A Primer to Parameter Estimation

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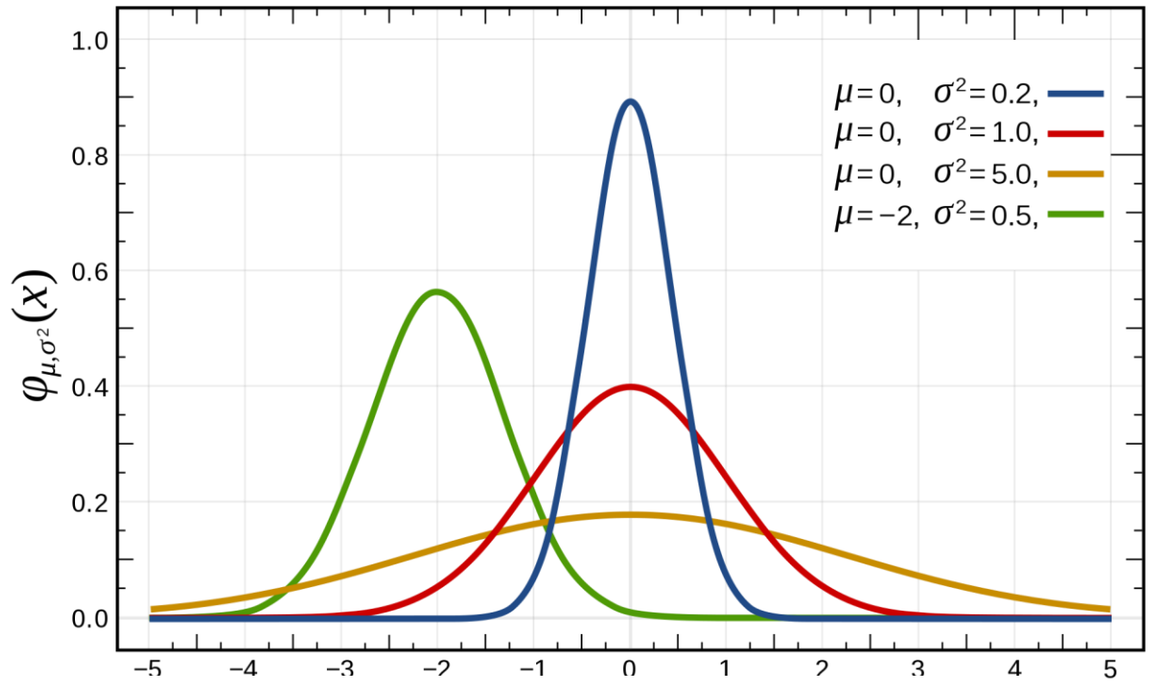
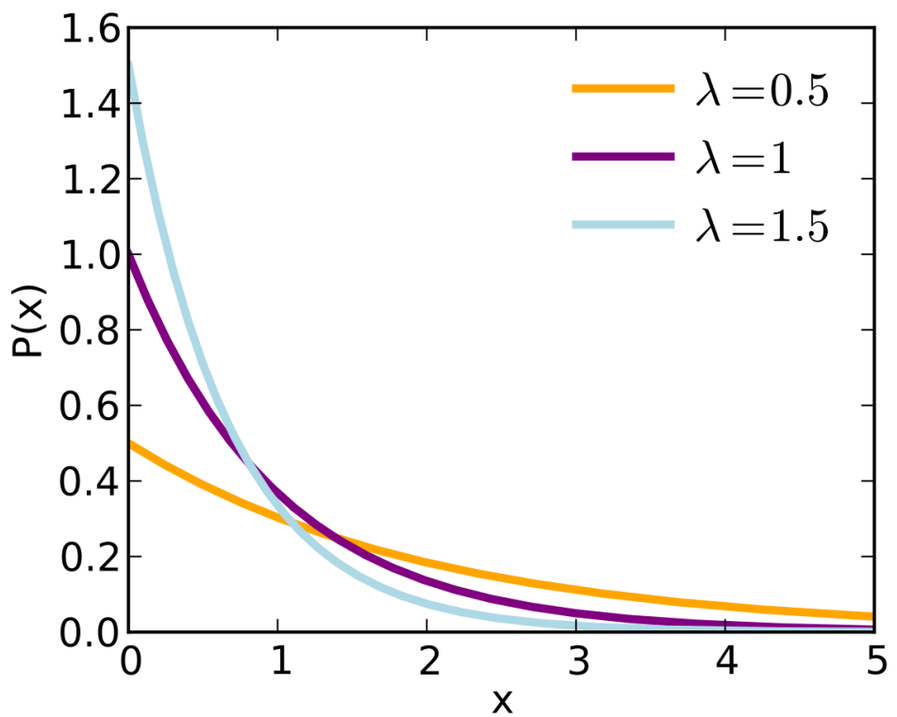
Sampling and Estimation

