UNIT - I

RANDOM VARIABLES

1.

a). Define discrete and continuous random variable. [1M]

Sol:- Discrete Random Variable;

A random variable 'X' which can take only a finite no. of discrete values (integers) in an interval of domain is called a 'discrete random variable'.

Ex:- Consider a random experiment consisting of tossing a coin twice.

The sample space is $\Rightarrow S = \{TT, TH, HT, HH\}$

Define a function $X: S \to R$ by $X(S) =$ no. of heads

$\Rightarrow X(TT) = 0,\ X(TH) = 1,\ X(HT) = 1,\ X(HH) = 2$

$\therefore$ The range of X is $\{0, 1, 2\}$

$\therefore$ X is discrete random variable.

Continuous Random Variable:

A random variable 'X' which can take values continuously i.e. which takes all possible values (integers & fractions) in an interval is called as 'continuous random variable'.

Ex:- The height, age and weight of individuals (or) students in a class are continuous random variables

b). Define mean and variance of a continuous random variable. [1M]

Sol:- The mean of a continuous probability distribution is given by $\Rightarrow \mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

If X is defined from a to b

$$\Rightarrow \mu = E(X) = \int_a^b x f(x) dx.$$

The variance of a continuous probability distribution is given by

$$\Rightarrow \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx \text{ (or) } \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

Suppose, the variant 'X' is defined from a to b.

$$\Rightarrow \sigma^2 = \int_a^b (x-\mu)^2 f(x) dx \text{ (or) } \int_a^b x^2 f(x) dx - \mu^2$$

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c). A random sample with replacement of size 2 is taken from $S = \{1, 2, 3\}$. Let the random variable X denote the sum of the two numbers taken, write the probability distribution. [1M]

Sol:- Given,

$S = \{1, 2, 3\}$

Since, we are sampling with replacement, each of the two numbers can be any of 1, 2 or 3.

The possible pairs are: $\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Let 'X' be the sum of the two numbers

$\Rightarrow X(1,1) = 1+1 = 2$, $X(1,2) = 3$, $X(1,3) = 4$, $X(2,1) = 3$, $X(2,2) = 4$, $X(2,3) = 5$, $X(3,1) = 4$, $X(3,2) = 5$, $X(3,3) = 6$.

The possible values of 'X' are 2, 3, 4, 5, 6

$\therefore$ The probability distribution of 'X' is

X	2	3	4	5	6
P(X)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

d). Define expectation and variance of discrete random variable. [1M]

Sol: Suppose random variable 'x' assumes values $x_1, x_2, x_3 \cdots\cdots x_n$ with probabilities $P_1, P_2 \cdots\cdots P_n$.

Mean : The mean is denoted by $E(X)$, is defined as sum of products of different values of X and their corresponding probabilities.

$$\Rightarrow \mu = \frac{\sum_{i=1}^{n} x_i P_i}{\sum_{i=1}^{n} P_i} = \sum_{i=1}^{n} x_i P_i$$

$$\Rightarrow var(x) = \sum_{i=1}^{n} x_i^2 P_i - \mu^2$$

e). If a random variable has the probability density $f(x)$ as $f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$. Find the probability between 1 and 3. [1M]

Sol: Given, $f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

$$\therefore P(1 \leq X \leq 3) = \int_1^3 f(x)dx$$

$$= \int_1^3 2e^{-2x}dx = 2\int_1^3 e^{-2x}dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$$

$$= -\left[e^{-2x}\right]_1^3$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

Q2:

a). Prove that if X is a discrete random variable and K is a constant, then $E(X+K) = E(X)+K$. [3M]

Sol: Suppose that 'X' is a random variable and K is a constant.

