Q. No	Question (s)	Marks	BL	CO
	UNIT – V			
1	a) Define line integral.	1M	L1	C121.5
	b) Define surface integral.	1M	L1	C121.5
	c) Define volume integral.	1M	L1	C121.5
	d) State Gauss Divergence Theorem	1M	L1	C121.5
	e) State Greens Theorem in a plane.	1M	L1	C121.5
2	a) State Stoke's theorem.	3M	L2	C121.5
	b) Find $\int_c \vec{F}.d\vec{r}$, where $\vec{F}=3xy\vec{i}-y^2\vec{j}$ and c is the curve $y=2x^2$ in the $xy-$plane from $(0,0)$ to $(1,2)$.	3M	L2	C121.5
	c) Show that the work done by the force $\vec{F}=(4xy-3x^2z^2)\vec{i}+2x^2\vec{j}-2x^3z\vec{k}$ is independent of the curve joining two points.	3M	L2	C121.5
	d) Show that $\int \vec{r}.\vec{n}ds=3V$ by using Gauss's theorem.	3M	L2	C121.5
	e) Find the work done in moving a particle in the force field $\vec{F}=3x^2\vec{i}+\vec{j}+z\vec{k}$ along the straight line joining $(0,0,0)$ and $(2,1,3)$	3M	L2	C121.5
3	a) Find $\int_s \vec{F}.\vec{n}ds$ where $\vec{F}=18z\vec{i}-12\vec{j}+3y\vec{k}$ and s is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant.	5M	L3	C121.5
	b) If $\vec{F}=4xz\vec{i}-y^2\vec{j}+yz\vec{k}$, find $\int \vec{F}.\vec{n}ds$, where S is the surface of the cube bounded by $x=0,x=a,y=0,y=a,z=0$.	5M	L3	C121.5
	c) Find $\int \vec{F}.\vec{n}ds$, where $\vec{F}=z\vec{i}+x\vec{j}-3y^2z\vec{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.	5M	L3	C121.5
	d) If $\vec{F}=2xz\vec{i}-x\vec{j}+y^2\vec{k}$. Then find $\int_V \vec{F}dv$, where V is the region bounded by the surfaces $x=0,x=2,y=0,y=6,z=x^2,z=4$.	5M	L3	C121.5
	e) A vector field is given by $\vec{F}=(\sin y)\vec{i}+x(1+\cos y)\vec{j}$, evaluate the line integral over the circular path $x^2+y^2=a^2,z=0$.	5M	L3	C121.5

4.	**a) Verify Green's theorem in plane for** $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ **where C is the region bounded by** $y = \sqrt{x}$ and $y = x^{2.}$	**10M**	L4	C121.5
	b) Verify Gauss's divergence theorem for $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + z\vec{k}$ **taken over the surface of the cube bounded by the planes** $x = y = z = a$ **and coordinate planes.**	**10M**	L4	C121.5
	c) Verify Stoke's theorem for $\vec{F} = -y^3\vec{i} + x^3\vec{j}$**, where** S **is the circular disc** $x^2 + y^2 \le 1, z = 0$.	**10M**	L4	C121.5

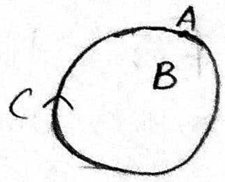
UNIT-V
VECTOR INTEGRATION

1. a.) Define line integral

Any integral which is to be evaluated along a curve is called a line integral

If $\overline{F}(t)$ is a vector point function defined and continuous along C, a smooth curve defined by $\overline{r} = f(t)$. Then the line integral of $\overline{F}$ taken along C is given by

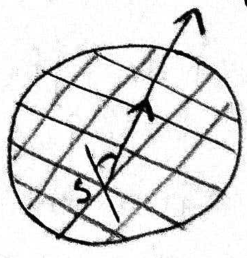
$$\int_A^B \overline{F}\cdot\overline{t}\,ds = \int_C \left(\overline{F}\cdot\frac{d\overline{r}}{ds}\right)ds = \int_C \overline{F}\cdot d\overline{r}$$



b.) Define surface integral.

Any integral which is to be evaluated over a surface is called a surface integral

let $\overline{F}(\overline{r})$ be a continuous vector point function defined over the smooth surface $\overline{r} = \overline{f}(u,v)$. Let S be the region of the surface and $\overline{n}$ be the unit normal to the surface at a point P. Then the surface integral of $\overline{F}(\overline{r})$ over the region S is given by



$$\int_S \overline{F}\cdot\overline{n}\,ds$$

If $\overline{F}(\overline{r}) = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$ then

$$\int_S \overline{F}\cdot\overline{n}\,ds = \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$$

c.) Define volume integral.