- What you have learnt : X is a random variable following a distribution
  - what does this mean ?
- Given the parameter value of a distribution how the density/distribution function look.
- Sampling : If X follows a distribution how can we obtain different value X will take
- Parameter estimation : Given different values X take, how can we obtain the parameters of the underlying distribution

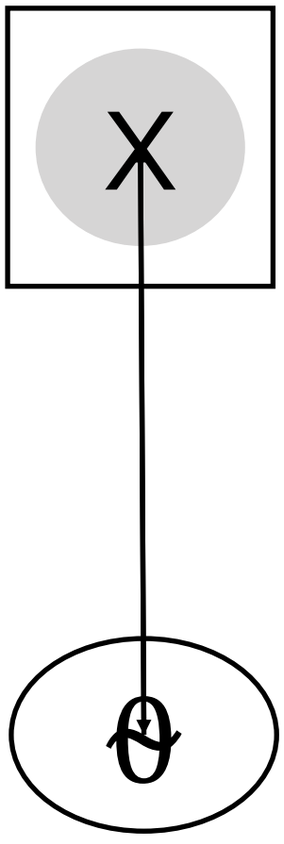


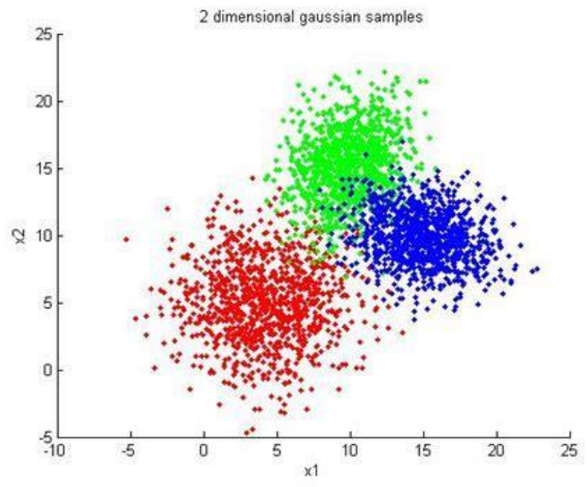


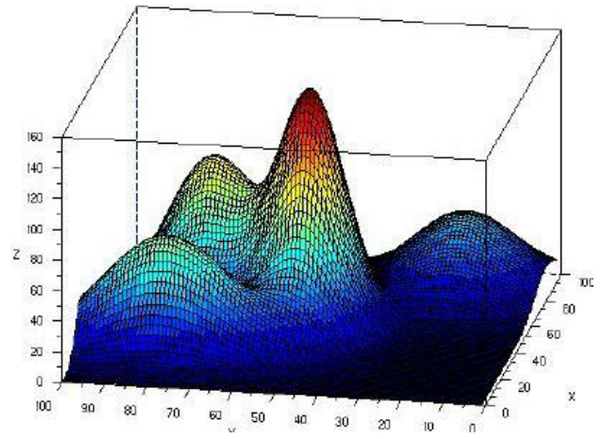
Sampling and Estimation

- Given data Statistics/ML aim to learn parameters of the underlying distribution from data
  - Gaussian mixture model, probabilistic graphical models, linear regression, logistic regression etc.
- Sampling is essential in probabilistic machine learning and Bayesian statistics
  - Latent dirichlet allocation, Gaussian process, probabilistic graphical model
  - Also used in computational physics, biology and many engineering disciplines.