By definition of expectation, we have

$$\Rightarrow E(X+K) = \sum_{i=1}^{n} (x_i+k)p_i \quad \left[\because E(X) = \sum_{i=1}^{n} x_i p_i\right]$$

$$= \sum_{i=1}^{n} (x_i p_i + k p_i)$$

$$= \sum_{i=1}^{n} x_i p_i + \sum_{i=1}^{n} k p_i$$

$$= E(X) + K\sum_{i=1}^{n} p_i \quad \left(\because \sum_{i=1}^{n} p_i = 1\right)$$

$$= E(X) + K(1)$$

$$= E(X) + K$$

$$\therefore E(X+K) = E(X)+K$$

Hence proved.

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b). A fair coin is tossed until a head or five tails occurs. Find the expected number E of tosses of the coin. [3M].

Sol: → If head appears first time there will be only one toss. $(\because n(S) = 2^1 = 2)$.

→ If first one is tail and second one is head then there will be two tosses. $(\because n(S) = 2^2 = 4)$.

→ If first two times is tail and third time is head then there will be three tosses.

$(\because n(S) = 2^3 = 8)$

-) If first three times is tail and fourth time is head then there will be four tosses.

$(\because n(S) = 2^4 = 16)$.

-) If first four times is tail and fifth time is head or five times is tail then there will be five tosses. $(\because n(S) = 2^5 = 32)$.

Let,

X be the number of tosses;

the the possible values of 'X' are; $\{1, 2, 3, 4, 5\}$.

$\therefore P(1) = P(H) = \frac{1}{2^1} = \frac{1}{2}$

$P(2) = P(TH) = \frac{1}{2^2} = \frac{1}{4}$

$P(3) = P(TTH) = \frac{1}{2^3} = \frac{1}{8}$

$P(4) = P(TTTH) = \frac{1}{2^4} = \frac{1}{16}$

$P(5) = P(TTTTH)$ (or) $P(TTTTT) = \frac{1}{2^5} = \frac{1}{32}$

$\therefore$ The probability distribution of 'X' is

X	1	2	3	4	5
P(X)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

The expected value 'E' is

$$\Rightarrow E = \sum_{i=1}^{5} x_i P_i = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{32}\right)$$

$$= 1.937$$

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c). If the probability density of a random variable is given by $f(x) = \begin{cases} K(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. Find the value of K. [3M]

Sol:- We know that,

The total probability is unity.

i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$\Rightarrow \int_{0}^{1} f(x)\,dx = 1$

$\Rightarrow \int_{0}^{1} K(1-x^2)\,dx = 1 \quad \Rightarrow K\left[x - \frac{x^3}{3}\right]_0^1 = 1$

$\Rightarrow K\left[\left(1-\frac{1}{3}\right) - (0-0)\right] = 1$

$\Rightarrow K\left(\frac{2}{3}\right) = 1$

$\therefore \boxed{K = \frac{3}{2}}$

d). If X is a continuous random variable and $Y = aX+b$. Prove that $E(Y) = aE(X)+b$ and $V(Y) = a^2 V(X)$ where V stands for variance and a, b are constants. [3M].

Sol: Given that;

X is continuous random variable.

and $Y = ax+b$.

By definition of expectation, we have

$\Rightarrow E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

then $E(Y) = E(ax+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$

$= \int_{-\infty}^{\infty} (ax f(x) + b f(x))\,dx$

$= \int_{-\infty}^{\infty} ax \cdot f(x)\,dx + \int_{-\infty}^{\infty} b f(x)\,dx$

$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$

$= aE[X] + b[1]$

$= aE(X) + b \quad \left(\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right)$

$\therefore E(ax+b) = aE(X) + b$

We have,

$E(Y) = E(ax+b) = aE(x)+b$ - ①

where, $Y = ax+b$ - ②

② - ①

$Y - E(Y) = (ax+b) - (aE(x)+b)$

$= ax+b-aE(x)-b$

$Y - E(Y) = a(x - E(x))$

Squaring on bothsides

$[Y-E(Y)]^2 = a^2[x-E(x)]^2$

Taking expectation, we get

$\Rightarrow E\{[Y-E(Y)]^2\} = a^2 E\{(x-E(x))^2\}$

$\Rightarrow V(Y) = a^2 V(X)$

$\because V(x) = E[x-E(x)]^2$

Hence proved.

