.5 Laplace transform of derivatives :— If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$$

Proof :— By the definition of Laplace transform

$$L\{f'(t)\} = \int_0^\infty \underset{I}{e^{-st}}\, \underset{II}{f'(t)}\, dt$$

$$= \left[e^{-st} \cdot f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s\, L\{f(t)\}$$

or $$L\{f'(t)\} = s\bar{f}(s) - f(0) \qquad \text{——— (1)}$$

Again $$L\{f''(t)\} = \int_0^\infty \underset{I}{e^{-st}}\, \underset{II}{f''(t)}\, dt$$

$$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f'(t)\, dt$$

$$= -f'(0) + s\int_0^\infty e^{-st} f'(t)\, dt$$

$$= s\, L\{f'(t)\} - f'(0) \qquad \text{(from equation (1))}$$

$$= s\left[s\bar{f}(s) - f(0)\right] - f'(0)$$

$$L\{f''(t)\} = s^2 \bar{f}(s) - s f(0) - f'(0) \qquad \text{——— (2)}$$

Similarly $$L\{f'''(t)\} = s^3 \bar{f}(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$L\{f^{iv}(t)\} = s^4 \bar{f}(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

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$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \cdots - f^{n-1}(0)$$

3.6 Laplace transform of integrals — If $L\{f(t)\} = \bar{f}(s)$, then

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{\bar{f}(s)}{s}$$

Proof — Let $g(t) = \int_0^t f(t)\,dt$, then $g'(t) = f(t)$ and $g(0) = 0$

Taking Laplace transform on both sides,

$$L\{g'(t)\} = L\{f(t)\}$$

$$\Rightarrow \quad sL\{g(t)\} - g(0) = \bar{f}(s)$$

$$\Rightarrow \quad s\,L\{g(t)\} - 0 = \bar{f}(s)$$

$$\Rightarrow \quad L\{g(t)\} = \frac{\bar{f}(s)}{s}$$

$$\Rightarrow \quad L\left\{\int_0^t f(t)\,dt\right\} = \frac{\bar{f}(s)}{s}$$

3.7 Laplace transform of multiplication of t^n :— If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$$

Proof :— By definition of Laplace transform

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

differentiating above equation w.r.t. s on both sides, we get

$$\frac{d}{ds}\left[\bar{f}(s)\right] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\,dt$$

(Using Leibnitz's rule i.e. differentation under integral sign)

$$= \int_0^\infty e^{-st}(-t)\, f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\left[t\, f(t)\right] dt$$

$$= -\,L\{t f(t)\}$$

$$\therefore \quad L\{t\, f(t)\} = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

which proves the theorem is true for $n = 1$

Now assume the theorem is true for $n = m$, so that

$$L\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)]$$

$$\Rightarrow (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^m f(t)]\,dt$$

differentating w.r.t. s on both sides, we get

$$(-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)] = \frac{d}{ds}\int_0^\infty e^{-st}[t^m f(t)]\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st})\,[t^m f(t)]\,dt$$

$$= \int_0^\infty e^{-st}(-t)\,[t^m f(t)]\,dt$$

$$= -\int_0^\infty e^{-st}[t^{m+1} f(t)]\,dt$$

$$\Rightarrow (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)] = L\{t^{m+1} f(t)\}$$

This is shows that the theorem is true for $n = m+1$. Since the theorem is true for $n=1$, $n=m$, $n=m+1$, hence by mathematical induction the theorem is true for all possitive integer.

3.8 Laplace transform of division by t :— If $L\{f(t)\} = \bar{f}(s)$ then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

Proof :— We know that $L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

integrating both sides w.r.t. s from s to ∞, we get

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt \quad \text{(By change of order of integration)}$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt = L\left\{\frac{f(t)}{t}\right\}$$

$$\Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds.$$

Questions based on the formula (3.7) i.e. $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

Que (4) :- Find the Laplace transform of

(a) $t \cos at$ (b) $t \sin^2 t$ (c) $t \sin 3t \cos 2t$ (d) $t^2 \sinh at$

(e) $t e^{-t} \sin 2t$ (f) $t e^{-t} \cosh t$ (g) $t e^{-2t} \cos t$ (h) $t^2 e^t \sin 4t$

Sol :- (a) We know that $L\{\cos at\} = \frac{s}{s^2+a^2}$

$$\therefore L\{t \cos at\} = (-1) \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= (-1) \frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(b)

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right] = \frac{2}{s(s^2+4)}$$

$$\therefore L\{t \sin^2 t\} = (-1) \frac{d}{ds}\left[\frac{2}{s(s^2+4)}\right]$$

$$= (-1)\times 2\left[\frac{-(s^2+4)\cdot 1 - s\cdot 2s}{s^2(s^2+4)^2}\right]$$

$$= \frac{2(3s^2+4)}{s^2(s^2+4)^2}$$

(c) $\sin 3t \cos 2t = \frac{1}{2} \cdot 2 \sin 3t \cos 2t = \frac{1}{2}[\sin 5t + \sin t]$

$$\therefore L\{\sin 3t \cos 2t\} = \frac{1}{2}\left[L\{\sin 5t\} + L\{\sin t\}\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$\therefore L\{t \sin 3t \cos 2t\} = (-1) \frac{d}{ds}\left[\frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)\right]$$

$$= -\frac{5}{2} \frac{d}{ds}\left(\frac{1}{s^2+25}\right) - \frac{1}{2} \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{5}{2} \cdot \frac{(-1)\cdot 2s}{(s^2+25)^2} - \frac{1}{2} \cdot \frac{(-1)\cdot 2s}{(s^2+1)^2}$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

(d) We know that $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

$$\therefore L\{t^2 \sinh at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2-a^2}\right)$$

$$= a \frac{d}{ds}\left[\frac{-2s}{(s^2-a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2-a^2)^2\cdot 1 - s\cdot 2(s^2-a^2)\cdot 2s}{(s^2-a^2)^4}\right]$$

$$= \frac{-2a}{(s^2-a^2)^4}\left[s^4+a^4-2a^2s^2-4s^4+4a^2s^2\right]$$

$$= \frac{2a}{(s^2-a^2)^4}\left(3s^4-a^4-2a^2s^2\right)$$

(e) Since $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$$\therefore L\{t \sin 2t\} = (-1)\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{4s}{(s^2+4)^2}$$

$$\therefore L\{e^{-t}\, t \sin 2t\} = \frac{4(s+1)}{\{(s+1)^2+4\}^2}$$

(f) Since $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

$$\therefore L\{t \cosh at\} = (-1)\frac{d}{ds}\left[\frac{s}{s^2-a^2}\right]$$

$$= (-1)\left[\frac{(s^2-a^2)\cdot 1 - s\cdot 2s}{(s^2-a^2)^2}\right] = \frac{s^2+a^2}{(s^2-a^2)^2}$$

$$\therefore L\{e^{-t}\, t \cosh at\} = \frac{(s+1)^2+a^2}{\{(s+1)^2-a^2\}^2}$$

(g) Since $L\{\cos t\} = \dfrac{s}{s^2+1}$

$$\therefore L\{t \cos t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

$$\therefore L\{e^{-2t}\, t \cos t\} = \frac{(s+2)^2-1}{\{(s+2)^2+1\}^2}$$

(h) $\because L\{\sin 4t\} = \dfrac{4}{s^2+16}$

$$\therefore L\{e^t \sin 4t\} = \frac{4}{(s-1)^2+16} = \frac{4}{s^2-2s+17}$$

$$\therefore L\{t^2 e^t \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} \frac{4}{(s^2-2s+17)}$$

$$= 4\frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s^2-2s+17}\right)\right) = 4\frac{d}{ds}\left(\frac{-2s+2}{(s^2-2s+17)^2}\right)$$

$$= 8\frac{d}{ds}\left[\frac{1-s}{(s^2-2s+17)^2}\right]$$

$$= 8\left[\frac{(s^2-2s+17)^2(-1) - (1-s)\cdot 2(s^2-2s+17)(2s-2)}{(s^2-2s+17)^4}\right]$$

$$= 8\left[\frac{-(s^2-2s+17) + 4(s-1)(s-1)}{(s^2-2s+17)^3}\right]$$

$$= \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

Questions based on the formula (3.8) i.e. $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s)\,ds$

Que ⑤ Find the Laplace transform of

(a) $\frac{\sin at}{t}$ (b) $\frac{e^{-at} - e^{-bt}}{t}$ (c) $\frac{\cos 2t - \cos 3t}{t}$

(d) $\frac{e^{at} - \cos bt}{t}$ (e) $\frac{1-\cos 2t}{t}$ (f) $\frac{e^{-t}\sin t}{t}$ (g) $\frac{1-\cos t}{t^2}$

Sol:— (a) Since $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\frac{s}{a}\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

(b) $L\{e^{-at} - e^{-bt}\} = L\{e^{-at}\} - L\{e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^{\infty} = \left[\log\frac{s+a}{s+b}\right]_s^{\infty}$$

$$= \left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^{\infty} = \log 1 - \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{s+b}{s+a}\right)$$

(c) $L\{\cos 2t - \cos 3t\} = \frac{s}{s^2+4} - \frac{s}{s^2+9}$

$$\therefore L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^{\infty}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^{\infty} = \frac{1}{2}\left[\log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right)\right] = \frac{1}{2}\log\frac{s^2+9}{s^2+4}$$

(d) $L\left\{e^{at} - \cos bt\right\} = \frac{1}{s-a} - \frac{s}{s^2+b^2}$

$$\therefore L\left\{\frac{e^{at} - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{(s-a)^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\left[\log\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{(s-a)^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$$

(e) $L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

$$\therefore L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{4}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{s^2}{s^2+4}\right]$$

$$= \frac{1}{2}\log\frac{s^2+4}{s^2}$$

(f) Since $L\{\sin t\} = \frac{1}{s^2+1}$

$$\therefore L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$\therefore L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty \frac{1}{(s+1)^2+1}\, ds = \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(g) $L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$

$$\therefore L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2}{s^2+1}\right] = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

$$\therefore L\left\{\frac{1-\cos t}{t^2}\right\} = \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left[\left(\log(s^2+1) - 2\log s\right) \cdot 1\, ds\right.$$

$$= \frac{1}{2}\left[\{\log(s^2+1) - 2\log s\}\cdot s - \int\left(\frac{2s}{s^2+1} - \frac{2}{s}\right)\cdot s\, ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log\frac{s^2+1}{s^2}\right]_s^\infty - \int_s^\infty \left(\frac{s^2}{s^2+1} - 1\right) ds$$

$$= \frac{1}{2}\left[s\log\left(1+\frac{1}{s^2}\right)\right]_s^\infty + \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= 0 - \frac{1}{2}\cdot s\log\left(1+\frac{1}{s^2}\right) + \left[\tan^{-1}s\right]_s^\infty$$

$$= -\frac{s}{2}\log\left(1+\frac{1}{s^2}\right) + \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s - \frac{s}{2}\log\left(1+\frac{1}{s^2}\right)$$

Questions based on the formula (3.6) i.e. $L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

Que ⑥ Find the Laplace transform of

(a) $\int_0^t \frac{\sin t}{t}\,dt$ (b) $\int_0^t \frac{e^{-t}\sin t}{t}\,dt$ (c) $\int_0^t e^{t}\,\frac{\sin^3 t}{t}\,dt$

(d) $\int_0^t e^{-2t}\,t\,\sin^3 t\,dt$ (e) $\int_0^t e^{-t}\cos t\,dt$

Sol — (a) Since $L\{\sin t\} = \frac{1}{s^2+1}$

$$\therefore\ L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty = \cot^{-1} s$$

$$\therefore\ L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{\cot^{-1} s}{s}$$

(b) Since $L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s$

$$\therefore\ L\left\{e^{-t}\,\frac{\sin t}{t}\right\} = \cot^{-1}(s+1)$$

$$\therefore\ L\left\{\int_0^t e^{-t}\,\frac{\sin t}{t}\,dt\right\} = \frac{\cot^{-1}(s+1)}{s}$$

(c) $\sin 3t = 3\sin t - 4\sin^3 t$ or $\sin^3 t = \frac{1}{4}(3\sin t - \sin 3t)$

$$\therefore\ L\{\sin^3 t\} = \frac{3}{4}L\{\sin t\} - \frac{1}{4}L\{\sin 3t\}$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$$\therefore\ L\left\{\frac{\sin^3 t}{t}\right\} = \int_s^\infty \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)ds$$

$$= \frac{3}{4}\left[\tan^{-1} s - \frac{1}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{3}{4}\left[\frac{\pi}{2} - \tan^{-1} s - \frac{1}{3}\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)\right]$$

$$= \frac{3}{4}\left[\cot^{-1} s - \frac{1}{3}\cot^{-1}\frac{s}{3}\right] = \frac{1}{4}\left(3\cot^{-1} s - \cot^{-1}\frac{s}{3}\right)$$

$$\therefore\ L\left\{e^{t}\,\frac{\sin^3 t}{t}\right\} = \frac{1}{4}\left[3\cot^{-1}(s-1) - \cot^{-1}\frac{(s-1)}{3}\right]$$

$$\therefore\ L\left\{\int_0^t e^{t}\,\frac{\sin^3 t}{t}\,dt\right\} = \frac{\left[3\cot^{-1}(s-1) - \cot^{-1}\left(\frac{s-1}{3}\right)\right]}{4s}$$

(d) Since $L\{\sin^3 t\} = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$

$$\therefore \; L\{t \sin^3 t\} = (-1)\frac{d}{ds}\left[\frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)\right]$$

$$= \frac{(-3)}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

$$\therefore \; L\{e^{-2t} t \sin^3 t\} = \frac{3(s+2)}{2}\left[\frac{1}{\{(s+2)^2+1\}^2} - \frac{1}{\{(s+2)^2+9\}^2}\right]$$

$$\therefore \; L\left\{\int_0^t e^{-2t} t \sin^3 t \, dt\right\} = \frac{3(s+2)}{2s}\left[\frac{1}{\{(s+2)^2+1\}^2} - \frac{1}{\{(s+2)^2+9\}^2}\right]$$

(e) Since $L\{\cos t\} = \frac{s}{s^2+1}$

$$\therefore \; L\{e^{-t} \cos t\} = \frac{s+1}{(s+1)^2+1}$$

$$\therefore \; L\left\{\int_0^t e^{-t} \cos t \, dt\right\} = \frac{(s+1)}{(s+1)^2+1} \Big/ s$$

Que ⑦ Evaluate the following integrals using Laplace transform

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(a) $\int_0^{\infty} e^{-2t}\, t \sin t \, dt$ (b) $\int_0^{\infty} e^{t} \frac{\sin^2 t}{t} \, dt$

(c) $\int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} \, dt$ (d) $\int_0^{\infty} e^{-3t}\, t^3 \cos t \, dt$

Sol (a) — $L\{\sin t\} = \frac{1}{s^2+1}$

$$L\{t \sin t\} = (-1) \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$\therefore \int_0^{\infty} e^{-2t}\, t \sin t \, dt = L\{t \sin t\}_{\text{at } s=2}$$

By the definition $\int_0^{\infty} e^{-st} f(t)\,dt = L\{f$

$$= \left[\frac{2s}{(s^2+1)^2}\right]_{\text{at } s=2} = \frac{4}{25}$$

(b) $$L\{\sin^2 t\} = \frac{1}{2} L\{1 - \cos 2t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_s^{\infty}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right.$$

$$= 0 - \frac{1}{2}\log\left(\frac{s^2}{s^2+4}\right) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

By the definition

$$\therefore \int_0^{\infty} e^{t} \frac{\sin^2 t}{t} \, dt = L\left\{\frac{\sin^2 t}{t}\right\}_{\text{at } s=-1}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2}\right)\right]_{\text{at } s=-1} = \frac{1}{2}\log 5$$

(c) $$L\{e^{-t} - e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\therefore L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds = \left[\log(s+1) - \log(s+3)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s+1}{s+3}\right)\right]_s^{\infty} = \left[\log \frac{1+\frac{1}{s}}{1+\frac{3}{s}}\right]_s^{\infty} = \log 1 - \log \frac{s+1}{s+3} = \log($$

$\therefore$ By the definition

$$\int_0^{\infty} e^{-0t}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\}_{\text{at } s=0} = \left[\log\left(\frac{s+3}{s+1}\right)\right]_{\text{at } s=0}$$

$$= \log 3$$

(d) $L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{t^3\cos t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$$

$$= -\frac{d^2}{ds^2}\left[\frac{(s^2+1)\cdot 1 - s\cdot 2s}{(s^2+1)^2}\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{(s^2+1)^2(-2s) - (1-s^2)\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\right] = -2\frac{d}{ds}\left[\frac{s^3-3s}{(s^2+1)^3}\right]$$

$$= -2\left[\frac{(s^2+1)^3(3s^2-3) - (s^3-3s)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}\right]$$

$$= \frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

$\therefore$ By definition

$$\int_0^\infty e^{-3t}\, t^3\cos t\, dt = L\{t^3\cos t\}_{\text{at } s=3}$$

$$= \left[\frac{6(s^4-6s^2+1)}{(s^2+1)^4}\right]_{\text{at } s=3}$$

$$= \frac{21}{1250}$$

Que (8) Find the Laplace transform of (a) $\sin\sqrt{t}$ (b) $\frac{\cos\sqrt{t}}{\sqrt{t}}$

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Sol — we know that $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$

$$\therefore \sin t^{1/2} = t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \frac{t^{7/2}}{\lfloor 7} + \cdots$$

$$\therefore L\{\sin t^{1/2}\} = L\{t^{1/2}\} - \frac{1}{\lfloor 3} L\{t^{3/2}\} + \frac{1}{\lfloor 5} L\{t^{5/2}\} - \frac{1}{\lfloor 7} L\{t^{7/2}\} + \cdots$$

$$= \frac{\lceil 3/2}{s^{3/2}} - \frac{1}{\lfloor 3} \cdot \frac{\lceil 5/2}{s^{5/2}} + \frac{1}{\lfloor 5} \frac{\lceil 7/2}{s^{7/2}} - \frac{1}{\lfloor 7} \cdot \frac{\lceil 9/2}{s^{9/2}} + \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{\lfloor 3} \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{\lfloor 5} \cdot \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{1}{\lfloor 7} \cdot \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{9/2}} +$$

$$= \frac{\sqrt{\pi}}{2 s^{3/2}} \left[1 - \frac{1}{2^2 s} + \frac{1}{\lfloor 2} \frac{1}{(2^2 s)^2} - \frac{1}{\lfloor 3 (2^2 s)^3} + \cdots \right]$$

$$= \frac{1}{2s}\sqrt{\frac{\pi}{s}} \cdot e^{-\frac{1}{2^2 s}}$$

$$= \frac{1}{2s}\sqrt{\frac{\pi}{s}} \cdot e^{-\frac{1}{4s}}$$

(b) Let $f(t) = \sin\sqrt{t}$ $\quad \therefore L\{f(t)\} = L\{\sin\sqrt{t}\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}} \cdot e^{-\frac{1}{4s}} = \bar{f}(s)$

