UNIT - II

1a) Define Binomial Distribution?

Binomial Distribution: let the number of trials be n. let p be the probability of success and q be the probability of failure. Then $p+q=1$.

If x is random variable representing the number of successes, the probability of getting all successes and n-r failures, if n trails is given by the probability function.

$P(X=r) = nC_r \cdot p^r \cdot q^{n-r}$, where $r = 0, 1, 2, ---, n$.

This probability function is known as Binomial distribution.

1b) Write the conditions of Binomial distribution.

Conditions of Binomial Distribution:

The Binomial Distribution holds under the following conditions:

1) Trials are repeated under identical conditions for a fixed number of times, say n times.
2) There are only two possible outcomes, e.g. success or failure for each trial.
3) The probability of success in each trial remains constant and does not change from trial to trial.
4) The trials are independent i.e the probability of an event in any trial is not affected by the results of any other tials.

c) Define Poisson Distribution?

Poisson Distribution: The poisson distribution can be derived as a limiting case of binomial distribution under the following conditions:

i) p, the probability of occurance of an event is every small.

ii) n is very very large i.e $n \to \infty$ (n means the no. of trails or sample size)

iii) np is finite quantity, say $np = \lambda$ then λ is called parameter of Poisson distribution.

d) Write the application of Normal Distribution?

Applications of Normal Distribution:

Normal distribution plays a very important role in statistical theory because of the following reasons:

1) Data obtained from psychological, physical and biological measurements approximately follows normal distribution.

2) Since the normal distribution is a limiting case of the binomial distribution to exceptionally large numbers, it is applicable to many applied problems in kinetic theory of gases.

3) for large samples, any statistics (i.e sample mean, sample S.D etc) approximately follows normal distribution.

1e) The mean and variance of a binomial distribution are 4 and $4/3$ respectively. find $P(X \geq 1)$

Given that the mean of binomial distribution is

$$np = 4 \longrightarrow (1)$$

Variance of binomial distribution is

$$npq = 4/3 \longrightarrow (2)$$

$$\text{Eqn } (2)/(1) = \frac{\cancel{n}\cancel{p}q}{\cancel{n}\cancel{p}} = 4/3 \times 1/4$$

$$\boxed{q = \frac{1}{3}}$$

W.K.T $\quad p+q = 1$

$$p + \frac{1}{3} = 1$$

$$p = 1 - \frac{1}{3}$$

$$p = \frac{3-1}{3} \qquad \boxed{p = \frac{2}{3}}$$

from (1) $\quad np = 4$

$$n \times \frac{2}{3} = 4$$

$$n = \frac{\overset{2}{\cancel{4}} \times 3}{\cancel{2}}$$

$$\boxed{n = 6}$$

The binomial distribution is,

$$P(X=r) = nC_r \, p^r q^{n-r}, \quad r = 0, 1, 2, ---, n.$$

$$P(X=r) = 6C_r \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{6-r}, \quad r = 0, 1, ---, 6$$

$$P(X \geq 1) = 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - 6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0}$$

$$= 1 - (1)(1)\left(\frac{1}{3}\right)^6$$

$$= 1 - \frac{1}{3^6}$$

$$P(X \geq 1) = 0.998.$$

2a) A fair coin is tossed ten times. find the probability of getting at least 6 heads.

Given that $n=10$

The probability of getting head is $p = 1/2$.

Then the probability of getting head is $q = 1-p$

$= 1 - 1/2$

$= 1/2$

let x be the no. of heads.

Then the binomial distribution is

$p(x=r) = p(x) = nC_r \, p^r \cdot q^{n-r}$, $r = 0, 1, \dots, n.$

$p(x=r) = 10C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r}$ $r = 0, 1, 2, \dots, 10.$

$= 10C_r \left(\frac{1}{2}\right)^{10}$

$p(x=r) = \frac{1}{2^{10}} \cdot 10C_r$ $r = 0, 1, 2, \dots, 10.$

getting at least 6 heads

$p(x \geq 6) = p(x=6) + p(x=7) + p(x=8) + p(x=9) + p(x=10)$

$= 10C_6 \frac{1}{2^{10}} + 10C_7 \frac{1}{2^{10}} + 10C_8 \frac{1}{2^{10}} + 10C_9 \frac{1}{2^{10}} + 10C_{10} \frac{1}{2^{10}}$

$= \frac{1}{2^{10}} (10C_6 + 10C_7 + 10C_8 + 10C_9 + 10C_{10})$

$= \frac{1}{2^{10}} (386)$

$= 0.3769.$

2b) If the probability of a defective bolt is 1/8.
i) Find mean,
ii) The variance for the distribution of defective bolts of 640.

