Unit-1

1M

1) a) Define Discrete and Continuous random variable

A: **Discrete Random Variable:** A random variable 'x' which can take only a finite no. of discrete values in an interval of domain is called a discrete random variable. In other words; if the random variable takes the values only on the set $\{0, 1, 2, \ldots n\}$ is called discrete random variable.

Continuous Random Variable: A random variable 'x' which can take both integer and fractional values continuously in an interval is called a continuous random variable

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1) b). Define mean and Variance of continuous random variable

A: Let $f(x)$ be the probability density function of continuous random variable x.

Mean: Mean of a distribution is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

If x is defined from a to b, then

$$\mu = E(X) = \int_{a}^{b} x\, f(x)\, dx$$

Variance: Variance of a distribution is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

(or) $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

Suppose that the variate X is defined from a to b then

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)\, dx \quad \text{(or)}$$

$$\sigma^2 = \int_a^b x^2 f(x)\, dx - \mu^2$$

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1) c) A random sample with replacement of size 2 is taken from $S = \{1, 2, 3\}$. Let the random variable X denote the sum of the two numbers taken, write the probability distribution.

A: Let 'S' be the Sample Space

$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$X(S) = \{2, 3, 4, 5, 6\}$

$P(2) = P(X=2) = P[(1,1)] = \frac{1}{9}$

$P(3) = P(X=3) = P[(1,2), (2,1)] = \frac{2}{9}$

$P(4) = P(X=4) = P[(1,3), (2,2), (3,1)] = \frac{3}{9}$

$P(5) = P(X=5) = P[(2,3), (3,2)] = \frac{2}{9}$

$P(6) = P(X=6) = P[(3,3)] = \frac{1}{9}$

The probability of X is

X	2	3	4	5	6
$P(X)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

1) d) Define Expectation and Variance of discrete random variable

A: <u>Expectation</u>: Suppose a random variable X assumes the values $x_1, x_2, \ldots x_n$ with respective probabilities $P_1, P_2, \ldots P_n$ then the mathematical expectation or mean or expected value of X, denoted by $E(X)$ is defined as the sum of products of different values of X and the corresponding probabilities

$$E(X) = x_1 P_1 + x_2 P_2 + \ldots + x_n P_n$$

<u>Variance</u>: Variance of the probability distribution of a random variable X is the mathematical expectation of $[X - E(X)]^2$

then, $Var(X) = E[X - E(X)]^2$

1) e) If a random variable has the probability density $f(x)$ as

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0 ; & x \leq 0 \end{cases}$$ Find the probability between 1 and 3

Sol: $P(1 \leq X \leq 3) = \int_1^3 f(x)dx$

$$= \int_1^3 2e^{-2x} dx$$

$$= \left[\frac{\cancel{2}e^{-2x}}{-\cancel{2}}\right]_1^3$$

$$= \left[-e^{-2(3)} - \left(-e^{-2(1)}\right)\right]$$

$$= -e^{-6} + e^{-2}$$

$$= e^{-2} - e^{-6}$$

2) a) Prove that If x is a discrete random variable and k is a constant, then $E(x+k) = E(x)+k$.

Sol: Suppose x is random Variable & k is constant

From the definition of Expectation,

$$E(X) = \sum_{i=1}^{n} x_i P_i$$

$$E(x+k) = \sum_{i=1}^{n} (x_i + k) P_i$$

$$= \sum_{i=1}^{n} (x_i P_i + k P_i)$$

$$= \sum_{i=1}^{n} x_i P_i + \sum_{i=1}^{n} k P_i$$

$$= E(X) + k(1) \quad \left(\because \sum_{i=1}^{n} P_i = 1\right)$$

$$= E(X) + k$$

$$\therefore E(x+k) = E(X) + k.$$

2) b). A fair coin is tossed until a head or five tails occurs. Find the expected number E of tosses of the coin.

Sol: If head occurs 1^{st} time there will be only one toss on the otherhand, if first one is tail, second occurs

If head occurs there will be only two tosses.

suppose second one is also tail third occurs

If head occurs there will be three tosses

& soon.