$\therefore f(0) = 0$

Now $f'(t) = \frac{d}{dt}\sin\sqrt{t} = \frac{1}{2t^{1/2}}\cos t^{1/2}$

we know that

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$\Rightarrow L\left\{\frac{\cos t^{1/2}}{2t^{1/2}}\right\} = s \cdot \frac{1}{2s}\sqrt{\frac{\pi}{s}} \cdot e^{-\frac{1}{4s}} - 0$$

$$\Rightarrow L\left\{\frac{\cos t^{1/2}}{t^{1/2}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

3.9 Laplace transform of some special functions —

(a) Periodic function — If $f(t)$ is a periodic function with period T, i.e.

$$f(t) = f(t+T) = f(t+2T) = f(t+3T) \; ---$$

then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Proof :— We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

Putting $t = x$ in first integral, $t = x+T$ in second integral, $t = x+2T$ in third integral and so on, we get

$$L\{f(t)\} = \int_0^T e^{-sx} f(x)\,dx + \int_0^T e^{-s(x+T)} f(x+T)\,dx + \int_0^T e^{-s(x+2T)} f(x+2T)\,dx + \cdots$$

$$= \int_0^T e^{-sx} f(x)\,dx + \int_0^T e^{-s(x+T)} f(x)\,dx + \int_0^T e^{-s(x+2T)} f(x)\,dx + \cdots$$

$$\left(\text{Since } f(x) = f(x+T) = f(x+2T) = \cdots\right)$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right) \int_0^T e^{-sx} f(x)\,dx$$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Questions based on the periodic function :-

Que ⑨ :- Find the Laplace transform of

(a) The square wave function of period a defined as

$$f(t) = \begin{cases} 1 & \text{when } 0 < t < a/2 \\ -1 & \text{when } a/2 < t < a \end{cases}$$

(b) The triangular wave of period 2a given by

$$f(t) = \begin{cases} t & , 0 < t < a \\ 2a - t & , a < t < 2a \end{cases}$$

(c) The function $f(t) = \begin{cases} \sin \omega t & , 0 < t < \frac{\pi}{\omega} \\ 0 & , \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$
(half wave rectifire)

(d) The function $f(t) = E \sin \omega t$, $0 < t < \frac{\pi}{\omega}$,
(full wave rectifire)

(e) The function $f(t) = \frac{t}{T}$, $0 < t < T$, $f(t+T) = f(t)$
(Saw tooth wave)

Sol (a) We know that the Laplace transform of a periodic function is

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt \qquad (\text{Since time period } T = a)$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} \cdot 1\, dt + \int_{a/2}^{a} e^{-st} (-1)\, dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} + \left(\frac{e^{-st}}{s} \right)_{a/2}^{a} \right]$$

$$= \frac{1}{1-e^{-as}} \left[\frac{\left(e^{-\frac{as}{2}} - 1\right)}{-s} + \frac{\left(e^{-as} - e^{-\frac{as}{2}}\right)}{s} \right]$$

$$= \frac{1}{s\left(1-e^{-as}\right)} \left(1 - 2e^{-\frac{as}{2}} + e^{-as}\right) = \frac{\left(1 - e^{-\frac{as}{2}}\right)^2}{s\left(1 - e^{-\frac{as}{2}}\right)\left(1 + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s} \left(\frac{1 - e^{-\frac{as}{2}}}{1 + e^{-\frac{as}{2}}} \right) = \frac{1}{s} \cdot \frac{e^{-\frac{as}{4}}}{e^{-\frac{as}{4}}} \left(\frac{e^{\frac{as}{4}} - e^{-\frac{as}{4}}}{e^{\frac{as}{4}} + e^{-\frac{as}{4}}} \right)$$

$$= \frac{1}{s} \tanh\left(\frac{as}{4}\right)$$

(25)

(b) given that $T = 2a$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} \cdot f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a \underset{II}{e^{-st}} \cdot \underset{I}{t}\,dt + \int_a^{2a} \underset{II}{e^{-st}} \underset{I}{(2a-t)}\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2} \right)_0^a + \left((2a-t)\,\frac{e^{-st}}{-s} + \frac{e^{-st}}{(-s)^2} \right)_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{a e^{-as}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{a e^{-as}}{(-s)} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right] = \frac{(1-e^{-as})^2}{s^2(1-e^{-2as})}$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})}{(1+e^{-as})} = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

(c) The time period $T = \frac{2\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \cdot \sin\omega t\,dt + \int_{\frac{\pi}{\omega}}^{2\pi/\omega} e^{-st} \cdot 0\,dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} \left(-s\sin\omega t - \omega\cos\omega t \right) \right]_0^{\pi/\omega}$$

$\left(\text{Since } \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \right.$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\frac{\pi s}{\omega}}}{s^2+\omega^2} (-\omega)(-1) + \frac{1}{s^2+\omega^2} (\omega) \right]$$

$$= \frac{\omega}{s^2+\omega^2} \left(\frac{1 + e^{-\frac{\pi s}{\omega}}}{1 - e^{-\frac{2\pi s}{\omega}}} \right)$$

$$= \frac{\omega}{s^2+\omega^2} \left(\frac{1}{1 - e^{-\frac{\pi s}{\omega}}} \right)$$

(d) The time period $T = \frac{\pi}{\omega}$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} \cdot E \sin \omega t \, dt$$

$$= \frac{E}{1-e^{-\frac{\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} \left(-s \sin \omega t - \omega \cos \omega t\right) \right]_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-\frac{\pi s}{\omega}}} \left[\frac{e^{-\frac{\pi s}{\omega}}}{s^2+\omega^2}(\omega) - \frac{e^0}{s^2+\omega^2}(-\omega) \right]$$

$$= \frac{E\omega}{s^2+\omega^2} \frac{\left(1+e^{-\frac{\pi s}{\omega}}\right)}{\left(1-e^{-\frac{\pi s}{\omega}}\right)}$$

$$= \frac{E\omega}{s^2+\omega^2} \cdot \frac{e^{-\frac{\pi s}{2\omega}}\left(e^{\frac{\pi s}{2\omega}} + e^{-\frac{\pi s}{2\omega}}\right)}{e^{-\frac{\pi s}{2\omega}}\left(e^{\frac{\pi s}{2\omega}} - e^{-\frac{\pi s}{2\omega}}\right)}$$

$$= \frac{E\omega}{s^2+\omega^2} \cdot \coth\left(\frac{\pi s}{2\omega}\right)$$

(e) given that time period $= T$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot \frac{t}{T}\, dt$$

$$= \frac{1}{T(1-e^{-sT})} \int_0^T t\, e^{-st}\, dt$$

$$= \frac{1}{T(1-e^{-sT})} \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2} \right]_0^T$$

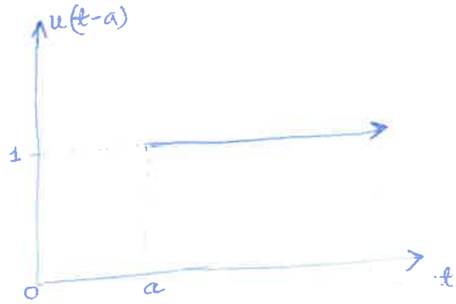
$$= \frac{1}{T(1-e^{-sT})} \left[\frac{T e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{1}{T(1-e^{-sT})} \left[\frac{(1-e^{-sT})}{s^2} - \frac{T e^{-sT}}{s} \right]$$

$$= \frac{1}{s^2 T} - \frac{e^{-sT}}{s(1-e^{-sT})}$$

(b) Unit Step Function (or Heaviside's Unit Step Function) :-

Definition :- The unit step function $U(t-a)$ is defined as



$$U(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

where a is always positive.

Laplace transform of unit step function :-

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$$

$$\therefore L\{U(t-a)\} = \frac{e^{-as}}{s}$$

Second shifting theorem :- If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{f(t-a)\, u(t-a)\} = e^{-as}\, \bar{f}(s)$$

Proof :-

$$L\{f(t-a)\, U(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, U(t-a)\, dt$$

$$= \int_0^a e^{-st} f(t-a) \cdot 0\, dt + \int_a^\infty e^{-st} f(t-a) \cdot 1\, dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\, dt$$

put $t - a = x$

$\therefore dt = dx$

$$= \int_0^\infty e^{-s(a+x)} f(x)\, dx$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)\, dx$$

$$L\{f(t-a)\, U(t-a)\} = e^{-as}\, \bar{f}(s)$$

Cor : $L\{f(t)\, U(t-a)\} = e^{-as}\, L\{f(t+a)\}$

Proof :-

$$L\{f(t)\, U(t-a)\} = \int_0^\infty e^{-st} f(t)\, U(t-a)\, dt$$

$$= \int_0^a e^{-st} f(t) \cdot 0\, dt + \int_a^\infty e^{-st} f(t) \cdot 1\, dt$$

$$= 0 + \int_a^\infty e^{-st} f(t)\, dt$$

put $t - a = x$

$\therefore dt = dx$

$$= \int_0^\infty e^{-s(a+x)} f(a+x)\, dx$$

$$= e^{-as} \int_0^\infty e^{-sx} f(a+x)\, dx = e^{-as}\, L\{f(t+a)\}$$

Questions based on the unit step function :-

Que ⑩ – Find the Laplace transform of

(a) $f(t) = k(t-2)[U(t-2) - U(t-3)]$ (b) $t\, U(t-2)$ or $t\, U_2(t)$

(c) $e^{-2t} U_\pi(t)$ or $e^{-2t} U(t-\pi)$ (d) $t^2 U(t-3)$

(e) $(t-1)^2 U(t-1)$ (f) $e^{-t}[1 - U(t-2)]$

Sol —(a) $f(t) = k[(t-2)U(t-2) - (t-3+1)U(t-3)]$

$= k[(t-2)U(t-2) - (t-3)U(t-3) - U(t-3)]$

$\therefore L\{f(t)\} = k[L\{(t-2)U(t-2)\} - L\{(t-3)U(t-3)\} - L\{U(t-3)\}]$

$= k[e^{-2s}L\{t\} - e^{-3s}L\{t\} - \frac{e^{-3s}}{s}]$

{ Since $L\{f(t-a)U(t-a)\} = e^{-as}L\{f(t)\}$
and $L\{U(t-a)\} = \frac{e^{-as}}{s}$

$= k\left[\frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}\right]$

(b) $L\{t\, U(t-2)\} = e^{-2s} L\{(t+2)\}$ (Since $L\{f(t)U(t-a)\} = e^{-as}L\{f(t+a)\}$
(here $a = 2$ and $f(t) = t$

$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$

(c) $L\{e^{-2t}U(t-\pi)\} = e^{-\pi s}L\{e^{-2(t+\pi)}\}$, here $a = \pi$ and $f(t) = e^{-2t}$

$= e^{-\pi s} L\{e^{-2t}\} \cdot e^{-2\pi}$

$= e^{-\pi(s+2)}\left(\frac{1}{s+2}\right)$

(d) $L\{t^2 U(t-3)\} = e^{-3s}L\{(t+3)^2\}$, here $a = 3$ and $f(t) = t^2$

$= e^{-3s}L\{t^2 + 6t + 9\}$

$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$

(e) $L\{(t-1)^2 U(t-1)\} = e^{-s} L\{t^2\}$, here $a=1$, $f(t)=t^2$ and using formula $L\{f(t-a)U(t-a)\} = e^{-as}L\{f(t)\}$

$$= e^{-s} \cdot \frac{2}{s^3} = \frac{2e^{-s}}{s^3}$$

(f) $L\{e^{-t}[1-U(t-2)]\} = L\{e^{-t}\} - L\{e^{-t}U(t-2)\}$

$$= \frac{1}{s+1} - e^{-2s} L\{e^{-(t+2)}\}$$

$$= \frac{1}{s+1} - e^{-2s} \cdot e^{-2} L\{e^{-t}\}$$

$$= \frac{1 - e^{-2(s+1)}}{s+1}$$

Que (11) – Express the following functions interms of unit step function and find its Laplace transform –

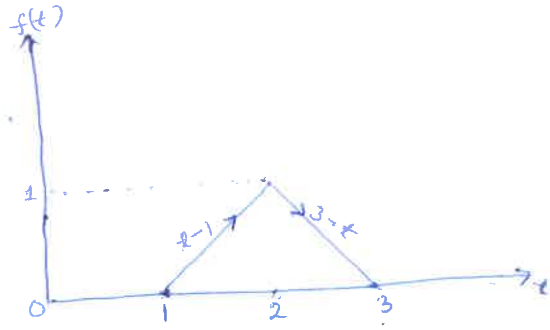
(a) $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$ (b) $f(t) = \begin{cases} t, & 0<t<\pi \\ \pi-t, & \pi<t<2\pi \end{cases}$

(c) $f(t) = \begin{cases} t^2, & 0<t<2 \\ 4t, & t>2 \end{cases}$ (d) $f(t) = \begin{cases} 2, & 0<t<\pi \\ 0, & \pi<t<2\pi \\ \sin t, & t>2\pi \end{cases}$

Sol (a) Since $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$

hence $f(t)$ can be written as in terms of unit step function ie.

$$f(t) = (t-1)[U(t-1) - U(t-2)] + (3-t)[U(t-2) - U(t-3)]$$



$$= (t-1)U(t-1) - (t-2+1)U(t-2) - (t-2-1)U(t-2) + (t-3)U(t-3)$$

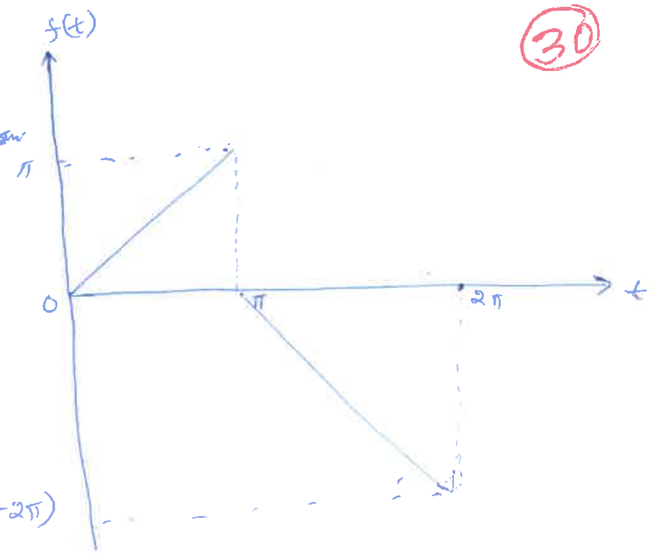
$$= (t-1)U(t-1) - (t-2)U(t-2) \cancel{-U(t-2)} - (t-2)U(t-2) \cancel{+U(t-2)} + (t-3)U(t-3)$$

$$f(t) = (t-1)U(t-1) - 2(t-2)U(t-2) + 3(t-3)U(t-3)$$

$$L\{f(t)\} = e^{-s}L\{t\} - 2e^{-2s}L\{t\} + e^{-3s}L\{t\}$$

$$= \frac{(e^{-s} - 2e^{-2s} + e^{-3s})}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2}$$

(b) Since $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$



hence $f(t)$ can be written as interms of Unit step function ie

$$f(t) = t\left[U(t-0) - U(t-\pi)\right] + (\pi - t)\left[U(t-\pi) - U(t-2\pi)\right]$$

$$= t\,U(t) - (t-\pi+\pi)U(t-\pi) - (t-\pi)\,U(t-\pi) + (t-2\pi+\pi)\,U(t-2\pi)$$

$$= t\,U(t) - (t-\pi)\,U(t-\pi) - \pi\,U(t-\pi) - (t-\pi)\,U(t-\pi) + (t-2\pi)\,U(t-2\pi) + \pi\,U(t-2\pi)$$

$$= (t-0)\,U(t-0) - 2(t-\pi)\,U(t-\pi) + (t-2\pi)\,U(t-2\pi) + \pi\left[U(t-2\pi) - U(t-\pi)\right]$$

$$\therefore L\{f(t)\} = e^{0s}L\{t\} - 2e^{-\pi s}L\{t\} + e^{-2\pi s}L\{t\} + \pi\left[\frac{e^{-2\pi s}}{s} - \frac{e^{-\pi s}}{s}\right]$$

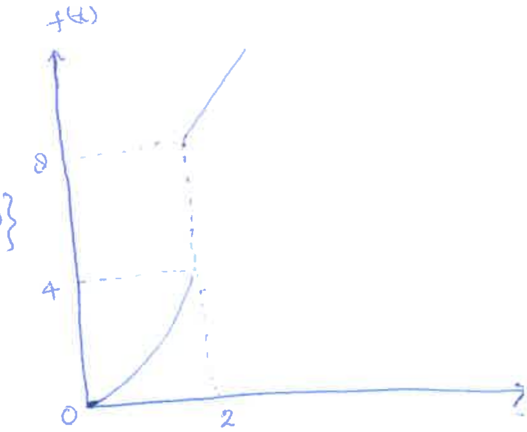
$$= \frac{1}{s^2} - \frac{2e^{-\pi s}}{s^2} + \frac{e^{-2\pi s}}{s^2} + \pi\left[\frac{e^{-2\pi s}}{s} - \frac{e^{-\pi s}}{s}\right]$$

$$= \frac{(1-e^{-\pi s})^2}{s^2} + \frac{\pi}{s}e^{-\pi s}\left[e^{-\pi s} - 1\right]$$

(c) $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4t & t > 2 \end{cases}$

$\therefore f(t)$ can be written as in terms of unit step function ie

$$f(t) = t^2\left[U(t-0) - U(t-2)\right] + 4t\,U(t-2)$$



$$\therefore L\{f(t)\} = L\{t^2 U(t)\} - L\{t^2 U(t-2)\} + 4\{t\,U(t-2)\}$$

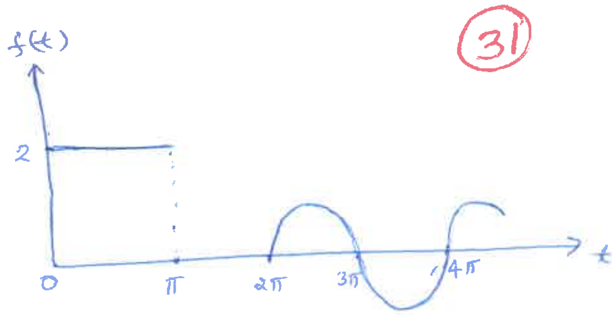
$$= e^{0s}L\{t^2\} - e^{-2s}L\{(t+2)^2\} + 4e^{-2s}L\{(t+2)\}$$

$\left(\text{Since } L\{f(t)\,U(t-a)\} = e^{-as}L\{f(t+a)\}\right)$

$$= \frac{2}{s^3} - e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] + 4e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

$$= \frac{2}{s^3} + e^{-2s}\left[\frac{4}{s} - \frac{2}{s^3}\right]$$

(d) $f(t)$ can be written as in terms of unit step function i.e.



$$f(t) = 2[U(t-0) - U(t-\pi)] + \sin t \; U(t-2\pi)$$

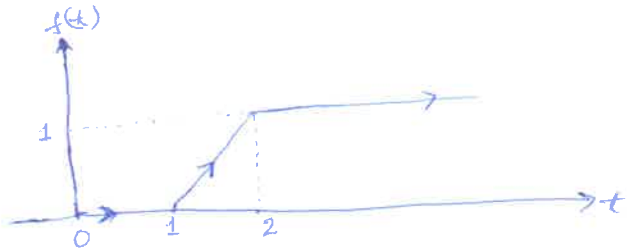
$$\therefore L\{f(t)\} = 2[L\{U(t)\} - L\{U(t-\pi)\}] + L\{\sin t \; U(t-2\pi)\}$$

$$= \frac{2}{s} - \frac{2e^{-\pi s}}{s} + e^{-2\pi s} L\{\sin(t+2\pi)\}$$

$$= \frac{2}{s}(1 - e^{-\pi s}) + e^{-2\pi s} L\{\sin t\}$$

$$= \frac{2}{s}(1 - e^{-\pi s}) + e^{-2\pi s} \frac{1}{(1+s^2)}$$

Que (12) – Express the following function in terms of unit step function and find its Laplace transform.