Given that $n = 640$.

The probability of a defective bolt $p = 1/8$.

Then $q = 1 - p$

$= 1 - 1/8$

$q = 7/8$

i) Mean $(\mu) = np$

$= \cancel{\frac{1}{8} \times \frac{7}{8}}$

$= 640 \times \frac{1}{8}$

$\mu = 80$

ii) Variance $(\sigma^2) = npq$

$= 640 \times \frac{1}{8} \times \frac{7}{8}$

$\sigma^2 = 70$

2c) Derive mean of normal distribution.

<u>Mean of Normal Distribution</u>:

The normal distribution is $f(x, b, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-b)^2}{2\sigma^2}}$

The mean of normal distribution is.

$$\mu = \int_{-\infty}^{\infty} x f(x) dx.$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-b)^2}{2\sigma^2}} dx.$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-b}{\sigma}\right)^2} dx.$$

Put $z = \frac{x-b}{\sigma}$

Here, $x = \sigma z + b$ and $dz = \frac{1}{\sigma} dx$; $dx = \sigma dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + b) e^{-\frac{1}{2}(z^2)} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + b) e^{-\frac{1}{2}(z^2)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sigma \int_{-\infty}^{\infty} z e^{-z^2/2} dz + b \int_{-\infty}^{\infty} e^{-z^2/2} dz \right]$$

$\therefore z e^{-z^2/2}$ is odd and $e^{-z^2/2}$ is even.

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & f \text{ is even} \\ 0, & f \text{ is odd} \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sigma(0) + b\left(2\int_{0}^{\infty} e^{-z^2/2} dz\right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + 2b \int_{0}^{\infty} e^{-z^2/2} dz \right] \quad \left[\because \int_{0}^{\infty} e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(2b \sqrt{\frac{\pi}{2}} \right)$$

$$= \frac{1}{\sqrt{2\cancel{\pi}}} \, 2b \cdot \frac{\sqrt{\cancel{\pi}}}{\sqrt{2}}$$

$$= \frac{\cancel{2}b}{\cancel{2}}$$

$$= b$$

$\therefore$ Mean of Normal distribution is "b".

2d) 20% of items produced from a factory are defective. Find the probability that in a sample of 5 chosen at random

(i) none is defective (ii) one is defective

(iii) lies between 1 and 4.

Given that $n = 5$

The probability of having defective items is

$P = 20\% = 0.2$

Then $q = 1 - p$

$= 1 - 0.2$

$q = 0.8$

let x denotes the number of defective items.

The binomial distribution is

$P(x = r) = {}^{n}C_{r}\, p^{r} q^{n-r}, \quad r = 0, 1, 2, \ldots, n.$

$P(x = r) = {}^{5}C_{r}\,(0.2)^{r}(0.8)^{5-r} \quad r = 0, 1, 2, 3, 4, 5.$

i) none is defective

$P(x=0) = {}^{5}C_{0}\,(0.2)^{0}(0.8)^{5-0}$

$= (1)(1)(0.8)^{5}$

$= 0.327.$

ii) one is defective

$P(x=1) = {}^{5}C_{1}\,(0.2)^{1}(0.8)^{4}$

$= 5(0.2)(0.8)^{4} = 0.409.$

iii) lies between 1 and 4.

$P(1 < x < 4) = P(x=2) + P(x=3)$

$= {}^{5}C_{2}\,(0.2)^{2}(0.8)^{3} + {}^{5}C_{3}\,(0.2)^{3}(0.8)^{2}$

$= 0.2048 + 0.0512$

$= 0.256.$

9e) If x is a normal variate with mean 30 and S.D 5. Find the probabilities that (i) $26 \le x \le 40$.
ii) $P(x \ge 45)$.

