UNIT - II MULTIVARIABLE CALCULUS:

1 Mark Questions:-

1. Define Euler's Theorem
2. Write the formula for Jacobian of three variables
3. Define Maxima and Minima of function of two variable.
4. Define Saddle point.
5. Write Lagrange's Method of Undetermined Multipliers.

3 Marks Question.

1. If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$
2. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that $xu_x + yu_y = \sin 2u$
3. If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functionally dependent. Find the relation between them
4. Show that the function $u = e^x \sin y$ $v = e^x \cos y$ are functionally independent.
5. Find the maximum and minimum values of $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

5 Marks Question

1. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$
2. If $r^2 = x^2+y^2+z^2$ and $u = r^m$ then prove that $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$$
3. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ Show that $$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$$

(4.) Show that the functions $u = x+y+z$, $V = x^2+y^2+z^2-2xy-2yz-2xz$ and $w = x^3+y^3+z^3-3xyz$ are functionally related.

(5) Find the minimum value of $x^2+y^2+z^2$, Given $x+y+z=3a$

10 Marks.

(1) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ and find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$.

(2) Find the maximum and minimum of $f(x,y) = x^3+3xy^2-3x^2-3y^2+4$.

(3) A Rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

— x —

4. Define Saddle Point

Saddle point in a multivarible function are those critical point where the function attains neither a local maximum value nor a minimum value

5. Write Lagrange's Method of Undertemined Multipliers.

Let $f(x,y,z)$ be a function in x,y,z where x,y,z are connected by $\phi(x,y,z)=0$

Step 1:- Form Lagrangean function $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$

Where λ is called the Lagrange multiplier, which is determined by the following conditions. Suppose it is required to find the extremum for the function $f(x,y,z)$ subject to the condition

$$\phi(x,y,z) = 0 \quad \text{--- (1)}$$

Step 2: Obtain the equation

$$\frac{\partial F}{\partial x} = 0 \quad \text{i.e,} \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial y} = 0 \quad \text{i.e,} \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial y} = 0 \quad \text{i.e,} \quad \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad \text{--- (4)}$$

Step 3: Solve equation (1), (2), (3), (4)

The values of x,y,z so obtained will give the stationary point $f(x,y,z)$

3 Marks Question.

1. If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

Given that,

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(e^{xyz})$$

$$= e^{xyz}\, xy(1)$$

$$= xy\, e^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial y}(xy\, e^{xyz})$$

$$= x\left[y e^{xyz} xz + e^{xyz}(1)\right]$$

$$= x^2yz\, e^{xyz} + x e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right)$$

$$= \frac{\partial}{\partial x}\left[x^2yz\, e^{xyz} + x e^{xyz}\right]$$

$$= yz \frac{\partial}{\partial x}(x^2 e^{xyz}) + \frac{\partial}{\partial x}(x e^{xyz})$$

$$= yz\left[x^2 \cdot e^{xyz} \frac{\partial}{\partial x}(xyz) + e^{xyz}(2x)\right] + e^{xyz}\frac{\partial}{\partial x}(xyz)x + e^{xyz}(1)$$

$$= yz\left[x^2 e^{xyz} yz + 2x e^{xyz}\right] + xyz\, e^{xyz} + e^{xyz}$$

$$= x^2y^2z^2 e^{xyz} + 2xyz\, e^{xyz} + xyz\, e^{xyz} + e^{xyz}$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

Hence Proved.

(2) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that $xu_x + yu_y = \sin 2u$.

Given that, $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

So, $\tan u = \frac{x^3+y^3}{x+y}$

Let $\tan u = z$ —— (1)

$$z = \frac{x^3+y^3}{x+y}$$

$$= \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1+\frac{y}{x}\right)}$$

$$z = \frac{x^2\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1+y/x\right]}$$

It is clear that z is homogeneous function of the form $x^n f(y/x)$ Here $n=2$

We know, A/C to Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{—— (2)}$$

Differentiate partially equation (1) with respect to x and y

$$\frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

putting the above values in eq (2) we get

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$xu_x + yu_y = 2\tan u\cos^2 u$$

$$xu_x + yu_y = 2\frac{\sin u}{\cancel{\cos u}} \times \cos^{\cancel{2}} u = 2\sin u\cos u = \sin 2u$$

So, $x u_x + y u_y = \sin 2u$ Hence proved.

3) If $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. Find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Hence proved that u and v are functionally dependent. Find the relation between them.

Given,

$$u = \frac{x+y}{1-xy} \quad \text{and} \quad v = \tan^{-1}x + \tan^{-1}y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2} \quad \frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}, \quad \frac{\partial v}{\partial x} = \frac{1}{1+x^2} \text{ and } \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$\therefore$ u and v are functionally dependent.

Now,

$$V = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$$

$\therefore v = \tan^{-1}u$ is the functional relation between u and v.

