UNIT – III
MULTIPLE INTEGRALS

1 Mark Question

1. Evaluate $\int_0^2 \int_0^y x \, dx \, dy$

2. Explain the double integral
3. Explain the triple integral
4. Find $\int_0^a \int_x^a f(x,y) \, dy \, dx$ by changing the order of integration
5. Evaluate $\int_0^2 \int_0^x y \, dy \, dx$

3 Marks Question

1. Explain the change of variables in double integrals.
2. Evaluate $\int_0^2 \int_0^x e^{x+y} \, dx \, dy$

3. Evaluate $\int_0^3 \int_1^2 xy(1+x+y) \, dy \, dx$

4. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy \, dx}{1+x^2+y^2}$

5. Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2) \, dx \, dy$

5 Marks Question

1. Evaluate triple integral $\int_0^1 \int_y^1 \int_0^{1-x} x \, dz \, dx \, dy$

2. Evaluate $\int_0^{\pi} \int_0^{a\sin\theta} r \, dr \, d\theta$

3. Find $\int_0^{\frac{\pi}{2}} \int_0^{a} r^2 \, dr \, d\theta$

4. Find the area between the circle $r = 2\sin\theta$ and $r = 4\sin\theta$

5. Evaluate the triple integral $\int_0^{1} \int_0^{1-x} \int_0^{1-x-y} e^{x} \, dx \, dy \, dz$.

UNIT-III

MULTIPLE INTEGRALS

1. Evaluate $\int_0^2 \int_0^y x \, dx \, dy$

The given integral is

$$\int_{y=0}^{2} \int_{x=0}^{y} x \, dx \, dy$$

First, we have to evaluate the inner integral keeping y as constant

$$\int_{y=0}^{2} \left[\frac{x^2}{2}\right]_0^y dy = \int_{y=0}^{2} \frac{y^2}{2} dy$$

$$= \left(\frac{y^3}{6}\right)_0^2 = \left(\frac{2^3}{6} - \frac{0}{6}\right) = \frac{8}{6} = \frac{4}{3}$$

2. Explain the double integral

Consider a region R in the xy plane bounded by one or more curves. Let $f(x,y)$ be a function defined at all points of R. Let the region R be divided into small subregions each of area δR_1, δR_2 ----- δR_n which are pairwaise non-Overlapping. Let (x_i, y_i) be an arbitary point within the subregion δR_i.

Consider the sum

$$f(x_1, y_1)\delta R_1 + f(x_2, y_2)\delta R_2 + \text{-----} + f(x_n, y_n)\delta R_n.$$

If this sum tends to a finite limit as $n \to \infty$ such that $\max(\delta R_i) \to 0$ irrespective of the choice of (x_i, y_i), the limit is called the double integral of $f(x,y)$ over the region R and is denoted by the symbol

$$\iint_R f(x,y)\, dR \quad (\text{OR}) \quad \iint_R f(x,y)\, dx\, dy$$

3. Explain the triple integral.

let $f(x,y,z)$ be a function defined over a 3-dimensional finite region V. Divide the region V into 'n' elementary volumes $\delta V_1, \delta V_2 \dots \delta V_n$. let (x_r, y_r, z_r) be any point in the r^{th} sub-division δV_r. Consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$$

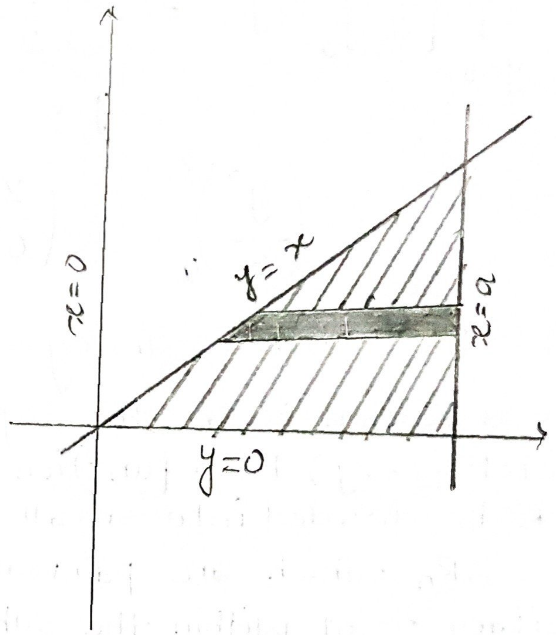
The limit of this sum, if it exists as $n \to \infty$ and $\delta V_r \to 0$ is called triple integral of $f(x,y,z)$ over the region V and is denoted by $\iiint f(x,y,z)\,dV$.