Any integral which is to be evaluated over a volume V is called a volume integral

Let $\bar{F}(\bar{r})$ be a vector point function defined over a volume V bounded by the surface $\bar{r} = \bar{f}(u,v)$, then the volume integral of $\bar{F}(\bar{r})$ in the region V is given by

$$\int_V \bar{F}(r)\,dr \quad \text{or} \quad \int_V \bar{F}\,dv$$

If $\bar{F}(\bar{r}) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$, where F_1, F_2, F_3 are functions of x, y, z and $dv = dx\,dy\,dz$. Then,

$$\int_V \bar{F}\,dv = \iiint (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\,dx\,dy\,dz$$

$$= \bar{i}\iiint F_1\,dx\,dy\,dz + \bar{j}\iiint F_2\,dx\,dy\,dz + \bar{k}\iiint F_3\,dx\,dy\,dz$$

d.) Define Gauss Divergence Theorem

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuosly differentiable vector point function, then

$$\int_V \operatorname{div}\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds$$

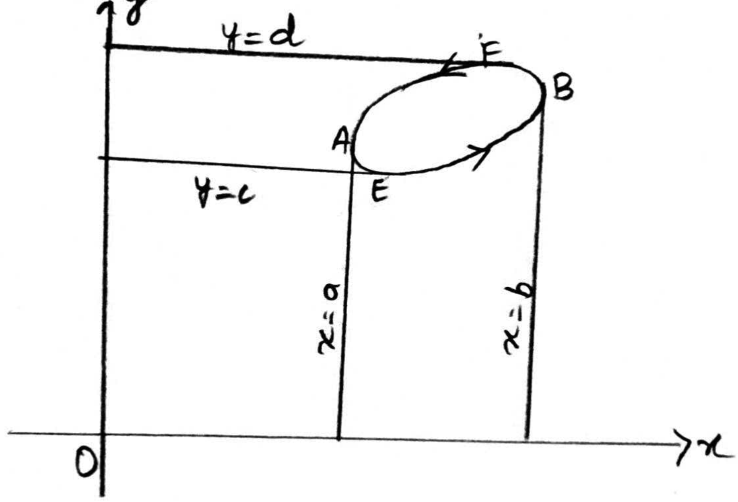
where $\bar{n}$ is the outward drawn normal vector at any point of S

e) Define Green's Theorem in a Plane

If S is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivatives in R, then

$$\oint M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Where c is traversed in the positive (anti-clockwise) direction.



2. a.) Define Stoke's Theorem.

let S be a open surface bounded by a closed non-intersecting curve C. If $\bar{F}$ is any differentiable vector point function $\oint_C \bar{F}\cdot dr = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$, where c is traversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

2b) Find $\int_C \overline{F} \cdot d\overline{r}$, where $\overline{F} = 3xy\overline{i} - y^2\overline{j}$ and C is the curve $y = 2x^2$ in the xy plane from (0,0) to (1,2)

Given $\overline{F} = 3xy\overline{i} - y^2\overline{j}$

$d\overline{r} = dx\overline{i} + dy\overline{j}$

$\overline{F} \cdot d\overline{r} = 3xy\,dx - y^2 dy$

The given curve is $y = 2x^2$

$dy = 4x\,dx$

$$\int_C \overline{F} \cdot d\overline{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2 4x\,dx$$

$$\Rightarrow \int_{x=0}^{1} (6x^3 - 16x^5)\,dx$$

$$\Rightarrow \int_{x=0}^{1} 6x^3\,dx - \int_{x=0}^{1} 16x^5\,dx$$

$$\Rightarrow \left[\frac{6x^4}{4} - \frac{16x^6}{4}\right]_{x=0}^{1}$$

$$\Rightarrow \left[\frac{3x^4}{2} - \frac{8x^6}{3}\right]_{x=0}^{1}$$

$$\Rightarrow \frac{3}{2} - \frac{8}{3} \Rightarrow \frac{9-16}{6} \Rightarrow \frac{-7}{6}$$

2c.) Show that the work done by the force $\overline{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$ is independent of the curve joining two points.

We know that if the force is conservative, then the work done is independent of the path and conversly. So we have to prove that $\overline{F} = \nabla\phi \Rightarrow \text{curl}\,\overline{F} = 0$

Now,

$$\text{curl}\,\overline{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$\Rightarrow \bar{i}(0-0) - \bar{j}(-6x^2z + 6x^2z) + \bar{k}(4x - 4x)$$

$$= \bar{0}$$

2d.) Show that $\int \bar{r}.\bar{n}\,ds = 3V$ by using Gauss's Theorem.