Given that mean $(\mu) = 30$

standard deviation $(\sigma) = 5$

W.K.T The z-score is $z = \frac{x-\mu}{\sigma}$

$z = \frac{x-30}{5}$

i) When $x=26$ then $z = \frac{26-30}{5} = \frac{-4}{5} = -0.8 = z_1$ (say)

when $x = 40$ then $z = \frac{40-30}{5} = 2 = z_2$ (say)

$\therefore z_1 < 0$ and $z_2 > 0$

$p(26 \le x \le 40) = p(z_1 \le x \le z_2)$

$= A(z_2) + A(z_1)$

$= A(2) + A(-0.8)$

$= A(2) + A(0.8)$ (due to symmetry)

$= 0.4772 + 0.2881$

$= 0.7653$.

ii) $x \ge 45$

when $x = 45$ then $z = \frac{45-30}{5} = 3 = z$ (say)

$\therefore z > 0$.

$P(x \ge 45) = p(z \ge 3)$

$= 0.5 - A(z)$

$= 0.5 - A(3)$

$= 0.5 - 0.4987$

$= 0.0013$

3a) Six dice are thrown 729 times. How many times do you except at least three dice to show a 5 or 6.

The probability of occurance of 5 or 6 in one throw $p = \frac{2}{6} = \frac{1}{3}$.

Then $q = 1-p = 1-\frac{1}{3} = \frac{2}{3}$

$n = 6$

$P(X=r) = nC_r \, p^r q^{n-r} \quad r = 0,1,2,3,----,n.$

$P(X=r) = 6C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r}, \quad r = 0,1,----,6.$

The probability of getting atleast three dice to show a 5 or 6.

$= P(X \geq 3)$

$= P(X=3) + P(X=4) + P(X=5) + P(X=6).$

$= 6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + 6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + 6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1 + 6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0$

$= \cancel{\frac{1}{3^6}}\, 6C_3 \frac{2^3}{3^6} + 6C_4 \frac{2^2}{3^6} + 6C_5 \frac{2}{3^6} + 6C_6 \frac{1}{3^6}.$

$= \frac{1}{3^6}\left(8 \times 6C_3 + 4 \times 6C_4 + 2 \times 6C_5 + 6C_6\right)$

$= \frac{1}{3^6}(233)$

$= \frac{233}{729}$

$\therefore$ The expected number of each such cases in 729 times.

$= \cancel{729} \times \frac{233}{\cancel{729}}$

$= 233.$

3b) Derive mean and variance of poisson distribution.

Mean of poisson distribution:

The poisson distribution is $P(x=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}$

The mean of the poisson distribution is

$$E(X) = \sum_{r=0}^{\infty} r \cdot P(r)$$

$$= \sum_{r=1}^{\infty} r \cdot P(r)$$

$$= \sum_{r=1}^{\infty} r \cdot \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}$$

$$= \sum_{r-1=0}^{\infty} \cancel{r} \cdot \frac{e^{-\lambda}\lambda^{r}}{\cancel{r}(r-1)!}$$

$$= \sum_{r-1=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{r}}{(r-1)!}$$

Put $x = r-1$

Then $r = x+1$

$$E(X) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x+1}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x} \cdot \lambda^{1}}{x!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$= \lambda e^{-\lambda}(e^{\lambda})$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda e^{-\lambda+\lambda}$$

$$= \lambda e^{0}$$

$$= \lambda$$

$\therefore$ The mean of poisson distribution is $\lambda = np$.