    MCMC was placed in the top 10 most important algorithms of the 20th century





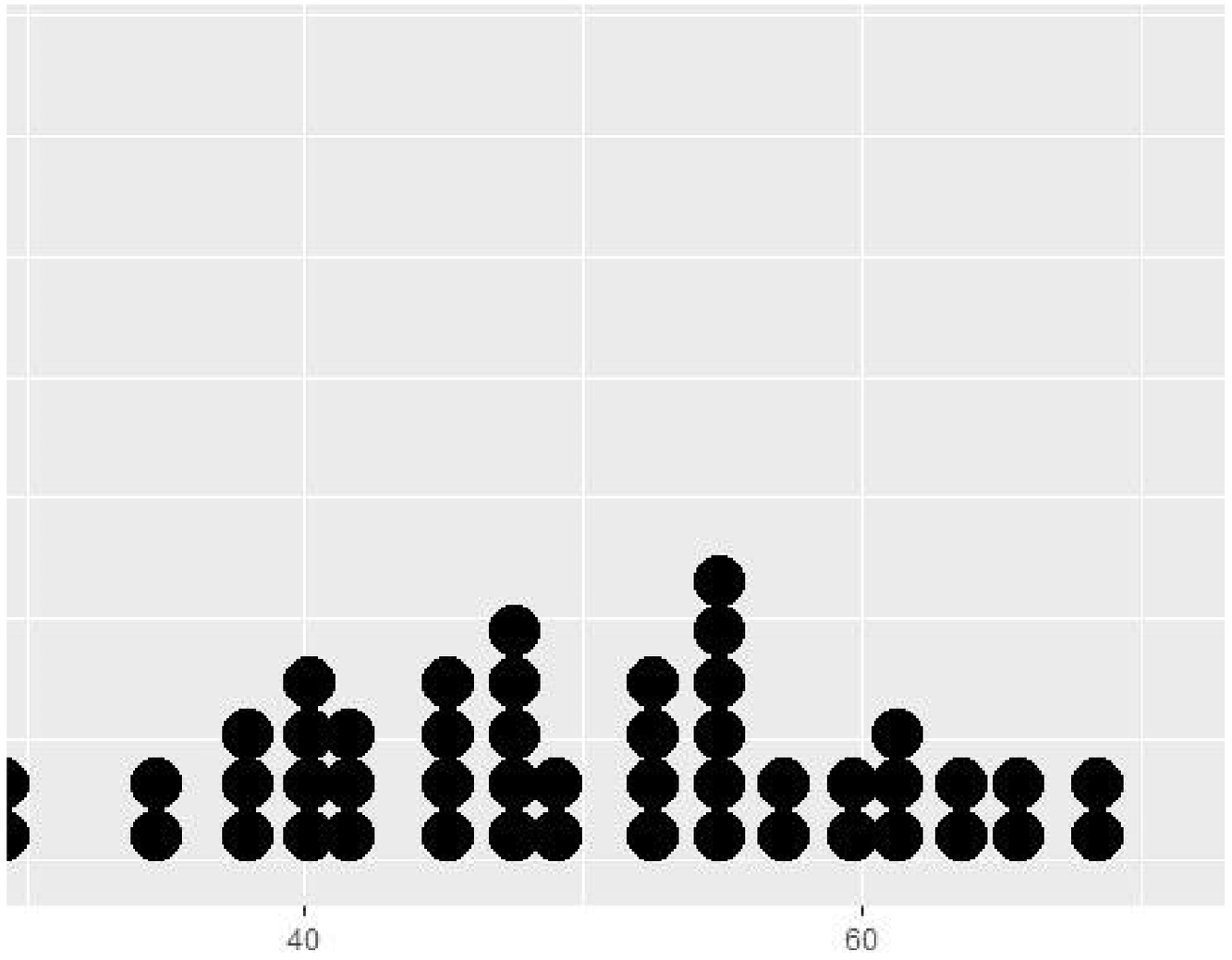


Parameter Estimation

- Data points representing the weight (in kgs) of students in a class.

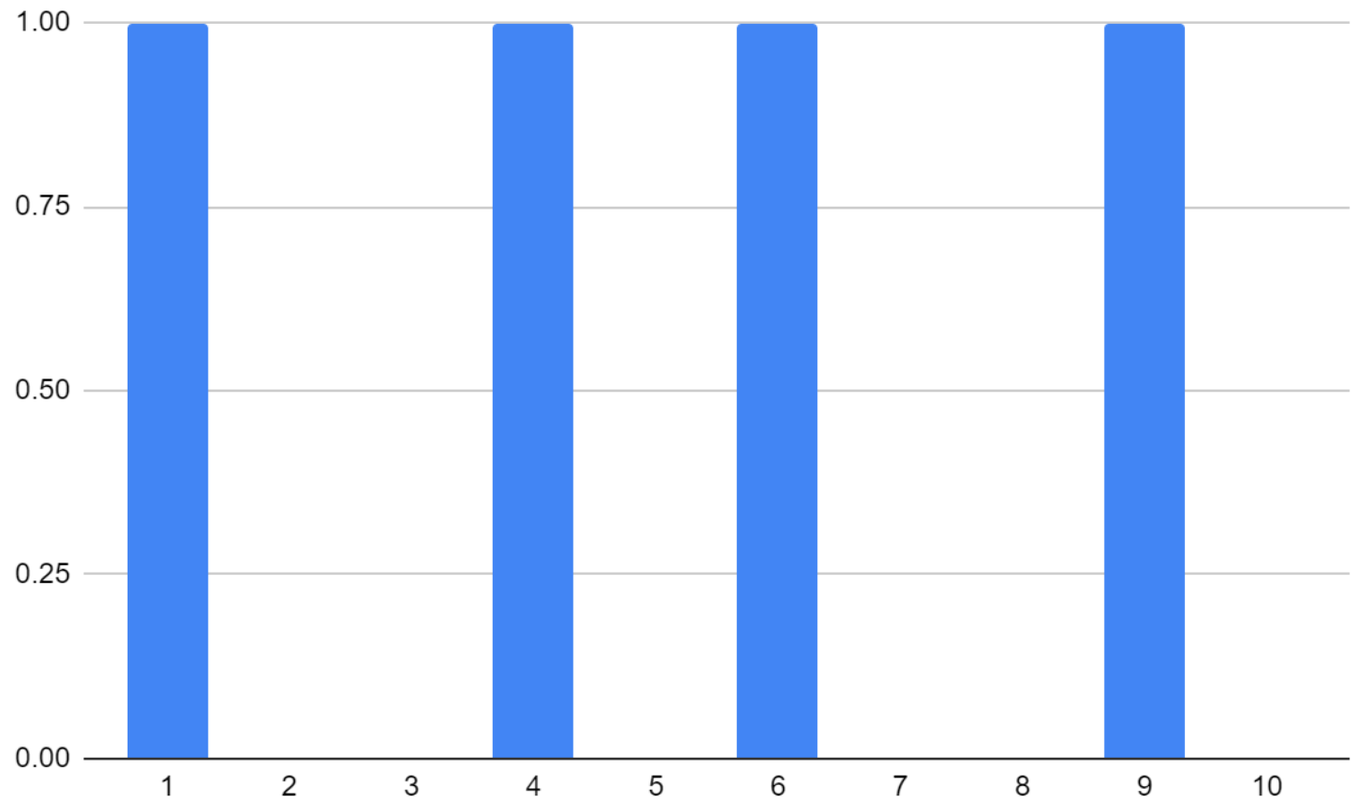
- Whats mean and std deviation of the data ?