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

We know that $\text{div}\,\bar{r} = 3$

$\therefore$ By Gauss divergence theorem,

$$\int_S F.\bar{n}\,ds = \int_V \text{div}\,F\,dv$$

Take $\overline{F} = \bar{r}$, we have

$$\int_S \bar{r}.\bar{n}\,ds = \int_V 3\,dv = 3V.$$

Hence Proved.

2e) Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$ along the straight line joining $(0,0,0)$ and $(2,1,3)$

let $P_1 = (0,0,0)$ and $P_2 (2,1,3)$

Also, Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

Since,

$\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$

$\therefore \bar{F} \cdot d\bar{r} = (3x^2\bar{i} + \bar{j} + z\bar{k}) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k})$

$= 3x^2 dx + dy + dz\bar{k}$

$\therefore$ Work done by $\bar{F} = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{r} = \int_{(0,0,0)}^{(2,1,3)} \bar{F} \cdot d\bar{r}$

$$= \int_{(0,0,0)}^{(2,1,3)} 3x^2 dx + dy + z dz$$

$$= \int_{(0,0,0)}^{(2,1,3)} 3x^2 dx + \int_{(0,0,0)}^{(2,1,3)} dy + \int_{(0,0,0)}^{(2,1,3)} z dz$$

$$\Rightarrow \left[x^3 + y + \frac{z^2}{2} \right]_{(0,0,0)}^{(2,1,3)}$$

$$\Rightarrow 8 + 1 + \frac{9}{2} \Rightarrow \frac{27}{2}$$

3 a) Find $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and s is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant.

let $\phi = 2x + 3y + 6z - 12$

Normal to ϕ is $\nabla\phi$

Now,

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= 2\bar{i} + 3\bar{j} + 6\bar{k}$$

unit normal, $\bar{n} = \dfrac{2\bar{i} + 3\bar{j} + 6\bar{k}}{\sqrt{2^2+3^2+6^2}} = \dfrac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$

let R be the projection of S on xy-plane, then

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

Since $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

$\therefore \bar{F}\cdot\bar{n} = (18z\bar{i} - 12\bar{j} + 3y\bar{k})\cdot\dfrac{(2\bar{i}+3\bar{j}+6\bar{k})}{7}$

$$= \frac{6}{7}(6z - 6 + 3y)$$

Also $\bar{n}\cdot\bar{k} = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})\cdot\bar{k} \Rightarrow \frac{6}{7}$

Given surface is $2x+3y+6z=12$

$\therefore$ R of xy-plane is $2x + 3y = 12 \Rightarrow y = \dfrac{12-2x}{3}$

$\therefore y = 0 \Rightarrow x = 6$

Now x varies from 0 to 6 and y varies from 0 to $\frac{12-2x}{3}$

$\therefore$ The surface integral $= \iint_R \frac{\bar{F}\cdot\bar{n}\,dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$= \iint_R \frac{6}{7}(6z-6-3y)\cdot\frac{7}{6}\,dx\,dy$$

$$\Rightarrow \iint_R (12-2x-3y-6+3y)\,dx\,dy \qquad (\because 6z = 12-2x-3y$$

$$\Rightarrow \iint_R (6-2x)\,dx\,dy$$

$$\Rightarrow 2\int_0^6 (3-x)\left[\int_0^{\frac{1}{3}(12-2x)} dy\right] dx$$

$$\Rightarrow 2\int_0^6 (3-x)\,[y]_0^{\frac{12-2x}{3}}\,dx$$

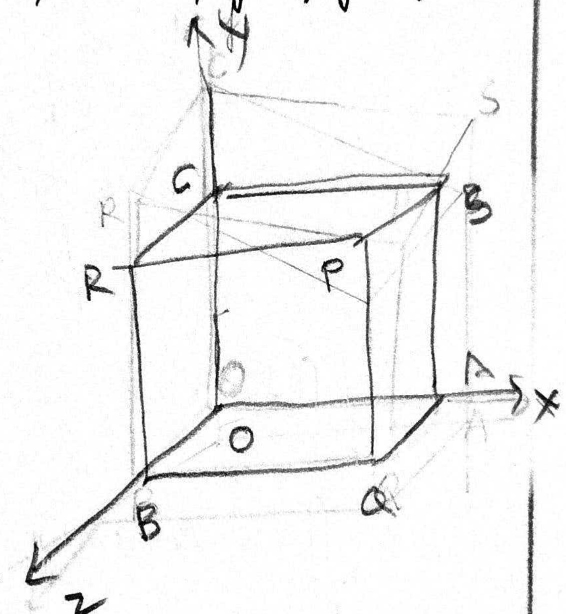
$$\Rightarrow 2\int_0^6 (3-x)\,\frac{1}{3}(12-2x)\,dx$$

$$\Rightarrow \frac{4}{3}\int_0^6 (18-9x+x^2)\,dx$$

$$\Rightarrow \frac{4}{3}\left[18x-\frac{9x^2}{2}+\frac{x^3}{3}\right]_0^6 \Rightarrow \underline{24}$$