Variance of Poisson distribution:

The poisson distribution is $P(X=r) = \frac{e^{-\lambda}(\lambda)^r}{r!}$

The variance of poisson distribution is.

$$V(X) = \sum_{r=0}^{\infty} r^2 p(r) - \mu^2$$

$$= \sum_{r=0}^{\infty} r^2 p(r) - \lambda^2 \quad (\because \lambda \text{ is mean of P.D})$$

$$= \sum_{r=0}^{\infty} (r^2 - r + r) p(r) - \lambda^2$$

$$= \sum_{r=0}^{\infty} r(r-1) r p(r) - \lambda^2$$

$$= \sum_{r=0}^{\infty} (r(r-1) p(r) + p(r)) - \lambda^2$$

$$= \sum_{r=0}^{\infty} r(r-1) p(r) + \sum_{r=1}^{\infty} r p(r) - \lambda^2$$

$$= \sum_{r=0}^{\infty} r(r-1) p(r) + \sum_{r=1}^{\infty} r(p(r)) - \lambda^2$$

$$= \sum_{r=1}^{\infty} r(r-1) \frac{e^{-\lambda} \lambda^r}{r!} + \sum_{r=1}^{\infty} r \frac{e^{-\lambda} \lambda^r}{r!} - \lambda^2$$

$$= \sum_{r=2}^{\infty} \cancel{r(r-1)} \frac{e^{-\lambda} \lambda^r}{\cancel{r(r-1)}(r-2)!} + \sum_{r=1}^{\infty} \cancel{r} e^{-\lambda} \frac{\lambda^r}{\cancel{r}(r-1)!} - \lambda^2$$

$$= \sum_{r=2}^{\infty} \frac{e^{-\lambda} \lambda^r}{(r-2)!} + \sum_{r-1=0}^{\infty} \frac{e^{-\lambda} \lambda^r}{(r-1)!} - \lambda^2.$$

Put $x = r-2$ in I^{st} expression.
& $y = r-1$ in II^{nd} expression.
Then $r = x+2$ & $r = y+1$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x+2}}{x!} + \sum_{y=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{y+1}}{y!} - \lambda^2$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^x \cdot \lambda^2}{x!} + \sum_{y=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^y \cdot \lambda}{y!} - \lambda^2$$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} + e^{-\lambda} \cdot \lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} - \lambda^2$$

$$= e^{-\lambda} \lambda^2 (e^{\lambda}) + e^{-\lambda} \cdot \lambda (e^{\lambda}) - \lambda^2$$

$$= \lambda^2 (e^{-\lambda+\lambda}) + \lambda (e^{\lambda-\lambda}) - \lambda^2$$

$$= \lambda^2 (e^0) + \lambda (e^0) - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$V(x) = \lambda$

3c) If a poisson distribution is such that ~~p(x=~~ $p(x=1) \cdot \frac{3}{2} = p(x=3)$, find i) $p(x \geq 1)$ ii) $p(x \leq 3)$ and iii) $p(2 \leq x \leq 5)$.

W.K.T the poisson distribution is

$$p(x=r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

Given that $p(x=1) \frac{3}{2} = p(x=3)$

$$\left(\cancel{e^{-\lambda}} \frac{\lambda^1}{1!}\right) \frac{3}{2} = \frac{\cancel{e^{-\lambda}} \cdot \lambda^3}{3!}$$

$$\frac{3\lambda}{\cancel{2}} = \frac{\lambda^3}{\cancel{6}\,3}$$

$$9\lambda = \lambda^3$$

$$\lambda^3 - 9\lambda = 0$$

$$\lambda(\lambda^2 - 9) = 0 \quad \therefore \lambda \neq 0$$

$$\lambda^2 - 9 = 0$$

$$\lambda = \pm 3$$

$$\lambda = 3$$

$$P(x=r) = \frac{e^{-3} 3^r}{r!}, \quad r = 0, 1, 2, \ldots$$

i) $p(x \geq 1) = 1 - p(x < 1)$

$$= 1 - p(x=0)$$

$$= 1 - \frac{e^{-3} 3^0}{0!}$$

$$= 1 - e^{-3} = 0.9502$$

ii) $P(X\leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!}$$

$$= e^{-3}\left(1+3+\frac{9}{2}+\frac{27}{6}\right)$$

$$= 0.6472.$$

iii) $P(2\leq X\leq 5)$

$= P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} + \frac{e^{-3}3^5}{5!}$$

$$= e^{-3}\left(\frac{3^2}{2} + \frac{3^3}{6} + \frac{3^4}{24} + \frac{3^5}{120}\right)$$

$$= 0.716$$

3d) The marks obtained in mathematics by 1000 Students is normally distributed with mean 78% and standard deviation 11%.

i) Determine how many students got marks above 90%.

ii) What was the highest mark obtained by the lowest 10% of the Students.