- Whats the probability that weight > 60



Parameter Estimation

- Data points representing the if it has rained or not in last 10 days.
- Whats the probability that it will rain tomorrow ?
- How many days will it rain in next 5 days ?



Parameter Estimation

- Any statistic used to estimate the value of an unknown parameter θ is called an estimator of θ.
  - mean and variance for Normal, rate (lambda) for Poisson, etc.

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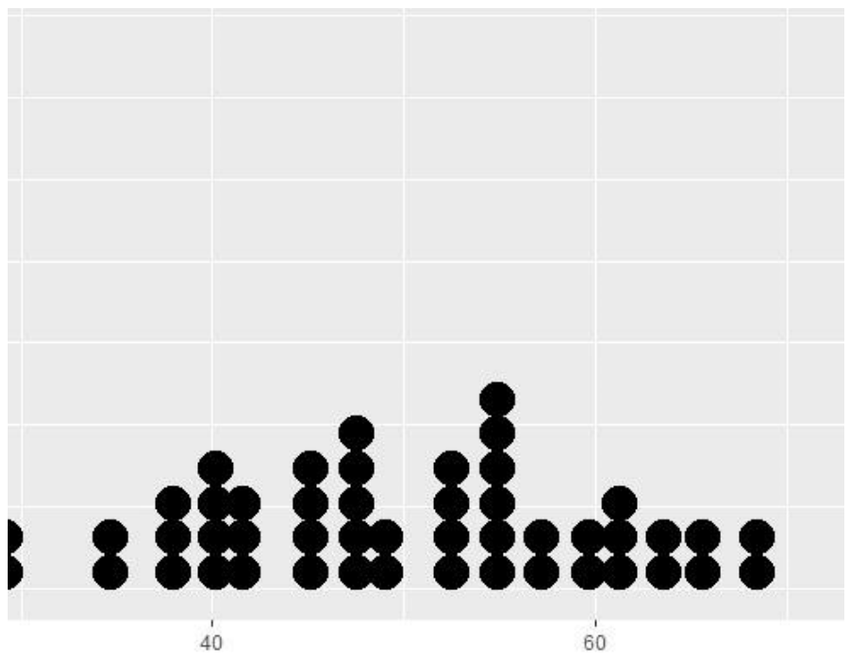
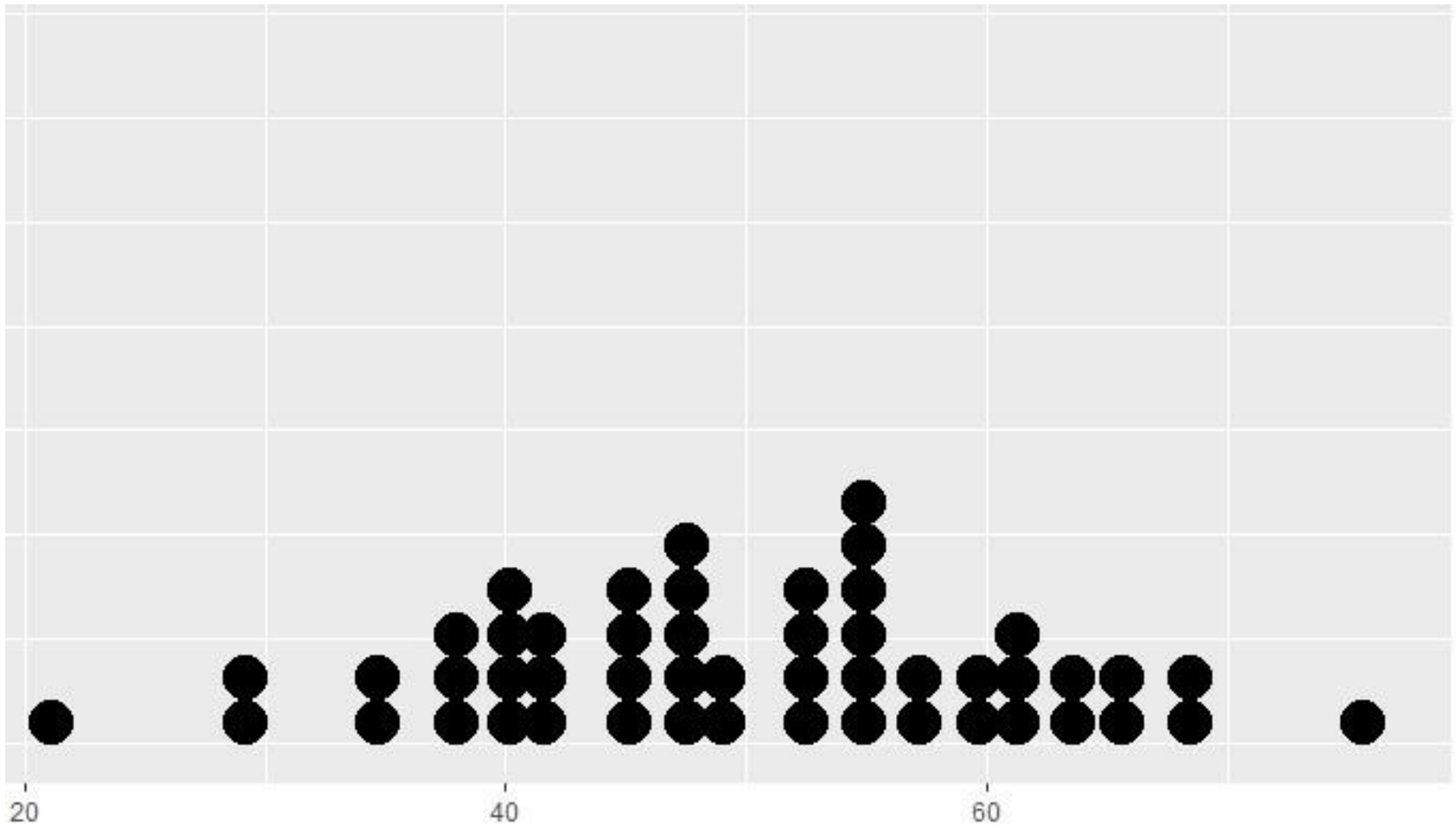
- **Maximum likelihood** estimator
- MLE can be defined as a method for estimating parameters of a distribution from sample data such that the likelihood of obtaining the observed data is maximized.
- Provides optimal way to fit a distribution to the data



