UNIT-IV

1 Mark Questions :-

1) Define vector differential Operator ?
2) Define gradient of a scalar point function ?
3) Define divergence of a vector.
4) Define curl of a vector.
5) Define Scalar potential.

3 Mark Questions :-

1) What is solenoidal vector. If $\bar{f} = (x+3y)\bar{i} + (y-2x)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find the value of p.
2) Find the directional derivatives of $f = xy + yz + zx$ in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point $(1, 2, 0)$.
3) Prove that $\nabla[f(r)] = \frac{f(r)}{r}\bar{r}$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.
4) Find a unit normal to the given surface $x^2y + 2xz = 4$ at $(-1, 2, 1)$.
5) Define irrotational vector. Also if $\bar{f} = \bar{r}$, then prove that $\bar{f}$ is irrotational.

5 Mark Questions :-

1) Discuss the angle between the normals to the surface $xy = z^2$ at the point $(4, 1, 2)$ and $(3, 3, -3)$.

2) Find the angle of intersection of the spheres $x^2+y^2+z^2 = 29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$

3) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a direction of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

4) Find the directional derivative of $f = x^2-y^2+2z^2$ at the point $(1,2,3)$ in the direction of the line PQ where $Q = (5,0,4)$.

5) Prove that if $\bar{r}$ is the position vector of any point in space, then $\bar{f} = r^n.\bar{r}$ is solenoidal, then $n = ?$

10 Mark Questions :-

1) If $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ is the position vector of any point in space and $r = |\bar{r}|$. Then prove that the vector $\frac{\bar{r}}{r^3}$ is solenoidal and the vector $r^n\bar{r}$ is irrotational.

2) Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

3) Find the constants a, b, c such that the vectors $\bar{A} = (x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

1 Mark Questions :-

1) Define vector differential operator ?

A) The vector differential operator $\overline{\nabla}$ is defined as $\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. That possess the properties of vectors as well as differentiation.

i.e., $\overline{\nabla} = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

2) Define gradient of a scalar point function ?

A) Let $\phi(x, y, z)$ be a scalar point function, then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ and it is denoted by grad ϕ (or) $\overline{\nabla}\phi$.

i.e., $\text{grad}\,\phi = \overline{\nabla}\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

3) Define divergence of a vector ?

A) Let $\bar{f}$ be any continuously differentiable vector point function, then $\bar{i}\frac{\partial \bar{f}}{\partial x} + \bar{j}\frac{\partial \bar{f}}{\partial y} + \bar{k}\frac{\partial \bar{f}}{\partial z}$ is called divergence of $\bar{f}$ and it is denoted as div $\bar{f}$ (or) $\overline{\nabla}\cdot\bar{f}$.

i.e., $\text{div}\,\bar{f} = \overline{\nabla}\cdot\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{f}$

$$\bar{\nabla}\cdot\bar{f} = \bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$$

4) Define curl of a vector.

A) If $\bar{f}$ is continously differentiable vector point function, then $\bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of a vector and it is denoted as curl $\bar{f}$ (or) $\bar{\nabla}\times\bar{f}$

i.e, $\text{curl } \bar{f} = \bar{\nabla}\times\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times\bar{f}$

$$= \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$$

5) Define scalar potential.

A) If $\bar{f}$ is irrotational, there will always exist a scalar function $\phi(x, y, z)$ such that $\bar{f} = \text{grad } \phi$. This ϕ is called Scalar potential of $\bar{f}$.

3 Mark Questions :-

1) What is solenoidal vector. If $\bar{f} = (x+3y)\bar{i} + (y-2x)\bar{j} + (x+pz)\bar{k}$ is solenoidal find the value of P.

A) Solenoidal vector :- A vector $\bar{f}$ is said to be solenoidal iff $\text{div}\,\bar{f} = 0$.