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e). A random variable X is defined as the sum of the numbers on the faces when two dice are thrown. Find the mean of X. [3M]

Sol:- Given, X is a random variable.

The sample space is

$$S = \left\{\begin{array}{l}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1)\ (3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1)\ (4,2),(4,3),(4,4),(4,5),(4,6)\\(5,1)\ (5,2),(5,3),(5,4)\ (5,5),(5,6)\\(6,1)\ (6,2),(6,3),(6,4),(6,5),(6,6)\end{array}\right\}$$

Define

$X: S \rightarrow R$ by $x(i,j) = i+j \ \forall (i,j) \in S$

i.e. 'x' is the sum of the numbers appearing on the faces of dice.

then,

$x(1,1) = 2$, $x(1,2) = 3$, $x(1,3) = 4$, $x(1,4) = 5$,

$x(1,5) = 6$, $x(1,6) = 7$.

$x(2,1)=3$, $x(2,2)=4$, $x(2,3)=5$, $x(2,4)=6$,
$x(2,5)=7$, $x(2,6)=8$.
$x(3,1)=4$, $x(3,2)=5$, $x(3,3)=6$, $x(3,4)=7$, $x(3,5)=8$,
$x(3,6)=9$.
$x(4,1)=5$, $x(4,2)=6$, $x(4,3)=7$, $x(4,4)=8$, $x(4,5)=9$
$x(4,6)=10$
$x(5,1)=6$, $x(5,2)=7$, $x(5,3)=8$, $x(5,4)=9$, $x(5,5)=10$,
$x(5,6)=11$.
$x(6,1)=7$, $x(6,2)=8$, $x(6,3)=9$, $x(6,4)=10$, $x(6,5)=11$,
$x(6,6)=12$

$\therefore$ The range of 'x' is $\{2,3,4,5,6,7,8,9,10,11,12\}$.

The probability distribution of x is

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\therefore$ Mean, $\mu = \sum_{i=2}^{12} x_i p_i$

$$= 2\left(\frac{1}{36}\right)+3\left(\frac{2}{36}\right)+4\left(\frac{3}{36}\right)+5\left(\frac{4}{36}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{6}{36}\right)+ 8\left(\frac{5}{36}\right)+9\left(\frac{4}{36}\right)+10\left(\frac{3}{36}\right)+11\left(\frac{2}{36}\right)+12\left(\frac{1}{36}\right)$$

$$= 7$$

3.

a). A player tosses two fair coins. He wins Rs. 100/- if a head appears, Rs. 200/- if two heads appear. On the other hand, he loses Rs. 500/- if no head appears. Determine the expected value E of the game and is the game favourable to the player? [5M]

Sol:- Given,

The sample space 'S' = $\{TT, HT, TH, HH\}$

The range of 'x' is = $\{-500, 100, 200\}$

then,

$P(x=-500) = \frac{1}{4}$ ($\because$ {TT})

$P(x=100) = \frac{2}{4}$ ($\because$ {TH, HT})

$P(x=200) = \frac{1}{4}$ ($\because$ {HH})

$\therefore$ The probability distribution of 'x' is

x	-500	100	200
P(x)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$\therefore$ The expected value is

$$E = -500\left(\frac{1}{4}\right)+100\left(\frac{2}{4}\right)+200\left(\frac{1}{4}\right)$$

$$= -25 < 0$$

$\therefore$ The gameis not favourable to the player.

b). A continuous random variable has the probability density function $f(x) = \begin{cases} kxe^{-x\lambda} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{, otherwise.} \end{cases}$

Determine i) k, ii). Mean, iii). Variance. [5M]

Sol:- We know that,

The total probability is unity.

i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{0}^{\infty} kxe^{-\lambda x}\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} kxe^{-\lambda x}\,dx = 1 \Rightarrow k\left[x\cdot\frac{e^{-\lambda x}}{(-\lambda)} - \frac{(1)\cdot e^{-\lambda x}}{(-\lambda)^2}\right]_0^{\infty} = 1$$