3b) If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$, Find $\int \bar{F}\cdot\bar{n}\,ds$, where S is the surface of the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.

Consider the volume within the cube PQASCRBO in figure bounded by $x=0, x=a, y=0, y=a, z=0, z=a$



Here, $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

let us calculate $\iint_S \bar{F}\cdot\bar{n}\,ds$ for each face of the cube.

(1) For R_1 = PQAS, $x=a$, $ds = dy\,dz$

$\because$ PQAS must be projected on yz plane

Unit outward drawn normal $\bar{n} = \bar{i}$ (OR) unit Normal $\bar{n} = \bar{i}$

Also $0 \le y \le a$ & $0 \le z \le a$

$$\therefore \bar{F}\cdot\bar{n} = (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\bar{i}$$

$$\Rightarrow 4xz$$

$$\Rightarrow 4az \qquad (\because x=a)$$

$$\therefore \iint_{R_1} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{a} \int_{z=0}^{a} 4az\,dz\,dy$$

$$\Rightarrow 4a \int_{y=0}^{a} \left(\frac{z^2}{2}\right)_{z=0}^{a} dy \Rightarrow 4a\cdot\frac{a^2}{2}\int_{y=0}^{a} dy$$

$$\Rightarrow \frac{4a^3}{2}a \Rightarrow \frac{4a^4}{2} = 2a^4$$

II) For $R_2 = OCRB$, $x=0$, $ds = dy\,dz$

unit normal $\bar{n} = -\bar{i}$

$\bar{F}\cdot\bar{n} = -4xz = 0 \qquad (\because x = 0$

$\therefore \iint_{R_2} \bar{F}\cdot\bar{n}\,ds = 0$

III) For $R_3 = RBQP$, $z=a$, $ds = dx\,dy$

$\bar{n} = \bar{k} \qquad 0 \le x \le a \qquad , 0 \le y \le a$

$\Rightarrow \bar{F}\cdot\bar{n} = yz = ay \qquad (\because z = a)$

$$\iint_{R_3} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{a}\int_{y=0}^{a} ay\,dy\,dx = a\cdot\frac{a^2}{2}\cdot a = \frac{a^4}{2}$$

IV) For $R_4 = OASC$, $z=0$, $ds = dx\,dy$

$\bar{n} = -\bar{k}$ and $0 \le x \le a$, $0 \le y \le a$

$\bar{F}\cdot\bar{n} = -yz = 0 \qquad (\because z = 0)$

$\therefore \iint_{R_4} \bar{F}\cdot\bar{n}\,ds = 0$

V) For $R_5 = RPSC$, $y=a$, $ds = dz\,dx$

$\bar{n} = \bar{j} \qquad 0 \le z \le a \quad , 0 \le x \le a$

$\therefore \bar{F}\cdot\bar{n} = -y^2 \Rightarrow -a^2$

$$\iint_{R_5} \bar{F}\cdot\bar{n}\,ds = \iint_{R_5} -a^2\,ds = \int_{x=0}^{a}\int_{z=0}^{a} (-a^2)\,dz\,dx = -a^2\cdot a\cdot a$$

$$= -a^4$$

vi) For $R_6 = OBQA$, $y=0$, $ds = dz\,dx$

$\bar{n} = -\bar{j}$ $\quad 0 \leq z \leq a \quad , \quad 0 \leq x \leq a$

$\bar{F}\cdot\bar{n} = y^2 = 0 \qquad (\because y = 0)$

$$\therefore \iint_{R_6} \bar{F}\cdot\bar{n}\, ds = 0$$

Thus,

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\,ds + \iint_{R_2} \bar{F}\cdot\bar{n}\,ds + \iint_{R_3} \bar{F}\cdot\bar{n}\,ds + \iint_{R_4} \bar{F}\cdot\bar{n}\,ds + \iint_{R_5} \bar{F}\cdot\bar{n}\,ds$$