Sol :– The above figure can be written as

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ t-1 & 1 < t < 2 \\ 1 & t > 2 \end{cases}$$

hence in terms of unit step function

$$f(t) = (t-1)[U(t-1) - U(t-2)] + U(t-2)$$

$$= (t-1)\,U(t-1) - (t-2+1)\,U(t-2) + U(t-2)$$

$$= (t-1)\,U(t-1) - (t-2)\,U(t-2)$$

$$\therefore L\{f(t)\} = L\{(t-1)\,U(t-1)\} - L\{(t-2)\,U(t-2)\}$$

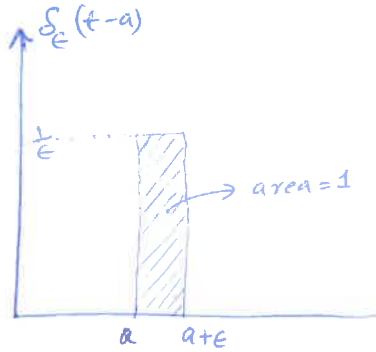
$$= \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} \quad \text{(By second shifting theorem)}$$

(c) Impulse function or Dirac-delta function :—

Definition :— When a very large force acts for a very small time, then the product of force and time is called impul[se].

Unit impulse function is denoted by $\delta_\epsilon(t-a)$ and is defined as

$$\delta_\epsilon(t-a) = \begin{cases} \frac{1}{\epsilon} & \text{for } a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$



Laplace transform of unit impulse function —

$$L\{\delta_\epsilon(t-a)\} = \int_0^\infty e^{-st}\,\delta_\epsilon(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^{a+\epsilon} e^{-st}\cdot\frac{1}{\epsilon}\,dt + \int_{a+\epsilon}^\infty e^{-st}\cdot 0\,dt$$

$$= \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,dt = \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon}$$

$$= \frac{1}{s\epsilon}\left[e^{-as} - e^{-(a+\epsilon)s}\right]$$

$$= \frac{e^{-as}}{s\epsilon}\left[1 - e^{-s\epsilon}\right]$$

$$= \frac{e^{-as}}{s\epsilon}\left[\cancel{1} - \left(\cancel{1} - s\epsilon + \frac{(s\epsilon)^2}{\lfloor 2} - \frac{(s\epsilon)^3}{\lfloor 3} + \cdots\right)\right]$$

$$= e^{-as}\left[1 - \frac{s\epsilon}{\lfloor 2} + \frac{(s\epsilon)^2}{\lfloor 3} - \cdots\right]$$

As $\epsilon \to 0$ we get

$$L\{\delta(t-a)\} = e^{-as}$$

In particular case of $a=0$ then

$$L\{\delta(t)\} = e^0 = 1$$

Cor :— If $f(t)$ be a continuous at $t=a$ then

$$\int_0^\infty f(t)\,\delta_\epsilon(t-a)\,dt = \int_a^{a+\epsilon} f(t)\cdot\frac{1}{\epsilon}\,dt$$

or $\int_0^\infty f(t)\,\delta_\epsilon(t-a)\,dt = (a+\epsilon-a)\cdot f(c)\cdot$

$= f(c)$

(where $a < c < a+\epsilon$)

(and mean value theore[m] $\int_a^b f(t)\,dt = (b-a)\,f(c)$)

As $\epsilon \to 0$ we get

$$\boxed{\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)}$$

Que (13) :— Find the Laplace transform of

(a) $t^3\,\delta(t-3)$ (b) $e^{-4t}\,\delta(t-3)$ (c) $\frac{\delta(t-4)}{t}$

Sol :— (a) $L\{t^3\,\delta(t-3)\} = \int_0^\infty e^{-st}\cdot t^3\,\delta(t-3)\,dt = \left[e^{-st}t^3\right]_{at\ t=3}$

$= (3)^3 e^{-3s}$ $\left(\text{Since } \int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)\right.$
here $\left. f(t) = e^{-st}t^3 \right)$

(b) $L\{e^{-4t}\,\delta(t-3)\} = \int_0^\infty e^{-st}\cdot e^{-4t}\,\delta(t-3)\,dt$

$= \int_0^\infty e^{-(s+4)t}\,\delta(t-3)\,dt$

$= \left[e^{-(s+4)t}\right]_{at\ t=3}$ $\left(\text{here } f(t) = e^{-(s+4)t}\right)$

$= e^{-3(s+4)}$

(c) $L\left\{\frac{\delta(t-4)}{t}\right\} = \int_0^\infty e^{-st}\cdot\frac{\delta(t-4)}{t}\,dt$

$= \left[\frac{e^{-st}}{t}\right]_{at\ t=4}$ here $f(t) = \frac{e^{-st}}{t}$

$= \frac{e^{-4s}}{4}$

Que (14) :— Evaluate the following integrals —

(a) $\int_0^\infty e^{-3t}\,\delta(t-4)\,dt$ (b) $\int_0^\infty \sin 2t\,\delta\left(t-\frac{\pi}{4}\right)dt$

Sol — (a) :— $\int_0^\infty e^{-3t}\,\delta(t-4)\,dt = L\{\delta(t-4)\}_{at\ s=3}$ (By definition of Laplace transform)

$= \left[e^{-4s}\right]_{at\ s=3}$

$= e^{-12}$

(b) $\int_0^\infty \sin 2t\,\delta\left(t-\frac{\pi}{4}\right)dt$

$= \int_0^\infty \left(\frac{e^{2it}-e^{-2it}}{2i}\right)\delta\left(t-\frac{\pi}{4}\right)dt = \int_0^\infty \frac{1}{2i}e^{2it}\,\delta\left(t-\frac{\pi}{4}\right)dt - \int_0^\infty \frac{e^{-2it}}{2i}\,\delta\left(t-\frac{\pi}{4}\right)$

$= \frac{1}{2i}\,L\left\{\delta\left(t-\frac{\pi}{4}\right)\right\}_{at\ s=-2i} - \frac{1}{2i}\,L\left\{\delta\left(t-\frac{\pi}{4}\right)\right\}_{at\ s=2i}$

$= \frac{1}{2i}\left[e^{-\frac{\pi}{4}s}\right]_{at\ s=-2i} - \frac{1}{2i}\left[e^{-\frac{\pi}{4}s}\right]_{at\ s=2i} = \frac{e^{\frac{\pi}{2}i}-e^{-\frac{\pi}{2}i}}{2i}$

$= \sin\frac{\pi}{2} = 1$

(d) Error function :-

The error function is denoted by $\operatorname{erf}\sqrt{t}$ and defined as

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - x^2 + \frac{x^4}{\lfloor 2} - \frac{x^6}{\lfloor 3} + \cdots\right)dx$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\cdot\lfloor 2} - \frac{t^{7/2}}{7\cdot\lfloor 3} + \cdots\right]$$

Now $$L\{\operatorname{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3\,s^{5/2}} + \frac{\Gamma 7/2}{5\cdot\lfloor 2\, s^{7/2}} - \frac{\Gamma 9/2}{7\cdot\lfloor 3\, s^{9/2}} + \cdots\right]$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2\cdot s^{5/2}} + \frac{1\cdot 3}{2\cdot 4\, s^{7/2}} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\, s^{9/2}} + \cdots$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{s^2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{s^3} + \cdots\right]$$

$$= \frac{1}{s^{3/2}}\left[1 + \frac{1}{s}\right]^{-1/2}$$

$$= \frac{1}{s^{3/2}}\left[\frac{s+1}{s}\right]^{-1/2} = \frac{1}{s^{3/2}}\cdot\frac{s^{1/2}}{(s+1)^{1/2}}$$

$$\therefore\ L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$$

(e) Complementary error function — The complementary error function is denoted as $\operatorname{erf}_c\sqrt{t}$ and defined as

$$\operatorname{erf}_c\sqrt{t} = 1 - \operatorname{erf}\sqrt{t}$$

Now $$L\{\operatorname{erf}_c\sqrt{t}\} = L\{1 - \operatorname{erf}\sqrt{t}\}$$

$$= L\{1\} - L\{\operatorname{erf}\sqrt{t}\}$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$

$$\therefore\ L\{\operatorname{erf}_c\sqrt{t}\} = \frac{\sqrt{s+1} - 1}{s\sqrt{s+1}}$$

(f) The Bessel function :—

The Bessel function of order n is given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \; \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

putting $n=0$, then the Bessel function of order zero is

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \; \Gamma(r+1)} \left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \cdot \lfloor r} \left(\frac{t}{2}\right)^{2r}$$

(Since $\Gamma(r+1) = \lfloor r$

$$= 1 - \frac{1}{(\lfloor 1)^2}\left(\frac{t}{2}\right)^2 + \frac{1}{(\lfloor 2)^2}\left(\frac{t}{2}\right)^4 - \frac{1}{(\lfloor 3)^2}\left(\frac{t}{2}\right)^6 + \cdots$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Now

$$L\{J_0(t)\} = L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right\}$$

$$= \frac{1}{s} - \frac{1}{2^2} \cdot \frac{\lfloor 2}{s^3} + \frac{1}{2^2 \cdot 4^2} \cdot \frac{\lfloor 4}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{\lfloor 6}{s^6} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{s^4}\right) - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{s^6}\right) + \cdots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2}$$

$$= \frac{1}{\sqrt{1+s^2}}$$

Questions based on error function.

Que 15 — Find the Laplace transform of

(a) $e^{3t}\,\text{erf}\sqrt{t}$ (b) $t\cdot\text{erf}\sqrt{t}$ (c) $t\cdot\text{erf}(2\sqrt{t})$

(d) $\int_0^t t\cdot\text{erf}(2\sqrt{t})\,dt$ (e) $\int_0^t \text{erfc}\sqrt{t}\,dt$

Sol(a):- We know that $L\{\text{erf}\sqrt{t}\} = \dfrac{1}{s\sqrt{s+1}}$

$$\therefore L\{e^{3t}\,\text{erf}\sqrt{t}\} = \frac{1}{(s-3)\sqrt{(s-3)+1}} = \frac{1}{(s-3)\sqrt{s-2}}$$

(b) Since $L\{\text{erf}\sqrt{t}\} = \dfrac{1}{s\sqrt{s+1}}$

$$\therefore L\{t\,\text{erf}\sqrt{t}\} = (-1)\frac{d}{ds}\left[\frac{1}{s\sqrt{s+1}}\right]$$

$$=(-1)\left[\frac{-(s+1)^{1/2} - s\cdot\frac{1}{2}\cdot(s+1)^{-1/2}}{s^2(s+1)}\right]$$

$$= \frac{(s+1)^{-1/2}}{2}\left[\frac{2(s+1)+s}{s^2(s+1)}\right] = \frac{(3s+2)}{2s^2(s+1)^{3/2}}$$

(c) Since $L\{\text{erf}\sqrt{t}\} = \dfrac{1}{s\sqrt{s+1}}$

$$\therefore L\{\text{erf}\,2\sqrt{t}\} = L\{\text{erf}\sqrt{4t}\} = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$$

{By change of scale property i.e. $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$\therefore L\{t\cdot\text{erf}\,2\sqrt{t}\} = -\frac{d}{ds}\left[\frac{2}{s\sqrt{s+4}}\right]$$

$$=(-2)\left[\frac{-(s+4)^{1/2} - s\cdot\frac{1}{2}(s+4)^{-1/2}}{s^2(s+4)}\right]$$

$$= \frac{(s+4)^{-1/2}}{s^2(s+4)}\left[2(s+4)+s\right]$$

$$= \frac{3s+8}{s^2(s+4)^{3/2}}$$

(d) From the above question $L\{t \cdot \text{erf}(2\sqrt{t})\} = \dfrac{3s+8}{s^2(s+4)^{3/2}}$

$$\therefore \; L\left\{\int_0^t t \,\text{erf}(2\sqrt{t})\,dt\right\} = \frac{3s+8}{s^3(s+4)^{3/2}}$$

(e) We know that $L\{\text{erf}_c\sqrt{t}\} = \dfrac{\sqrt{s+1}-1}{s\sqrt{s+1}}$

$$\therefore \; L\left\{\int_0^t \text{erf}_c\sqrt{t}\right\} = \frac{\sqrt{s+1}-1}{s^2\sqrt{s+1}}$$

Que (16) — Evaluate the following integrals

(a) $\int_0^\infty e^{-2t}\,\text{erf}\sqrt{t}\,dt$ (b) $\int_0^\infty e^{-t}\,\text{erf}_c\sqrt{t}\,dt$

Sol (a) Using the definition of ~~Lo~~ laplace transform

$$\int_0^\infty e^{-2t} \cdot \text{erf}\sqrt{t}\,dt = L\{\text{erf}\sqrt{t}\}_{at\; s=2} = \left[\frac{1}{s\sqrt{s+1}}\right]_{at\; s=2} = \frac{1}{2\sqrt{3}}$$

(b)

$$\int_0^\infty e^{-t} \cdot \text{erf}_c\sqrt{t}\,dt = L\{\text{erf}_c\sqrt{t}\}_{at\; s=1} = \left[\frac{\sqrt{s+1}-1}{s\sqrt{s+1}}\right]_{at\; s=1} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

Que (17) – Find the Laplace transform of

(a) $t\, J_0(2t)$ (b) $\int_0^t e^{-bt} J_0(at)\, dt$

Sol(a) We know that $L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$

$$\therefore L\{J_0(2t)\} = \frac{1}{2} \frac{1}{\sqrt{1+\left(\frac{s}{2}\right)^2}} = \frac{1}{\sqrt{4+s^2}}$$ (By change of scale property)

$$\therefore L\{t\, J_0(2t)\} = -\frac{d}{ds}\left(\frac{1}{\sqrt{4+s^2}}\right) = (-1)\left(-\tfrac{1}{2}\right)(4+s^2)^{-3/2} \cdot 2s = \frac{s}{(4+s^2)^{3/2}}$$

(b) $$L\{J_0(at)\} = \frac{1}{a} \frac{1}{\sqrt{1+\left(\frac{s}{a}\right)^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

$$\therefore L\{e^{-bt} J_0(at)\} = \frac{1}{\sqrt{(s+b)^2+a^2}}$$

$$L\left\{\int_0^t e^{-bt} J_0(at)\, dt\right\} = \frac{1}{s\sqrt{(s+b)^2+a^2}}$$

Que (18): – Evaluate the following integral

(a) $\int_0^\infty J_0(t)\, dt$ (b) $\int_0^\infty e^{-2t} \cdot t\, J_0(2t)\, dt$

Sol(a) By the definition of Laplace transform

$$\int_0^\infty e^{-0\cdot t} J_0(t)\, dt = L\{J_0(t)\}_{\text{at } s=0} = \left[\frac{1}{\sqrt{1+s^2}}\right]_{\text{at } s=0} = 1$$

(b) again $$\int_0^\infty e^{-2t} \cdot t\, J_0(2t)\, dt = L\{t\, J_0(2t)\}_{\text{at } s=2}$$

$$= \left[\frac{s}{(s^2+4)^{3/2}}\right]_{\text{at } s=2}$$

using above question (17) (a)

$$= \frac{1}{8\sqrt{2}}$$

Question based on the formula (3.5) i.e. derivative of Laplace Transform —

Que (19) :— If $L\{t \sin \omega t\} = \dfrac{2\omega s}{(s^2+\omega^2)^2}$, evaluate

(i) $L\{\omega t \cos \omega t + \sin \omega t\}$ (ii) $L\{2\cos \omega t - \omega t \sin \omega t\}$

Sol :— Let $f(t) = t \sin \omega t$, and $\bar{f}(s) = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

$\therefore f'(t) = \omega t \cos \omega t + \sin \omega t$

and $f''(t) = 2\omega \cos \omega t - \omega^2 t \sin \omega t$

putting $t = 0$ we get $f(0) = 0$, $f'(0) = 0$, $\cancel{f''(0) = 2\omega}$

(i) $L\{f'(t)\} = s\bar{f}(s) - f(0)$

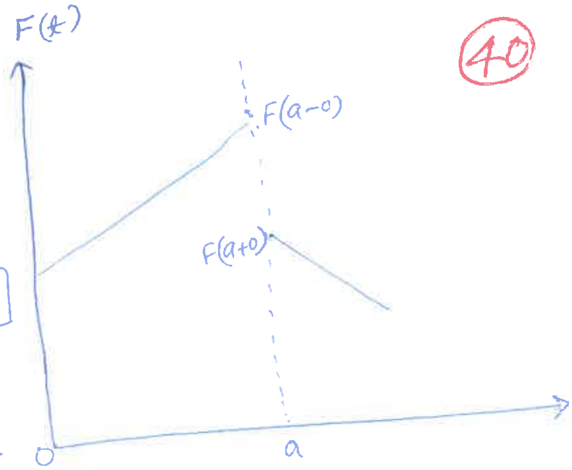
$$\Rightarrow L\{\omega t \cos \omega t + \sin \omega t\} = s \cdot \frac{2\omega s}{(s^2+\omega^2)^2} - 0 = \frac{2\omega s^2}{(s^2+\omega^2)^2}$$

(ii) $L\{f''(t)\} = s^2\bar{f}(s) - s f(0) - f'(0)$

$$\therefore L\{2\omega \cos \omega t - \omega^2 t \sin \omega t\} = s^2 \cdot \frac{2\omega s}{(s^2+\omega^2)^2} - s \cdot 0 - 0$$

$$\Rightarrow L\{2\cos \omega t - \omega t \sin \omega t\} = \frac{2s^3}{(s^2+\omega^2)^2}$$

Que [illegible] :— If $F(t)$ is continuous except for an ordinary discontinuity at $t=a\,(a>0)$ as given below:



Then $L\{F'(t)\} = s\,L\{F(t)\} - F(0) - e^{-as}[F(a+0) - F(a-0)]$

where $F(a+0)$ and $F(a-0)$ are the limits of F at $t = a$ as t approaches a from right and from left respectively. The quantity $F(a+0) - F(a-0)$ is called the jump at the discontinuity $t = a$, and $e^{-st}F(t) \to 0$ as $t \to \infty$.