4. Show that The function $u = e^x \sin y$ $v = e^x \cos y$ are functionally independent.

Given that

$$u = e^x \sin y \quad v = e^x \cos y$$

$$u_x = \frac{\partial u}{\partial x} = e^x \sin y, \quad v_x = \frac{\partial v}{\partial x} = e^x \cos y$$

$$u_y \ \frac{\partial u}{\partial y} = e^x \cos y, \quad v_y = \frac{\partial v}{\partial y} = -e^x \sin y$$

$$\text{Jacobian} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$= e^x(-\sin^2 y - \cos^2 y) = -e^x \neq 0$$

So, u, v are functionally independent.

(5) Find the maximum and minimum values of $xy + \frac{a^3}{x} + \frac{a^3}{y}$

Given function is $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ —— (1)

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}, \quad \frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

and $\frac{\partial^2 f}{\partial x \partial y} = 1$.

The condition for $f(x,y)$ to have minimum or maximum is

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$$

$$\Rightarrow y = \frac{a^3}{x^2} \quad \text{—— (2)}$$

$$x = \frac{a^3}{y^2} \quad \text{—— (3)}$$

Substituting (3) in (2) we get

$$y = \frac{a^3 y^4}{a^6} = \frac{y^4}{a^3}$$

$$y(y^3 - a^3) = 0$$

$$y = 0 \text{ or } y = a$$

Now, $y = 0 \Rightarrow x = \infty$

$\therefore$ It is not possible.

Now, $y = a \Rightarrow x = a$

$\therefore$ The extremum point is (a, a)

$f(x, y)$ will have max. or min at (a, a)

At (a, a) $\quad l = \frac{\partial^2 f}{\partial x^2} = 2, \; m = 1, \; n = 2$

$$ln - m^2 = 4 - 1 = 3 > 0, \; l = 2 > 0$$

$\therefore f(x, y)$ has minimum at (a, a)

The minimum value is $f(a, a)$

$$= a^2 + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= 3a^2$$

5 Marks Questions

(1.) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Given that,

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z} \quad \text{———(1)}$$

Now,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad [\text{From } 1]$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$=\frac{-9}{(x+y+z)^2}$$

Hence Proved.

(2) If $r^2=x^2+y^2+z^2$ and $u=r^m$ then prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=m(m+1)r^{m-2}$$

We have

$$r^2=x^2+y^2+z^2 \quad \text{———(1)}$$

Differentiating partially w.r.t 'x' we get,

$$2r\frac{\partial r}{\partial x}=2x \quad \text{or} \quad \frac{\partial r}{\partial x}=\frac{x}{r} \quad \text{———(2)}$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$

Also, we have, $u = r^m$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} = m r^{m-1}\cdot\frac{x}{r} = m r^{m-2}\cdot x \quad [\text{using } (2)]$$

and,

$$\frac{\partial^2 u}{\partial x^2} = m\left[x(m-2)r^{m-3}\cdot\frac{\partial r}{\partial x} + r^{m-2}\cdot 1\right]$$

$$= m r^{m-2}\left[(m-2)\frac{x^2}{r^2} + 1\right]$$

$$= \frac{m r^{m-2}}{r^2}\left[(m-2)x^2 + r^2\right]$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{m r^{m-2}}{r^2}\left[(m-2)\Sigma x^2 + \Sigma r^2\right]$$

$$= \frac{m r^{m-2}}{r^2}\left[(m-2)r^2 + 3r^2\right]$$

$$= m r^{m-2}\left[(m-2) + 3\right]$$

$$= m(m+1)r^{m-2}.$$

3) If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$

Show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Given, $u = x^2 - y^2$ and $v = 2xy$

Since, $x = r\cos\theta$ and $y = r\sin\theta$, we have,

$$u = r^2\cos^2\theta - r^2\sin^2\theta$$

$$= r^2(\cos^2\theta - \sin^2\theta)$$

$$= r^2\cos 2\theta \quad \text{——— (1)}$$

and, $v = 2(r\cos\theta)(r\sin\theta) = r^2\sin 2\theta$ ——— (2)

Differentiating (1) & (2) partially w.r.t 'r' and θ we have

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta \quad \text{and} \quad \frac{\partial v}{\partial r} = 2r\sin 2\theta$$

$$\frac{\partial v}{\partial \theta} = 2r^2 \cos 2\theta$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix}$$

$$= 4r^3 (\cos^2 2\theta + \sin^2 2\theta)$$

$$= 4r^3$$

Hence Proved.

(4) Show that the functions $u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2xz$ and $w = x^3+y^3+z^3-3xyz$ are functionally related.

Given,

$u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2zx$ and

$w = x^3+y^3+z^3-3xyz$.

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-y-x) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

Appyling $C_1 \to C_1 - C_2$
$C_2 \to C_2 - C_3$

$$6\begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & z-y-x \\ (x-y)(x+y+z) & (y-z)(x+y+z & z^2-xy \end{vmatrix}$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 12\begin{vmatrix} x-y & y-z \\ (x-y)(x+y+z) & (y-z)(x+y+z) \end{vmatrix}$$

$$= 12(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y+z & x+y+z \end{vmatrix}$$

$$= 12(x-y)(y-z)(0) \quad \left[\because C_1 \text{ and } C_2 \text{ are identical}\right]$$

$$= 0$$

Hence, the functional relationship exists between u, v, and w.