$$+ \iint_{R_6} \bar{F}\cdot\bar{n}\,ds$$

$$\Rightarrow 2a^4 + 0 + \frac{a^4}{2} + 0 - a^4 + 0$$

$$\Rightarrow a^4 + \frac{a^4}{2}$$

$$\Rightarrow \frac{3a^4}{2}$$

3 c.) find $\int \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

The surface S is the curved surface ABCEA

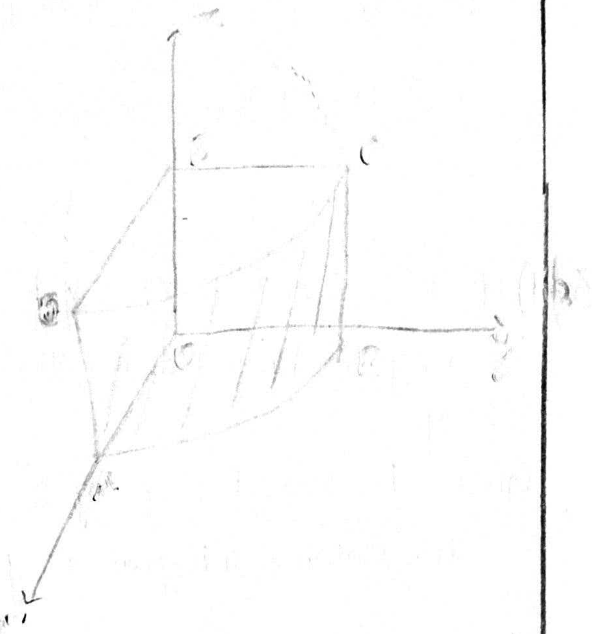
let $\phi = x^2+y^2-16$

$\therefore$ The Normal to the surface $S = \nabla\phi$

$= 2x\bar{i} + 2y\bar{j}$

Unit normal $\bar{n} = \dfrac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2 + 4y^2}} = \dfrac{x\bar{i} + y\bar{j}}{4}$ $\quad (\because x^2+y^2=16)$

let R be the projection of S on yz plane. Then R is the rectangle OBCD.



$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_S \bar{F}\cdot\bar{n}\, \frac{dydz}{|\bar{n}\cdot\bar{i}|}$$

Now,

$$\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$$

$$\bar{F}\cdot\bar{n} = (z\bar{i} + x\bar{j} - 3y^2z\bar{k})\left(\frac{x\bar{i}+y\bar{j}}{4}\right)$$

$$= \frac{1}{4}(xz + xy)$$

and,

$$\bar{n}\cdot\bar{i} = \frac{1}{4}(x\bar{i}+y\bar{j})\bar{i} = \frac{x}{4}$$

For the surface $x^2+y^2=16$ in the yz plane, $x=0 \Rightarrow y=4$

$\therefore$ In the first octant, y varies from 0 to 4, z varies from 0 to 5. Then the surface integral

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dydz}{(\bar{n}\cdot\bar{i})} = \iint_R \left(\frac{xz+xy}{4}\right)\frac{4}{x}\, dy\,dz$$

$$\Rightarrow \int_{y=0}^{4}\int_{z=0}^{5} (y+z)\,dy\,dz$$

$$\int_{y=0}^{4} y\,dy \int_{z=0}^{5} dz \quad + \quad \int_{y=0}^{4} dy \int_{z=0}^{5} z\,dz$$

$$\left[\frac{y^2}{2}\right]_0^4 [z]_0^5 \quad + \quad [y]_0^4 \left[\frac{z^2}{2}\right]_0^5$$

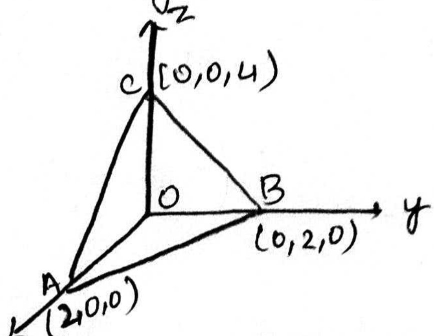
$\Rightarrow 40 + 50$

$\Rightarrow 90$

3. d) If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$. Then evaluate $\int_V \bar{F}\,dv$, where V is the region bounded by the surface $x=0$, $x=2$, $y=6$, $z=x^2$, $z=4$