Given that,

$$\bar{f} = (x+3y)\bar{i} + (y-2x)\bar{j} + (x+pz)\bar{k}$$

We know that, $\bar{f}$ is solenoidal, iff $\text{div}\,\bar{f} = 0$

i.e., $$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2x) + \frac{\partial}{\partial z}(x+pz) = 0$$

$$(1+0) + (1-0) + (0+p) = 0$$

$$1+1+p = 0$$

$$p = -2$$

2) Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$ at the point $(1,2,0)$.

A) Given that,

$$f(x,y,z) = xy+yz+zx$$

$$\text{grad}\, f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xy+yz+zx)$$

$$= (y+z)\bar{i} + (x+z)\bar{j} + (x+y)\bar{k}$$

$$(\text{grad } f)_{(1,2,0)} = (2+0)\bar{i} + (1+0)\bar{j} + (1+2)\bar{k}$$

$$= 2\bar{i} + 1\bar{j} + 3\bar{k}$$

Unit vector, $\bar{e} = \dfrac{\bar{i} + 2\bar{j} + 2\bar{k}}{|\bar{i} + 2\bar{j} + 2\bar{k}|}$

$$= \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$= \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}}$$

$$= \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{9}}$$

$$= \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$= \frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

$\therefore$ Directional derivative of f in the direction of $\bar{e}$ at $(1,2,0)$ is $\bar{e} \cdot \text{grad } f$.

i.e, $\frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k}) \cdot (2\bar{i} + \bar{j} + 3\bar{k})$

$$= \frac{1}{3}(2+2+6)$$

$$= \frac{10}{3} \text{ //}$$

3) Prove that $\bar{\nabla}[f(r)] = \frac{f'(r)}{r} \cdot \bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

A) Given that, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{r} \cdot \bar{r} = (x\bar{i} + y\bar{j} + z\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})$$

$$r^2 = x^2 + y^2 + z^2.$$

diff. partially w.r.t 'x' on b/s

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Ily, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\bar{\nabla}[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \Sigma\, \bar{i} \frac{\partial}{\partial x} f(r)$$

$$= \Sigma\, \bar{i}\, f'(r) \frac{\partial r}{\partial x}$$

$$= \Sigma\, \bar{i}\, f'(r) \cdot \frac{x}{r}$$

$$= \frac{f'(r)}{r} \cdot \Sigma\, \bar{i} x$$

$$= \frac{f'(r)}{r} \cdot (\bar{i}x + \bar{j}y + \bar{k}z)$$

$$= \frac{f'(r)}{r} \cdot \bar{r}$$

4) Find a unit normal to the given surface $x^2y + 2xz = 4$ at $(-1, 2, 1)$

A) Let the given surface be

$$f = x^2y + 2zx - 4$$

$$\text{grad } f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2y + 2xz - 4)$$

$$= (2xy + 2z)\bar{i} + x^2\bar{j} + 2x\bar{k}$$

Normal to the surface is grad f

i.e, $\text{grad } f = (2xy + 2z)\bar{i} + x^2\bar{j} + 2x\bar{k}$

$$\text{grad } f_{(-1,2,1)} = (2(-1)(2) + 2(1))\bar{i} + (-1)^2\bar{j} + 2(-1)\bar{k}$$

$$= (-4 + 2)\bar{i} + \bar{j} + (-2)\bar{k}$$

$$= -2\bar{i} + \bar{j} - 2\bar{k}$$

Unit Normal to the surface $= \dfrac{-2\bar{i} + \bar{j} - 2\bar{k}}{\sqrt{(-2)^2 + (1)^2 + (-2)^2}}$

$$= \frac{-2\bar{i} + \bar{j} - 2\bar{k}}{\sqrt{4 + 1 + 4}}$$

$$= \frac{-2\bar{i} + \bar{j} - 2\bar{k}}{\sqrt{9}}$$

$$= \frac{1}{3}(-2\bar{i} + \bar{j} - 2\bar{k})$$

$$= \frac{-1}{3}(2\bar{i} - \bar{j} + 2\bar{k})$$

5) Define irrotational vector. Also if $\bar{f} = \bar{r}$, then prove that $\bar{f}$ is irrotational.

