UNIT-I ACVC Question

1 Mark Question

1. Verify Rolle's theorem for $f(x) = \tan x$ in $[0, \pi]$
2. State Lagrange's Mean Value Theorem
3. State Cauchy's Mean Value Theorem
4. State Taylor's theorem
5. Write Maclaurin's Series expansion

3 Mark Questions

1. Write Geometric Interpretation of Rolle's Theorem
2. Verify Lagrange's Mean value theorem for $2x^2 - 7x + 10$ in $[0, 5]$
3. Find 'c' of Cauchy's Mean value theorem on $[a, b]$ for
$$f(x) = e^x, \quad g(x) = e^{-x} \quad (a, b) > 0$$
4. Obtain the Taylor's series expansion of $\sin x$ in power of
$$\left(x - \frac{\pi}{4}\right).$$
5. Obtain the Maclaurin series expansion for $f(x) = (1+x)^n$

5 Marks Question

1. Verify Rolle's theorem for $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
2. Show that for any $x > 0$, $1 + x < e^x < 1 + xe^x$
3. Verify Generalised mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b)$
4. Expand $\log_e^x$ in powers of $(x-1)$ and hence find $\log_e^{1.5}$ upto 4 decimal places.
5. Expand $e^{\sin x}$ in powers of x.

10 Marks Question

1. Prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ and hence deduce (i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}4/3 < \frac{\pi}{4} + \frac{1}{6}$

   (ii) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

2. Prove using Mean Value Theorem $|\sin u - \sin v| \leq |u-v|$.

3. Obtain Maclaurin's series expansion for (i) e^x (ii) $\sin x$ (iii) $\cosh x$

UNIT - I

SINGLE VARIABLE CALCULUS

One Mark Question

1. Verify Rolle's theorem for $f(x) = \tan x$ in $[0, \pi]$.

Given,

$f(x) = \tan x$ in $[0, \pi]$

Clearly, $\tan x$ is not continuous in $[0, \pi]$ since at $x = \pi/2$ $\tan x = \infty$.

Hence, Rolle's Theorem is not applicable for $f(x) = \tan x$ in $[0, \pi]$.

2. State Lagrange's mean value Theorem.

$f(x)$ is a function such that

i) $f(x)$ is continuous on $[a,b]$

ii) $f(x)$ is derivable on (a,b) then $\exists$ atleast one $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

3. State Cauchy's mean value Theorem.

If $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ such that

(i) f, g are continuous on $[a,b]$

(ii) f, g are derivable on (a,b)

(iii) $g'(x) \neq 0$ $\forall$ $x \in (a,b)$, then $\exists$ $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4. State Taylor's Theorem.

If $f : [a,b] \to R$ is such that

a) $f^{(n-1)}$ is continuous on $[a,b]$

b) $f^{(n-1)}$ is derivable on (a,b) or $f^{(n)}$ exists on (a,b) and $p \in Z^+$ then there exist a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

Where, $R_n = \dfrac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$

3 Marks Questions

1. Write the geometric Interpretation of Rolle's Theorem

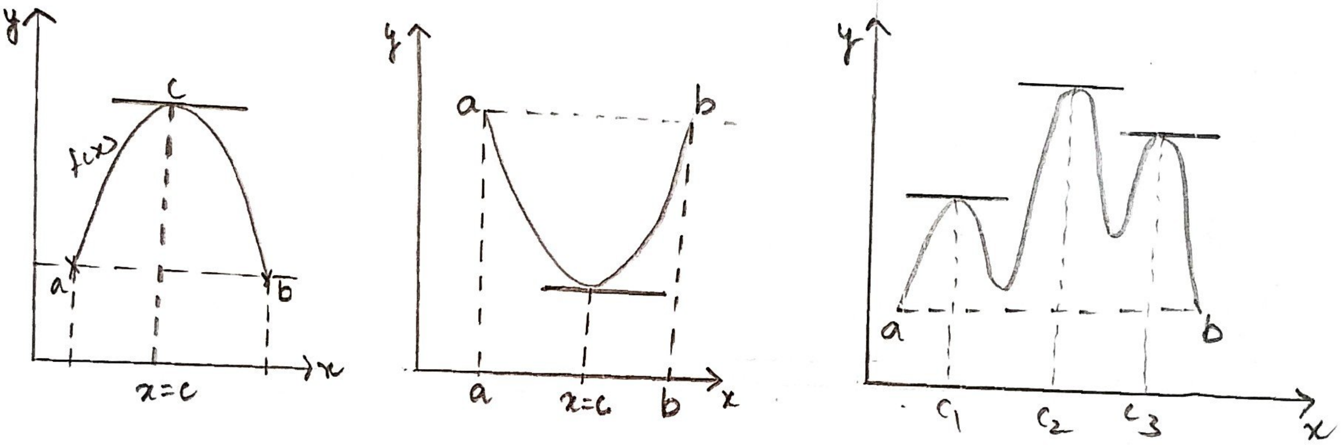
Geometrical Interpretation of Rolle's Theorem are :-