$P(1) = P(H) = \frac{1}{2}$

$P(2) = P(TH) = \frac{1}{4}$

$P(3) = P(TTH) = \frac{1}{8}$

$P(4) = P(TTTH) = \frac{1}{16}$

$P(5) = P(TTTTH) + P(TTTTT)$

$= \frac{1}{32} + \frac{1}{32} = \frac{1}{16}$

the probablity distribution of X is

X	1	2	3	4	5
P(X)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

$\therefore E(X) = \sum_{i=1}^{n} P_i x_i = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{16}\right)$

$= \frac{31}{16} = 1.937$

2) c) If the probability density of a random variable is given by

$$f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$ find the Value of k.

Sol: we know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{1} k(1-x^2)\,dx + 0 = 1$$

$$k\left(x - \frac{x^3}{3}\right)_0^1 = 1 \quad \text{(or)} \quad k\left(1-\frac{1}{3}\right) = 1$$

$$\therefore k = \frac{3}{2}$$

2) d) If X is a continuous random variable and $Y = aX+b$. Prove that $E(Y) = aE(X)+b$ and $V(Y) = a^2 V(X)$ where V stands for variance and a, b are constants.

Sol: Suppose that X is a continuous random variable and $Y = aX+b$

W.K.T the expectation is $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

then $E(Y) = E(aX+b)$

$$= \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (axf(x) + bf(x))\,dx$$

$$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= aE(X) + b(1)$$

$$E(Y) = aE(X)+b \quad \text{--- (i)}$$

W.K.T variance, $V(X) = E\{[X-E(X)]^2\}$

from (i), $E(Y) = aE(X) + b$ — ①

where $Y = ax + b$ — ②

② − ① $\quad Y - E(Y) = ax + b - (aE(X) + b)$

$Y - E(Y) = ax + \cancel{b} - aE(X) - \cancel{b}$

$Y - E(Y) = a(X - E(X))$

Squaring on Both sides

$(Y - E(Y))^2 = a^2 (X - E(X))^2$

Taking Expectation on B/S

$E\{[Y - E(Y)]^2\} = E\{a^2[X - E(X)]^2\}$

$E\{[Y - E(Y)]^2\} = a^2 E\{[X - E(X)]^2\}$

$V(Y) = a^2 V(X)$

2) e) A random variable X is defined as the sum of the numbers on the faces when two dice are thrown. Find the mean of X.

Sol: Sample space of two dice is

S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Let X denote $X(a,b) = a + b \;\; \forall (a,b) \in S$

$X(1,1) = 2$	$X(2,1) = 3$	$X(3,1) = 4$	$X(4,1) = 5$
$X(1,2) = 3$	$X(2,2) = 4$	$X(3,2) = 5$	$X(4,2) = 6$
$X(1,3) = 4$	$X(2,3) = 5$	$X(3,3) = 6$	$X(4,3) = 7$
$X(1,4) = 5$	$X(2,4) = 6$	$X(3,4) = 7$	$X(4,4) = 8$
$X(1,5) = 6$	$X(2,5) = 7$	$X(3,5) = 8$	$X(4,5) = 9$
$X(1,6) = 7$	$X(2,6) = 8$	$X(3,6) = 9$	$X(4,6) = 10$

$X(5,1) = 6$ $X(6,1) = 7$

$X(5,2) = 7$ $X(6,2) = 8$

$X(5,3) = 8$ $X(6,3) = 9$

$X(5,4) = 9$ $X(6,4) = 10$

$X(5,5) = 10$ $X(6,5) = 11$

$X(5,6) = 11$ $X(6,6) = 12$

$\therefore$ The possible values of X are $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

The probability distribution is

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The mean $\mu = \sum_{i=1}^{n} x_i p_i$

$= \sum_{i=1}^{12} 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right)$
$+ 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$

$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36}$
$+ \frac{22}{36} + \frac{12}{36}$

$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36}$

$= \frac{252}{36}$

$= 7$

$\therefore$ The Mean μ of $x = 7$

9) a) A player tosses two fair coins. He wins Rs. 100/- if a head appears. Rs. 200/- if two heads appear. On the other hand, he loses Rs. 500/- if no head appears. Determine the expected value E of the game and is the game favourable to player.