Given that mean is $\mu = 0.78$

& standard deviation is $\sigma = 0.11$

w.k.t the z-score is $z = \frac{x-\mu}{\sigma}$

$$z = \frac{X-0.78}{0.11}$$

i) for $X = 0.9$, $z = \frac{0.9-0.78}{0.11}$

$$z = 1.09 = z_1 \text{ (say)}$$

$P(X > 0.9) = P(z > 1.09)$

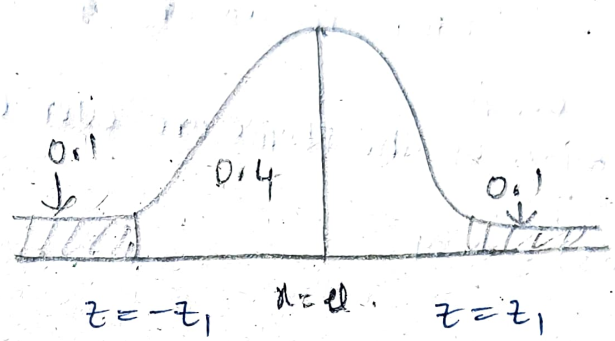
$= 0.5 - A(1.09)$

$= 0.5 - 0.3621$

$= 0.1379$

$\therefore$ The no. of Students with marks more than $90\% = 0.1379 \times 1000 = 137.9$

$\simeq 138.$



from figure

$P(0 < z < z_1) = \cancel{0.4}\ 0.4$

$z_1 = 1.29$

Then $\quad -1.29 = \dfrac{x - 0.78}{0.11}$

$x - 0.78 = (-1.29)(0.11)$

$x = 0.78 - [(1.29)(0.11)]$

$x = 0.6381$

$\doteq 63.8\%$

$x \simeq 64\%$

Hence the highest mark obtained by the lowest 10% of the students is 64%.

3e) Find the mean and variance of the distribution. In a normal distribution, 7% of the items are under 35 and 89% are over 63.

let μ and σ be the mean and standard deviation of the normal curve at $z=0$.

Given that 7% of the items are under 35

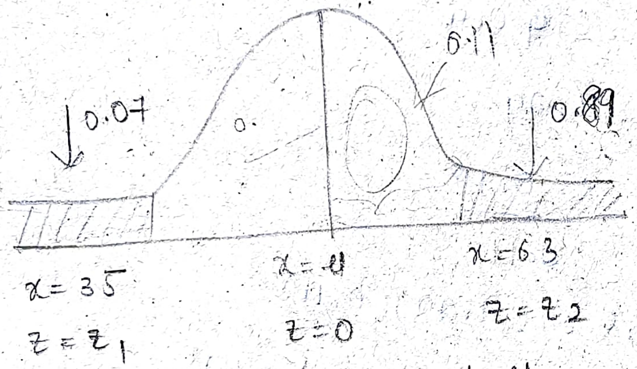
i.e $p(x<35) = 0.07$.

89% of the items are over 63.

i.e $p(x>63) = 0.89$

$p(x \le 63) = 1 - p(x>63)$

$= 1 - 0.89 = 0.11$



W.k.T z score is $z = \frac{x-\mu}{\sigma}$.

when $x=35$, $z = \frac{35-\mu}{\sigma} = -z_1$ (say) $\longrightarrow$ (1)

$x=63$, $z = \frac{63-\mu}{\sigma} = z_2$ (say) $\longrightarrow$ (2)

from figure,

$p(0<z<z_2) = 0.39 \Rightarrow z_2 = 1.23$ (from the table)