Parameter estimation

- which of the following would maximize the probability of observing the data
  - Mean = 100, SD = 10
  - Mean = 50, SD = 10



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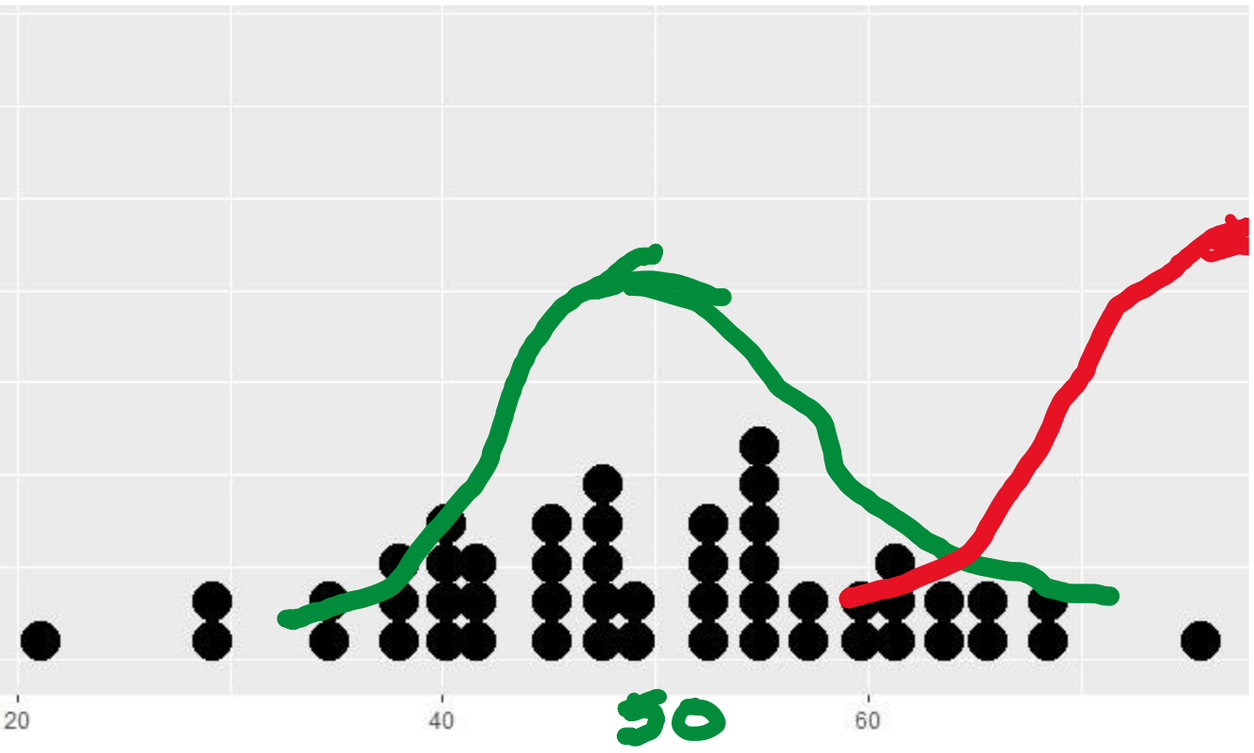