($\because \int fg\,dx = f\int g\,dx - f'\int\int g\,dx + f''\int\int\int g\,dx + \cdots$)

$$\Rightarrow k\left[\left(-\frac{x}{\lambda}-\frac{1}{\lambda^2}\right)e^{-\lambda x}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[0-\left(-\frac{1}{\lambda^2}\right)\right] = 1 \Rightarrow k\left(\frac{1}{\lambda^2}\right) = 1 \Rightarrow \boxed{k=\lambda^2}$$

$\therefore f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{, otherwise} \end{cases}$

ii) Mean, $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_{0}^{\infty} x \lambda^2 x e^{-\lambda x} dx$$

$$= \lambda^2 \int_{0}^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda^2 \left[x^2 \frac{e^{-\lambda x}}{(-\lambda)} - \frac{2x e^{-\lambda x}}{(-\lambda)^2} + \frac{2 e^{-\lambda x}}{(-\lambda)^3} \right]$$

$$= \lambda^2 \left[\left(-\frac{x^2}{\lambda} - \frac{2x}{\lambda} - \frac{2}{\lambda^3} \right) e^{-\lambda x} \right]_0^{\infty}$$

$$= \lambda^2 \left[0 - \left(-\frac{2}{\lambda^3} \right) \right] = \lambda^2 \left(\frac{2}{\lambda^3} \right)$$

$$\therefore \boxed{\mu = \frac{2}{\lambda}}$$

iii) Variance, $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

$$= \int_{0}^{\infty} x^2 (\lambda^2 x e^{-\lambda x}) dx - \left(\frac{2}{\lambda} \right)^2$$

$$= \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x} dx - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[x^3 \frac{e^{-\lambda x}}{(-\lambda)} - \frac{3x^2 e^{-\lambda x}}{(-\lambda)^2} + \frac{6x e^{-\lambda x}}{(-\lambda)^3} - \frac{6 e^{-\lambda x}}{(-\lambda)^4} \right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[\left(-\frac{x^3}{\lambda} - \frac{3x^2}{\lambda^2} - \frac{6x}{\lambda^3} - \frac{6}{\lambda^4} \right) e^{-\lambda x} \right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[0 - \left(-\frac{6}{\lambda^4} \right) \right] - \frac{4}{\lambda^2} = \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

$$\therefore \boxed{\sigma = \frac{2}{\lambda^2}}$$

c). A random variable X has following probability distribution.

X	0	1	2	3	4	5	6	7	8
P(X)	a	3a	5a	7a	9a	11a	13a	15a	17a

i). Determine the value of a.

ii). Find $P(x<3)$ and $P(0<X<5)$. [5M].

Sol:- We know that,

$\Rightarrow \sum_{i=0}^{8} P_i = 1$

$\Rightarrow a+3a+5a+7a+9a+11a+13a+15a+17a=1$

$\Rightarrow 81a=1$

$\Rightarrow \boxed{a=\frac{1}{81}}$

$\therefore$ The distribution function of X is

X	0	1	2	3	4	5	6	7	8
P(X)	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

$\therefore P(X<3) = P(0)+P(1)+P(2) = \frac{1}{81}+\frac{3}{81}+\frac{5}{81} = \frac{9}{81} = \frac{1}{9}$

$P(0<X<5) = P(1)+P(2)+P(3)+P(4)$

$= \frac{3}{81}+\frac{5}{81}+\frac{7}{81}+\frac{9}{81} = \frac{24}{81} = 0.296$

d). For a continuous probability density function is given by $f(x)=ce^{-|x|}, -\infty<x<\infty$. Find the value of c and hence mean. [5M].