4. Find $\int_0^a \int_x^a f(x,y)\,dy\,dx$ by changing the order of integration.

let $I = \int_0^a \int_x^a f(x,y)\,dy\,dx$

Given $y = x$, $y = a$
$x = 0$, $x = a$

$$I = \int_{y=0}^{a} \int_{x=y}^{x=a} f(x,y)\,dx\,dy$$



5. Evaluate $\int_0^2 \int_0^x y\,dy\,dx$

Given, $\int_{x=0}^{2} \int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y\,dy\right] dx$

$$\therefore \Rightarrow \int_{x=0}^{2} \left[\frac{y^2}{2}\right]_0^x dx = \int_{x=0}^{2} \frac{x^2}{2}\,dx = \left[\frac{x^3}{6}\right]_{x=0}^{2}$$

$$= \frac{8}{6} = \frac{4}{3}$$

3 Marks Question

1. Explain the change of variable in double integrals.

By choice of an appropriate coordinate system, a given integral can be transformed into a simpler integral involving the new variable.

Transformation of coordinates:-

Let $x = f(u,v)$ and $g(u,v)$ be the relation between the old variable (x,y) with the new variables (u,v) of the new coordinate system.

Then $$\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dy \qquad \text{—— (1)}$$

Where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ which is called the Jacobian of the coordinate transformation.

Change of variables from Cartesian to Polar co-ordinates:-

In this case we have $u = r$ and $v = \theta$

$$x = r\cos\theta, \quad y = r\sin\theta$$

and $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Hence equation (1) becomes

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

This corresponds to $$\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$$

Evaluate:

2. $\int_0^2\int_0^x e^{x+y}\, dx\, dy$

Given integral $= \int_0^2 [e^x e^y]_0^x dx = \int_0^2 e^x \left\{\int_0^x e^y dy\right\} dx$

$= \int_0^2 e^x \{(e^y)_0^x\} dx = \int_0^2 e^x(e^x - 1)\, dx$

$= \int_0^2 (e^{2x} - e^x) dx$

$= \left(\frac{e^{2x}}{2} - e^x\right)_0^2 = \left(\frac{e^4}{2} - e^2\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{e^4}{2} - e^2 + \frac{1}{2}$

$= \frac{1}{2}(e^2 - 1)^2$

3. Evaluate:

$\int_0^3\int_1^2 xy(1+x+y)\, dy\, dx$

Given integral $= \int_0^3\int_1^2 (xy + x^2y + xy^2)\, dy\, dx$

$= \int_0^3 \left\{\int_1^2 (xy + x^2y + xy^2) dy\right\} dx$

$= \int_0^3 \left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_1^2 dx$

$$\int_{x=0}^{3}\left(x\cdot\frac{2^2}{2}+x^2\cdot\frac{2^2}{2}+x\cdot\frac{2^3}{3}-x\cdot\frac{1^2}{2}-x\cdot\frac{1^2}{2}-x\cdot\frac{1^3}{3}\right)dx$$

$$\Rightarrow \int_{x=0}^{3}\left(2x+2x^2+\frac{8x}{3}-\frac{x}{2}-\frac{x^2}{2}-\frac{x}{3}\right)dx$$

$$\Rightarrow \left[\cancel{2}\cdot\frac{x^2}{\cancel{2}}+\frac{2x^3}{3}+\frac{8}{3}\cdot\frac{x^2}{2}-\frac{x^2}{4}-\frac{x^3}{6}-\frac{x^2}{6}\right]_{x=0}^{3}$$

$$\Rightarrow \left[x^2+\frac{2x^3}{3}+\frac{4x^2}{3}-\frac{x^2}{4}-\frac{x^3}{6}-\frac{x^2}{6}\right]_0^3$$

$$\Rightarrow 3^2+\frac{2(3)^3}{3}+\frac{4}{3}(3)^2-\frac{3^2}{4}-\frac{3^3}{6}-\frac{3^2}{6}$$

$$\Rightarrow 9+18+12-\frac{9}{4}-\frac{9}{2}-\frac{3}{2}$$

$$\Rightarrow \frac{36+72+48-9-18-6}{4}$$

$$\Rightarrow \frac{123}{4}$$

4. Evaluate:

$$\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dy\,dx}{1+x^2+y^2}$$

The given integral $= \displaystyle\int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}}\frac{dy}{(1+x^2)+y^2}\right]dx$

$$\Rightarrow \int_{x=0}^{1}\left[\int_{y=0}^{p} \frac{dy}{p^2+y^2}\right] dx \qquad \text{where } p = \sqrt{1+x^2}$$