Proof —

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt$$

$$= \int_0^a e^{-st} F'(t)\,dt + \int_a^\infty e^{-st} F'(t)\,dt$$

$$= \left[e^{-st} F(t)\right]_0^a + s\int_0^a e^{-st} F(t)\,dt + \left[e^{-st} F(t)\right]_a^\infty + s\int_a^\infty e^{-st} F(t)\,dt$$

$$= e^{-as} F(a-0) - F(0) + s\left[\int_0^a e^{-st} F(t)\,dt + \int_a^\infty e^{-st} F(t)\,dt\right]$$

$$+ \lim_{t\to\infty} e^{-st} F(t) - e^{-as} F(a+0)$$

$$= e^{-as}[F(a-0) - F(a+0)] - F(0) + s\int_0^\infty e^{-st} F(t)\,dt + 0$$

$$= e^{-as}[F(a-0) - F(a+0)] - F(0) + s\,L\{F(t)\}$$

$$= s\,L\{F(t)\} - F(0) - e^{-as}[F(a+0) - F(a-0)]$$

(3.10) Initial and final value theorem —

(a) Initial value theorem — If $L\{f(t)\} = \bar{f}(s)$ then

$$\lim_{s \to \infty} [s\bar{f}(s)] = \lim_{t \to 0} f(t),$$ provided that limits exist.

Proof :— We know that $L\{f'(t)\} = s\bar{f}(s) - f(0)$

$$\Rightarrow \int_0^\infty e^{-st} f'(t)\,dt = s\bar{f}(s) - f(0)$$

Taking limit $s \to \infty$ on both sides, we get

$$\lim_{s \to \infty} \left[\int_0^\infty e^{-st} f'(t)\,dt\right] = \lim_{s \to \infty} [s\bar{f}(s) - f(0)]$$

or $$\int_0^\infty \left(\lim_{s \to \infty} e^{-st}\right) f'(t)\,dt = \lim_{s \to \infty} [s\bar{f}(s) - f(0)]$$

or $$\int_0^\infty 0 \cdot f'(t)\,dt = \lim_{s \to \infty} [s\bar{f}(s)] - f(0)$$ $\left\{\text{Since } e^{-\infty} = 0 \text{ \& } e^0 = 1\right.$

$$\therefore \lim_{s \to \infty} [s\bar{f}(s)] - f(0) = 0$$

or $$\lim_{s \to \infty} [s\bar{f}(s)] = \lim_{t \to 0} f(t)$$

(b) Final value theorem :— If $L\{f(t)\} = \bar{f}(s)$ then

$$\lim_{s \to 0} [s\bar{f}(s)] = \lim_{t \to \infty} f(t)$$ provided that limits exist.

Proof :— We know that $L\{f'(t)\} = s\bar{f}(s) - f(0)$ or $\int_0^\infty e^{-st} f'(t)\,dt = s\bar{f}(s) -$

Taking limit $s \to 0$ on both sides, we get

$$\lim_{s \to 0} \left[\int_0^\infty e^{-st} f'(t)\,dt\right] = \lim_{s \to 0} [s\bar{f}(s) - f(0)]$$

$$\Rightarrow \int_0^\infty \left(\lim_{s \to 0} e^{-st}\right) f'(t)\,dt = \lim_{s \to 0} [s\bar{f}(s)] - f(0)$$

$$\Rightarrow \int_0^\infty 1 \cdot f'(t)\,dt = \lim_{s \to 0} [s\bar{f}(s)] - f(0)$$

$$\Rightarrow [f(t)]_0^\infty = \lim_{s \to 0} [s\bar{f}(s)] - f(0)$$

$$\Rightarrow \lim_{t \to \infty} f(t) - f(0) = \lim_{s \to 0} [s\bar{f}(s)] - f(0)$$

or $$\lim_{s \to 0} [s\bar{f}(s)] = \lim_{t \to \infty} f(t)$$

(3.11) Convolution Theorem :—

If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$ then

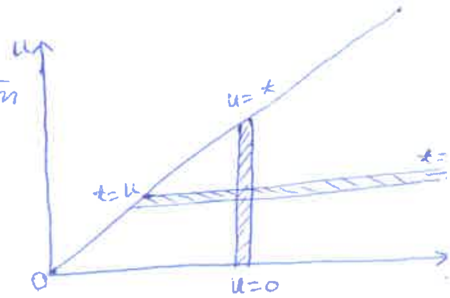
$$L\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \bar{f}(s)\cdot\bar{g}(s)$$

Proof :— Using the definition of Laplace transform

$$L\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u)\, g(t-u)\, du\right] dt$$

$$= \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st} f(u)\, g(t-u)\, du\, dt$$

using change of order of integration



$$= \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)\, g(t-u)\, dt\, du$$

put $(t-u) = x$

$\therefore dt = dx$

$$= \int_{u=0}^{\infty}\int_{x=0}^{\infty} e^{-s(u+x)} f(u)\, g(x)\, dx\, du$$

$$= \int_{u=0}^{\infty} e^{-su} f(u)\, du \int_{x=0}^{\infty} e^{-sx} g(x)\, dx$$

$$= L\{f(t)\}\cdot L\{g(t)\}$$

$$= \bar{f}(s)\cdot\bar{g}(s)$$

(3.12) Inverse Laplace Transform :— If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\{\bar{f}(s)\} = f(t)$, where L^{-1} is called inverse Laplace transform.

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(3.13) Formulae of Laplace and inverse Laplace transform :—

S.No.	Laplace transform	Inverse Laplace transform
1 -	$L\{f(t)\} = \bar{f}(s)$	$L^{-1}\{\bar{f}(s)\} = f(t)$
2 -	$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\{\frac{1}{s-a}\} = e^{at}$
3 -	$L\{1\} = \frac{1}{s}$	$L^{-1}\{\frac{1}{s}\} = 1$
-	$L\{t^n\} = \frac{n!}{s^{n+1}}$ or $\frac{\Gamma(n+1)}{s^{n+1}}$	$L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$ or $\frac{t^n}{\Gamma(n+1)}$
5 -	$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sin at}{a}$
-	$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
-	$L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\{\frac{1}{s^2-a^2}\} = \frac{\sinh at}{a}$
8 -	$L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$
9 -	$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$	$L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$
10 -	$L\{e^{at} f(t)\} = \bar{f}(s-a)$	$L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$
11 -	$L\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$	$L^{-1}\{\frac{1}{(s-a)^{n+1}}\} = \frac{e^{at} t^n}{n!}$
12 -	$L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$	$L^{-1}\{\frac{1}{(s-a)^2+b^2}\} = \frac{e^{at} \sin bt}{b}$
-	$L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$	$L^{-1}\{\frac{(s-a)}{(s-a)^2+b^2}\} = e^{at} \cos bt$
14 -	$L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2-b^2}$	$L^{-1}\{\frac{1}{(s-a)^2-b^2}\} = \frac{e^{at} \sinh bt}{b}$
15 -	$L\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$	$L^{-1}\{\frac{(s-a)}{(s-a)^2-b^2}\} = e^{at} \cosh bt$

S.No.	Laplace transform	Inverse Laplace transform
16-	$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$ If $n=1$ then $L\{t\, f(t)\} = -\frac{d}{ds}\bar{f}(s)$	$L^{-1}\left\{\frac{d^n}{ds^n}\bar{f}(s)\right\} = (-1)^n t^n f(t)$ $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -t f(t) = -t\, L^{-1}\{\bar{f}(s)\}$ or $L^{-1}\{\bar{f}(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}$
17-	$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds$	$L^{-1}\left\{\int_s^\infty \bar{f}(s)\, ds\right\} = \frac{f(t)}{t} = \frac{L^{-1}\{\bar{f}(s)\}}{t}$ or $L^{-1}\{\bar{f}(s)\} = t\, L^{-1}\left\{\int_s^\infty \bar{f}(s)\, ds\right\}$
18 -	$L\left\{\int_0^t f(t)\, dt\right\} = \frac{\bar{f}(s)}{s}$	$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\, dt$
19 -	$L\{U(t-a)\} = \frac{e^{-as}}{s}$	$L^{-1}\left\{\frac{e^{-as}}{s}\right\} = U(t-a)$
20 -	$L\{f(t-a)\, U(t-a)\} = e^{-as}\bar{f}(s)$	$L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)\, U(t-a)$
21 -	$L\{f(t)\, U(t-a)\} = e^{-as} L\{f(t+a)\}$	
22 -	$L\{\delta(t-a)\} = e^{-as}$	$L^{-1}\{e^{-as}\} = \delta(t-a)$
23 -	$L\{\delta(t)\} = 1$	$L^{-1}\{1\} = \delta(t)$
24 -	$L\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \bar{f}(s)\, \bar{g}(s)$	$L^{-1}\{\bar{f}(s)\, \bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$
25 -	$L\left\{\frac{d^n}{dt^n} f(t)\right\} = L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$	

Questions based on basic formulae of Inverse Laplace transform :- (45)

Que (21) :- Find the inverse Laplace transform of

(a) $\frac{1}{s-3}$ (b) $\frac{1}{s^2+4}$ (c) $\frac{s}{s^2+9}$ (d) $\frac{1}{s^2-5}$ (e) $\frac{s}{s^2-3}$ (f) $\frac{1}{3s+5}$

Sol — (a) $L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$ $\left(\text{Since } L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}\right)$

(b) $L^{-1}\left\{\frac{1}{s^2+4}\right\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$ $\left(\text{Since } L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}\right)$

(c) $L^{-1}\left\{\frac{s}{s^2+9}\right\} = L^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos 3t$ $\left(\text{Since } L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at\right)$

(d) $L^{-1}\left\{\frac{1}{s^2-5}\right\} = L^{-1}\left\{\frac{1}{s^2-(\sqrt{5})^2}\right\} = \frac{1}{\sqrt{5}}\sinh\sqrt{5}t$ $\left(\text{Since } L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}\right)$

(e) $L^{-1}\left\{\frac{s}{s^2-3}\right\} = L^{-1}\left\{\frac{s}{s^2-(\sqrt{3})^2}\right\} = \cosh\sqrt{3}t$ Since $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

(f) $L^{-1}\left\{\frac{1}{3s+5}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s+\frac{5}{3}}\right\} = \frac{1}{3}e^{-\frac{5}{3}t}$.

Que (22) — (a) Find the inverse Laplace transform of

(a) $\frac{3s-8}{4s^2+25}$ (b) $\frac{s^3}{s^4-a^4}$ (c) $\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}$

(d) $\frac{2s+3}{s(s+3)}$ (e) $\frac{(s^2-2)^2}{2s^5}$

Sol :— (a) $\frac{3s-8}{4s^2+25} = \frac{3s}{4\left(s^2+\frac{25}{4}\right)} - \frac{8}{4\left(s^2+\frac{25}{4}\right)}$

$\therefore L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\} = \frac{3}{4}L^{-1}\left\{\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right\} - 2L^{-1}\left\{\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right\}$

$= \frac{3}{4}\cos\frac{5}{2}t - 2\cdot\frac{2}{5}\sin\frac{5}{2}t$

$= \frac{3}{4}\cos\frac{5}{2}t - \frac{4}{5}\sin\frac{5}{2}t$

(b) $$L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\} = L^{-1}\left\{s\cdot\frac{s^2}{(s^2-a^2)(s^2+a^2)}\right\} = L^{-1}\left\{\frac{s}{2}\left(\frac{1}{s^2-a^2}+\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{s}{s^2-a^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{2}(\cosh at + \cos at)$$

(c) $$L^{-1}\left\{\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right\}$$

$$= L^{-1}\left\{\frac{6}{2(s-\frac{3}{2})} - \frac{3}{9(s^2-\frac{16}{9})} - \frac{4s}{9(s^2-\frac{16}{9})} + \frac{8}{16(s^2+\frac{9}{16})} - \frac{6s}{16(s^2+\frac{9}{16})}\right\}$$

$$= L^{-1}\left\{\frac{3}{s-\frac{3}{2}} - \frac{1}{3}\cdot\frac{1}{s^2-(\frac{4}{3})^2} - \frac{4}{9}\cdot\frac{s}{s^2-(\frac{4}{3})^2} + \frac{1}{2}\cdot\frac{1}{s^2+(\frac{3}{4})^2} - \frac{3}{8}\frac{s}{s^2+(\frac{3}{4})^2}\right\}$$

$$= 3\cdot e^{\frac{3}{2}t} - \frac{1}{3}\cdot\frac{3}{4}\sinh\frac{4}{3}t - \frac{4}{9}\cdot\cosh\frac{4}{3}t + \frac{1}{2}\cdot\frac{4}{3}\sin\frac{3}{4}t - \frac{3}{8}\cos\frac{3}{4}t$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4}\sinh\frac{4}{3}t - \frac{4}{9}\cosh\frac{4}{3}t + \frac{2}{3}\sin\frac{3}{4}t - \frac{3}{8}\cos\frac{3}{4}t$$

(d) $$L^{-1}\left\{\frac{2s+3}{s(s+3)}\right\} = L^{-1}\left\{\frac{s+3+s}{s(s+3)}\right\} = L^{-1}\left\{\frac{1}{s} + \frac{1}{s+3}\right\}$$

$$= 1 + e^{-3t}$$

(e) $$L^{-1}\left\{\frac{(s^2-2)^2}{2s^5}\right\} = L^{-1}\left\{\frac{s^4-4s^2+4}{2s^5}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{2s} - \frac{4}{s^3} + \frac{4}{s^5}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2} - 4\cdot\frac{t^2}{\underline{|2}} + 4\cdot\frac{t^4}{\underline{|4}}\right]$$

$$= \frac{1}{4} - t^2 + \frac{1}{12}t^4$$

Question based on first shifting theorem i.e. $L^{-1}\{F(s-a)\} = e^{at} f(t)$ 47

Que (23) :— Find the inverse Laplace transform of

(a) $\dfrac{1}{(s-1)^2+1}$ (b) $\dfrac{s}{(s+2)^2+4}$ (c) $\dfrac{1}{(s+1)^2-2}$ (d) $\dfrac{s+1}{(s-2)^2-4}$

(e) $\dfrac{s+1}{s^2+s+1}$ (f) $\dfrac{1}{\sqrt{s+1}}$ (g) $\dfrac{s+8}{s^2+4s+5}$. (h) $\dfrac{s}{(s+7)^4}$

(i) $\dfrac{1}{9s^2+6s+1}$ (j) $\dfrac{s}{s^2+6s+25}$ (k) $\dfrac{s+2}{s^2-2s-8}$

Sol — (a) $L^{-1}\left\{\dfrac{1}{(s-1)^2+1}\right\} = e^{t} L^{-1}\left\{\dfrac{1}{s^2+1^2}\right\} = e^{t}\sin t$

(b) $L^{-1}\left\{\dfrac{s+2-2}{(s+2)^2+4}\right\} = L^{-1}\left\{\dfrac{(s+2)}{(s+2)^2+2^2}\right\} - L^{-1}\left\{\dfrac{2}{(s+2)^2+2^2}\right\}$

$= e^{-2t} L^{-1}\left\{\dfrac{s}{s^2+2^2}\right\} - e^{-2t} L^{-1}\left\{\dfrac{2}{s^2+2^2}\right\}$

$= e^{-2t}\cos 2t - e^{-2t}\sin 2t = e^{-2t}(\cos 2t - \sin 2t)$

(c) $L^{-1}\left\{\dfrac{1}{(s+1)^2-2}\right\} = e^{-t} L^{-1}\left\{\dfrac{1}{s^2-(\sqrt{2})^2}\right\} = \dfrac{e^{-t}}{\sqrt{2}}\sinh\sqrt{2}t$

(d) $L^{-1}\left\{\dfrac{s+1}{(s-2)^2-4}\right\} = L^{-1}\left\{\dfrac{s+1-2+2}{(s-2)^2-2^2}\right\} = L^{-1}\left\{\dfrac{(s-2)}{(s-2)^2-2^2}\right\} + L^{-1}\left\{\dfrac{3}{(s-2)^2-2^2}\right\}$

$= e^{2t} L^{-1}\left\{\dfrac{s}{s^2-2^2}\right\} + 3e^{2t} L^{-1}\left\{\dfrac{1}{s^2-2^2}\right\}$

$= e^{2t}\cosh 2t + \dfrac{3}{2} e^{2t}\sinh 2t = \dfrac{e^{2t}}{2}(2\cosh 2t + 3\sinh$

(e) $\dfrac{s+1}{s^2+s+1} = \dfrac{s+\frac{1}{2}+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} = \dfrac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \dfrac{1}{2}\cdot\dfrac{1}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$\therefore L^{-1}\left\{\dfrac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\dfrac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + \dfrac{1}{2} L^{-1}\left\{\dfrac{1}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$

$= e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}e^{-\frac{1}{2}t}\cdot\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$

$= e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$

(f) $L^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\} = e^{-t} L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = e^{-t} \cdot \frac{t^{-1/2}}{\Gamma \frac{1}{2}} = \frac{e^{-t}}{\sqrt{\pi t}}$

(g) $L^{-1}\left\{\frac{s+8}{s^2+4s+5}\right\} = L^{-1}\left\{\frac{(s+2)+6}{(s+2)^2+1}\right\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} + 6 L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$

$= e^{-2t}\cos t + 6 \cdot e^{-2t} \cdot \sin t$

$= e^{-2t}(\cos t + 6 \sin t)$

(h) $L^{-1}\left\{\frac{s}{(s+7)^4}\right\} = L^{-1}\left\{\frac{s+7-7}{(s+7)^4}\right\} = L^{-1}\left\{\frac{1}{(s+7)^3}\right\} - 7 L^{-1}\left\{\frac{1}{(s+7)^4}\right\}$

