Engineering Mathematics-I

KAS-103

Module - IV

Lecture-4 5

Multiple Integrals

Content:- Triple Integrals.

Triple Integrals:- Consider a function $f(x,y,z)$ which is continuous at every point of region V.

then $\iiint_V f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$.

order of integration depends upon the limit.

Let z_1, z_2 be functions of x and y.

Let y_1, y_2 be functions of x.

Let x_1, x_2 be constants.

then $z_1 = \phi_1(x,y)$, $z_2 = \phi_2(x,y)$

$y_1 = \psi_1(x)$, $y_2 = \psi_2(x)$

$x_1 \le x \le x_2$

$$\iiint_V f(x,y,z)\,dV = \int_{x_1}^{x_2}\left[\int_{y_1=\psi_1(x)}^{y_2=\psi_2(x)}\left[\int_{z_1=\phi_1(x,y)}^{z_2=\phi_2(x,y)} dz\right] dy\right] dx$$

Remark:

Limits involving two variables are kept innermost, then limits involving one variable and finally the constant limits.

If all the limits are constant than order of integration is immaterial.

Q.1 Evaluate. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$.

Integrating w.r.t 'x'.

$$\int_0^1\int_0^1 \left[e^{x+y+z}\right]_0^1 dy\,dz = \int_0^1\int_0^1 \left[e^{1+y+z} - e^{y+z}\right] dy\,dz$$

$$= \int_0^1 \left[e^{1+y+z} - e^{y+z}\right]_0^1 dz = \int_0^1 \left[e^{2+z} - e^{1+z} - e^{1+z} + e^{z}\right] dz$$

$$\left[e^{2+z} - 2e^{1+z} + e^{z}\right]_0^1 \Rightarrow e^3 - 2e^2 + e - e^2 + 2e - e^0$$

$$= e^3 - 3e^2 + 3e - 1 \Rightarrow (e-1)^3 \text{ Ans.}$$

2009

Q.2 Evaluate. $\iiint_R (x-2y+z)\,dz\,dy\,dx$, where R is region determined by $0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le x+y$.

Sol. $$\int_0^1\int_0^{x^2}\int_0^{x+y} (x-2y+z)\,dz\,dy\,dx.$$

$$\int_0^1\int_0^{x^2} \left[xz - 2yz + \frac{z^2}{2}\right]_0^{(x+y)} dy\,dx.$$

$$= \int_0^1\int_0^{x^2} x^2 + xy - 2xy - 2y^2 + \frac{(x+y)^2}{2}\,dy\,dx.$$

$$\Rightarrow \int_0^1 \left[x^2y + \frac{xy^2}{2} - \frac{2xy^2}{2} - \frac{2y^3}{3} + \frac{(x+y)^3}{6}\right]_0^{x^2} dx$$

$$= \int_0^1 \left[x^4 + \frac{x^5}{2} - \frac{2x^5}{2} - \frac{2x^6}{3} + \frac{(x+x^2)^3}{6} - \frac{x^3}{6}\right]$$

$$\left[\frac{x^5}{5} + \frac{x^6}{12} - \frac{x^6}{6} - \frac{2x^7}{21} + \frac{1}{6}\left[\frac{x^4}{4} + \frac{x^7}{7} + \frac{3x^5}{5} + \frac{3x^6}{6} - \frac{x^4}{4}\right]\right]_0^1$$

$$\frac{1}{5} + \frac{1}{12} - \frac{1}{6} - \frac{2}{21} + \frac{1}{6}\left[\frac{1}{4} + \frac{1}{7} + \frac{3}{5} + \frac{1}{2} - \frac{1}{4}\right] = \frac{8}{35} \text{ Ans.}$$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$

$$\iiint \frac{1}{\sqrt{1-X-Y-Z}} \frac{dX\,dY\,dZ}{8\sqrt{XYZ}}$$

$0 \le z \le \sqrt{1-x^2-y^2}$

$0 \le z^2 \le 1-x^2-y^2$

or $x^2+y^2+z^2 \le 1$

$x^2 = X$, $x = \sqrt{X}$, $y = \sqrt{Y}$, $z = \sqrt{Z}$.

$$\Rightarrow \frac{1}{8} \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)} \int_0^1 \frac{1}{\sqrt{1-u}}\, u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\, du \quad X+Y+Z$$

$$\Rightarrow \frac{1}{8} \frac{\pi}{\frac{1}{2}} \int_0^1 (1-u)^{-1/2} \cdot u^{1/2}\, du$$

$$\Rightarrow \frac{\pi}{4} \beta\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{4} \frac{\Gamma\frac{1}{2}\Gamma\frac{3}{2}}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}$$

$$\Rightarrow \frac{\pi}{4} \frac{\frac{1}{2}\pi}{\Gamma 2} \Rightarrow \frac{\pi^2}{8} \text{ Ans}$$

Alternate.

$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\, dz\, dy\, dx.$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx.$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx \Rightarrow \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \ldots\right.$$

$$\Rightarrow \frac{\pi}{4}(\sin^{-1} 1) = \frac{\pi^2}{8} \text{ Ans}$$

Engineering Mathematics-I

Mathematics-I

Q. Evaluate $\iiint_R dx\,dy\,dz$, R is region bounded by $x+y+z \le a$, $x \ge 0$, $y \ge 0$, $z \ge 0$.

Sol. $\iiint_R dx\,dy\,dz$

$$\int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\,dy\,dx$$

$x \ge 0$

for limits of z

$z = a-x-y$

$0 \le z \le a-x-y$

for y put z=0 $0 \le y \le a-x$

for x put $y=0, z=0$

$0 \le x \le a$.

$$\int_0^a \int_0^{a-x} (a-x-y)\,dy\,dx$$

$$\int_0^a \left[(a-x)y - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$\int_0^a \left[(a-x)^2 - \frac{(a-x)^2}{2}\right] dx = \int_0^a \frac{(a-x)^2}{2}\,dx = \left[\frac{-(a-x)^3}{6}\right]_0^a$$

$\Rightarrow \frac{a^3}{6}$ Ans.

Problems for practice -

Q.1 Evaluate $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$, where R denotes region $x=0, y=0, z=0$ and $x+y+z=a$.

Q2 $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ where R denotes region bounded by Co-ordinate plane and $x+y+z=1$.

Q.3 Evaluate $\iiint x^2yz\,dx\,dy\,dz$, throughout volume bounded by planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

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Q4 Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$ bounded by the region. $0 \le x \le 1$, $1 \le y \le 2$, $2 \le z \le 3$