A) Irrotational Vector :-

If A vector $\bar{f}$ is said to be irrotational iff curl $\bar{f} = \bar{0}$.

Given that

$$\bar{f} = \bar{r}$$

We know that, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \bar{f} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \bar{i}[0-0] - \bar{j}[0-0] + \bar{k}[0-0]$$

$$= 0\bar{i} - 0\bar{j} + 0\bar{k}$$

$$= \bar{0}$$

$$\therefore \text{Curl } \bar{f} = 0.$$

Hence, $\bar{f}$ is irrotational.

5 Mark Questions :-

1) Discuss the angle between the normals to the surface $xy = z^2$ at the point $(4,1,2)$ and $(3,3,-3)$

A) Let $f(x,y,z) = xy - z^2$

Normal to this surface is ∇f

$$\nabla f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \right)(xy - z^2)$$

$$= \bar{i}(y) + \bar{j}(x) - 2z\bar{k}$$

$$= y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$(\nabla f)_{(4,1,2)} = \bar{n}_1 = \bar{i} + 4\bar{j} - 4\bar{k}$$

$$(\nabla f)_{(3,3,-3)} = \bar{n}_2 = 3\bar{i} + 3\bar{j} + 6\bar{k}$$

If θ is the angle between the two normals, then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$\cos\theta = \frac{(\bar{i} + 4\bar{j} - 4\bar{k}) \cdot (3\bar{i} + 3\bar{j} + 6\bar{k})}{\sqrt{1^2 + 4^2 + (-4)^2}\sqrt{3^2 + 3^2 + 6^2}}$$

$$\cos\theta = \frac{3 + 12 - 24}{\sqrt{1+16+16}\sqrt{9+9+36}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\sqrt{54}}$$

$$\cos\theta = \frac{-1}{\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$$

2) Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$

A) We know that angle between the two surfaces is equal to angle between their normals.

Let the given surfaces are

$$f(x,y,z) = x^2+y^2+z^2-29$$

$$g(x,y,z) = x^2+y^2+z^2+4x-6y-8z-47$$

$$\bar{\nabla} f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-29)$$

$$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\bar{\nabla} g = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2+4x-6y-8z-47)$$

$$= (2x+4)\bar{i} + (2y-6)\bar{j} + (2z-8)\bar{k}$$

$$\bar{\nabla} f_{(4,-3,2)} = \bar{n}_1 = 8\bar{i} - 6\bar{j} + 4\bar{k}$$

$$\bar{\nabla} g_{(4,-3,2)} = \bar{n}_2 = 12\bar{i} - 12\bar{j} - 4\bar{k}$$

The angle between curves is θ, then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|}$$

$$\cos\theta = \frac{(8\bar{i} - 6\bar{j} + 4\bar{k})\cdot(12\bar{i} - 12\bar{j} - 4\bar{k})}{\sqrt{8^2+(-6)^2+4^2}\ \sqrt{12^2+(-12)^2+4^2}}$$

$$\cos\theta = \frac{96+72-16}{\sqrt{116}\ \sqrt{304}}$$

$$\cos\theta = \frac{152}{\sqrt{116}\ \sqrt{304}} = \sqrt{\frac{19}{29}} \Rightarrow \theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right).$$

3) Find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$.

A) Let $f(x,y,z) = xyz^2 + xz$

$$\text{grad } f = \bar{\nabla} f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xyz^2 + xz)$$

$$= (yz^2 + z)\bar{i} + (xz^2)\bar{j} + (2xyz + x)\bar{k}$$

$$\bar{\nabla} f_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$$

Let $g(x,y,z) = 3xy^2 + y - z$

We know that, grad g is the normal to the surface $g(x,y,z)$.

$$\text{grad } g = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(3xy^2 + y - z)$$

$$= 3y^2\bar{i} + (6xy + 1)\bar{j} - \bar{k}$$

$$(\text{grad } g)_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k}$$

Unit vector, $\bar{e} = \dfrac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{3^2 + 1^2 + (-1)^2}}$

$$= \frac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{11}}$$

Directional derivative of $f(x,y,z)$ at $(1,1,1)$ in the direction of normal to the surface $g(x,y,z)$ at $(0,1,1)$ is

$$\bar{e} \cdot \text{grad } f = \frac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{11}} \cdot (2\bar{i} + \bar{j} + 3\bar{k})$$

$$= \frac{6 + 1 - 3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

4) Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $(1,2,3)$ in the direction of the line PQ where $Q = (5,0,4)$.