i) The curve $y = f(x)$ is continuous in $[a, b]$

ii) At every point $x = c$ where $a < c < b$ at the point $[c, f(c)]$ on the curve $y = f(x)$ there exist a unique tangent to the curve.

iii) $f(a) = f(b)$ i.e, the two end points of the curve $y = f(x)$ corresponding to $x = a$, $x = b$ or at the same height from the x-axis.

iv) Under these assumptions there is atleast one point on the curve where the tangent is parallel to x-axis according to Rolle's Theorem.



2. Verify Lagrange's Mean Value Theorem for $2x^2 - 7x + 10$ in $[0, 5]$

Given the function is

$$f(x) = 2x^2 - 7x + 10$$

Clearly $f(x)$ is polynomial function in x

So, (i) $f(x)$ is continuous in $[0, 5]$

$$f'(x) = 4x - 7 \quad \forall x\ (0, 5)$$

(ii) $f(x)$ is derivable in $(0, 5)$

Hence, All the two condition of Lagrange's Mean Value Theorem is satisfied

So, there exist at least one $c \in (0,5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$4c - 7 = \frac{f(5) - f(0)}{5-0}$$

$$f(b) = f(5) = 2(5)^2 - 7(5) + 10 = 25$$

$$f(a) = f(0) = 2(0)^2 - 7(0) + 10 = 10$$

$$\therefore \; 4c - 7 = \frac{f(5) - f(0)}{5-0}$$

$$4c - 7 = \frac{25 - 10}{5}$$

$$4c - 7 = \frac{15}{5}$$

$$4c - 7 = 3$$

$$4c = 10$$

$$c = \frac{5}{2} \in (0,5)$$

Hence, Langrange's Mean Value theorem is verified for $f(x)$ in $[0,5]$.

3. Find 'c' of Cauchy's Mean Value theorem on $[a,b]$ for $f(x) = e^x$ $g(x) = e^{-x}$ $(a,b) > 0$.

We know that,

(i) $f(x), g(x)$ are continuous on $[a,b]$

(ii) $f(x), g(x)$ are derivable on (a,b)

(iii) Also, $g'(x) = -e^{-x} \neq 0 \;\forall\; x \in (a,b)$

$\therefore$ Cauchy's mean value theorem conditions are satisfied.

Applying Cauchy's mean value theorem $\exists$ $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$-e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-e^{2c} = \frac{\cancel{e^b - e^a}}{\frac{\cancel{e^a - e^b}}{e^{a+b}}}$$

$$\cancel{+}e^{2c} = \cancel{+}e^{a+b}$$

$$e^{2c} = e^{a+b}$$

$$2c = a+b$$

$$\boxed{c = \frac{a+b}{2}} \in (a,b)$$

Hence, Cauchy's Mean Value Theorem is verified.

4. Obtain the Taylor's series expansion of $\sin x$ in powers of $\left(x - \frac{\pi}{4}\right)$

Here $a = \frac{\pi}{4}$; $f(x) = \sin x$

$f(x) = \sin x \Rightarrow f(a) = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'(a) = f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''(a) = f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x \Rightarrow f'''(a) = f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$\vdots$

The Taylor's series $f(x)$ in powers of $x-a$ is

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^2}{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots\cdots$$

5. Obtain the Maclaurin series expansion for $f(x) = (1+x)^n$?

Let $f(x) = (1+x)^n$. Then

$f'(x) = n(1+x)^{n-1}$

$f''(x) = n(n-1)(1+x)^{n-2}$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$

$\therefore f^{(k)}(x) = n(n-1)(n-2)\cdots\cdots(n-k+1)(1+x)^{n-k}$

$\therefore f^{(k)}(0) = n(n-1)\cdots\cdots(n-k+1)$

Hence,

$$f(x) = \sum_{k=0}^{\infty} f^{k}(0)\frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{n(n-1)\cdots\cdots(n-k+1)}{1.2.\cdots\cdots k}x^k$$

i.e, $$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^2 + \cdots\cdots$$

5 Marks Question

1. Verify Rolle's theorem for $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Given that, $f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Clearly,