5) Find the minimum value of $x^2+y^2+z^2$, given $x+y+z=3a$

Given that, $x+y+z=3a$

Let $f(x,y,z) = x^2+y^2+z^2$

$\phi(x,y,z) = x+y+z-3a=0$

Consider the Lagrange's function.

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$F(x,y,z) = x^2+y^2+z^2+\lambda[x+y+z-3a]$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x+\lambda-1=0 \Rightarrow x = -\lambda/2$ —— (1)

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda - 1 = 0 \Rightarrow y = -\lambda/2 \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda - 1 = 0 \Rightarrow z = -\lambda/2 \quad \text{--- (3)}$$

Substitute (1), (2), (3) in $\phi(x, y, z)$

we have

$$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a$$

$$-\frac{\cancel{3}\lambda}{2} = \cancel{3}a$$

$$\boxed{\lambda = -2a}$$

$$x = -\frac{1}{2}(-2a) = a$$

$$y = -\frac{1}{2}(-2a) = a$$

$$z = -\frac{1}{2}(-2a) = a$$

$\therefore$ The minimum value of $f(x, y, z)$ is

$$a^2 + a^2 + a^2 = 3a^2$$

10 Marks Questions

1) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Show that, $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$ and find $\dfrac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$

From the given spherical polar co-ordinates we have

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \quad , \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \quad , \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta \, , \quad \frac{\partial z}{\partial \theta} = -r\sin\theta \quad , \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[(r\cos\theta\cos\phi)(r\sin\theta\cos\phi) + (r\cos\theta\sin\phi)(r\sin\theta\sin\phi)\right] + r\sin\theta\left[(\sin\theta\cos\phi)(r\sin\theta\cos\phi) + (\sin\theta\sin\phi)(r\sin\theta\sin\phi)\right]$$

(expanding by 3rd row)

$$= \cos\theta\left[r^2\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)\right] + r\sin\theta\left[r\sin^2\theta(\cos^2\phi + \sin^2\phi)\right]$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta)$$

$$= r^2\sin\theta$$

Hence proved,

Since, $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \cdot \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1$

we have, $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2\sin\theta}$

2. Find the maximum and minimum of $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Given that,

$$f(x) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$$

We have,

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0 \quad \text{—} (1)$$

and, $$\frac{\partial f}{\partial y} = 6xy - 6y = 0 \quad \text{—} (2)$$

Solving (1) & (2) we get

$$x = 0, 1, 2 \text{ and } y = 0, \pm 1$$

Now,

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6 \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = 6y \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 6x - 6$$

At (0,0).

$$ln - m^2 = (6x-6)^2 - 36y^2 = 36 > 0 \text{ and } l = 6x - 6 < 0$$

$\therefore f(0,0) = 4$ is the maximum value

At (2,0).

$$ln - m^2 = (6x-6)^2 - 36y^2 = 36 > 0 \text{ and } l = 6x - 6 > 0$$

$\therefore f(2,0) = 0$ is the minimum value.

At $(1, \pm 1)$.

$$ln - m^2 = (6x-6)^2 - 36y^2 = -36 < 0$$

$\therefore f(1, \pm 1)$ is not an extreme value.

3. A Rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Let x ft, y ft and z ft be the dimensions of the box and let S be the surface area of the box

Then, we have

$S = xy + 2yz + 2zx$ (since the top is open) —— (1)

Given that, its volume, $xyz = 32$ —— (2)

From (2) $z = \frac{32}{xy}$

Substituting value of z in (1) we get

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x = xy + \frac{64}{x} + \frac{64}{y}$$

Now, $\frac{\partial s}{\partial x} = y - \frac{64}{x^2} = 0$

$\frac{\partial s}{\partial y} = x - \frac{64}{y^2} = 0$

solving these we get

$x = 4,\ y = 4$

Also, $l = \frac{\partial^2 s}{\partial x^2} = \frac{128}{x^3}$, $m = \frac{\partial^2 s}{\partial x \partial y} = 1$, $n = \frac{\partial^2 s}{\partial y^2} = \frac{128}{y^3}$

At $x = 4$ & $y = 4$.

$$ln - m^2 = \frac{128}{x^3} \times \frac{128}{y^3} - 1$$

$$= \frac{128}{(4)^3} \times \frac{128}{(4)^3} - 1 = 2 \times 2 - 1 = 3 > 0$$

and $l = \frac{128}{x^3} = 2 > 0$

Thus S is minimum when $x = 4$, $y = 4$

From (2) we get $z = \frac{32}{xy} = \frac{32}{4 \times 4} = 2$

So, The dimension of the box for least material for its construction are 4ft, 4ft, 2ft.

—— x ——