Sol: Let, $S = \{HH, HT, TH, TT\}$ be a sample

Let 'x' denote the value of money that he win or lose then

$X = \{-500, 100, 200\}$

$P(X=-500) = \frac{1}{4}$ (no head i.e, only tails)

$P(X=100) = \frac{2}{4}$ (TH, HT i.e, only one head)

$P(X=200) = \frac{1}{4}$ (HH, i.e two heads)

The probability distribution is

X	-500	100	200
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

The Expected Value is

$$E(X) = \sum_{i=1}^{3} x_i p_i$$

$$= -500\left(\frac{1}{4}\right) + 100\left(\frac{2}{4}\right) + 200\left(\frac{1}{4}\right)$$

$$= \frac{-500+200+200}{4}$$

$$= -25 < 0$$

∴ The game is not favourable

3) b) A continuous random variable has the probability density function

$$f(x) = \begin{cases} kxe^{-x\lambda} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{, otherwise} \end{cases}$$

Determine i) k ii) Mean iii) Variance

Sol: Given that,

$$f(x) = \begin{cases} kxe^{-\lambda x} & ; \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} (0)\,dx + \int_{0}^{\infty} k.xe^{-\lambda x}\,dx = 1$$

$$\Rightarrow 0 + k\int_{0}^{\infty} xe^{-\lambda x}\,dx = 1$$

$$\Rightarrow k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - (1)\frac{e^{-\lambda x}}{(-\lambda)^2}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[\left(\frac{1}{-\lambda} - \frac{1}{\lambda^2}\right)e^{-\lambda x}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[0 - \left(\frac{0}{-\lambda} - \frac{1}{\lambda^2}\right)e^{0}\right] = 1$$

$\Rightarrow K\left(\frac{1}{\lambda^2}\right)=1$

$\Rightarrow K=\lambda^2$

The probability distribution function is:

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ \infty & \text{otherwise} \end{cases}$$

Mean, $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x(0)\,dx + \int_{0}^{\infty} x\left(\lambda^2 x e^{-\lambda x}\right)dx$$

$$= 0 + \lambda^2 \int_{0}^{\infty} x^2 e^{-\lambda x}\,dx$$

$$= \lambda^2 \left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{(-\lambda)^2}\right) + 2\,\frac{e^{-\lambda x}}{(-\lambda)^3} \right]_0^{\infty}$$

$$= \lambda^2 \left[\left(\frac{x^2}{-\lambda} - \frac{2x}{\lambda^2} - \frac{2}{\lambda^3}\right) e^{-\lambda x} \right]_0^{\infty}$$

$$= \lambda^2 \left[0 - \left(0 - 0 - \frac{2}{\lambda^3}\right) e^{0} \right]$$

$$= \lambda^2 \left(\frac{2}{\lambda^3}\right)$$

$$= \frac{2}{\lambda}$$

3) c) A random variable X has following probability distribution

X	0	1	2	3	4	5	6	7	8
P(X)	a	3a	5a	7a	9a	11a	13a	15a	17a

i) Determine the value of a.

ii) Find $P(X<3)$ and $P(0<X<5)$

Sol:

i) W.K.T $\sum_{i=1}^{n} P_i = 1$

$a+3a+5a+7a+9a+11a+13a+15a+17a=1$

$81a=1$

$\boxed{a=\frac{1}{81}}$

The probability distribution is

X	0	1	2	3	4	5	6	7	8
P(X)	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

ii) $P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$= \frac{1}{81}+\frac{3}{81}+\frac{5}{81}$

$= \frac{9}{81}$

$= \frac{1}{9}$

$P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= \frac{3}{81}+\frac{5}{81}+\frac{7}{81}+\frac{9}{81} = \frac{24}{81} = 0.296$