$$\Rightarrow \int_{x=0}^{1}\left[\frac{1}{p}\tan^{-1}\left(\frac{y}{p}\right)\right]_{y=0}^{p} dx \Rightarrow \int_{x=0}^{1} \frac{1}{p}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$\Rightarrow \int_{x=0}^{1} \frac{\pi}{4}\cdot\frac{1}{\sqrt{1+x^2}}\, dx \qquad \left(\because p = \sqrt{1+x^2}\right)$$

$$\Rightarrow \frac{\pi}{4}\int_{x=0}^{1} \frac{dx}{\sqrt{1+x^2}} \Rightarrow \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_{x=0}^{1}$$

$$\Rightarrow \frac{\pi}{4}\left[\log\left(1+\sqrt{1^2+1}\right) - \log\left(0+\sqrt{0^2+1}\right)\right] = \frac{\pi}{4}\log\left(1+\sqrt{2}\right) - \log 1$$

$$\Rightarrow \frac{\pi}{4}\log\left(1+\sqrt{2}\right) \qquad (\because \log 1 = 0)$$

5. Evaluate:

$$\int_{0}^{5}\int_{0}^{x^2} x(x^2+y^2)\, dx\, dy$$

$$\int_{0}^{5}\int_{0}^{x^2} x(x^2+y^2)\, dx\, dy = \int_{x=0}^{5}\left[\int_{y=0}^{x^2} (x^3+xy^2)\, dy\right] dx$$

$$\Rightarrow \int_{0}^{5} \left(x^3 y + \frac{xy^3}{3}\right)_{0}^{x^2} dx$$

$$\Rightarrow \int_{0}^{5} \left(x^5 + \frac{x^7}{3}\right) dx$$

$$\Rightarrow \left(\frac{x^6}{6} + \frac{x^8}{24}\right)_0^5$$

$$\Rightarrow \frac{5^6}{6} + \frac{5^8}{24}$$

$$\Rightarrow 5^6\left(\frac{1}{6} + \frac{25}{24}\right) = \frac{29\,(5^6)}{24}$$

5 Marks Questions

1. Evaluate the triple integral

$$\int_0^1 \int_y^1 \int_0^{1-x} x \, dz \, dx \, dy$$

$$\int_{y=0}^{1} \int_{x=y}^{1} \int_{z=0}^{1-x} x \, dz \, dx \, dy$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} x[z]_0^{1-x} \, dx \, dy$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} x\,[1-x] \, dx \, dy$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} (x - x^2) \, dx \, dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_y^1 dy$$

$$\Rightarrow \int_{y=0}^{1} \left(\frac{1^2}{2} - \frac{1^3}{3} - \frac{y^2}{2} + \frac{y^8}{3} \right) dy$$

$$\Rightarrow \int_{y=0}^{1} \left(\frac{1}{2} - \frac{1}{3} - \frac{y^2}{2} + \frac{y^3}{3} \right) dy$$

$$\Rightarrow \left[\frac{1}{2}y - \frac{1}{3}y - \frac{1}{2} \cdot \frac{y^3}{3} + \frac{1}{3} \cdot \frac{y^4}{4} \right]_0^1$$

$$\Rightarrow \frac{1}{2} - \frac{1}{3} - \frac{1}{6} + \frac{1}{12} - [0 - 0 - 0 + 0]$$

$$\Rightarrow \frac{6 - 4 - 2 + 1}{12} = \frac{1}{12}$$

2. $$\int_0^{\pi} \int_0^{a\sin\theta} r \, dr \, d\theta$$

$$\int_{\theta=0}^{\pi} \int_{r=0}^{r=a\sin\theta} r \, dr \, d\theta \Rightarrow \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a\sin\theta} d\theta = \frac{1}{2} \int_{\theta=0}^{\pi} a^2 \sin^2\theta \, d\theta$$

$$\Rightarrow \frac{a^2}{4} \int 2\sin^2\theta \, d\theta \Rightarrow \frac{a^2}{4} \int_{\theta=0}^{\pi} [1 - \cos 2\theta] \, d\theta$$

$$\Rightarrow \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_{\theta=0}^{\pi}$$

$$\Rightarrow \frac{a^2}{4} \left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 0}{2} \right]$$

$$= \frac{\pi a^2}{4}$$

3. Find $\int_0^{\pi/2}\int_0^a r^2\,dr\,d\theta$

$$\Rightarrow \int_{\theta=0}^{\frac{\pi}{2}}\int_0^a r^2\,dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{\frac{\pi}{2}}\left(\frac{r^3}{3}\right)_{r=0}^{a} d\theta$$