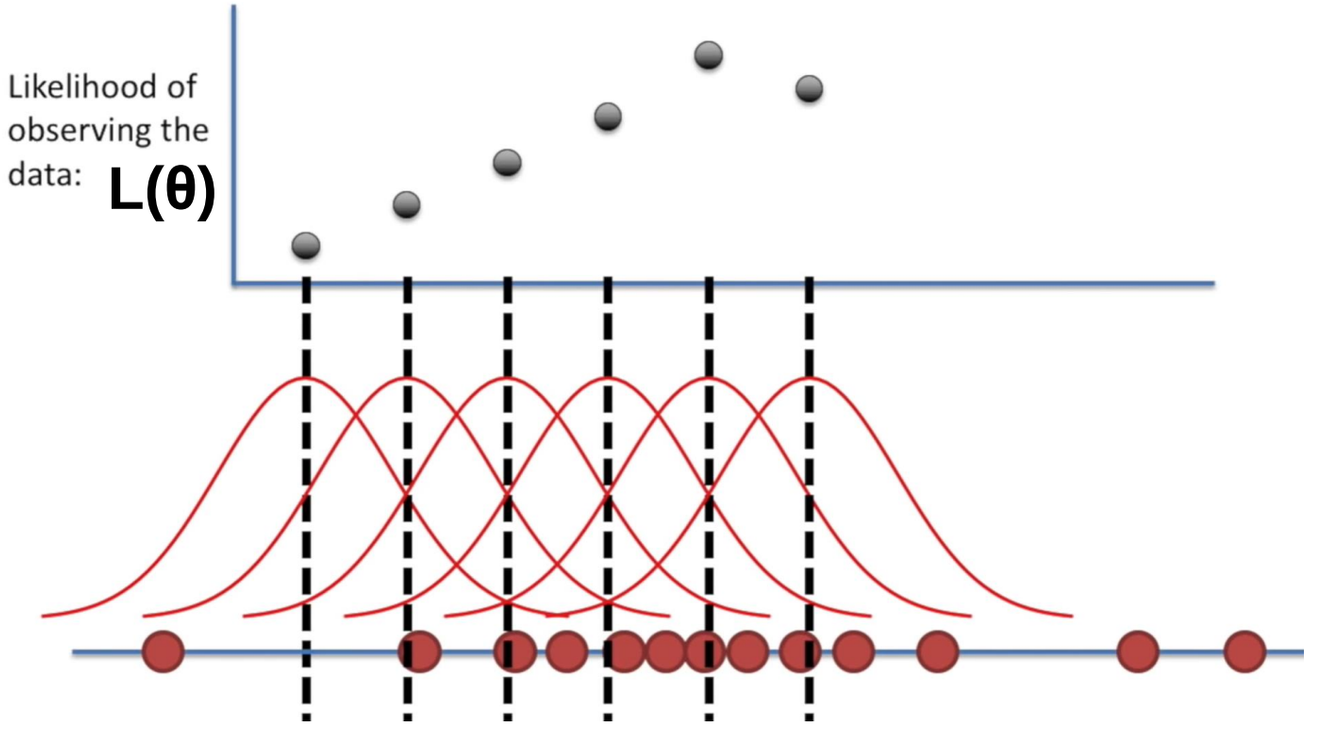

Parameter Estimation

Maximum likelihood estimator

- f(x1, . . . , xn|θ) represents the probability that the values x1, x2, . . . , xn will be observed when θ is the true value of the parameter

- Maximum Likelihood estimation : maximum likelihood estimate θ is defined to be that value of θ maximizing L(θ) = f (x1, . . . , xn|θ)

argmax_θ L(θ) = $f(x_1, \ldots, x_n|\theta)$ = argmax_θ $\log[f(x_1, \ldots, x_n|\theta)]$.