(i) $f(x)$ is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

(ii) $f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)$

(ii) $f(x)$ is differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(iii) $f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$

$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$

$f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$

Thus, all condition of Rolle's Theorem are satisfied

So, there exist at least one $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that

$$f'(c) = 0$$

$$e^c(\sin c - \cos c) + e^c(\cos c + \sin c) = 0$$

$$\sin c - \cancel{\cos c} + \cancel{\cos c} + \sin c = 0$$

$$2\sin c = 0$$

$$\sin c = 0$$

$$c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Hence, Rolle's Theorem is verified.

(2) Show that for any $x > 0$ $\quad 1+x < e^x < 1+xe^x$

let $f(x) = e^x$ in $[0, x]$

Clearly, (i) $f(x)$ is continuous on $[0, x]$

(ii) $f(x)$ is derivable on $(0, x)$, then by Lagrange's Mean Value Theorem $\exists$ atleast one $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$e^c = \frac{e^x - e^0}{x - 0}$$

$$e^c = \frac{e^x - 1}{x} \quad \text{--- (1)}$$

since $c \in (0, x)$

$\Rightarrow 0 < c < x$

$$e^0 < e^c < e^x$$

$$e^0 < \frac{e^x - 1}{x} < e^x \qquad \text{[From eq (1)]}$$

$$1.x < e^x - 1 < e^x.x$$

$$1+x < e^x < 1 + xe^x$$

(3) Verify Generalised Mean value Theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$.

Clearly (i) $f(x)$ and $g(x)$ are continuous function on $[a, b]$

(ii) $f'(x) = \frac{1}{2\sqrt{x}}$ and $g'(x) = \frac{-1}{2x\sqrt{x}}$ $\forall\, x \in (a, b)$

(ii) $f(x)$ and $g(x)$ are derivable on (a, b)

(iii) $g'(x) = \frac{-1}{2x\sqrt{x}} \neq 0 \quad \forall\, x \in (a, b)$

So, Conditions of Cauchy's Mean Value theorem are satisfied on (a,b)

So, There exist $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\therefore \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}}$$

$$\cancel{-}\sqrt{ab} = \cancel{-}c$$

$c = \sqrt{ab} \in (a,b)$ which verifies Cauchy's Generalised Mean Value Theorem.

(4.) Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e 1.5$ upto 4 decimal places.

Here $a = 1$

Let $f(x) = \log_e^x \Rightarrow f(1) = \log_e^1 = 0$

$f'(x) = \log x = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -\frac{1}{1} = -1$

$f'''(x) = -\left(\frac{-2}{x^3}\right) = \frac{2}{x^3} \Rightarrow f'''(1) = \frac{2}{1^3} = 2$

$f^{IV}(x) = \frac{-6}{x^4} \Rightarrow f^{IV}(1) = -6$

By Taylor series,

$$f(x) = f(1) + \frac{x-1}{1!} f'(x) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$$= x-1 + \frac{-(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Put $x = 1.5$

$$\log_e 1.5 = (1.5-1) - \frac{(1.5-1)^2}{2} + \frac{(1.5-1)^3}{3} - \frac{(1.5-1)^4}{4} + \cdots$$

$$= 0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} = 0.4166$$

5. Expand $e^{\sin x}$ in power of x. (By Maclaurin Series)

let $f(x) = e^{\sin x}$

$$f'(x) = e^{\sin x} . \cos x \Rightarrow f'(0) = f(0)\cos 0 = 1 \times 1 = 1$$

$$f''(x) = f(x)(-\sin x) + f'(x)\cos x \Rightarrow f''(0) = 1 \times 1 = 1$$

$$f'''(x) = f(x)(-\cos x) - f'(x)\sin x + f'(x)(-\sin x) + f''(x)\cos x$$

$$= f(x)(-\cos x) - 2f'(x)\sin x + f''(x)\cos x$$

$$f'''(0) = -f(0)\cos 0 - 2f'(0)\sin 0 + f''(0)\cos 0$$

$$= -1 \times 1 + 1 \times 1 = 0$$

$$f^{IV}(x) = -f(x)(-\sin x) - f'(x)\cos x - 2f'(x)\cos x - 2f''(x)\sin x + f''(x)(-\sin x) + f'''(x)\cos x$$

$$f^{IV}(0) = f'(0)\cos 0 - 2f'(0)\cos 0 + f'''(0)\cos 0$$

$$= -1 \times 1 - 2(1) + 0$$

$$= -3$$

$\therefore$ By Maclaurin's series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= 1 + x(1) + \frac{x^2}{2}(1) + 0 + \frac{x^4}{24}(-3) + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{-x^4}{8} + \cdots$$