Given $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$

$\therefore$ The volume integral is given by



$$\int_V \bar{F}\,dv = \iiint (2xz\bar{i} - x\bar{j} + y^2\bar{k})\,dx\,dy\,dz$$

$$= \bar{i} \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\,dy\,dz - \bar{j} \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{k} \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$\Rightarrow \bar{i} \int_{x=0}^{2}\int_{y=0}^{6} \left[\frac{2xz^2}{2}\right]_{x^2}^{4} dx\,dy - \bar{j} \int_{x=0}^{2}\int_{y=0}^{6} [xz]_{x^2}^{4}\,dx\,dy + \bar{k} \int_{x=0}^{2}\int_{y=0}^{6} [y^2 z]_{x^2}^{4}\,dx\,dy$$

$$\Rightarrow \bar{i} \int_{x=0}^{2}\int_{y=0}^{6} x(16-x^4)\,dx\,dy - \bar{j} \int_{x=0}^{2}\int_{y=0}^{6} x(4-x^2)\,dx\,dy + \bar{k} \int_{x=0}^{2}\int_{y=0}^{6} y^2(x^2-4)\,dx\,dy$$

$$\Rightarrow \bar{i} \int_{x=0}^{2} (16x - x^5)[y]_0^6\,dx - \bar{j} \int_{x=0}^{2} (4x - x^3)[y]_0^6\,dx + \bar{k} \int_{x=0}^{2} (x^2-4)\left[\frac{y^3}{3}\right]_0^6 dx$$

$$\Rightarrow \bar{i}\left(8x^2-\frac{x^6}{6}\right]_0^2(6)-\bar{j}\left(2x^2-\frac{x^4}{4}\right)_0^2(6)-\bar{k}\left(4x-\frac{x^3}{3}\right)_0^2\left(\frac{216}{3}\right)$$

$$\Rightarrow 128\bar{i}-24\bar{j}-384\bar{k}$$

3e) Find $\bar{F}=(\sin y)\bar{i}+x(1+\cos y)\bar{j}$ Where over the circular path $x^2+y^2=a^2$, $z=0$

Given $\bar{F}=(\sin y)\bar{i}+x(1+\cos y)\bar{j}$

let $M=\sin y$ $\quad N=x(1+\cos y)$

$\therefore \frac{\partial M}{\partial y}=\cos y \quad \frac{\partial N}{\partial x}=1+\cos y$

$\therefore$ By Green's Theorem

$$\Rightarrow \oint M\,dx+N\,dy=\iint_S\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$\Rightarrow \oint \sin y\,dx+x(1+\cos y)\,dy=\iint_S(1+\cos y-\cos y)\,dx\,dy$$

$$\Rightarrow \iint_S dx\,dy$$

$$\Rightarrow \iint_S dA$$

$$\Rightarrow A$$

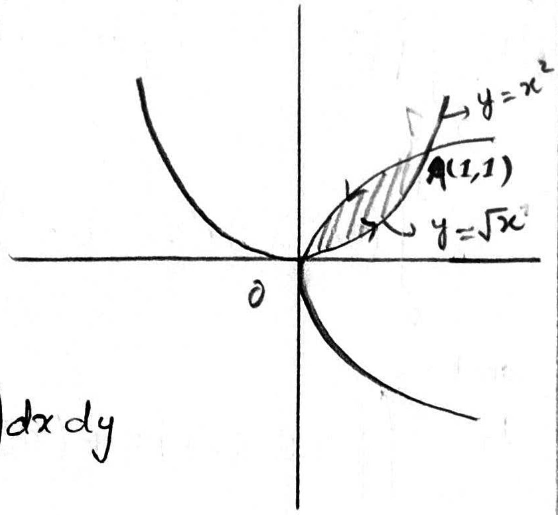
$$\Rightarrow \pi a^2 \quad \because (\text{Area of circle}=\pi a^2)$$

4 a) Verify Green's theorem in plane for $\oint (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$

Let $M = 3x^2 - 8y^2$

$N = 4y - 6xy$, Then

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$



We have by Green Theorem,

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Now,

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_S (16y - 6y)\,dx\,dy$$

$$= 10 \iint_S y\,dx\,dy$$

$$\Rightarrow 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y\,dy\,dx \Rightarrow 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$\Rightarrow 5 \int_{x=0}^{1} (x - x^4)\,dx$$

$$\Rightarrow 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right) = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2} \quad \text{—— (1)}$$

Verification:-

We can write the line integral along C

= [line integral along $y=x^2$ (from O to A)] + [line integral along $y^2 = x$ (from A to O)]

$$= I_1 + I_2 \quad (\text{say})$$

$$I_1 = \int_{x=0}^{1} \left\{ \left[3x^2 - 8(x^2)^2\right] dx + \left[4x^2 - 6x(x^2)\right] 2x\,dx \right\} \quad \left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$$

$$= \int_0^1 \left(3x^3 + 8x^3 - 20x^4\right) dx = -1$$

and,

$$I_2 = \int_0^1 \left[(3x^2 - 8x)\,dx + \left(4\sqrt{x} - 6x^{3/2}\right) \cdot \frac{1}{2\sqrt{x}}\,dx\right]$$

$$= \int_0^1 \left(3x^2 - 11x + 2\right) dx = \frac{5}{2}$$

$$I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2} \quad \text{——— } (2)$$

From (1) and (2)

we have,

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence the verification of the Green's Theorem.