$$\Rightarrow \frac{1}{3}\int_{\theta=0}^{\pi/2} a^3\,d\theta$$

$$= \frac{a^3}{3}\left[\theta\right]_{\theta=0}^{\pi/2}$$

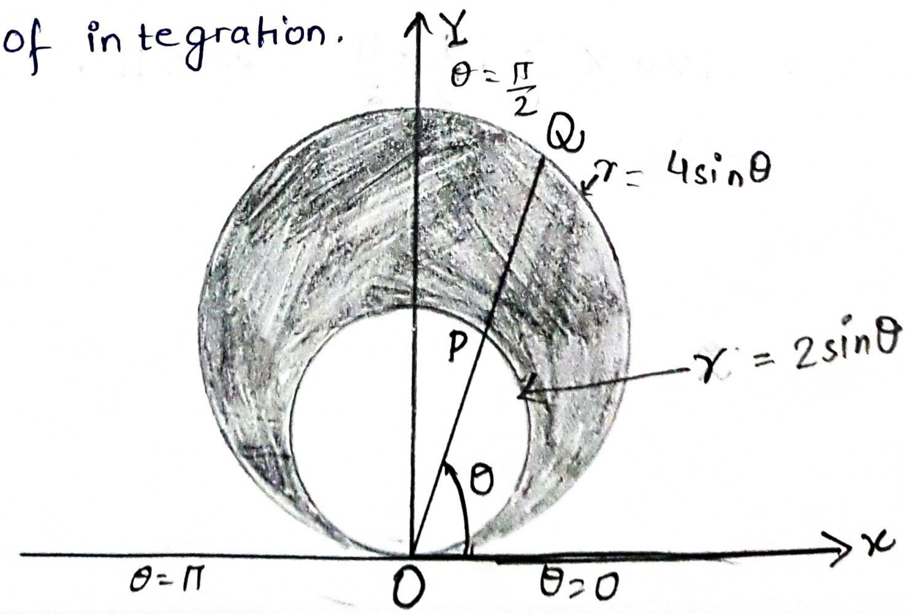
$$= \frac{a^3}{3}\left[\frac{\pi}{2}-0\right] = \frac{\pi a^3}{6}$$

4. Find the area between the circle $r = 2\sin\theta$ and $r = 4\sin\theta$.

Given circles $r = 2\sin\theta$

$r = 4\sin\theta$

The shaded area between these circles is the region of integration.



If we integrate first w.r.t 'r', then its limits are from P ($r = 2\sin\theta$) to Q ($r = 4\sin\theta$) and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3\, dr$$

$$= \int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} \Rightarrow \int_0^{\pi} d\theta \left[\frac{4^4\sin^4\theta - 2^4\sin^4\theta}{4}\right]$$

$$\Rightarrow \int_0^{\pi} \sin^4\theta \left[\frac{4^4 - 2^4}{4}\right] \times d\theta$$

$$= \int_0^{\pi} \sin^4\theta \left[\frac{256-16}{4}\right] d\theta$$

$$= \int_0^{\pi} \frac{240}{4} \sin^4\theta\, d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta\, d\theta \Rightarrow 60 \times 2 \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 120 \times \frac{3}{4} \cdot \frac{1}{2} \times \frac{\pi}{2} = 22.5\pi .$$

5. Evaluate the triple integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x \, dx \, dy \, dz$$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{1-x-y} e^x \, dz \, dy \, dx \Rightarrow \int_{x=0}^{x=1} \int_{y=0}^{1-x} e^x \left[z\right]_0^{1-x-y} dy \, dx$$

$$\int_{x=0}^{1} \int_{y=0}^{1-x} e^x \left[1-x-y-0\right] dy \, dx$$

$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{1-x} e^x (1-x-y) \, dy \, dx$$

$$\int_{x=0}^{1} e^x \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$\int_{x=0}^{1} e^x \left[1-x - x(1-x) - \frac{(1-x)^2}{2}\right] dx$$

$$\int_{x=0}^{1} e^x (1-x) \left[1 - x - \frac{(1-x)}{2}\right] dx$$

$$\int_{x=0}^{1} e^x (1-x) \frac{(1-x)}{2} \, dx$$

$$\frac{1}{2}\int_{x=0}^{1} e^x (1-x)^2 dx$$

$$\frac{1}{2}\left[(1+x^2-2x)e^x - (2x-2)e^x + 2e^x\right]_{x=0}^{1}$$

$$\Rightarrow \frac{1}{2}\left[(1+1-2)e^1 - (2-2)e^1 - 2e^1 - \left[(1+0-0)e^0 - (2(0)-2)e^0 + 2e^0\right]\right]$$

$$\Rightarrow \frac{1}{2}\left[2e - (1+2+2)\right]$$

$$\Rightarrow \frac{1}{2}(2e-5)$$