Ten Marks Questions

1.) Prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ and hence

deduce (i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

Soln:- Let $f(x) = \tan^{-1}x$ in $[a,b]$

Clearly,

$f(x)$ is continuous on $[a,b]$

$f(x)$ is derivable on (a,b)

then $\exists$ atleast one $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}, \quad \text{since } f'(x) = \frac{1}{1+x^2}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$$

since $c \in (a,b)$

$$\Rightarrow a < c < b$$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \quad \text{———} (1)$$

1.) put $a = 1$, $b = 4/3$ in eq (1)

$$\frac{4/3 - 1}{1+(4/3)^2} < \tan^{-1}\frac{4}{3} - \tan^{-1} < \frac{4/3 - 1}{1+1^2}$$

$$\frac{\frac{1}{3}}{\frac{9+16}{9}} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$$

$$\frac{1}{\cancel{3}} \times \frac{\overset{3}{\cancel{9}}}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$$

17.) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

Put $b=2$, $a=1$ in equation (1)

$$\frac{2-1}{1+4} < \tan^{-1}2 - \tan^{-1}1 < \frac{2-1}{1+1}$$

$$\frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\frac{1}{5} + \frac{\pi}{4} < \tan^{-1}2 < \frac{1}{2} + \frac{\pi}{4}$$

$$\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$$

(2.) Prove using Mean value Theorem $|\sin u - \sin v| \leq |u-v|$

Soln:- (i) If $u=v$, there is nothing to prove

(ii) If $u>v$, thus consider the function

$f(u) = \sin u$ on $[v,u]$

clearly $f(u)$ is continuous on $[v,u]$ and derivable on (v,u)

$\therefore$ By Lagrange's Mean value Theorem,

$\exists\ c \in (v,u)$ such that

$$f'(c) = \frac{f(u)-f(v)}{u-v}$$

$$\cos c = \frac{\sin u - \sin v}{u-v} \quad \text{—(1)}$$

But $|\cos c| \leq 1$

$$\therefore \left|\frac{\sin u - \sin v}{u-v}\right| \leq 1 \qquad [\text{From } ①].$$

$$|\sin u - \sin v| \leq |u-v|$$

(iii) If $v > u$, then in similar manner, we have

$$\therefore |\sin v - \sin u| \leq |v-u|$$

$$|\sin u - \sin v| \leq |u-v| \qquad [\because |-x| = |x|]$$

Hence for all $u, v \in \mathbb{R}$

$$|\sin u - \sin v| \leq |u-v|$$

3. Obtain the Maclaurin's series expansion for
i) e^x ii) $\sin x$ iii) $\cosh x$

i) e^x

let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

~~$f(0) = e^0 = 1$~~

$f'(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f(0) = e^0 = 1$

$\vdots$

$f^n(x) = e^x \Rightarrow f^n(0) = e^0 = 1$

By Maclaurins series of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(x) + \frac{x^2}{2!} f''(x) + \cdots\cdots + \frac{x^n}{n!} f^n(0) + \cdots\cdots$$

$$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \cdots + \frac{x^n}{n!} + \cdots\cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots \frac{x^n}{n!}.$$

ii) $\sin x$

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f(0) = -\cos 0 = -1$

$f^{IV}(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$\vdots$

$f^{n}(x) = \sin\left(\frac{n\pi}{2} + x\right)$

By Maclaurin's series of f(x) is

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots \frac{x^n}{n!}\sin\left(\frac{n\pi}{2}\right)$$

$$\sin x = x - \frac{x^3}{3!} + \cdots \frac{x^n}{n!}\sin\left(\frac{n\pi}{2}\right).$$

iii) $\cosh x$

let $f(x) = \cosh x \Rightarrow f(0) = 1$

$f'(x) = +\sinh x \Rightarrow f(0) = 0$

$f''(x) = \cosh x \Rightarrow f(0) = 1$

$f'''(x) = \sinh x \Rightarrow f(0) = 0$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

The Maclaurin's series expansion of ~~cosh~~ $\cosh x$ is given by

$$\cosh x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \cdots \frac{x^{2n}}{(2n)!} + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \frac{x^{2n}}{(2n)!}$$