4b) Verify Gauss divergence theorem for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = a$ and coordinate plane.

By Gauss divergence theorem, we have

$$\int_S \bar{F}.\bar{n}\,ds = \int_V \bar{\nabla}.\bar{F}\,dv$$

Now,

$$\text{RHS} = \int_V \bar{\nabla}.\bar{F}\,dv = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)\,dx\,dy\,dz$$

$$\Rightarrow \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\,dy\,dz$$

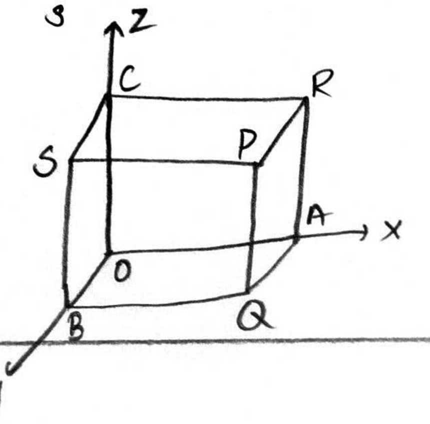
$$\Rightarrow \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\,dz \Rightarrow \int_0^a\int_0^a \left(\frac{a^3}{3} + a\right) dy\,dz$$

$$\Rightarrow \left(\frac{a^3}{3} + a\right)\int_0^a\int_0^a dy\,dz \qquad \Rightarrow \left(\frac{a^3}{3} + a\right)\int_0^a [y]_0^a\,dz$$

$$\Rightarrow \left(\frac{a^3}{3} + a\right)\int_0^a a\,dz \Rightarrow a\left(\frac{a^3}{3} + a\right)\int_0^a dz \Rightarrow a^2\left(\frac{a^3}{3} + a\right)$$

$$\Rightarrow \frac{a^5}{3} + a^3 \quad \text{—} (1)$$

Again, To find the LHS, we need to find $\int_S \bar{F}.\bar{n}\,ds$ for each face of the cube as follows



(i) For $S_1 = PQAR$, unit normal, $\bar{n} = \bar{i}$

$x = a$, $ds = dy\,dz$, $0 \le y \le a$, $0 \le z \le a$

$\bar{F}.\bar{n} = x^3 - yz \qquad \because x = a$

$= a^3 - yz$

$\therefore \int_S \bar{F}.\bar{n}ds = \iint_{S_1} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}(a^3-yz)\,dy\,dz$

$\Rightarrow \int_{z=0}^{a}\left[a^3y - \frac{y^2}{2}z\right]_{y=0}^{a} dz \Rightarrow \int_{y=0}^{a}\left(a^4 - \frac{a^2}{2}z\right)dz$

$\Rightarrow a^5 - \frac{a^4}{4}$ ———— (2)

(ii) For $S_2 = OCSB$, $\bar{n} = -\bar{i}$

$x=0$, $ds = dydz$, $0 \leq y \leq a$, $0 \leq z \leq a$

$\bar{F}.\bar{n} = -(x^3 - yz) = yz \quad (\because x=0)$

$\therefore \int_S \bar{F}.\bar{n}ds \Rightarrow \iint_{S_2} \bar{F}.\bar{n}ds \Rightarrow \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz$

$\Rightarrow \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_0^a dz = \frac{a^2}{2}\int_0^a z\,dz = \frac{a^2}{4}$ ———— (3)

(iii) For $S_3 = SBQP$, $\bar{n} = \bar{k}$

$z=a$, $ds = dx\,dy$ $\quad 0 \leq x \leq a$, $0 \leq y \leq a$

$\therefore \bar{F}.\bar{n} = z = a$

$\because z = a$

$\int_S \bar{F}.\bar{n}ds = \iint_{S_3} \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a^3$ — (4)

(iv) For $S_4 = OARC$, $z=0$, $\bar{n} = -\bar{k}$ $ds = dx\,dy$ $0 \le x \le a$,