3) d) For a continuos probability density function is given by $f(x) = ce^{-|x|}, -\infty < x < \infty$
Find the value of c & hence mean and variance

sol: $f(x) = ce^{-|x|}$

we know that the total probability is unity

i.e $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{\infty} ce^{-|x|}\,dx = 1$ $\quad \left[\because f(-x) = ce^{-|-x|} = ce^{-|x|} = f(x)\right.$

$\Rightarrow 2c\int_{0}^{\infty} e^{-x}\,dx = 1$ $\quad \Rightarrow f$ is even

$\Rightarrow 2c\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1.$ $\quad \left.\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right]$

$\Rightarrow -2c\left(e^{-x}\right)_0^{\infty} = 1$ $\quad \left[\because |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}\right]$

$\Rightarrow -2c(0-1) = 1$

$\Rightarrow 2c = 1$

$\Rightarrow c = \frac{1}{2}$

$\therefore f(x) = \frac{1}{2}e^{-|x|}, \; -\infty < x < 0$

3) e) Let X denote the minimum of two numbers that appear when a pair of fair dice is thrown once. Determine the

i) Discrete probability distribution

ii) Expectation

Sol: let $S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$
$(5,1)(5,2),(5,3),(5,4),(5,5),(5,6)$
$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

then

$X(1,1)=1$ $\quad X(2,1)=1$ $\quad X(3,1)=1$ $\quad X(4,1)=1$
$X(1,2)=1$ $\quad X(2,2)=2$ $\quad X(3,2)=2$ $\quad X(4,2)=2$
$X(1,3)=1$ $\quad X(2,3)=2$ $\quad X(3,3)=3$ $\quad X(4,3)=3$
$X(1,4)=1$ $\quad X(2,4)=2$ $\quad X(3,4)=3$ $\quad X(4,4)=4$
$X(1,5)=1$ $\quad X(2,5)=2$ $\quad X(3,5)=3$ $\quad X(4,5)=4$
$X(1,6)=1$ $\quad X(2,6)=2$ $\quad X(3,6)=3$ $\quad X(4,6)=4$

$X(5,1)=1$ $\quad X(6,1)=1$
$X(5,2)=2$ $\quad X(6,2)=2$
$X(5,3)=3$ $\quad X(6,3)=3$
$X(5,4)=4$ $\quad X(6,4)=4$
$X(5,5)=5$ $\quad X(6,5)=5$
$X(5,6)=5$ $\quad X(6,6)=6$

i) The Probability Distribution is

X	1	2	3	4	5	6
$P(X=x_i)$	$11/36$	$9/36$	$7/36$	$5/36$	$3/36$	$1/36$

Expected Value $= \frac{11}{36} + \frac{18}{36} + \frac{21}{36} + \frac{20}{36} + \frac{15}{36} + \frac{6}{36}$
$= \frac{91}{36} = 2.52$

Variance, $\sigma^2 \int_{\infty}^{-\infty} x^2 f(x)\,dx - \mu^2$

$$= \left[\int_{-\infty}^{\infty} x^2(0)\,dx + \int_{0}^{\infty} x^2(\lambda^2 x e^{-\lambda x})\,dx\right] - \left(\frac{2}{\lambda}\right)^2$$

$$= \left[0 + \lambda^2 \int_0^{\infty} x^3 e^{-\lambda x}\,dx\right] - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[\frac{x^3 e^{-\lambda x}}{(-\lambda)} - \frac{3x^2 e^{-\lambda x}}{(-\lambda)^2} + \frac{6x e^{-\lambda x}}{(-\lambda)^3} - \frac{6x^{1-\lambda x}}{(-\lambda)^4}\right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[\left(-\frac{x^3}{\lambda} - \frac{3x^2}{\lambda^2} - \frac{6x}{\lambda^3} - \frac{6}{\lambda^4}\right)e^{-\lambda x}\right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2\left[0 - \left(0 - 0 - 0 - \frac{6}{\lambda^4}\right)(1)\right] - \frac{4}{\lambda^2}$$

$$= \lambda^2\left(\frac{6}{\lambda^4}\right) - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$

$\therefore$ Variance $= \frac{2}{\lambda^2}$

4) a) A random variables X has the following probability function.