A) Given that $f = x^2 - y^2 + 2z^2$

$$\text{grad } f = \bar{\nabla} f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2)$$

$$= 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$$

$$(\bar{\nabla} f)_{(1,2,3)} = 2\bar{i} - 4\bar{j} + 12\bar{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 5\bar{i} + 4\bar{k} - (\bar{i} + 2\bar{j} + 3\bar{k})$$

$$= 4\bar{i} - 2\bar{j} + \bar{k}$$

unit vector, $$\bar{e} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{(4)^2 + (-2)^2 + 1^2}}$$

$$= \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$$

Directional derivative of f at $p(1,2,3)$ in the direction of $\overline{PQ}$ is

$$\bar{e} \cdot \bar{\nabla} f = \frac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}} \cdot (2\bar{i} - 4\bar{j} + 12\bar{k})$$

$$= \frac{8 + 8 + 12}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}}$$ //

5) Prove that if $\bar{r}$ is the position vector of any point in space, then $\bar{f} = r^n . \bar{r}$ is Solenoidal then $n = ?$

A). We know that, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{r}.\bar{r} = r^2 = x^2 + y^2 + z^2$$

diff. partially w.r. to 'x' on b/s

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

llly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Given, $\bar{f} = r^n . \bar{r}$

$$= r^n (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$$

Given that, $\text{div}\, \bar{f} = \bar{0}$

$$\frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n) = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$\Sigma \frac{\partial}{\partial x} xr^n = 0$$

$$\Sigma \left(1 r^n + xn r^{n-1}\frac{\partial r}{\partial x}\right) = 0$$

$$\Sigma \left(r^n + x . n r^{n-1} \frac{x}{r}\right) = 0$$

$$\Sigma (r^n + n r^{n-2} . x^2) = 0$$

$$3r^n + n r^{n-2}(x^2 + y^2 + z^2) = 0$$

$$3r^n + n . r^{n-2} . r^2 = 0$$

$$3r^n + n r^n = 0$$

10 Mark Questions :-

1) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ is the position vector of any point in space and $r = |\bar{r}|$. Then prove that the vector $\frac{\bar{r}}{r^3}$ is solenoidal and the vector $r^n \bar{r}$ is irrotational.

A) Given that, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{r}\cdot\bar{r} = r^2 = x^2 + y^2 + z^2$$

diff. partially w. r. to 'x' on b/s

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Ilusing. $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Then, $\bar{\nabla}\cdot\frac{\bar{r}}{r^3} = \Sigma\, \bar{i}\, \frac{\partial}{\partial x}\cdot\left(\frac{\bar{r}}{r^3}\right)$

$$= \Sigma\, \bar{i}\cdot \frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right)$$

$$= \Sigma\, \bar{i}\cdot \frac{\partial}{\partial x}\left[r^{-3}\bar{r}\right]$$

$$= \Sigma\, \bar{i}\cdot\left[-3r^{-4}\frac{\partial r}{\partial x}\bar{r} + r^{-3}\frac{\partial \bar{r}}{\partial x}\right]$$

$$= \Sigma\, \bar{i}\cdot\left[-3r^{-4}\frac{x}{r}\cdot\bar{r} + r^{-3}\bar{i}\right]$$

$$= \Sigma\, \bar{i}\cdot\left[-3r^{-5}x\,\bar{r} + r^{-3}\bar{i}\right]$$

$$= \Sigma\left[-3r^{-5}x\,(\bar{i}\cdot\bar{r}) + r^{-3}[\bar{i}\cdot\bar{i}]\right]$$

$$= \Sigma\left[-3r^{-5}x\cdot x + r^{-3}\right]$$

$$= \Sigma\left[-3r^{-5}x^2 + r^{-3}\right]$$

$$= -3r^{-5}\,\Sigma x^2 + \Sigma r^{-3}$$

$$= -3r^{-5}\, r^{2} + \Sigma r^{-3}$$

$$= -3r^{-3} + 3r^{-3}$$

$$= 0.$$

$\therefore \dfrac{\bar{r}}{r^3}$ is solenoidal.