$\&\ p(0<z<z_1) = 0.43 \Rightarrow z_1 = 1.48$

from (1). we have $z = \frac{35-\mu}{\sigma} = -z_1$

$\frac{35-\mu}{\sigma} = -1.48$

$35-\mu = -1.48\sigma \longrightarrow$ (3)

from ② we have $\frac{63-\mu}{\sigma} = z_2$

$\frac{63-\mu}{\sigma} = 1.23$

$63-\mu = 1.23\sigma \rightarrow ④$

Solving ③ & ④

③ - ④

$35-\cancel{\mu}-63+\cancel{\mu} = -1.48\sigma - 1.23\sigma$

$\cancel{-}28 = \cancel{-}2.71\sigma$

$\sigma = 10.332$

from ③ $35-\mu = -1.48\sigma$

$35-\mu = 1.48(10.332)$

$35-\mu = 15.29136$

$\mu = 35+15.29136$

$\mu = 50.2913$

$\mu = 50.3$

∴ mean is $\mu = 50.3$

variance $= \sigma^2$

$= (10.332)^2$

$= 106.750$

$\sigma^2 = 106.75$

4a)

I) Out of 800 families with 5 children each, how many would you expect to have a) 3 boys b) 5 girls c) either 2 or 3 boys? Assume equal probabilities for boys & girls.

Given that $n=5$

The probability of each boy is $\boxed{p=\frac{1}{2}}$, Then

$q=1-p$

$=1-\frac{1}{2}$

$\boxed{q=\frac{1}{2}}$

let x denotes the number of boys

The binomial distribution is

$P(x=r)=nC_r p^r q^{n-r}$, $r=0,1,2,---,n.$

$P(x=r)=5C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{5-r}$, $r=0,1,2,3,4,5$

$=5C_r\left(\frac{1}{2}\right)^{r+5-r}$

$=\left(\frac{1}{2}\right)^5 5C_r$

$P(x=r)=\frac{1}{2^5}5C_r$

a) 3 boys

$P(x=3)=\frac{1}{2^5}5C_3=\frac{10}{32}=0.3125$

b) 5 girls

$P(x=0)=\frac{1}{2^5}5C_0=\frac{1}{32}=0.03125$

c) either 2 or 3 boys

$P(x=2)+P(x=3)=\frac{1}{2^5}5C_2+\frac{1}{2^5}5C_3$

$=\frac{1}{2^5}(5C_2+5C_3)$

$=\frac{1}{2^5}(10+10)=\frac{20}{32}$

$=0.625$

The no. of families having 3 boys $= 0.3125 \times 800$
$= 250$

The no. of families having 5 girls $= 0.03125 \times 800$
$= 25$

The no. of families having either 2 (or) 3 boys
$= 0.625 \times 800 = 500.$

II) Derive mean and variance of binomial distribution.

Mean of the binomial distribution

The binomial ~~dis~~ probability distribution is given by

$p(r) = p(x=r) = nC_r p^r . q^{n-r}$ $r = 0,1,2,---n$ &

$q = 1-p$

mean of x, $\mu = \sum_{r=0}^{n} r(p(r))$

$= \sum_{r=0}^{n} r(nC_r p^r q^{n-r})$

$= 0 + (nC_1 p^r q^{n-1}) + 2(nC_2 p^2 q^{n-2}) + 3(nC_3 p^3 q^{n-3}) + ---- n(nC_n p^n q^{n-n})$

$= npq^{n-1} + \cancel{2}\left(\frac{n(n-1)}{\cancel{2}} p^2 q^{n-2}\right) + \cancel{3}\frac{n(n-1)(n-2)}{\cancel{6}_2} p^3 q^{n-3} + ---- + np^n$

$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2} p^2 q^{n-3} + --- + p^{n-1}\right]$

$= np(p+q)^{n-1}$ ($\therefore$ by using binomial theorem)

$= np(1)^{n-1}$

$\boxed{\mu = np}$

Variance of the binomial distribution:

The binomial probability distribution is given by

$p(r) = p(X=r) = {}^nC_r\, p^r q^{n-r} \quad r = 0, 1, 2, \ldots, n \;\&\; q = 1-p.$

Variance of x, $V(x) = \sum_{r=0}^{n} r^2 p(r) - \mu^2$.