Sol:- We know that,

The total probability is unity

i.e. $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} ce^{-|x|}dx = 1$

$\Rightarrow c\int_{-\infty}^{\infty} e^{-|x|}dx = 1$ $\left(\because f \text{ is even, } 2\int_{0}^{a} f(x)dx\right)$

$\Rightarrow C \cdot 2 \int_0^{\infty} e^{x} dx = 1$

$\Rightarrow 2C\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \Rightarrow -2 \cdot C\left(e^{-x}\right)_0^{\infty} = 1$

$\Rightarrow -2C(0-1) = 1 \Rightarrow 2C = 1 \Rightarrow \boxed{C = \frac{1}{2}}$

$\therefore f(x) = \frac{1}{2} e^{-|x|}, \ -\infty < x < \infty$

$\Rightarrow$ Mean, $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$= \int_{-\infty}^{\infty} \frac{x}{2} e^{-|x|} dx. \quad \left(\because \frac{x}{2} e^{-|x|} \text{ is odd}\right)$

$= 0$

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e). Let X denote the minimum of two numbers that appear when a pair of fair dice is thrown once. Determine the i). Discrete probability distribution, ii). Expectation. [5M]

Sol:- Given,

The sample space is

$S = \{(1,1), (1,2), (1,3) \cdots\cdots (6,5), (6,6)\}$

Given that,

$X(a,b) = \min(a,b) \ \forall \ (a,b) \in S.$

then,

$X(1,1) = 1, X(1,2) = 1, X(1,3) = 1, X(1,4) = 1, X(1,5) = 1, X(1,6) = 1$

$X(2,1) = 1, X(2,2) = 2, X(2,3) = 2, X(2,4) = 2, X(2,5) = 2, X(2,6) = 2$

$X(3,1) = 1, X(3,2) = 2, X(3,3) = 3, X(3,4) = 3, X(3,5) = 3, X(3,6) = 3$

$X(4,1) = 1, X(4,2) = 2, X(4,3) = 3, X(4,4) = 4, X(4,5) = 4, X(4,6) = 4$

$X(5,1) = 1, X(5,2) = 2, X(5,3) = 3, X(5,4) = 4, X(5,5) = 5, X(5,6) = 5$

$X(6,1) = 1, X(6,2) = 2, X(6,3) = 3, X(6,4) = 4, X(6,5) = 5, X(6,6) = 6$

The discrete probability distribution of X is

Since, the possible values of X are 1, 2, 3, 4, 5, 6

x	1	2	3	4	5	6
$P(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

Expectation, $E(x) = \sum_{i=1}^{6} x_i p_i$

$= 1\left(\frac{11}{36}\right) + 2\left(\frac{9}{36}\right) + 3\left(\frac{7}{36}\right) + 4\left(\frac{5}{36}\right) + 5\left(\frac{3}{36}\right) + 6\left(\frac{1}{36}\right)$

$\therefore E(x) = \frac{91}{36}$

4.

a). A random variable X has the following probability function:

x	0	1	2	3	4	5	6	7
$P(X=x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

i). Determine K

ii). $P(X \leq K) > \frac{1}{2}$, find the minimum value of K

iii). Determine the distribution function of x.

iv). Mean

v). Variance - [10M]

Sol: We know that,

Sum of the probabilities $= 1$

then, $\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow 10K^2 + 9K - 1 = 0$

$\Rightarrow 10K^2 + 10K - K - 1 = 0$

$\Rightarrow 10K(K+1) - 1(K+1) = 0$

$\Rightarrow (10K-1)(K+1) = 0$

$\Rightarrow K = -1,\ K = \frac{1}{10}$

$\therefore \boxed{K = \frac{1}{10}}$ $(\because P(x) \geq 0\ \forall x)$

The probability distribution of 'x' is

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

$\Rightarrow P(X \leq K) > \frac{1}{2}$

$P(X \leq 1) = P(0) + P(1) = 0 + \frac{1}{10} = 0.1 < 0.5$

$P(X \leq 2) = P(0) + P(1) + P(2) = 0 + \frac{1}{10} + \frac{2}{10} = 0.3 < 0.5$

$P(X \leq 3) = P(0) + P(1) + P(2) + P(3) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = 0.5 = 0.5$