Now, curl $r^n . \bar{r} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$$= x r^n \bar{i} + r^n y \bar{j} + r^n z \bar{k}$$

$$\text{curl } r^n . \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^n & y r^n & z r^n \end{vmatrix}$$

$$= \bar{i}\left[z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-1} \frac{\partial r}{\partial z}\right] - \bar{j}\left[z n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial z}\right] + \bar{k}\left[y n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial y}\right]$$

$$= n r^{n-1} \left[\bar{i}\left[z \frac{y}{r} - y \frac{z}{r}\right] - j\left[z \frac{x}{r} - x \frac{z}{r}\right] + \bar{k}\left[y . \frac{x}{r} - x . \frac{y}{r}\right]\right]$$

$$= n r^{n-1} \left[0\bar{i} - 0\bar{j} + 0\bar{k}\right]$$

$$= \bar{0}$$

Hence, $r^n . r$ is irrotational.

2) Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

A) Given that,

$$\bar{f}=(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$=\bar{i}\,[0-x-0+x]-\bar{j}\,[0-y-0+y]+\bar{k}\,[0-z-0+z]$$

$$=\bar{i}\,(0)+\bar{j}\,(0)+\bar{k}\,(0)$$

$$=\bar{0}$$

$\therefore$ curl $\bar{f}=0$, Hence $\bar{f}$ is irrotational.

If $\bar{f}$ is irrotational $\exists$ a scalar ϕ such that

$$\bar{f}=\bar{\nabla}\phi$$

$$(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}=\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

$\frac{\partial\phi}{\partial x}=x^2-yz$

$\phi=\int(x^2-yz)\,dx+c_1$

$\phi=\frac{x^3}{3}-xyz+c_1$

$\frac{\partial\phi}{\partial y}=y^2-zx$

$\phi=\int(y^2-zx)\,dy+c_2$

$\phi=\frac{y^3}{3}-xyz+c_2$

$\frac{\partial\phi}{\partial z}=z^2-xy$

$\phi=\int(z^2-xy)\,dz+c_3$

$\phi=\frac{z^3}{3}-xyz+c_3$

$$\therefore\ \phi(x,y,z)=\frac{x^3}{3}+\frac{y^3}{3}+\frac{z^3}{3}-xyz+k$$

which is the required scalar potential.

3) Find the constants a, b, c such that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

A) Given that,

$$\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$$

Also given $\bar{A}$ is irrotational, then

$$\text{curl } \bar{A} = 0$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\bar{i}\,[(c+1)] - j\,[4-a] + \bar{k}\,[b-2] = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

Comparing the coefficients of $\bar{i}, \bar{j}, \bar{k}$ on b/s.

$c+1=0$, $\quad +1-a=0$, $\quad b-2=0$

$c=-1$, $\quad a=4$, $\quad b=2$

$\therefore\ a=4,\ b=2,\ c=-1$

Now, $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

Also $\quad \bar{A} = \nabla\phi$

$$(x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = i\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}.$$

compare the coefficients of $\bar{i}, \bar{j}, \bar{k}$ on b/s.

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z$$

$$\phi = \int (x + 2y + 4z)\,dx + c_1$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz + c_1$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z$$

$$\phi = \int (2x - 3y - z)\,dy + c_2$$

$$\phi = 2xy - \frac{3y^2}{2} - yz + c_2$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z$$

$$\phi = \int (4x - y + 2z)\,dz + c_3$$

$$\phi = 4xz - yz + z^2 + c_3$$

Hence,

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + K$$ //