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Determine K

ii) $P(X \le K) > \frac{1}{2}$, find the minimum value of K

iii) Determine the distribution function of x.

iv) mean

v) Variance

Sol: i) we know that,

$$\sum_{i=0}^{7} P_i = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$10k^2 + 10k - k - 1 = 0$$

$$10k(k+1) - 1(k+1) = 0$$

$$(k+1)(10k-1) = 0$$

$$k = -1, \frac{1}{10}$$

$$\boxed{\therefore K = \frac{1}{10}}$$

ii) The probability distribution is

x	0	1	2	3	4	5	6	7
$P(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

$P(X < k) > 1/2$ find min value of k

$P(X \leq 6) = 1 - P(X > 6)$

$= 1 - \frac{17}{100}$

$= 0.83 > 0.5$

$P(X \leq 5) = 1 - P(X > 5)$

$= 1 - [P(X=6) + P(X=7)]$

$= 1 - \left[\frac{2}{100} + \frac{17}{100}\right]$

$= 1 - \frac{19}{100}$

$= 0.81 > 0.5$

$P(X \leq 4) = 1 - P(X > 4)$

$= 1 - [P(X=5) + P(X=6) + P(X=7)]$

$= 1 - \left[\frac{1}{100} + \frac{2}{100} + \frac{17}{100}\right]$

$= 1 - \frac{20}{100}$

$= 1 - 0.2$

$= 0.8 > 0.5$

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10}$

$= \frac{5}{10}$

$= 0.5$

iii) $P(X < 6) = 1 - P(X \geq 6)$

$= 1 - [P(X=6) + P(X=7)]$

$= 1 - \left[\frac{2}{100} + \frac{17}{100}\right] = 0.8$

$P(x \geq 6) = 1 - P(X < 6)$

$= 1 - \left(0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}\right)$

$= 1 - 0.81$

$= 0.19$

$P(0 < x < 5) = 1 - [P(X=1) + P(X=2) + P(X=3) + P(X=4)]$

$= 1 - \frac{8}{10}$

$= \frac{10-8}{10}$

$= \frac{2}{10}$

$P(0 < x < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{8}{10}$

$= 0.8$

$P(0 \leq X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$= \frac{8}{10}$

$= 0.8$

iv) Mean $= \sum_{i=0}^{7} x_i P_i$

$= 0(0) + 1\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 3\left(\frac{2}{10}\right) + 4\left(\frac{3}{10}\right) + 5\left(\frac{1}{100}\right) + 6\left(\frac{2}{100}\right) + 7\left(\frac{17}{100}\right)$

$= \frac{23}{10} + \frac{136}{100}$

$= 2.3 + 1.36 = 3.66$

v) variance $= \sum_{i=0}^{7} x_i^2 p_i - \mu^2$

$= 0^2(0) + 1^2\left(\frac{1}{10}\right) + 2^2\left(\frac{2}{10}\right) + 3^2\left(\frac{2}{10}\right)$
$+ 4^2\left(\frac{3}{10}\right) + 5^2\left(\frac{1}{100}\right) + 6^2\left(\frac{2}{100}\right) + 7^2\left(\frac{17}{100}\right)$
$- (3.66)^2$

$= \frac{75}{10} + \frac{930}{100} - 13.39$

$= 7.5 + 9.3 - 13.39$

$= 3.43$

4) b)

i) A sample of 4 items is selected at random from a box containing 12 items of which 5 are defected. Find the expected number E of defective items

ii) Find the mean & variance of the uniform probability distribution given by $f(x) = \frac{1}{n}$ for $x = 1, 2, \dots n$

Sol:

i) Given that

Total no. of items $= 12$

No. of defective items $= 5$

No. of Non-defective items $= 7$.