Note that L(θ) is not a distribution over θ but just a function of θ.

Independent and identically distributed (i.i.d.) assumption

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

Maximum likelihood estimation

- (Maximum Likelihood Estimator of a Bernoulli Parameter) Suppose you have data from n independent Bernoulli trials, *X*1, *. . .* , *Xn*. Assuming the success probability is p what is the maximum likelihood estimator of p?

$$X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{otherwise} \end{cases}$$

$$P\{X_i = 1\} = p = 1 - P\{X_i = 0\}$$

$$P\{X_i = x\} = p^x(1-p)^{1-x}, \quad x = 0, 1$$

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$$\begin{aligned} f(x_1, \ldots, x_n | p) &= P\{X_1 = x_1, \ldots, X_n = x_n | p\} \\ &= p^{x_1}(1-p)^{1-x_1} \cdots p^{x_n}(1-p)^{1-x_n} \\ &= p^{\Sigma_1^n x_i}(1-p)^{n-\Sigma_1^n x_i}, \quad x_i = 0, 1, \quad i = 1, \ldots, n \end{aligned}$$

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To determine the value of p that maximizes the likelihood,

$$\log f(x_1, \ldots, x_n|p) = \sum_1^n x_i \log p + \left(n - \sum_1^n x_i\right) \log(1-p)$$

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To determine the value of p that maximizes the likelihood,

proportion of the observed trials that result in successes.

$$\frac{d}{dp} \log f(x_1, \ldots, x_n | p) = \frac{\sum_1^n x_i}{p} - \frac{\left(n - \sum_1^n x_i\right)}{1 - p} \qquad \hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

Suppose that each RAM (random access memory) chip produced by a certain manufacturer is, independently, of acceptable quality with probability *p*. Then if out of a sample of 1,000 tested 921 are acceptable, what is the maximum likelihood estimate of *p* ?

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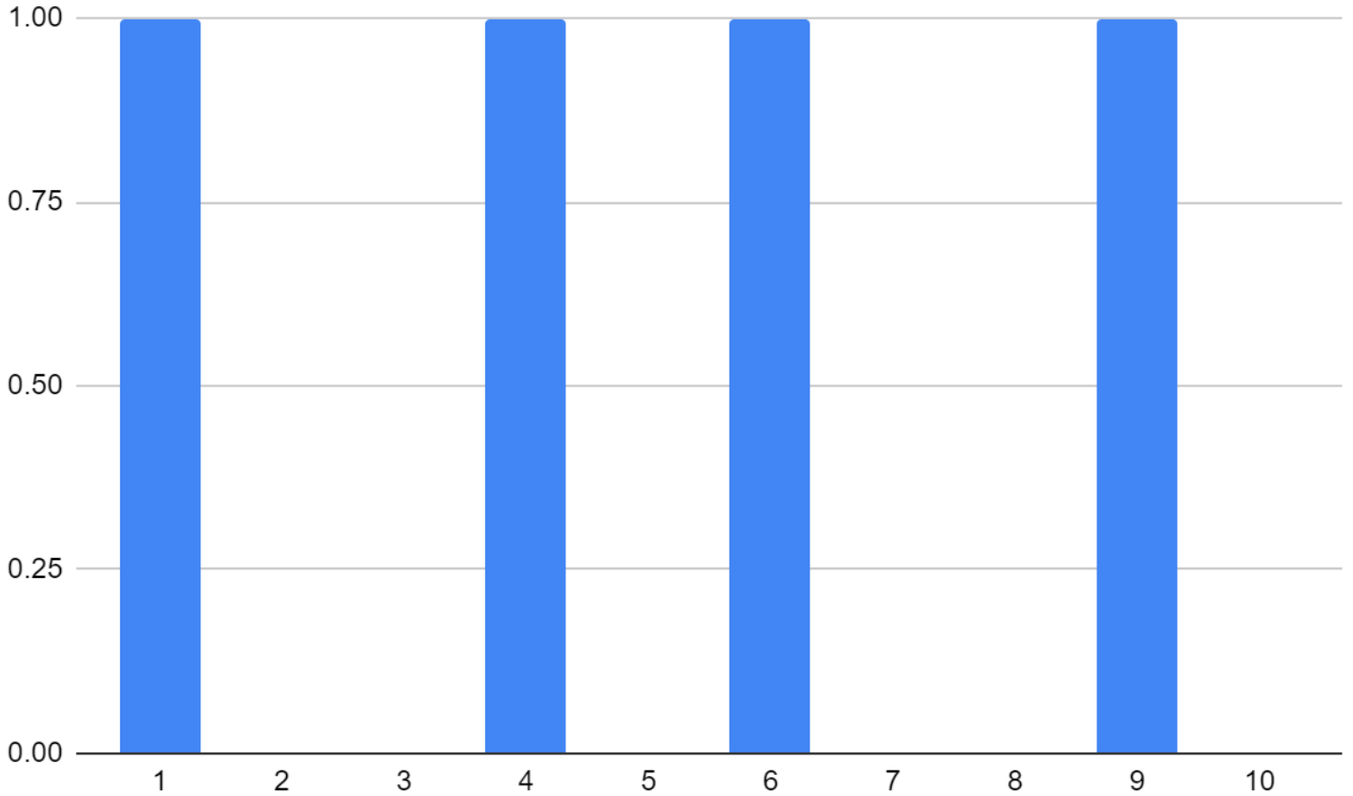