$P(X \leq 4) = P(0) + P(1) + P(2) + P(3) + P(4) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$= 0.8 > 0.5$

$\therefore$ The minimum value of K is 4.

iv). Mean $(\mu) = \sum_{i=0}^{7} x_i p_i$

$= 0 + \frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10} + \frac{5}{100} + \frac{12}{100} + \frac{119}{100}$

$= \frac{23}{10} + \frac{136}{100}$

$= 2.3 + 1.36$

$= 3.66$

v). $Var(X) = \sum_{i=0}^{7} x_i^2 p_i - \mu^2$

$= 0 + \frac{1}{10} + \frac{8}{10} + \frac{18}{10} + \frac{48}{10} + \frac{25}{100} + \frac{72}{100} + 7^2\left(\frac{17}{100}\right) - (3.66)^2$

$= 3.404$

b). I). A sample of 4 items is selected at random from a box containing 12 items of which 5 are defected. Find the expected number E of defective items.

II). Find the mean and variance of the uniform probability distribution given by $f(x) = \frac{1}{n}$ for $x = 1, 2 \cdots n$ - [10M].

Sol:- I). Given,

Number of items = 12

No. of defective items = 5

No. of non-defective items = 7

'X' = No. of defective items.

then, the possible values of 'X' are 0, 1, 2, 3, 4

$P(X=0) = P$ (no defective item)

$= \frac{7C_4}{12C_4} = \frac{35}{495}$

$P(X=1) = P$ (one defective item)

$= \frac{5C_1 \times 7C_3}{12C_4} = \frac{175}{495}$

$P(X=2) = P$ (two defective items)

$= \frac{5C_2 \times 7C_2}{12C_4} = \frac{210}{495}$

$P(X=3) = P$ (three defective items)

$= \frac{5C_3 \times 7C_1}{12C_4} = \frac{70}{495}$

$P(X=4) = P$ (four defective items)

$= \frac{5C_4}{12C_4} = \frac{5}{495}$

∴ The probability distribution of 'X' is

X	0	1	2	3	4
P(X)	35/495	175/495	210/495	70/495	5/495

∴ The expected number is

$E(X) = \sum_{i=0}^{4} x_i p_i$

$= 0\left(\frac{35}{495}\right) + 1\left(\frac{175}{495}\right) + 2\left(\frac{210}{495}\right) + 4\left(\frac{5}{495}\right) + 3\left(\frac{70}{495}\right)$

$= \frac{825}{495}$

$= 1.667$

II). The probability distribution of 'x' is

x	1 2 3 - - - - - - - n
P(X)	$\frac{1}{n}$ $\frac{1}{n}$ $\frac{1}{n}$ - - - - - - $\frac{1}{n}$

The mean, $\mu = \sum_{i=1}^{n} x_i p_i$

$$= 1\left(\frac{1}{n}\right)+2\left(\frac{1}{n}\right)+3\left(\frac{1}{n}\right)+ \cdots\cdots + n\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\left(1+2+3+\cdots+n\right)$$

$$= \frac{1}{n}\left(\frac{n(n+1)}{2}\right)$$

$$\Rightarrow \mu = \frac{n+1}{2}$$

Variance, $Var(X) = \sum_{i=1}^{n} x_i^2 p_i - \mu^2$

$$= \left(1^2\left(\frac{1}{n}\right)+2^2\left(\frac{1}{n}\right)+3^2\left(\frac{1}{n}\right)+\cdots+n^2\left(\frac{1}{n}\right)\right) - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\left(1^2+2^2+3^2+\cdots+n^2\right)-\frac{(n+1)^2}{4}$$

$$= \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right)-\frac{[n+1]^2}{4}$$

$$= \frac{(n+1)}{2}\left(\frac{2n+1}{3}-\frac{n+1}{2}\right)$$

$$= \frac{(n+1)}{2}\left(\frac{2(2n+1)-3(n+1)}{6}\right)$$

$$= \frac{(n+1)}{2}\left(\frac{4n+2-3n-3}{6}\right)$$

$$= \frac{(n+1)}{2}\left(\frac{n-1}{6}\right)$$

$$\Rightarrow Var(X) = \frac{n^2-1}{12}$$

c). I). If X is a continuous random variable and K is a constant. Then prove that;

i). $Var(X+K) = Var(X)$, ii) $Var(KX) = K^2 Var(X)$

II). For the continuous probability function $f(x) = Kx^2e^{-x}$ when $x \geq 0$, find mean. - [10M].