$= e^{-7t} L^{-1}\left\{\frac{1}{s^3}\right\} - 7 e^{-7t} L^{-1}\left\{\frac{1}{s^4}\right\}$

$= e^{-7t} \cdot \frac{t^2}{2} - 7 e^{-7t} \cdot \frac{t^3}{\lfloor 3} = e^{-7t} \cdot \frac{t^2}{6}(3 - 7t)$

(i) $L^{-1}\left\{\frac{1}{9s^2+6s+1}\right\} = L^{-1}\left\{\frac{1}{(3s+1)^2}\right\} = \frac{1}{9} L^{-1}\left\{\frac{1}{(s+\frac{1}{3})^2}\right\} = \frac{1}{9} e^{-\frac{1}{3}t} L^{-1}\left\{\frac{1}{s^2}\right\}$

$= \frac{1}{9} e^{-\frac{1}{3}t} \cdot t = \frac{t\, e^{-\frac{1}{3}t}}{9}$

(j) $L^{-1}\left\{\frac{s}{s^2+6s+25}\right\} = L^{-1}\left\{\frac{s+3-3}{(s+3)^2+16}\right\} = L^{-1}\left\{\frac{(s+3)}{(s+3)^2+4^2}\right\} - 3 L^{-1}\left\{\frac{1}{(s+3)^2+4^2}\right\}$

$= e^{-3t} L^{-1}\left\{\frac{s}{s^2+4^2}\right\} - 3 e^{-3t} L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$

$= e^{-3t}\cos 4t - 3 e^{-3t} \cdot \frac{1}{4}\sin 4t$

$= \frac{e^{-3t}}{4}(4\cos 4t - 3\sin 4t)$

(k) $L^{-1}\left\{\frac{s+2}{s^2-2s-8}\right\} = L^{-1}\left\{\frac{s+2+1-1}{(s-1)^2-9}\right\} = L^{-1}\left\{\frac{s-1}{(s-1)^2-3^2}\right\} + L^{-1}\left\{\frac{3}{(s-1)^2-3^2}\right\}$

$= e^{t} L^{-1}\left\{\frac{s}{s^2-3^2}\right\} + e^{+t} L^{-1}\left\{\frac{3}{s^2-3^2}\right\}$

$= e^{t}\cosh 3t + e^{+t} \cdot \sinh 3t$

$= e^{t}(\cosh 3t + \sinh 3t)$

Questions based on division by s i.e. $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt = \int_0^t L^{-1}\{F(s)\}\,dt$

Que (24) :— Find the inverse Laplace transform of

(a) $\frac{1}{s(s+2)}$ (b) $\frac{s^2+2}{s(s^2+4)}$ (c) $\frac{1}{s(s^2+1)}$ (d) $\frac{1}{s^2(s+1)}$

(e) $\frac{1}{s(s^2-4)}$

Sol (a) :— $L^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s+2}\right\}dt = \int_0^t e^{-2t}dt = \left[\frac{e^{-2t}}{-2}\right]_0^t = \frac{1-e^{-2t}}{2}$

(b) $L^{-1}\left\{\frac{s^2+2}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{s^2+4-2}{s(s^2+4)}\right\}$

$= L^{-1}\left\{\frac{1}{s} - \frac{2}{s(s^2+4)}\right\}$

$= 1 - 2\int_0^t L^{-1}\left\{\frac{1}{s^2+2^2}\right\}dt$

$= 1 - 2\int_0^t \frac{\sin 2t}{2}\,dt$

$= 1 + \left[\frac{\cos 2t}{2}\right]_0^t \qquad = 1 + \frac{1}{2}(\cos 2t - 1)$

$= \frac{2 + 2\cos^2 t - 1 - 1}{2} \qquad = \cos^2 t$

(c) $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s^2+1}\right\}dt = \int_0^t \sin t\,dt = \left[-\cos t\right]_0^t = 1-\cos t$

(d) Now first we find $L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s+1}\right\}dt = \int_0^t e^{-t}dt = \left[\frac{e^{-t}}{-1}\right]_0^t = 1-$

$\therefore L^{-1}\left\{\frac{1}{s\cdot s(s+1)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s(s+1)}\right\}dt = \int_0^t (1-e^{-t})\,dt = \left[t + e^{-t}\right]_0^t$

$= t + e^{-t} - 1$

(e) $L^{-1}\left\{\frac{1}{s(s^2-4)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s^2-4}\right\}dt = \int_0^t \frac{\sinh 2t}{2}\,dt = \frac{1}{2\cdot 2}\left[\cosh 2t\right]_0^t$

$= \frac{1}{4}(\cosh 2t - 1)$

Que (25) :– Find the inverse Laplace transform of

(a) $\dfrac{1}{s^2-7s+12}$ (b) $\dfrac{s^2+2s-3}{s(s-3)(s+2)}$ (c) $\dfrac{3s+1}{(s-1)(s^2+1)}$

(d) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ (e) $\dfrac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ (f) $\dfrac{s}{s^4+s^2+1}$

Sol (a) :– $\dfrac{1}{s^2-7s+12} = \dfrac{1}{(s-3)(s-4)} = \dfrac{1}{s-4} - \dfrac{1}{s-3}$ (By partial fraction)

$\therefore L\left\{\dfrac{1}{s^2-7s+12}\right\} = L^{-1}\left\{\dfrac{1}{s-4}\right\} - L^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{4t} - e^{+3t}$

(b) $\dfrac{s^2+2s-3}{s(s-3)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s-3} + \dfrac{C}{s+2} = \dfrac{A(s-3)(s+2)+Bs(s+2)+Cs(s-3)}{s(s-3)(s+2)}$

$\Rightarrow s^2+2s-3 = A(s^2-s-6) + B(s^2+2s) + C(s^2-3s)$

$\Rightarrow A+B+C=1$, $-A+2B-3C=2$ and $-6A=-3$

$\Rightarrow A=\frac{1}{2}$, $B=\frac{4}{5}$, $C=\frac{-3}{10}$

$\therefore \dfrac{s^2+2s-3}{s(s-3)(s+2)} = \dfrac{1}{2s} + \dfrac{4}{5}\dfrac{1}{s-3} - \dfrac{3}{10}\dfrac{1}{s+2}$

$\therefore L^{-1}\left\{\dfrac{s^2+2s-3}{s(s-3)(s+2)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{5}L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{3}{10}L^{-1}\left\{\frac{1}{s+2}\right\}$

$= \frac{1}{2}\cdot 1 + \frac{4}{5}e^{3t} - \frac{3}{10}e^{-2t}$

(c) $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1} = \dfrac{A(s^2+1)+(Bs+C)(s-1)}{(s-1)(s^2+1)}$

$\Rightarrow 3s+1 = A(s^2+1) + Bs^2 + (-B+C)s - C$

Comparing the coefficients we get

$A+B=0$, $-B+C=3$ and $A-C=1$

On solving we get $A=2$, $B=-2$ and $C=1$

$\therefore \dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{2}{s-1} + \dfrac{-2s+1}{s^2+1} = \dfrac{2}{s-1} - \dfrac{2s}{s^2+1} + \dfrac{1}{s^2+1}$

$\therefore L^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\} = 2L^{-1}\left\{\frac{1}{s-1}\right\} - 2L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= 2e^{t} - 2\cos t + \sin t$

(d) $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2} = \frac{(As+B)(s^2+b^2)+(Cs+D)(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}$$

$$\Rightarrow s^2 = As^3 + Bs^2 + Ab^2s + Bb^2 + Cs^3 + Ds^2 + Ca^2s + Da^2$$

Comparing on both sides, we get

$A + C = 0$, $B + D = 1$, $Ab^2 + Ca^2 = 0$ and $Bb^2 + Da^2 = 0$

On solving, we get $A = 0$, $C = 0$, $B = \frac{a^2}{a^2-b^2}$, $D = -\frac{b^2}{a^2-b^2}$

$$\therefore \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{a^2}{a^2-b^2}\cdot\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\cdot\frac{1}{s^2+b^2}$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{a^2-b^2}\left[a^2 L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - b^2 L^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right]$$

$$= \frac{1}{a^2-b^2}\left\{a \sin at - b \sin bt\right\}$$

(e) $$\frac{11s^2-2s+5}{2s^3-3s^2-3s+2} = \frac{11s^2-2s+5}{(s+1)(2s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{2s-1} + \frac{C}{s-2}$$

$$= \frac{A(2s-1)(s-2) + B(s+1)(s-2) + C(s+1)(2s-1)}{(s+1)(2s-1)(s-2)}$$

$$\Rightarrow 11s^2 - 2s + 5 = A(2s^2-5s+2) + B(s^2-s-2) + C(2s^2+s-1)$$

Comparing on both sides we get

$2A + B + 2C = 11$, $-5A - B + C = -2$, $2A - 2B - C = 5$

On solving we get $A = 2$, $B = -3$ and $C = 5$

$$\therefore \frac{11s^2-2s+5}{2s^3-3s^2-3s+2} = \frac{2}{s+1} - \frac{3}{2s-1} + \frac{5}{s-2}$$

$$\therefore L^{-1}\left\{\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}\right\} = 2L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{s-\frac{1}{2}}\right\} + 5L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 2e^{-t} - \frac{3}{2}e^{\frac{1}{2}t} + 5e^{2t}$$

(f) $$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1)^2-s^2} = \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$$

$$\Rightarrow \quad s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

Comparing bothsides we get

$A+C=0$, $-A+B+C+D=0$, $A-B+C+D=1$, $B+D=0$

on solving above equations we get

$A=0$, $B=-\frac{1}{2}$, $C=0$, $D=\frac{1}{2}$

$$\therefore \frac{s}{s^4+s^2+1} = -\frac{1}{2}\cdot\frac{1}{s^2+s+1} + \frac{1}{2}\cdot\frac{1}{s^2-s+1} = -\frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{2}\frac{1}{(s-\frac{1}{2})^2+\frac{3}{4}}$$

$$\therefore L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$= -\frac{1}{2}e^{-\frac{t}{2}}\frac{\sin(\frac{\sqrt{3}}{2})t}{\frac{\sqrt{3}}{2}} + \frac{1}{2}e^{\frac{t}{2}}\frac{\sin(\frac{\sqrt{3}}{2})t}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\right)t\left[e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right]$$

$$= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\right)t \cdot \sinh\frac{t}{2}$$

Question based on second shifting theorem i.e. $L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)\,U(t-a)$

Que (26) :— Find the inverse Laplace transform

(a) $\dfrac{e^{-2s}}{s^3}$ (b) $\dfrac{e^{-2s}}{s-3}$ (c) $\dfrac{e^{-s}}{(s+1)^3}$ (d) $\dfrac{s\,e^{-as}}{s^2-w^2}$, $a>0$

(e) $\dfrac{s\,e^{-\frac{s}{2}} + \pi e^{-s}}{s^2+\pi^2}$ (f) $\dfrac{e^{-s} - 3e^{-3s}}{s^2}$ (g) $\dfrac{e^{-s}}{\sqrt{s+1}}$ (h) $\dfrac{e^{-cs}}{s^2(s+a)}$, $c>0$

Sol :— (a) Let $\bar{f}(s) = \dfrac{1}{s^3}$ $\therefore$ $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\dfrac{1}{s^3}\right\} = \dfrac{t^2}{2} = f(t)$

$$\therefore\ L^{-1}\left\{e^{-2s}\cdot\frac{1}{s^3}\right\} = L^{-1}\{e^{-2s}\bar{f}(s)\} = f(t-2)\,U(t-2)$$
$$= \frac{(t-2)^2}{2}\,U(t-2)$$

(b) Let $\bar{f}(s) = \dfrac{1}{s-3}$ $\therefore$ $L^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t} = f(t)$

$$\therefore\ L^{-1}\left\{\frac{e^{-2s}}{s-3}\right\} = L^{-1}\{e^{-2s}\cdot\bar{f}(s)\} = f(t-2)\;U(t-2)$$
$$= e^{3(t-2)}\;U(t-2)$$

(c) Let $\bar{f}(s) = \dfrac{1}{(s+1)^3}$ $\therefore$ $L^{-1}\left\{\dfrac{1}{(s+1)^3}\right\} = e^{-t}L^{-1}\left\{\dfrac{1}{s^3}\right\} = e^{-t}\dfrac{t^2}{2} = f(t)$

$$\therefore\ L^{-1}\left\{\frac{e^{-s}}{(s+1)^3}\right\} = L^{-1}\{e^{-s}\cdot\bar{f}(s)\} = f(t-1)\;U(t-1)$$
$$= \frac{1}{2}(t-1)^2\,e^{-(t-1)}\;U(t-1)$$

(d) Let $\bar{f}(s) = \dfrac{s}{s^2-w^2}$ $\therefore$ $L^{-1}\left\{\dfrac{s}{s^2-w^2}\right\} = \cosh wt = f(t)$

$$\therefore\ L^{-1}\left\{e^{-as}\cdot\frac{s}{s^2-w^2}\right\} = L^{-1}\{e^{-as}\cdot\bar{f}(s)\} = f(t-a)\;U(t-a)$$
$$= \cosh w(t-a)\;U(t-a)$$

(e) Let $\bar{f}(s) = \dfrac{s}{s^2+\pi^2}$ $\therefore$ $L^{-1}\left\{\dfrac{s}{s^2+\pi^2}\right\} = \cos\pi t = f(t)$

$$\therefore\ L^{-1}\left\{e^{-\frac{s}{2}}\cdot\frac{s}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-\frac{s}{2}}\cdot\bar{f}(s)\right\} = f\left(t-\tfrac{1}{2}\right)\,U\left(t-\tfrac{1}{2}\right)$$
$$= \cos\pi\left(t-\tfrac{1}{2}\right)\cdot U\left(t-\tfrac{1}{2}\right)$$
$$= \sin\pi t\ \cdot\ U\left(t-\tfrac{1}{2}\right)$$

again let $\bar{f}(s) = \frac{\pi}{s^2+\pi^2}$ $\therefore L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} = \sin \pi t = f(t)$

$$\therefore L^{-1}\left\{e^{-s}\cdot\frac{\pi}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s}\cdot\bar{f}(s)\right\} = f(t-1)\, U(t-1)$$
$$= \sin \pi(t-1)\; U(t-1)$$
$$= -\sin(\pi t)\cdot U(t-1)$$

Hence
$$L^{-1}\left\{\frac{s e^{-\frac{s}{2}} + \pi e^{-s}}{s^2+\pi^2}\right\} = \sin\pi t\; U(t-\tfrac{1}{2}) - \sin\pi t\cdot U(t-1)$$
$$= \sin\pi t\left[U(t-\tfrac{1}{2}) - U(t-1)\right]$$

(f) Let $\bar{f}(s) = \frac{1}{s^2}$ $\therefore L^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$

$$\therefore L^{-1}\left\{\frac{e^{-s}-3e^{-3s}}{s^2}\right\} = L^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\} - 3L^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2}\right\}$$
$$= L^{-1}\left\{e^{-s}\cdot\bar{f}(s)\right\} - 3L^{-1}\left\{e^{-3s}\cdot\bar{f}(s)\right\}$$
$$= f(t-1)\; U(t-1) - 3 f(t-3)\; U(t-3)$$
$$= (t-1)\; U(t-1) - 3(t-3)\; U(t-3)$$

(g) Let $\bar{f}(s) = \frac{1}{(s+1)^{1/2}}$ $\therefore L^{-1}\left\{\frac{1}{(s+1)^{1/2}}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = e^{-t}\cdot\frac{t^{-1/2}}{\sqrt{\frac{1}{2}}} = \frac{e^{-t}}{\sqrt{\pi t}} = f(t)$

$$\therefore L^{-1}\left\{\frac{e^{-s}}{\sqrt{s+1}}\right\} = L^{-1}\left\{e^{-s}\cdot\bar{f}(s)\right\} = f(t-1)\; U(t-1)$$
$$= \frac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}\; U(t-1)$$

(h) $L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{-\frac{e^{-cs}}{a^2 s} + \frac{e^{-cs}}{a s^2} + \frac{e^{-cs}}{a^2(s+a)}\right\}$ (by partial fractions)

$$= -\frac{1}{a^2} L^{-1}\left\{e^{-cs}\cdot\bar{f}_1(s)\right\} + \frac{1}{a} L^{-1}\left\{e^{-cs}\bar{f}_2(s)\right\} + \frac{1}{a^2} L^{-1}\left\{e^{-cs}\bar{f}_3(s)\right\}$$

where $\bar{f}_1(s) = \frac{1}{s}$ $\therefore L^{-1}\left\{\frac{1}{s}\right\} = 1 = f_1(t)$

$\bar{f}_2(s) = \frac{1}{s^2}$ $\therefore L^{-1}\left\{\frac{1}{s^2}\right\} = t = f_2(t)$

$\bar{f}_3(s) = \frac{1}{s+a}$ $\therefore L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} = f_3(t)$

$$\therefore L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = -\frac{1}{a^2} f_1(t-c)\; U(t-c) + \frac{1}{a} f_2(t-c)\; U(t-c) + \frac{1}{a^2} f_3(t-c)\; U(t-c)$$
$$= \left[-\frac{1}{a^2}\cdot 1 + \frac{1}{a}(t-c) + \frac{1}{a^2}\cdot e^{-a(t-c)}\right] U(t-c)$$

Question based on derivatives i.e. $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -t\, f(t) = -t\, L^{-1}\{\bar{f}(s)\}$

$$\Rightarrow L^{-1}\{\bar{f}(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}$$

Que (27) :- Find the inverse Laplace transform of

(a) $\log\left(\frac{s+a}{s+b}\right)$, (b) $\log\left(\frac{s+1}{s-1}\right)$, (c) $\log\left(1+\frac{\omega^2}{s^2}\right)$

(d) $\tan^{-1}(s+1)$, (e) $\tan^{-1}\left(\frac{2}{s^2}\right)$, (f) $\cot^{-1}\left(\frac{s+3}{2}\right)$

(g) $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$, (h) $\frac{1}{2}\log\left(\frac{s^2+b^2}{(s-a)^2}\right)$

Sol (a) :-
$$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log\left(\frac{s+a}{s+b}\right)\right]\right\}$$
$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log(s+a) - \log(s+b)\right]\right\}$$
$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$
$$= -\frac{1}{t}\left(e^{-at} - e^{-bt}\right) = \frac{e^{-bt} - e^{-at}}{2}$$

(b)
$$L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log(s+1) - \log(s-\right.\right.$$
$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}$$
$$= -\frac{1}{t}\left\{e^{-t} - e^{t}\right\} = \frac{1}{t}\left(e^{t} - e^{-t}\right) = \frac{2}{t}\sinh t$$