$$= \sum_{r=0}^{n} (r^2 - r + r)\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} (r(r-1) + r)\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} (r(r-1)\, p(r) + r\, p(r)) - \mu^2$$

$$= \sum_{r=0}^{n} r(r-1)\, p(r) + \sum_{r=0}^{n} r\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} r(r-1)({}^nC_r\, p^r q^{n-r}) + \mu - \mu^2$$

$$= 0 + 0 + 2\cdot 1({}^nC_2\, p^2 q^{n-2}) + 3\cdot 2({}^nC_3\, p^3 q^{n-3}) + \cdots + n(n-1)({}^nC_n\, p^n q^{n-n}) + \mu - \mu^2$$

$$= \cancel{2}\left(\frac{n(n-1)}{\cancel{2}}\right) p^2 q^{n-2} + \cancel{6}\cdot\frac{n(n-1)(n-2)}{\cancel{6}}\, p^3 q^{n-3} + \cdots + n(n-1)(p^n) + \mu - \mu^2$$

$$= n(n-1)p^2 \left[q^{n-2} + (n-2)pq^{n-3} + \cdots + p^{n-2}\right] + \mu - \mu^2$$

[$\because$ by using binomial theorem ~~distribu~~]

$$= n(n-1)p^2 (p+q)^{n-2} + np - (np)^2$$

$$= n(n-1)p^2 (1)^{n-2} + np - n^2p^2$$

$$= (n^2 - n)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$= np - np^2$

$= np(1-p) \quad [\because q = 1-p]$

$= np(q)$

$\boxed{V(X) = npq}$

4b)

I) If x is a poisson variate such that $3p(x=4) = \frac{1}{2}p(x=2) + p(x=0)$ find i) the mean of x ii) $p(x \leq 2)$

Given that

'x' is a poisson variate with parameters λ

The poission distribution is,

$$p(x=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}, \quad r = 0, 1, 2, ---$$

$$3p(x=4) = \frac{1}{2}P(x=2) + p(x=0)$$

$$3\frac{e^{-\lambda} \cdot \lambda^4}{4!} = \frac{1}{2}\frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda} \cdot \lambda^0}{0!}$$

$$e^{-\lambda}\frac{3\lambda^4}{24} = e^{-\lambda}\left(\frac{1}{2} \cdot \frac{\lambda^2}{2} + 1\right)$$

$$\frac{\lambda^4}{8} = \frac{\lambda^2 + 4}{4}$$

$$\lambda^4 = 2\lambda^2 + 8$$

$$(\lambda^2)^2 - 2\lambda^2 - 8 = 0$$

$$(\lambda^2 - 4)(\lambda^2 + 2) = 0$$

$$\lambda^2 = 4 \text{ or } \lambda^2 = -2$$

$$\lambda = \pm 2 \quad (\because \lambda = \sqrt{-2} \notin R)$$

$$\lambda = 2 \quad (\because \lambda > 0)$$

The mean is $\lambda = 2$

$$P(X=r) = \frac{e^{-2} \cdot 2^r}{r!}, \quad r = 0, 1, 2$$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-2} \cdot 2^0}{0!} + \frac{e^{-2} \cdot 2^1}{1!} + \frac{e^{-2} \cdot 2^2}{2!}$$

$$= e^{-2}(1+2+2)$$

$$= 5e^{-2}.$$

II) In a sample of 1000 cases, the mean of certain test is 14 and standard deviation of 2.5. Assuming distribution is normal. find (i) how many students score between 12 and 15 (ii) how many score above 18%. iii) how many score below 18%.

Given that, $\mu = 14$

& $\sigma = 2.5$

let 'x' denotes the score in a test

Then, the z-score is $z = \frac{x-\mu}{\sigma} = \frac{x-14}{2.5}$

i) for $x=12$, $z = \frac{12-14}{2.5} = -0.8 = z_1$ (say)

for $x=15$, $z = \frac{15-14}{2.5} = 0.4 = z_2$ (say)

$$P(12 \le x \le 15) = A(z_2) + A(z_1)$$

$$= A(0.4) + A(-0.8)$$

$$= A(0.4) + A(0.8)$$ ($\because A(-0.8) = A(0.8)$ Due to symmetry)