Let X denotes the no. of defective items then the possible values of X are 0, 1, 2, 3, 4

$P(X=0) = P(\text{non defective items})$
$= {}^7C_4 / {}^{12}C_4 = 35/495 = 0.07$

$P(X=1) = P(1$ defective & 3 non defective$)$

$$= \frac{5c_1 \times 7c_3}{12c_4}$$

$$= \frac{175}{495}$$

$$= 0.35$$

$P(X=2) = P(2$ defective & 2 non-defective$)$

$$= \frac{5c_2 \times 7c_2}{12c_4}$$

$$= \frac{210}{495}$$

$$= 0.42$$

$P(X=3) = P(3$ defective & 1 non-defective$)$

$$= \frac{5c_3 \times 7c_2}{12c_4}$$

$$= \frac{70}{495}$$

$$= 0.14$$

$P(X=4) = P($All defective$)$

$$= \frac{5c_4}{12c_4}$$

$$= \frac{5}{495} = \frac{1}{99}$$

The probability distribution is

$X = x_i$	0	1	2	3	4
$P(X=x_i)$	$\frac{7}{99}$	$\frac{35}{99}$	$\frac{42}{99}$	$\frac{14}{99}$	$\frac{1}{99}$

Expected number of defective items

$= E(X) = \Sigma x_i p_i$

$= 0\left(\frac{7}{99}\right) + 1\left(\frac{35}{99}\right) + 2\left(\frac{42}{99}\right) + 3\left(\frac{14}{99}\right)$

$+ 4\left(\frac{1}{99}\right)$

$= \frac{165}{99}$

$= 1.66.$

ii) The probability Distribution is

x	1	2	3	n
$f(x)$	$1/n$	$1/n$	$1/n$	$1/n$

i) Mean $= \sum_{i=1}^{n} x_i f(x_i)$

$= 1\left(\frac{1}{n}\right) + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + n\left(\frac{1}{n}\right)$

$E(X) = \mu = \frac{1}{n}(1+2+\dots+n)$

$= \frac{n(n+1)}{2n} = \frac{n+1}{2}$

ii) Variance $= \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$

$= 1\left(\frac{1}{n}\right) + 2^2\left(\frac{1}{n}\right) + 3^2\left(\frac{1}{n}\right) + \dots + n^2\left(\frac{1}{n}\right) - \left(\frac{n+1}{2}\right)^2$

$$= \frac{1}{n}\left[1^2+2^2+\cdots+n^2\right] - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{n}\,\frac{n(n+1)(2n+1)}{6} - \frac{1}{4}(n+1)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{(n+1)(n-1)}{12}$$

$$= \frac{n^2-1}{12}$$

If x is continuous random variable and K is constant then i) $Var(x+K) = Var(x)$

ii) $Var(Kx) = K^2 Var(x)$

Sol: We know that

$$Var(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$\Rightarrow Var(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$$

Then $Var(x+K) = \int_{-\infty}^{\infty} (x+K)^2 f(x)\,dx -$

$$\left(\int_{-\infty}^{\infty} (x+K) f(x)\,dx\right)^2$$

$$= \int_{-\infty}^{\infty} (x^2+K^2+2Kx) f(x)\,dx - \left(\int_{-\infty}^{\infty} (x f(x) + K f(x))\,dx\right)^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + K^2 \int_{-\infty}^{\infty} f(x)\,dx + 2K \int_{-\infty}^{\infty} x f(x)\,dx$$

$$- \left[\int_{-\infty}^{\infty} x f(x)\,dx + K \int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= E(x^2) + K^2(1) + 2KE(x) - (E(x) + K(1))^2$$

$$\left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= E(x^2) + K^2 + 2K E(x) - \left[(E(x))^2 + K^2 + 2K E(x) \right]$$

$$= E(x^2) + \cancel{K^2} + \cancel{2K E(x)} - (E(x))^2 - \cancel{K^2} - \cancel{2K E(x)}$$