(c)
$$L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s^2+\omega^2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+\omega^2) - \log s^2\right]\right\}$$
$$= -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$
$$= -\frac{2}{t}\left(\cos\omega t - 1\right)$$
$$= \frac{2}{t}\left(1 - \cos\omega t\right)$$

(d) $$L^{-1}\left\{\tan^{-1}(s+1)\right\} = -\frac{1}{t}\, L^{-1}\left\{\frac{d}{ds}\left[\tan^{-1}(s+1)\right]\right\}$$

$$= -\frac{1}{t}\, L^{-1}\left\{\frac{1}{1+(s+1)^2}\right\}$$

$$= -\frac{1}{t}\, e^{-t}\sin t$$

(e) $$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = -\frac{1}{t}\, L^{-1}\left\{\frac{d}{ds}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]\right\}$$

$$= -\frac{1}{t}\, L^{-1}\left\{\frac{1}{1+\frac{4}{s^4}}\cdot\left(\frac{-4}{s^3}\right)\right\}$$

$$= -\frac{1}{t}\, L^{-1}\left\{\frac{s^4}{s^4+4}\left(\frac{-4}{s^3}\right)\right\}$$

$$= \frac{4}{t}\, L^{-1}\left\{\frac{s}{s^4+4}\right\} \quad \text{———— (i)}$$

Now $$\frac{s}{s^4+4} = \frac{s}{s^4+4+4s^2-4s^2} = \frac{s}{(s^2+2)^2-(2s)^2} = \frac{s}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{1}{4}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right] \quad \text{(By partial fractio}$$

$$= \frac{1}{4}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\left\{\frac{s}{s^4+4}\right\} = \frac{1}{4}\, L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{4}\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= \frac{\sin t}{4}\left[e^{t}-e^{-t}\right]$$

$$= \frac{\sin t\,\sinh t}{2} \quad \text{———— (ii)}$$

from (i) and (ii) we get

$$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = \frac{4}{t}\,\frac{\sin t\,\sinh t}{2} = \frac{2}{t}\sin t\,\sinh t\,.$$

(f) $L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]\right\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{1+\left(\frac{s+3}{2}\right)^2}\cdot\left(-\frac{1}{2}\right)\right\}$

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$$= \frac{1}{2t} L^{-1}\left\{\frac{4}{(s+3)^2+4}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}$$

$$= \frac{1}{t} e^{-3t} \sin 2t$$

(g) $L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\} = -\frac{1}{2t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}$

$$= -\frac{1}{2t} L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]\right\}$$

$$= -\frac{1}{2t} L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\}$$

$$= -\frac{1}{t}\left[\cos bt - \cos at\right]$$

$$= \frac{\cos at - \cos bt}{t}$$

(h) $L^{-1}\left\{\frac{1}{2}\cdot\log\frac{s^2+b^2}{(s-a)^2}\right\} = -\frac{1}{2t} L^{-1}\left\{\frac{d}{ds}\log\frac{s^2+b^2}{(s-a)^2}\right\}$

$$= -\frac{1}{2t} L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+b^2) - 2\log(s-a)\right]\right\}$$

$$= -\frac{1}{2t} L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2}{s-a}\right\}$$

$$= -\frac{1}{t}\left[\cos bt - e^{at}\right]$$

$$= \frac{e^{at} - \cos bt}{t}$$

Questions based on integral formula i.e. $L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t} = \frac{1}{t} L^{-1}\{\bar{f}(s)\}$

$\Rightarrow L^{-1}\{\bar{f}(s)\} = t\, L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\}$

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Que 28 :– Find the inverse Laplace transform of

(a) $\frac{2as}{(s^2+a^2)^2}$, (b) $\frac{1}{(s+1)^2}$, (c) = $\frac{s+2}{(s^2+4s+5)^2}$

Sol :– (a)

$$L^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\} = t\, L^{-1}\left\{\int_s^\infty \frac{2as}{(s^2+a^2)^2}\,ds\right\} = t\, L^{-1}\left\{(-a)\left[\frac{1}{(s^2+a^2)}\right]_s^\infty\right\}$$

$$= t\, L^{-1}\left\{(-a)\left[0 - \frac{1}{s^2+a^2}\right]\right\} = t\, L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = t \sin at$$

(b)

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t\, L^{-1}\left\{\int_s^\infty \frac{1}{(s+1)^2}\,ds\right\} = t\, L^{-1}\left\{(-1)\left[\frac{1}{s+1}\right]_s^\infty\right\}$$

$$= t\, L^{-1}\left\{(-1)\left[0 - \frac{1}{s+1}\right]\right\} = t\, L^{-1}\left\{\frac{1}{s+1}\right\} = t\, e^{-t}$$

(c)

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = L^{-1}\left\{\frac{s+2}{[(s+2)^2+1]^2}\right\} = e^{-2t} L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\cancel{= e^{-2t} L^{-1}\left\{\left(-\tfrac{1}{2}\right)\left[\frac{1}{s^2+1}\right]_s\right\}}$$

$$= e^{-2t} \cdot t\, L^{-1}\left\{\int_s^\infty \frac{s}{(s^2+1)^2}\,ds\right\}$$

$$= e^{-2t}\, t\, L^{-1}\left\{\left(-\tfrac{1}{2}\right)\left[\frac{1}{s^2+1}\right]_s^\infty\right\}$$

$$= e^{-2t} \cdot t\, L^{-1}\left\{\left(-\tfrac{1}{2}\right)\left[0 - \frac{1}{s^2+1}\right]\right\}$$

$$= e^{-2t} \cdot \frac{t}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-2t} \cdot \frac{t}{2} \sin t$$

$$= \frac{t\, e^{-2t} \sin t}{2}$$

$$= t\, L^{-1}\left\{\frac{1}{2} \cdot \frac{1}{s^2+4s+5}\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \frac{t}{2} e^{-2t} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{t}{2} e^{-2t} \sin t$$

Questions based on convolution theorem i.e. $L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$

Que (29) :— Find the inverse Laplace transform by convolution theorem —

(a) $\dfrac{s}{(s^2+1)(s^2+4)}$, (b) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$, (c) $\dfrac{1}{s(s^2-a^2)}$

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(d) $\dfrac{1}{(s+1)(s^2+1)}$, (e) $\dfrac{s}{(s^2+a^2)^2}$

Sol (a) :— Let $\bar{f}(s) = \dfrac{s}{s^2+1}$ and $\bar{g}(s) = \dfrac{1}{s^2+4}$

$$\therefore \quad L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t = f(t)$$

$$\text{and} \quad L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t = g(t)$$

$$\therefore \quad L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+1)}\cdot\frac{1}{(s^2+4)}\right\} = \frac{1}{2}\int_0^t \cos u \,\sin 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2(t-u)\cdot\cos u\,du$$

$$= \frac{1}{4}\int_0^t \left[\sin\{2(t-u)+u\} + \sin\{2(t-u)-u\}\right] du$$

$$= \frac{1}{4}\int_0^t \left[\sin(2t-u) + \sin(2t-3u)\right] du$$

$$= \frac{1}{4}\left[\cos(2t-u) + \frac{1}{3}\cos(2t-3u)\right]_0^t$$

$$= \frac{1}{4}\left[\cos t + \frac{1}{3}\cos t - \cos 2t - \frac{1}{3}\cos 2t\right]$$

$$= \frac{1}{4}\left[\frac{4}{3}\cos t - \frac{4}{3}\cos 2t\right]$$

$$= \frac{1}{3}\left[\cos t - \cos 2t\right]$$

(b) Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+b^2}$

$\therefore \quad L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$

and $\quad L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt = g(t)$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at - \sin bt}{a+b}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

(c) Let $\bar{f}(s) = \frac{1}{s^2-a^2}$ and $\bar{g}(s) = \frac{1}{s}$

$\therefore \quad L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at = f(t)$

and $\quad L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$

$$\therefore \quad L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \sinh au \cdot 1\, du$$

$$= \frac{1}{a}\left[\frac{\cosh au}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left(\cosh at - 1\right)$$

(d) Let $\bar{f}(s) = \frac{1}{s^2+1}$ and $\bar{g}(s) = \frac{1}{s+1}$

$$\therefore L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$$

and $$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} = g(t)$$

$$\therefore L^{-1}\left\{\left(\frac{1}{s^2+1}\right)\cdot\left(\frac{1}{s+1}\right)\right\} = \int_0^t \sin u \cdot e^{-(t-u)}\,du$$

$$= e^{-t}\int_0^t e^{u}\sin u\,du$$

$$= e^{-t}\left[\frac{e^u}{2}(\sin u - \cos u)\right]_0^t$$

$$= e^{-t}\left[\frac{e^t}{2}(\sin t - \cos t) - \frac{e^0}{2}(0-1)\right]$$

$$= \frac{1}{2}\left(\sin t - \cos t + e^{-t}\right)$$

(e) Let $\bar{f}(s) = \frac{1}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+a^2}$

$$\therefore L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at = f(t)$$

and $$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = g(t)$$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+a^2)}\,\frac{s}{(s^2+a^2)}\right\} = \frac{1}{a}\int_0^t \sin au\,\cos a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t \left[\sin(au+at-au) + \sin(au-at+au)\right]du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(2au-at)\right]du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at - \frac{\cos at}{2a} - 0 + \frac{\cos at}{2a}\right]$$

$$= \frac{1}{2a}\,t\sin at$$

Que (30) :– Solve the integral equation by Laplace transform

(a) $y(t) = t^2 + \int_0^t y(u)\sin(t-u)\,du$ (b) $\int_0^t \frac{y(u)}{\sqrt{t-u}}\,du = 1 + t + t^2$

Sol :– (a) Taking Laplace transform on both sides, we get

$$L\{y(t)\} = L\{t^2\} + L\left\{\int_0^t y(u)\sin(t-u)\,du\right\}$$

$$\Rightarrow \bar{y}(s) = \frac{2}{s^3} + L\{y(t)\}\cdot L\{\sin t\} \quad (\text{by convolution theorem})$$

$$\Rightarrow \bar{y}(s) = \frac{2}{s^3} + \bar{y}(s)\cdot\frac{1}{s^2+1}$$

$$\Rightarrow \left[1 - \frac{1}{s^2+1}\right]\bar{y}(s) = \frac{2}{s^3}$$

$$\Rightarrow \bar{y}(s) = \frac{2}{s^3}\left(\frac{s^2+1}{s^2}\right) = \frac{2}{s^3} + \frac{2}{s^5}$$

Taking inverse Laplace transform, we get

$$L^{-1}\{\bar{y}(s)\} = y(t) = L^{-1}\left\{\frac{2}{s^3}\right\} + L^{-1}\left\{\frac{2}{s^5}\right\} = t^2 + \frac{t^4}{12}$$

(b) $$L\left\{\int_0^t \frac{y(u)}{\sqrt{t-u}}\,du\right\} = L\{1\} + L\{t\} + L\{t^2\}$$

$$\Rightarrow L\left\{\int_0^t y(u)\,(t-u)^{-1/2}\,du\right\} = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}$$

$$\Rightarrow L\{y(t)\}\cdot L\{t^{-1/2}\} = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}$$

$$\Rightarrow \bar{y}(s)\cdot\frac{\Gamma\frac{1}{2}}{s^{1/2}} = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3} \quad \left(\text{Since } \Gamma\tfrac{1}{2} = \sqrt{\pi}\right)$$

$$\Rightarrow \bar{y}(s) = \frac{1}{\sqrt{\pi}}\left[\frac{1}{s^{1/2}} + \frac{1}{s^{3/2}} + \frac{2}{s^{5/2}}\right]$$

Taking inverse Laplace transform

$$y(t) = \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\Gamma\frac{1}{2}} + \frac{t^{1/2}}{\Gamma\frac{3}{2}} + \frac{2t^{3/2}}{\Gamma\frac{5}{2}}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} + \frac{2t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\right]$$

$$= \frac{1}{\pi}\left[t^{-1/2} + 2t^{1/2} + \frac{8}{3}t^{3/2}\right]$$

(3.14) Application of Laplace transform :—

Working Rule :— (1) Taking the Laplace transform on both sides to the given differential equation (using the formula derivative of Laplace transform) and put the values of given initial conditions,

(2) Solve the algebraic equation, we get $\bar{y}(s)$ in terms of s.

(3) Taking inverse Laplace transform on both sides, this gives y in terms of t.

Que (31) :— Solve the differential equations by the method of Laplace transfo

(a) $y''' + 2y'' - y' - 2y = 0$, given $y(0) = y'(0) = 0$ and $y''(0) = 6$

(b) $y'' + 4y' + 3y = e^{-t}$, given $y(0) = y'(0) = 1$

(c) $y''' - 3y'' + 3y' - y = t^2 e^t$, given that $y(0) = 1$, $y'(0) = 0$, $y''(0) = -$

(d) $\dfrac{d^2x}{dt^2} + 9x = \cos 2t$ if $x(0) = 1$, $x\left(\frac{\pi}{2}\right) = -1$

(e) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^t$, with $x = 2$, $\dfrac{dx}{dt} = -1$ at $t = 0$

(f) $(D^2 + n^2)x = a \sin(nt + \alpha)$, $x = Dx = 0$ at $t = 0$

(g) $(D^2 + 2D + 5)y = e^{-t} \sin t$ where $y(0) = 0$, $y'(0) = 1$

(h) $y''' - 2y'' + 5y' = 0$, $y = 0$, $y' = 1$ at $t = 0$ and $y = 1$ at $t = \frac{\pi}{8}$

Sol (a) :- $y''' + 2y'' - y' - 2y = 0$ ——— (1)

Taking Laplace transform on both sides we get

$$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = 0$$

$$\Rightarrow [s^3\bar{y}(s) - s^2 y(0) - s y'(0) - y''(0)] + 2[s^2\bar{y}(s) - s y(0) - y'(0)]$$
$$- [s\bar{y}(s) - y(0)] - 2\bar{y}(s) = 0$$

given that $y(0) = y'(0) = 0$ and $y''(0) = 6$, we get

$$\therefore s^3\bar{y}(s) - 6 + 2s^2\bar{y}(s) - s\bar{y}(s) - 2\bar{y}(s) = 0$$

$$\Rightarrow (s^3 + 2s^2 - s - 2)\bar{y}(s) = 6$$

or $$\bar{y}(s) = \frac{6}{(s^3+2s^2-s-2)} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}$$

(By partial fraction)

Taking inverse Laplace transform, we get

$$L^{-1}\{\bar{y}(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{s+1}\right\} + 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$$

(b) $y'' + 4y' + 3y = e^{-t}$ ——— (1)

Taking Laplace transform on both sides we get

$$L\{y''\} + 4L\{y'\} + 3L\{y\} = L\{e^{-t}\}$$

$$\Rightarrow [s^2\bar{y}(s) - s y(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$$

using the give condition $y(0)=1$ and $y'(0) = 1$

$$[s^2\bar{y}(s) - s - 1] + 4[s\bar{y}(s) - 1] + 3\bar{y}(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)\bar{y}(s) = (s+5) + \frac{1}{(s+1)}$$

$$\Rightarrow \bar{y}(s) = \frac{(s+5)}{(s+1)(s+3)} + \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{2}{(s+1)} - \frac{1}{(s+3)} + \frac{1}{4}\cdot\frac{1}{(s+3)} - \frac{1}{4(s+1)} + \frac{1}{2}\cdot\frac{1}{(s+1)^2}$$

(By partial fraction)

$$= \frac{7}{4}\cdot\frac{1}{s+1} - \frac{3}{4}\frac{1}{(s+3)} + \frac{1}{2}\cdot\frac{1}{(s+1)^2}$$

$$\therefore y(t) = \frac{7}{4}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{3}{4}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} + \frac{1}{2}te^{-t}$$

(c) $y''' - 3y'' + 3y' - y = t^2 e^t$ ——— (1)

Taking Laplace transform on both sides, we get

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\{t^2 e^t\}$$

$$\Rightarrow [s^3\bar{y}(s) - s^2 y(0) - s y'(0) - y''(0)] - 3[s^2\bar{y}(s) - s y(0) - y'(0)]$$

$$+ 3[s\bar{y}(s) - y(0)] - \bar{y}(s) = \frac{2}{(s-1)^3}$$

putting the given condition i.e. $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$

$$[s^3\bar{y}(s) - s^2 + 2] - 3[s^2\bar{y}(s) - s] + 3[s\bar{y}(s) - 1] - \bar{y}(s) = \frac{2}{(s-1)^3}$$

$$[s^3 - 3s^2 + 3s - 1]\bar{y}(s) = (s^2 - 3s + 1) + \frac{2}{(s-1)^3}$$

$$\Rightarrow (s-1)^3\bar{y}(s) = (s^2 - 2s + 1 - s) + \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y}(s) = \frac{(s-1)^2 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2}{(s-1)^3} - \frac{(s-1+1)}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\{\bar{y}(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^3}\right\} + 2L^{-1}\left\{\frac{1}{(s-1)^6}\right\}$$

$$= e^t - t e^t - \frac{1}{2}t^2 e^t + \frac{2}{\lfloor 5} t^5 e^t$$

$$= e^t\left[1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right]$$

(d) $x'' + 9x = \cos 2t$ —— (1) given that $x(0) = 1$ and $x\left(\frac{\pi}{2}\right) = -1$

Taking Laplace transform on both sides, we get

$$L\{x''\} + 9\,L\{x\} = L\{\cos 2t\}$$

$$\Rightarrow \quad s^2 \bar{x}(s) - s\,x(0) - x'(0) - 9\,\bar{x}(s) = \frac{s}{s^2+4}$$

put $x(0) = 1$ and let $x'(0) = a$, we get

$$s^2 \bar{x}(s) - s - a + 9\,\bar{x}(s) = \frac{s}{s^2+4}$$

$$(s^2+9)\,\bar{x}(s) = s + a + \frac{s}{s^2+4}$$

$$\Rightarrow \quad \bar{x}(s) = \frac{s}{s^2+9} + \frac{a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$= \frac{s}{s^2+9} + \frac{a}{s^2+9} + \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

(By partial fraction)

$$= \frac{s}{s^2+9} + \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{(s^2+4)} - \frac{1}{5}\cdot\frac{s}{(s^2+9)}$$

$$= \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{(s^2+4)} + \frac{4}{5}\cdot\frac{s}{(s^2+9)} \quad \text{—— (2)}$$