$0 \le y \le a$

$\therefore \bar{F}.\bar{n} = -z = 0 \quad \because z = 0$

$$\iint_{S_4} \bar{F}.\bar{n}\,ds = 0 \quad \text{———} (5)$$

(v) For $S_5 = PSCR$, $y = a$, $\bar{n} = \bar{j}$, $ds = dz\,dx$, $0 \le x \le a$,

$0 \le z \le a$

$\therefore \bar{F}.\bar{n} = -2x^2y = -2ax^2 \quad (\because y = a)$

$$\iint_{S_5} \bar{F}.\bar{n}\,ds = \int_{x=0}^{a} \int_{z=0}^{a} (-2ax^2)\,dz\,dx$$

$$= \int_{x=0}^{a} \left[-2ax^2z\right]_{z=0}^{a} dx = \int_{x=0}^{a} -2ax^2a\,dx$$

$$\Rightarrow -2a^2\left[\frac{x^3}{3}\right]_0^a \Rightarrow -\frac{2a^5}{3} \quad \text{———} (6)$$

(vi) For $S_6 = OBQA$, $y = 0$, $\bar{n} = -\bar{j}$, $ds = dz\,dx$

$0 \le x \le a$, $0 \le y \le a$

$\bar{F}.\bar{n} = 2x^2y = 0 \quad \because y = 0$

$$\iint_{S_6} \bar{F}.\bar{n}\,ds = 0 \quad \text{———} (7)$$

Hence,

$$\iint_{S} \bar{F}.\bar{n}\,ds = \iint_{S_1} \bar{F}.\bar{n}\,ds + \iint_{S_2} \bar{F}.\bar{n}\,ds + \iint_{S_3} \bar{F}.\bar{n}\,ds + \iint_{S_4} \bar{F}.\bar{n}\,ds$$

$$+ \iint_{S_5} \bar{F}.\bar{n}\,ds + \iint_{S_6} \bar{F}.\bar{n}\,ds$$

$$\Rightarrow a^5 - \frac{a^4}{4} + \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3$$

$\Rightarrow$ LHS = RHS (using 1)

Hence, Gauss Divergence Theorem is verified

4.c) Verify Stoke's Theorem, for $\overline{F} = -y^3\bar{i} + x^3\bar{j}$, where S is the circular disc $x^2+y^2 \leq 1,\ z=0$

Given $\overline{F} = -y^3\bar{i} + x^3\bar{j}$

By Stoke's Theorem,

$$\int_C \overline{F}\cdot d\bar{r} = \int_C \text{curl}\,\overline{F}\cdot\bar{n}\,ds$$

The boundary of C of S is a circle in xy plane, $x^2+y^2=1$, $z=0$.

We use the parametric co-ordinates

$x = \cos\theta \qquad y = \sin\theta, \qquad 0 \leq \theta \leq 2\pi$

$dx = -\sin\theta\,d\theta \qquad dy = \cos\theta\,d\theta$

$$\text{LHS} = \oint \overline{F}\cdot d\bar{r} = \int F_1\,dx + F_2\,dy + F_3\,dz = \int -y^3dx + x^3dy$$

$$= \int_0^{2\pi} \left[-\sin^3\theta\,(-\sin\theta) + \cos^3\theta\cos\theta\right]d\theta$$

$$= \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\,d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} \sin^2 2\theta\,d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi}(1-\cos4\theta)\,d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}\sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4}$$

$$= \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now,

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}\left(3x^2 + 3y^2\right)$$

$$\therefore RHS = \int (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3\int_S (x^2+y^2)\, \bar{k} \cdot \bar{n}\, ds$$

We have $(\bar{k} \cdot \bar{n})\, dS = dx\, dy$ and R is the region in xy plane

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3\iint_R (x^2+y^2)\, dx\, dy$$

Put $x = r\cos\phi$, $y = r\sin\phi$ $\therefore dx\, dy = r\, dr\, d\phi$

r is varying from 0 to 1 and $0 \leq \phi \leq 2\pi$

$$\therefore \int (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3\int_{\phi=0}^{2\pi} \int_{r=0}^{1} r^2 \cdot r\, dr\, d\phi$$

$$= 3\int_{\phi=0}^{2\pi} \int_{r=0}^{1} r^3\, dr\, d\phi = 3\int_{\phi=0}^{2\pi} \left[\frac{r^4}{4}\right]_{r=0}^{1} d\phi = 3\int_{\phi=0}^{2\pi} \frac{1}{4}\, dr$$

$$= \frac{3}{4}\int_{\phi=0}^{2\pi} d\phi = \frac{3}{4}\left[2\pi - 0\right]$$

$$= \frac{3\pi}{2} = LHS$$

$\therefore LHS = RHS$

Hence, Stoke's Theorem is verified