$$\Rightarrow E(x^2) - (E(x))^2 = Var(x)$$

$$\therefore \boxed{Var(X+K) = Var(X)}$$

ii) $$Var(KX) = \int_{-\infty}^{\infty} (Kx)^2 f(x) dx - \left[\int_{-\infty}^{\infty} (Kx) f(x) dx \right]^2$$

$$= \int_{-\infty}^{\infty} K^2 x^2 f(x) dx - K^2 \left[\int_{-\infty}^{\infty} x f(x) dx \right]^2$$

$$= K^2 \int_{-\infty}^{\infty} x^2 f(x) dx - K^2 \left[\int_{-\infty}^{\infty} x f(x) dx \right]^2$$

$$= K^2 E(x^2) - K^2 (E(x))^2$$

$$= K^2 \left[E(x^2) - (E(x))^2 \right]$$

$$\Rightarrow \boxed{K^2 Var(x) = Var(Kx)}$$

3) For the continuous probability function $f(x) = kx^2e^{-x}$ where $x \geq 0$ find

i) k ii) mean iii) variance.

Sol:- Given that

$$f(x) = Kx^2e^{-x} \quad \text{when } x \geq 0$$

Treat $f(x) = 0$ when $x \leq 0$

i) We know that the total probability is unity i.e

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} (0)dx + \int_{0}^{\infty} Kx^2e^{-x}dx = 1$$

$$\Rightarrow 0 + K\int_{0}^{\infty} \underset{f}{x^2}\underset{g}{e^{-x}}dx = 1$$

$$\Rightarrow K\left[x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 1$$

$$\Rightarrow K\left[(-x^2 - 2x - 2)e^{-x}\right]_0^{\infty} = 1$$

$$\Rightarrow K[0 - (-2)] = 1$$

$$\Rightarrow K[2] = 1$$

$$\Rightarrow K = \frac{1}{2} \qquad \therefore f(x) = \frac{1}{2}x^2e^{-x} \text{ when } x \geq 0$$

ii) Mean:

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{0} x(0)dx + \int_{0}^{\infty} x\left(\frac{1}{2}x^2e^{-x}\right)dx$$

$$= 0 + \frac{1}{2} \int_0^\infty x^3 e^{-x} dx$$

$$= \frac{1}{2} \left[x^3 \frac{e^{-x}}{(-1)} - 3x^2 \frac{e^{-x}}{(-1)^2} + 6x \frac{e^{-x}}{(-1)^3} - 6 \frac{e^{-x}}{(-1)^4} \right]_0^\infty$$

$$= \frac{1}{2} \left[(-x^3 - 3x^2 - 6x - 6) e^{-x} \right]_0^\infty$$

$$= \frac{1}{2} \left[(0 - (-6)) \right]$$

$$= \frac{6}{2}$$

$$= 3$$

Mean $= 3$

iii) <u>Variance:</u>

The variance, $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

$$= \int_{-\infty}^{0} x^2 (0)\, dx + \int_0^\infty x^2 \left(\frac{1}{2} x^2 e^{-x} \right) dx - 3^2$$

$$= 0 + \frac{1}{2} \int_0^\infty x^4 e^{-x} dx - 9$$

$$= \frac{1}{2} \left[x^4 \frac{e^{x}}{(-1)} - 4 \frac{x^3 e^{-x}}{(-1)^2} + \frac{12x^2 e^{-x}}{(-1)^3} - 24x \frac{e^{-x}}{(-1)^4} + 24 \frac{e^{-x}}{(-1)^5} \right]_0^\infty - 9$$

$$= \frac{1}{2}\left[(-x^4 - 4x^3 - 12x^2 - 24x - 20)e^{-x}\right]_0^{\infty} - 9$$

$$= \frac{1}{2}\left[(0 - (-24))\right] - 9$$

$$= \frac{1}{2}(24) - 9$$

$$= 12 - 9$$

$$= 3$$

$\therefore$ variance $= 3$.