Taking inverse Laplace transform we get

$$L^{-1}\{\bar{x}(s)\} = a\,L^{-1}\left\{\frac{1}{s^2+9}\right\} + \frac{1}{5}\,L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{4}{5}\,L^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$\Rightarrow \quad x(t) = \frac{a}{3}\sin 3t + \frac{1}{5}\cdot\cos 2t + \frac{4}{5}\cos 3t \quad \text{—— (3)}$$

putting $t = \frac{\pi}{2}$ ~~in~~ equation (3) we get

$$-1 = \frac{a}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\cos\pi + \frac{4}{5}\cos\frac{3\pi}{2}$$

$$= \frac{a}{3}(-1) + \frac{1}{5}(-1)$$

$$\Rightarrow \quad \frac{a}{3} = \frac{4}{5} \quad \text{or} \quad a = \frac{12}{5} \quad \text{—— (4)}$$

~~or a =~~

Hence the solution is

$$x = \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$$

$$= \frac{1}{5}\left(4\sin 3t + \cos 2t + 4\cos 3t\right)$$

(e) $x'' - 2x' + x = e^{t}$, given that $x(0) = 2$, $x'(0) = -1$

Taking Laplace transform on both sides, we get

$$L\{x''\} - 2L\{x'\} + L\{x\} = L\{e^{t}\}$$

$$\Rightarrow \left[s^2 \bar{x}(s) - s\,x(0) - x'(0)\right] - 2\left[s\,\bar{x}(s) - x(0)\right] + \bar{x}(s) = \frac{1}{s-1}$$

putting the values of given condition, we get

$$(s^2 - 2s + 1)\,\bar{x}(s) - 2s + 5 = \frac{1}{s-1}$$

or $$(s-1)^2\,\bar{x}(s) = (2s-5) + \frac{1}{s-1}$$

or $$\bar{x}(s) = \frac{(2s-2-3)}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$= \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Taking L^{-1} on both sides we get

$$x(t) = 2L^{-1}\left\{\frac{1}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= 2e^{t} - 3te^{t} + \frac{t^2}{2}e^{t} = e^{t}\left(2 - 3t + \frac{t^2}{2}\right)$$

(f) $x'' + n^2 x = a\sin(nt+\alpha)$, given that $x(0) = x'(0) = 0$

Taking Laplace transform on both sides, we get

$$L\{x''\} + n^2 L\{x\} = a\,L\{\sin nt\cos\alpha + \cos nt\sin\alpha\}$$

$$\Rightarrow s^2\bar{x}(s) - s\,x(0) - x'(0) + n^2\bar{x}(s) = a\cos\alpha\cdot\frac{n}{s^2+n^2} + a\sin\alpha\cdot\frac{s}{s^2+n^2}$$

$$\Rightarrow (s^2+n^2)\,\bar{x}(s) = an\cos\alpha\cdot\frac{1}{s^2+n^2} + a\sin\alpha\cdot\frac{s}{s^2+n^2}$$

$$\Rightarrow \bar{x}(s) = an\cos\alpha\cdot\frac{1}{(s^2+n^2)^2} + a\sin\alpha\cdot\frac{s}{(s^2+n^2)^2}$$

Taking L^{-1} on both sides, we get

$$L^{-1}\{\bar{x}(s)\} = an\cos\alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha\, L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}$$

Using convolution theorem, we get

$$x(t) = an\cos\alpha\cdot\frac{1}{2n^3}\left(\sin nt - nt\cos nt\right) + a\sin\alpha\cdot\frac{t}{2n}\sin nt$$

$$= \frac{a}{2n^2}\left(\sin nt\cos\alpha - nt\cos(nt+\alpha)\right)$$

(g) $y'' + 2y' + 5y = e^{-t}\sin t$, given that $y(0)=0$ and $y'(0)=1$

Taking Laplace transform on both sides, we get

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$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}$$

$$\Rightarrow [s^2\bar{y}(s) - s\,y(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] + 5\bar{y}(s) = \frac{1}{(s+1)^2+1}$$

putting the value of $y(0)=0$ and $y'(0)=1$, we get

$$(s^2+2s+5)\bar{y}(s) - 1 = \frac{1}{s^2+2s+2}$$

or $$(s^2+2s+5)\bar{y}(s) = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

or $$\bar{y}(s) = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{2}{3}\cdot\frac{1}{(s^2+2s+5)} + \frac{1}{3}\frac{1}{(s^2+2s+2)}$$

(By partial fraction)

$$= \frac{2}{3}\cdot\frac{1}{(s+1)^2+4} + \frac{1}{3}\frac{1}{(s+1)^2+1}$$

$$\therefore L^{-1}\{\bar{y}(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$\therefore y(t) = \frac{1}{3}e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

(h) $y''' - 2y'' + 5y' = 0$ ——(1) given that $y(0)=0$, $y'(0)=1$ and $y=1$ at $t=\frac{\pi}{8}$

Taking Laplace transform on both sides, we get

$$L\{y'''\} - 2L\{y''\} + 5L\{y'\} = 0$$

$$\Rightarrow [s^3\bar{y}(s) - s^2y(0) - s\,y'(0) - y''(0)] - 2[s^2\bar{y}(s) - s\,y(0) - y'(0)] + 5[s\bar{y}(s) - y(0)] = 0$$

putting $y(0)=0$ and $y'(0)=1$ and suppose $y''(0)=a$, then

$$(s^3 - 2s^2 + 5s)\bar{y}(s) - s - a + 2 = 0$$

$$\Rightarrow \bar{y}(s) = \frac{(a-2)+s}{s(s^2-2s+5)} = \frac{a-2}{s(s^2-2s+5)} + \frac{1}{(s^2-2s+5)}$$

$$= \frac{(a-2)}{5}\left[\frac{1}{s} - \frac{s-2}{s^2-2s+5}\right] + \frac{1}{s^2-2s+5} \quad \text{(By partial fract}$$

$$= \frac{(a-2)}{5}\cdot\frac{1}{s} - \frac{(a-2)}{5}\left[\frac{s-1}{(s-1)^2+4}\right] + \frac{(a+3)}{10}\left\{\frac{2}{(s-1)^2+4}\right\}$$

$$\therefore y(t) = \frac{(a-2)}{5} - \frac{(a-2)}{5}e^{t}\cos 2t + \frac{(a+3)}{10}e^{t}\sin 2t \quad \text{——(2)}$$

putting $t=\frac{\pi}{8}$ in eqn (2) we get

$$1 = \frac{(a-2)}{5} - \frac{(a-2)}{5}e^{\pi/8}\cdot\frac{1}{\sqrt{2}} + \frac{(a+3)}{10}e^{\pi/8}\cdot\frac{1}{\sqrt{2}} \Rightarrow A=7$$

Hence sol is $y(t) = 1 + e^{t}(\sin 2t - \cos 2t)$

Que 32 :- Using Laplace transform, solve the differential equation

(a) $y''+2ty'-y=t$, when $y(0)=0$ and $y'(0)=1$

(b) $ty''+2y'+ty=\cos t$ given that $y(0)=1$

(c) $ty''+y'+4ty=0$ given that $y=3$ and $y'=0$ when $t=0$

Sol — (a) $y''+2ty'-y=t$ —— (1)

Taking Laplace transform on both sides, we get

$$L\{y''\}+2L\{ty'\}-L\{y\}=L\{t\}$$

$$\Rightarrow \left[s^2\bar{y}-sy(0)-y'(0)\right]+2\left[-\frac{d}{ds}L\{y'\}\right]-\bar{y}=\frac{1}{s^2}$$

$$\Rightarrow s^2\bar{y}-1-2\frac{d}{ds}\left[s\bar{y}-y(0)\right]-\bar{y}=\frac{1}{s^2}$$ (Since $y(0)=0$ and $y'(0)=1$

$$\Rightarrow s^2\bar{y}-1-2\frac{d}{ds}(s\bar{y})-\bar{y}=\frac{1}{s^2}$$

$$\Rightarrow s^2\bar{y}-2\left(s\frac{d\bar{y}}{ds}+\bar{y}\right)-\bar{y}=1+\frac{1}{s^2}$$

$$\Rightarrow -2s\frac{d\bar{y}}{ds}+(s^2-3)\bar{y}=1+\frac{1}{s^2}$$

$$\Rightarrow \frac{d\bar{y}}{ds}+\left(\frac{3-s^2}{2s}\right)\bar{y}=-\frac{1}{2s}-\frac{1}{2s^3}$$ —— (2)

equation (2) is a linear differential equation of I order hence

$$\text{I.F.}=e^{\int\left(\frac{3-s^2}{2s}\right)ds}=e^{\frac{1}{2}\int\left(\frac{3}{s}-s\right)ds}=e^{\frac{1}{2}\left(3\log s-\frac{s^2}{2}\right)}=s^{\frac{3}{2}}\cdot e^{-\frac{s^2}{4}}$$ —— (3)

∴ solution of eqn (2) is

$$\bar{y}\cdot s^{\frac{3}{2}}e^{-\frac{s^2}{4}}=-\frac{1}{2}\int\left(\frac{1}{s}+\frac{1}{s^3}\right)s^{\frac{3}{2}}e^{-\frac{s^2}{4}}ds+C=-\frac{1}{2}\int\left(s^{1/2}+\frac{1}{s^{3/2}}\right)e^{-\frac{s^2}{4}}$$

put $s^2=4x \Rightarrow s=2\sqrt{x} \Rightarrow ds=2\cdot\frac{1}{2}\cdot x^{-1/2}dx=$

$$\therefore \bar{y}\cdot s^{\frac{3}{2}}e^{-\frac{s^2}{4}}=-\frac{1}{2}\int\left(\sqrt{2}\,x^{1/4}+\frac{1}{2\sqrt{2}}x^{-\frac{3}{4}}\right)e^{-x}\frac{dx}{\sqrt{x}}+C$$

$$=-\frac{1}{\sqrt{2}}\int\left(x^{-1/4}+\frac{1}{4}x^{-\frac{5}{4}}\right)e^{-x}dx+C$$

$$=-\frac{1}{\sqrt{2}}\left[x^{-\frac{1}{4}}\left(\frac{e^{-x}}{-1}\right)+\int\left(-\frac{1}{4}\right)x^{-\frac{5}{4}}e^{-x}dx+\frac{1}{4}\int x^{-\frac{5}{4}}e^{-x}dx\right]$$

$$=-\frac{1}{\sqrt{2}}\left[x^{-\frac{1}{4}}\frac{e^{-x}}{-1}\right]+C$$

$$\bar{y}\cdot s^{\frac{3}{2}}\cdot e^{-\frac{s^2}{4}} = \frac{1}{\sqrt{2}}\left(\frac{s^2}{4}\right)^{-1/4} e^{-\frac{s^2}{4}} + c = \frac{1}{\sqrt{s}}\, e^{-\frac{s^2}{4}} + c$$

$$\Rightarrow \quad \bar{y} = \frac{1}{s^2} + \frac{c}{s^{\frac{3}{2}}}\cdot e^{\frac{s^2}{4}}$$

, where c is a constant, c must be vanish if $\bar{y}$ is transform since $\bar{y} \to 0$ as $s \to \infty$, hence

$$\bar{y} = \frac{1}{s^2} \qquad (4)$$

Taking inverse Laplace transform on both sides, we get

$$y = L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

(b) $$t\,y'' + 2y' + t\,y = \cos t \qquad (1)$$

Taking Laplace transform on both sides, we get

$$L\{ty''\} + 2L\{y'\} + L\{ty\} = L\{\cos t\}$$

$$\Rightarrow -\frac{d}{ds}L\{y''\} + 2\left[s\bar{y} - y(0)\right] - \frac{d}{ds}L\{y\} = \frac{s}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds}\left(s^2\bar{y} - s\,y(0) - y'(0)\right) + 2s\bar{y} - 2y(0) - \frac{d}{ds}\bar{y} = \frac{s}{s^2+1}$$

$$\Rightarrow -\left(s^2\frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d\bar{y}}{ds} = \frac{s}{s^2+1}$$

$$\Rightarrow -(s^2+1)\frac{d\bar{y}}{ds} - 1 = \frac{s}{s^2+1} \qquad (\text{Since } y(0) = 1)$$

or $$\frac{d\bar{y}}{ds} = -\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2} \qquad (2)$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\left\{\frac{d\bar{y}}{ds}\right\} = -L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\Rightarrow -t\,y = -\sin t - \frac{1}{2}\,t\sin t$$

or $$y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t$$

(c) $t y'' + y' + 4 t y = 0$ ——— (1)

Taking Laplace transform on both sides we get

$L\{t y''\} + L\{y'\} + 4 L\{t y\} = 0$

$\Rightarrow -\frac{d}{ds} L\{y''\} + [s\bar{y} - y(0)] - 4 \frac{d}{ds} L\{y\} = 0$

$\Rightarrow -\frac{d}{ds}\left[s^2 \bar{y} - s y(0) - y'(0)\right] + (s\bar{y} - 3) - 4\frac{d\bar{y}}{ds} = 0$

$\Rightarrow -\left(s^2 \frac{d\bar{y}}{ds} + 2s\bar{y} - 3\right) + (s\bar{y} - 3) - 4\frac{d\bar{y}}{ds} = 0$

$\Rightarrow (s^2+4)\frac{d\bar{y}}{ds} + s\bar{y} = 0$ ——— (2)

using separation of variables method

$$\frac{d\bar{y}}{\bar{y}} + \frac{s}{s^2+4}\, ds = 0$$

on integration we get

$$\log \bar{y} + \frac{1}{2}\log(s^2+4) = \log c$$

or $\bar{y} = \dfrac{c}{\sqrt{s^2+4}}$ ——— (3)

Taking Laplace transform on both sides we get

$y(t) = c\, J_0(2t)$ ——— (4)

putting the condition at $t = 0$, $y = 3$, we get

$3 = c\, J_0(0) = c \cdot 1 \Rightarrow c = 3$

$\therefore y(t) = 3\, J_0(2t)$

Que (33) :— Using Laplace transform solve the simultaneous differential equations

(a) $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given that $x(0) = 1$, $y(0) = 0$

(b) $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, given that $x(0) = y(0) = 0$

(c) $(D-2)x - (D+1)y = 6e^{3t}$, $(2D-3)x + (D-3)y = 6e^{3t}$

given $x = 3$, $y = 0$ when $t = 0$

Sol (a) :—

$$x' - y = e^t \quad \text{——— (1)}$$

$$y' + x = \sin t \quad \text{——— (2)}$$

Taking Laplace transform of eq^n (1) and (2), we get

$$s\bar{x} - x(0) - \bar{y} = \frac{1}{s-1}$$

$$\Rightarrow s\bar{x} - 1 - \bar{y} = \frac{1}{s-1} \quad (\text{Since } x(0) = 1)$$

$$\text{or} \quad s\bar{x} - \bar{y} = \frac{s}{s-1} \quad \text{——— (3)}$$

and

$$s\bar{y} - y(0) + \bar{x} = \frac{1}{s^2+1}$$

$$\text{or} \quad \bar{x} + s\bar{y} = \frac{1}{s^2+1} \quad \text{——— (4)} \quad (\text{Since } y(0) = 0)$$

Solving (3) & (4) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \begin{vmatrix} \frac{s}{s-1} & -1 \\ \frac{1}{s^2+1} & s \end{vmatrix} \div \begin{vmatrix} s & -1 \\ 1 & s \end{vmatrix}$$

$$= \left(\frac{s^2}{s-1} + \frac{1}{s^2+1}\right) \div (s^2+1) = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$\text{or} \quad \bar{x} = \frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2} \quad \text{——— (5)}$$

Taking L^{-1} on both sides

$$x = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}\right] + L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$= \frac{1}{2}\left[e^t + \cos t + \sin t\right] + \frac{1}{2}(\sin t - t\cos t) \quad (\text{By Convolution th}$$

$$= \frac{1}{2}\left[e^t + \cos t + 2\sin t - t\cos t\right] \quad \text{——— (6)}$$

and $\bar{y} = \begin{vmatrix} s & \frac{s}{s-1} \\ 1 & \frac{1}{s^2+1} \end{vmatrix} \div \begin{vmatrix} s & -1 \\ 1 & s \end{vmatrix}$

$$= \left(\frac{s}{s^2+1} - \frac{s}{s-1}\right) \div (s^2+1) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] \quad \text{——— (7)}$$

Taking L^{-1} on both sides we get

$$y = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2} t \sin t - \frac{1}{2}\left(e^t - \cos t + \sin t\right)$$

$$= \frac{1}{2}\left(t \sin t - e^t + \cos t - \sin t\right) \quad \text{——— (8)}$$

(b) $\quad x' + 5x - 2y = t \quad$ ——— (1)

and $\quad y' + 2x + y = 0 \quad$ ——— (2)

Taking Laplace transform on both sides of eqn (1) & (2)

$$s\bar{x} - x(0) + 5\bar{x} - 2\bar{y} = \frac{1}{s^2}$$

or $\quad (s+5)\bar{x} - 2\bar{y} = \frac{1}{s^2} \quad$ ——— (3) $\quad (\because x(0) = 0)$

and $\quad s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0$

or $\quad 2\bar{x} + (s+1)\bar{y} = 0 \quad$ ——— (4) $\quad (\because y(0) = 0)$

On solving equation (3) & (4) for $\bar{x}$ and, we get

$$\bar{x} = \begin{vmatrix} \frac{1}{s^2} & -2 \\ 0 & s+1 \end{vmatrix} \div \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix}$$

$$= \frac{(s+1)}{s^2} \div \left((s+5)(s+1) + 4\right)$$

$$= \frac{(s+1)}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2} \quad \text{——— (5)}$$

Putting the value of $\bar{x}$ in eqn. (4) we get

$$\bar{y} = -\frac{2}{(s+1)}\bar{x} = -\frac{2}{(s+1)}\cdot\frac{(s+1)}{s^2(s+3)^2} = -\frac{2}{s^2(s+3)^2}$$

$$= \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2} \quad \text{———} (6)$$

Taking L^{-1} of eqⁿ (5) and (6) we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}t\,e^{-3t} \quad \text{———} (7)$$

and $y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}t\,e^{-3t}$ ——— (8)

(c) $x' - 2x - y' - y = 6e^{3t}$ ——— (1)

and $2x' - 3x + y' - 3y = 6e^{3t}$ ——— (2)

Taking Laplace transform of eqⁿ (1) and (2) we get

$$s\bar{x} - x(0) - 2\bar{x} - s\bar{y} + y(0) - \bar{y} = \frac{6}{s-3}$$

or $(s-2)\bar{x} - (s+1)\bar{y} - 3 = \frac{6}{s-3}$ (Since $x(0) = 3$,

or $(s-2)\bar{x} - (s+1)\bar{y} = \frac{3s-3}{s-3}$ ——— (3)

and $2(s\bar{x} - x(0)) - 3\bar{x} + s\bar{y} - y(0) - 3\bar{y} = \frac{6}{s-3}$

$$(2s-3)\bar{x} + (s-3)\bar{y} - 6 = \frac{6}{s-3}$$

or $(2s-3)\bar{x} + (s-3)\bar{y} = \frac{6s-12}{s-3}$ ——— (4)

Solving eqⁿ (3) & (4) for $\bar{x}$ and $\bar{y}$ we get

$$\bar{x} = \begin{vmatrix} \frac{3(s-1)}{s-3} & -(s+1) \\ \frac{6(s-2)}{s-3} & (s-3) \end{vmatrix} \div \begin{vmatrix} (s-2) & -(s+1) \\ (2s-3) & (s-3) \end{vmatrix}$$

$$= \left[3(s-1) + \frac{6(s+1)(s-2)}{s-3}\right] \div \left[(s-2)(s-3) + (2s-3)(s+1)\right]$$

$$= \left[3(s-1) + \frac{6(s+1)(s-2)}{s-3}\right] \div \left[3(s-1)^2\right]$$

$$= \frac{3(s-1)}{3(s-1)^2} - \frac{6(s+1)(s-2)}{3(s-3)(s-1)^2} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{s-3} \quad \text{———} ($$

$$\bar{y} = \begin{vmatrix} (s-2) & \frac{3(s-1)}{s-3} \\ (2s-3) & \frac{6(s-2)}{s-3} \end{vmatrix} \div \begin{vmatrix} (s-2) & -(s+1) \\ (2s-3) & (s-3) \end{vmatrix}$$

$$= \left[\frac{6(s-2)^2}{s-3} - \frac{3(s-1)(2s-3)}{s-3} \right] \div \left[3(s-1)^2 \right]$$

$$= \frac{5-3s}{(s-1)^2(s-3)} = \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-3)} \quad \text{——} (6)$$

(By partial fraction)

Taking L^{-1} of equation (5) & (6) we get

$$x = e^t + 2te^t + 2e^{3t}$$

and $y = e^t - te^t - e^{3t}$

Que (34) :- A resistance R in series with inductance L is connected with e.m.f. E(t). The current i is given by

(76)

$$L \frac{di}{dt} + Ri = E(t)$$

If the switch is connected at $t=0$ and disconnected at $t=a$, find the current i in terms of t.