Sol:- I). Given that,

X is continuous random variable and K is constant.

By definition of variance, we have

$$Var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} x f(x)dx\right)^2$$

then,

$$var(X+K) = \int_{-\infty}^{\infty} (x+K)^2 f(x)dx - \left(\int_{-\infty}^{\infty}(x+K)f(x)dx\right)^2$$

$$= \int_{-\infty}^{\infty} (x^2+K^2+2Kx)f(x)dx - \left(\int_{-\infty}^{\infty} xf(x)+Kf(x)dx\right)^2$$

$$= \int_{-\infty}^{\infty}(x^2f(x)+K^2f(x)+2Kxf(x))dx - \left(\int_{-\infty}^{\infty} xf(x)dx + \int_{-\infty}^{\infty} Kf(x)dx\right)^2$$

$$= \int_{-\infty}^{\infty} x^2f(x)dx + K^2\int_{-\infty}^{\infty} f(x)dx + 2K\int_{-\infty}^{\infty} xf(x)dx - \left(E(X)+K\int_{-\infty}^{\infty} f(x)dx\right)^2$$

$$= E(X^2) + K^2(1) + 2KE(X) - (E(X)+K(1))^2 \quad \left(\because \int_{-\infty}^{\infty} f(x)dx = 1\right)$$

$$= E(X^2) + K^2 + 2KE(X) - ((E(X))^2 + K^2 + 2KE(X))$$

$$= E(X^2) + K^2 + 2KE(X) - (E(X))^2 - K^2 - 2KE(X)$$

$$= E(X^2) - (E(X))^2$$

$$= Var(X)$$

$\therefore var[X+K] = var[X]$

ii). $$var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} xf(x)dx\right)^2$$

then,

$$var(KX) = \int_{-\infty}^{\infty} (Kx)^2 f(x)dx - \left(\int_{-\infty}^{\infty}(Kx)f(x)dx\right)^2$$

$$= \int_{-\infty}^{\infty} k^2 x^2 f(x)\,dx - \left(k \int_{-\infty}^{\infty} x f(x)\,dx\right)^2$$

$$= k^2 \int_{-\infty}^{\infty} x^2 f(x)\,dx - k^2 \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$$

$$= k^2 \left[\int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2\right]$$

$$= k^2 \left[E(x^2) - (E(x))^2\right]$$

$\therefore$ $var(kx) = k^2 var(x)$

Hence proved.

II). Given that,

$f(x) = kx^2 e^{-x}, \; x \geq 0$

We know that,

The total probability is unity.

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \int_{-\infty}^{\infty} kx^2 e^{-x}\,dx = 1$$

$$\Rightarrow k\left[x^2 \frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + \frac{2e^{-x}}{(-1)^3}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[(-x^2 - 2x - 2)e^{-x}\right] = 1 \quad \Rightarrow k[0 - (-2)] = 1$$

$$\Rightarrow 2k = 1 \Rightarrow \boxed{k = \frac{1}{2}}$$

$$\Rightarrow \text{Mean}, \mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \cdot \frac{1}{2} x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} x^3 e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^3 \frac{e^{-x}}{(-1)} - 3x^2 \frac{e^{-x}}{(-1)^2} + 6x\frac{e^{-x}}{(-1)^3} + 6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[(-x^3 - 3x^2 - 6x - 6)e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}[0 - (-6)]$$

$$\therefore \boxed{\mu = 3}$$