$$= 0.1554 + 0.2881$$

$$= 0.4435.$$

$\therefore$ The no. of students scoring between 12 and 15

$$= 0.4435 \times 1000$$

$$= 443.5$$

ii) for $x = 18$, $z = \frac{18-14}{2.5} = 1.6$

$\therefore P(x > 18) = P(z > 1.6) = 0.5 - A(1.6)$
$= 0.5 - 0.4452$
$= 0.0548$

The no. of students scoring above 18 $= 0.0548 \times 1000$
$= 54.8 \simeq 55$

iii) $P(x < 18) = P(z < 1.6)$
$= 0.5 + A(1.6)$
$= 0.5 + 0.4452$
$= 0.9452$

The no. of students scoring below 18 $= 0.9452 \times 1000$
$= 945.2 \simeq 945$

4c)
I) seven coins are tossed and the number of heads are ~~tas~~ noted. The experiment is ~~reap~~ repeated 128 times and the following distribution is obtained.

x	0	1	2	3	4	5	6	7
F	7	6	19	35	30	23	7	1

Fit a binomial distribution assuming the coin is unbiased.

The coin is unbiased.

$p = \frac{1}{2}$, $q = \frac{1}{2}$ and $n = 7$

$N = \Sigma f_i = 7 + 6 + 19 + 35 + 30 + 23 + 7 + 1 = 128$

By binomial distribution, $P(x) = nC_x p^x q^{n-x}$

we have the recurrence relation

$P(x+1) = \frac{(n-x)p}{(x+1)q} \cdot P(x) = \frac{7-x}{x+1} \cdot P(x)$ $\left[\because n = 7, \frac{p}{q} = 1\right]$

$\therefore P(0) = {}^7C_0\, p^0 q^7 = {}^7C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^7 = \frac{1}{2^7}$

No. of heads (x)	observed frequency(f)	Probability $P(x)$	Expected or Theoretical frequency $f(x) = N \cdot P(x)$.
0	7	$P(0) = \frac{1}{2^7}$	$f(0) = 128 \cdot P(0) = 128 \times \frac{1}{2^7} = 1$
1	6	$P(1) = 7 \cdot P(0) = \frac{7}{2^7}$	$f(1) = 128 \cdot P(1) = 128 \times \frac{7}{2^7} = 7$
2	10	$P(2) = 3 \cdot P(1) = \frac{21}{2^7}$	$f(2) = 128 \cdot P(2) = 128 \times \frac{21}{2^7} = 21$
3	35	$P(3) = \frac{35}{2^7}$	$f(3) = 128 \cdot P(3) = 35$
4	30	$P(4) = \frac{35}{2^7}$	$f(4) = 128 \cdot P(4) = 35$
5	23	$P(5) = \frac{21}{2^7}$	$f(5) = 128 \cdot P(5) = 21$
6	7	$P(6) = \frac{7}{2^7}$	$f(6) = 128 \cdot P(6) = 7$
7	1	$P(7) = \frac{1}{2^7}$	$f(7) = 128 \cdot P(7) = 1$

II) Fit a poisson distribution to the following data:

x	0	1	2	3	4	5	Total
F	142	156	69	27	5	1	400

~~142+156+69+~~

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i}$$

$$= \frac{0(142) + 1(156) + 2(69) + 3(27) + 4(5) + 5(1)}{400}.$$

$$= \frac{0+156+138+81+20+5}{400}$$

$$= \frac{400}{400}$$

$$= 1$$

$\therefore$ mean of poisson distribution i.e, $\lambda = 1$

Hence the theoretical frequency for x successes is given by

$N \cdot p(x)$ where $x = 0, 1, 2, 3, 4, 5$

i.e $400 \frac{e^{-1}(1)^x}{x!}$, where $x = 0, 1, 2, 3, 4, 5$

i.e $400(e^{-1}), 400(e^{-1}), 200(e^{-1}), 66.67(e^{-1}), 16.67(e^{-1}), 3.33(e^{-1})$

i.e $147.15, 147.15, 73.58, 24.53, 6.13, 1.23$

$\therefore$ The theoretical frequencies are

x	0	1	2	3	4	5
f	147	147	74	25	6	1