Sol :- The given condition is the form of unit step function i.e.

$$E(t) = \begin{cases} E & , 0 < t < a \\ 0 & , t \geq a \end{cases} \quad \text{——— (1)}$$

and when $i=0$ at $t=0$

$$L \frac{di}{dt} + Ri = E(t) \quad \text{——— (2)}$$

Taking Laplace transform on both sides of equation (2), we get

$$L[s\bar{i} - i(0)] + R\bar{i} = \int_0^\infty e^{-st} E(t)\,dt$$

$$\Rightarrow L s\bar{i} + R\bar{i} = \int_0^a e^{-st}.E\,dt + \int_a^\infty e^{-st}.0\,dt$$

$$\Rightarrow (Ls+R)\bar{i} = E\left[\frac{e^{-st}}{-s}\right]_0^a + 0$$

$$= \frac{E}{-s}\left(e^{-as} - 1\right) = \frac{E}{s} - \frac{E}{s}e^{-as}$$

$$\Rightarrow \bar{i} = \frac{E}{s(Ls+R)} - \frac{E e^{-as}}{s(Ls+R)} \quad \text{——— (3)}$$

Taking inverse Laplace transform, we get

$$i = L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} - L^{-1}\left\{\frac{E e^{-as}}{s(Ls+R)}\right\} \quad \text{——— (4)}$$

Now $L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} = \frac{E}{L} L^{-1}\left\{\frac{1}{s(s+\frac{R}{L})}\right\} = \frac{E}{L}\cdot\frac{L}{R} L^{-1}\left\{\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right\} = \frac{E}{R}\left[1 - e\right.$

again $L^{-1}\left\{\frac{E e^{-as}}{s(Ls+R)}\right\} = \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right] U(t-a)$ (where $U(t-a)$ is unit step fu

hence from eqn (4)

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right] U(t-a) \quad \text{——— (5)}$$

Case I – For $0 < t < a$ then $U(t-a) = 0$, hence

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] \quad \text{——— (6)}$$

Case II :- For $t \geq a$ then $U(t-a) = 1$, hence

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]$$

$$= \frac{E}{R}\left[1 - e^{-\frac{R}{L}t} - 1 + e^{-\frac{R}{L}(t-a)}\right]$$

$$= \frac{E}{R}\, e^{-\frac{R}{L}t}\left[e^{\frac{Ra}{L}} - 1\right] \quad \text{——— (7)}$$

Que (5) :– A condenser of capacity c is charged to potential E and discharge at $t=0$ through an inductive resistance L, R. Show that the charge at time t is given by

$$q = \frac{cE}{n} e^{-\mu t}(\mu \sin nt + n \cos nt), \text{ where } \mu = \frac{R}{2L} \text{ and } n^2 = \left(\frac{1}{Lc} - \frac{R^2}{4L^2}\right)$$

Sol :– Let q be the charge and i be the current in the circuit at time t, then by voltange law the potential drop across L, R and c are $L\frac{di}{dt}$, Ri and $\frac{q}{c}$ is zero.

$$\therefore \quad L\frac{di}{dt} + Ri + \frac{q}{c} = 0 \quad \text{or} \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0 \quad \text{——— (1)} \left(\because i = \frac{dq}{dt}\right)$$

Taking Laplace transform of eq^n (1), we get

$$L\left[s^2\bar{q} - s\,q(0) - q'(0)\right] + R\left[s\bar{q} - q(0)\right] + \frac{1}{c}\bar{q} = 0$$

$$\Rightarrow \left(Ls^2 + Rs + \frac{1}{c}\right)\bar{q} = (Ls + R)EC$$

$$(\text{Since } q = EC \text{ and } i = 0 \text{ at } t = 0)$$

$$\Rightarrow \bar{q} = \frac{(Ls+R)EC}{L\left(s^2 + Rs + \frac{1}{c}\right)} = \frac{EC\left(s + \frac{R}{L}\right)}{s^2 + \frac{R}{L}s + \frac{1}{Lc}} = \frac{EC(s + 2\mu)}{s^2 + 2\mu s + \mu^2 + n^2} \quad \text{——— (2)}$$

Taking inverse Laplace transform, we get

$$q = EC\,L^{-1}\left[\frac{s+\mu}{(s+\mu)^2 + n^2} + \frac{\mu}{(s+\mu)^2 + n^2}\right]$$

$$= EC\,\frac{e^{-\mu t}}{n}\left(n\cos nt + \mu \sin nt\right)$$

Find the Laplace transform (of questions 1 to 6) —

Que ① (a) $1+2\sqrt{t}+3/\sqrt{t}$ Ans :- $\frac{1}{s}+\frac{\sqrt{\pi}}{s^{3/2}}+3\cdot\sqrt{\frac{\pi}{s}}$

(b) $\cos(at+b)$ Ans :- $\frac{s\cos b - a\sin b}{s^2+a^2}$

(c) $t-\sinh 2t$ — Ans :— $\frac{4+s^2}{s^2(4-s^2)}$

② (a) $f(t)=\begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & t>3 \end{cases}$ Ans — $\frac{2}{s^3}-\frac{e^{-2s}}{s^3}(2+3s+3s^2)+\frac{e^{-3s}}{s^2}(5s-1)$

(b) $f(t)=\begin{cases} \sin t, & 0<t<\pi \\ 0 & t>\pi \end{cases}$ Ans :— $\frac{1+e^{-\pi s}}{1+s^2}$

(c) $f(t)=\begin{cases} \sin(t-\frac{\pi}{3}), & t>\frac{\pi}{3} \\ 0, & t<\frac{\pi}{3} \end{cases}$ Ans :— $e^{-\frac{\pi s}{3}}\cdot\frac{1}{s^2+1}$

③ (a) $e^{-3t}(2\cos 5t-3\sin 5t)$ Ans :- $\frac{2s-9}{s^2+6s+34}$

(b) $\sin 2t\ \sin 3t$ Ans :- $\frac{12s}{(s^2+1)(s^2+25)}$

(c) $e^{3t}\sin^2 t$ Ans :- $\frac{1}{2}\left[\frac{1}{s-3}-\frac{s-3}{(s-3)^2+4}\right]$

④ (a) $t^2\cos at$ Ans :- $\frac{2s^3-6a^2s}{(s^2+a^2)^3}$

(b) $t\sinh at$ Ans — $\frac{2as}{(s^2-a^2)^2}$

(c) $t^2\sin t$ Ans :- $\frac{2(3s^2-1)}{(s^2+1)^3}$

⑤ (a) $\frac{1-e^{-t}}{t}$ Ans :- $\log\left(\frac{s-1}{s}\right)$

(b) $\frac{\cos at-\cos bt}{t}$, Ans :- $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

(c) $\frac{e^{-t}\sin t}{t}$, Ans :- $\cot^{-1}(s+1)$

⑥ (a) $\int_0^t \frac{e^t\sin t}{t}\,dt$ Ans — $\frac{1}{s}\cot^{-1}(s-1)$

(b) $\int_0^t \frac{1-\cos 2t}{t}\,dt$ Ans :- $\frac{1}{2s}\log\left(1+\frac{4}{s^2}\right)$

(c) $\int_0^t \frac{e^{-at}-e^{-bt}}{t}\,dt$ Ans — $\frac{1}{s}\log\left(\frac{s+b}{s+a}\right)$

Que (7) Evaluate the following integrals using Laplace transform —

(a) $\int_0^\infty \frac{\sin t}{t}\,dt$, Ans:- $\frac{\pi}{2}$, (b) $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$ Ans:- $\pi/2$

(c) $\int_0^\infty e^{-2t}\, t \cos t\, dt$, Ans:- $\frac{3}{25}$

Que (8) Find the Laplace transform of the periodic functions —

(a) $f(t) = \sin\left(\frac{\pi t}{a}\right)$ for $0 < t < a$, (Rectified sine wave of period a)
Ans:— $a\pi \coth\left(\frac{as}{2}\right) / (a^2 s^2 + \pi^2)$

(b) $f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$ and $f(t+2) = f(t)$, Ans:- $\frac{1 - e^{-s}(s+1)}{s^2(1-e^{-2s})}$

(c) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ and $f(t+2\pi) = f(t)$, Ans:- $\frac{1}{(1-e^{-\pi s})(1+s^2)}$

Que (9) Find the Laplace transform of unit step function

(a) $e^{-3t}\, U(t-2)$ Ans:- $\frac{e^{-2(s+3)}}{s+3}$, (b) $\sin t\; U(t-\pi)$ Ans:- $\frac{e^{-\pi s}}{s^2+1}$

(c) $t^2\, U(t-2)$, Ans - $\frac{2e^{-2s}}{s^3}(2s^2+2s+1)$

Que (10) Express the following functions in terms of unit step function and find Laplace transform —

(a) $f(t) = \begin{cases} E, & a < t < b \\ 0, & t \geq b \end{cases}$ Ans:- $E[U(t-a) - U(t-b)]$, $\frac{E}{s}\left(e^{-as} - e^{-bs}\right)$

(b) $f(t) = \begin{cases} t & 0 < t < 2 \\ 0 & t \geq 2 \end{cases}$ Ans:- $[U(t) - U(t-2)]$, $\frac{1-(2s+1)e^{-2s}}{s^2}$

Find the inverse Laplace transform (of question 11 to 16)

Que (11) (a) $\frac{2s+9}{s^2+9} + \frac{12}{4-3s} + \frac{1}{\sqrt{s}}$ Ans:- $2\cos 3t - 3\sin 3t - 4e^{\frac{4}{3}t} + \frac{1}{\sqrt{\pi t}}$

(b) $\frac{1}{s} + \frac{1}{1-s^2}$, Ans:- $1 - \sinh t$, (c) $\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}$

Ans:- $3\cos 2t - 4\sin 2t - 4\cosh 4t + 6\sinh 4t$

(12) (a) $\frac{1}{\sqrt{2s+3}}$ Ans:- $\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{3t}{2}}$

(b) $\frac{s}{s^2+4s+13}$, Ans:- $e^{-2t}\cos 3t - \frac{2}{3} e^{-2t} \sin 3t$

(c) $\frac{s^2+6}{(s^2+1)(s^2+4)}$ Ans:- $\frac{1}{3}(5\sin t - \sin 2t)$

(d) $\frac{s}{(s-3)(s^2+4)}$ Ans:- $\frac{1}{13}(3e^{3t} - 3\cos 2t + 2\sin 2t)$

(e) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ Ans:- $\frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$

Que (13) :–

(a) $\dfrac{1}{s(s^2-a^2)}$ Ans :– $\dfrac{1}{a^2}(\cosh at - 1)$

(b) $\dfrac{1}{s^2(s+2)}$ Ans :– $\dfrac{1}{4}\left(e^{-2t} + 2t - 1\right)$

(c) $\dfrac{1}{s(s+2)^3}$ Ans :– $\dfrac{1}{8} - \dfrac{1}{4}\left(t^2 + t + \dfrac{1}{2}\right) e^{-2t}$

(d) $\dfrac{s+2}{s^2(s+1)(s-2)}$ Ans :– $\dfrac{1}{3}\left(e^{2t} - e^{-t} - t\right)$

Que (14)

(a) $\dfrac{e^{-\pi s}}{s^2}$ Ans. $(t-\pi)\, U(t-\pi)$, (b) $\dfrac{e^{-\pi s}}{s^2+1}$ Ans :– $-\sin t \cdot U(t-$

(c) $\dfrac{e^{-s}}{(s-1)(s-2)}$ Ans :– $\left[e^{-2(t-1)} - e^{t-1}\right] U(t-1)$

Que (15)

(a) $\cot^{-1}\left(\dfrac{s}{2}\right)$ Ans :– $\dfrac{\sin 2t}{t}$, (b) $\log\left\{\dfrac{s^2+1}{s(s+1)}\right\}$ Ans :– $\dfrac{1}{t}\left(1 + e^{-t} - 2\cos\right.$

(c) $\log\left(\dfrac{1+s}{s}\right)$ Ans :– $\dfrac{1-e^{-t}}{t}$, (d) $\log\left\{\dfrac{s+1}{(s+2)(s+3)}\right\}$ Ans :– $e^{-t} - e^{-2t} - e^{-3t}$

Que (16)

(a) $\dfrac{s}{(s^2-1)^2}$ Ans :– $\dfrac{t}{2}\sinh t$, (b) $\dfrac{s^2}{(s-2)^3}$ Ans :– $e^{2t}(1 + 4t + 2t^2)$

(c) $\dfrac{16}{(s^2+2s+5)^2}$ Ans :– $e^{-t}(\sin 2t - 2t\cos 2t)$, (d) $\dfrac{s+3}{(s^2+6s+13)^2}$, Ans :– $\dfrac{1}{4} t\, e^{-3t} \sin$

Que (17) Using convolution theorem, find the inverse Laplace transform of the following functions

(a) $\dfrac{1}{(s+a)(s+b)}$ Ans :– $\dfrac{e^{-bt} - e^{-at}}{a-b}$, (b) $\dfrac{1}{(s^2+a^2)^2}$ Ans :– $\dfrac{1}{2a^3}(\sin at - at\cos at)$

(c) $\dfrac{1}{s^2(s+1)^2}$ Ans :– $t(e^{-t}+1) + 2(e^{-t} - 1)$

(d) $\dfrac{1}{s(s^2+4)}$ Ans :– $\dfrac{1}{4}(1 - \cos 2t)$

(e) $\dfrac{64}{(s+1)(s+9)^2}$ Ans :– $e^{-t}\left[1 - e^{-8t}(1 + 8t)\right]$

Que (18) :– Solve the following differential equation by Laplace transform –

(a) $y''' + 2y'' - y' - 2y = 0$, given that $y = 1$, $y' = 2$, $y'' = 2$ at $t = 2$

Ans :– $y = \dfrac{1}{3}\left(5e^{t} + e^{-2t}\right) - e^{-t}$

(b) $y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1$, $y'(0) = -1$

Ans :– $y = 3 + 2t + \dfrac{1}{2}\left(e^{3t} - e^{t}\right) - 2e^{2t}$

(c) $y'' + y = t$, given that $y(0) = 1$, $y'(0) = -2$

Ans :– $y = t - 3\sin t + \cos t$

(d) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, $y = \frac{dy}{dt} = 0$ when $t=0$.

Ans:- $\frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$

(e) $t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = \sin t$ where $y(0) = 1$

Ans:- $y = \frac{1}{2}\left(\frac{3\sin t}{t} - \cos t\right)$

(f) $t\frac{d^2y}{dt^2} + (1-2t)\frac{dy}{dt} - 2y = 0$ when $y(0) = 1$, $y'(0) = 2$

Ans — $y = e^{2t}$

Que (19) Solve the following simultaneous equation by the Laplace transform

(a) $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$

Ans:- $x = e^{t} + e^{-t}$, and $y = e^{-t} - e^{t} + \sin t$

(b) $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$, $\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$, given $x = 3$, $y = 0$ when $t = 0$

Ans:- $x = \frac{1}{10}\left(5 - 2e^{-t} - 3e^{-\frac{6t}{11}}\right)$, $y = \frac{1}{5}\left(e^{-t} - e^{-\frac{6t}{11}}\right)$

(c) $(D^2 - 3)x - 4y = 0$, $x + (D^2+1)y = 0$, given that $x = y = \frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 2$ at $t =$

Ans:- $x = 2t\cosh t$, $y = (1-t)\sin t$

Que (20) :- A voltage Ee^{-at} is applied at $t=0$ to a circuit of inductance L and resistance R. Show that the current at time t is $\frac{E}{R-aL}\left[e^{-at} - e^{-\frac{Rt}{L}}\right]$.

Que (21) :- If $L\frac{di}{dt} + \frac{q}{c} = E_0\delta(t)$, where $i = \frac{dq}{dt}$ at any instant t. Initially the circuit has no current and $q = 0$ but at $t = 0$ an emf of very large voltage is applied for a very short time so that it may be represented by $E_0\delta(t)$. Find it at an instant t.

Ans:- $i = \frac{E_0}{L}\cos\frac{t}{\sqrt{LC}}$

Que (22) :- A body falls from rest in a liquid whose density is one fourth that of the body. If the liquid offers resistance proportional to the velocity and the velocity approaches a limiting value of 9 meters/sec, find the distance fallen in 5 seconds.

Ans:- $x = 9\left[t - \frac{12\left(1 - e^{-gt/12}\right)}{g}\right]$

put $t = 5$ sec. $g = 9.8\ m/sec^2$. $x = 34.17$ ~~meters~~ metres.