Maximum likelihood estimation

- (Maximum Likelihood Estimator of a Poisson Parameter) Suppose X1, . . . , Xn are independent Poisson random variables each having mean λ. Determine the maximum likelihood estimator of λ.

Maximum likelihood estimation

- (Maximum Likelihood Estimator in a Normal Population) Suppose X1, . . . , Xn are independent, normal random variables each with unknown mean μ and unknown standard deviation σ.

Bayes' Theorem

$$P(model|data, I) = P(model, I) \frac{P(data|model, I)}{P(data,I)}$$

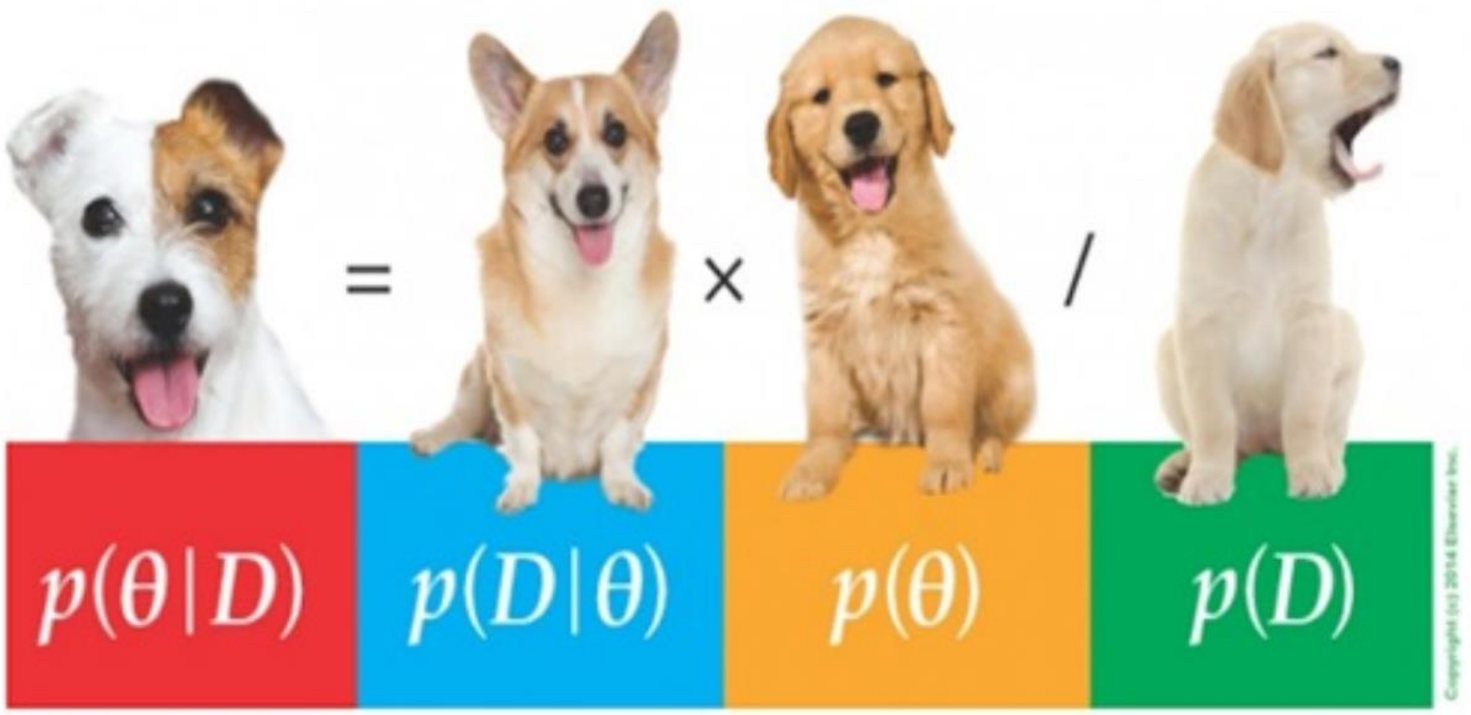
Likelihood

describes how well the model predicts the data

Posterior Probability

Prior Probability

Normalizing constant



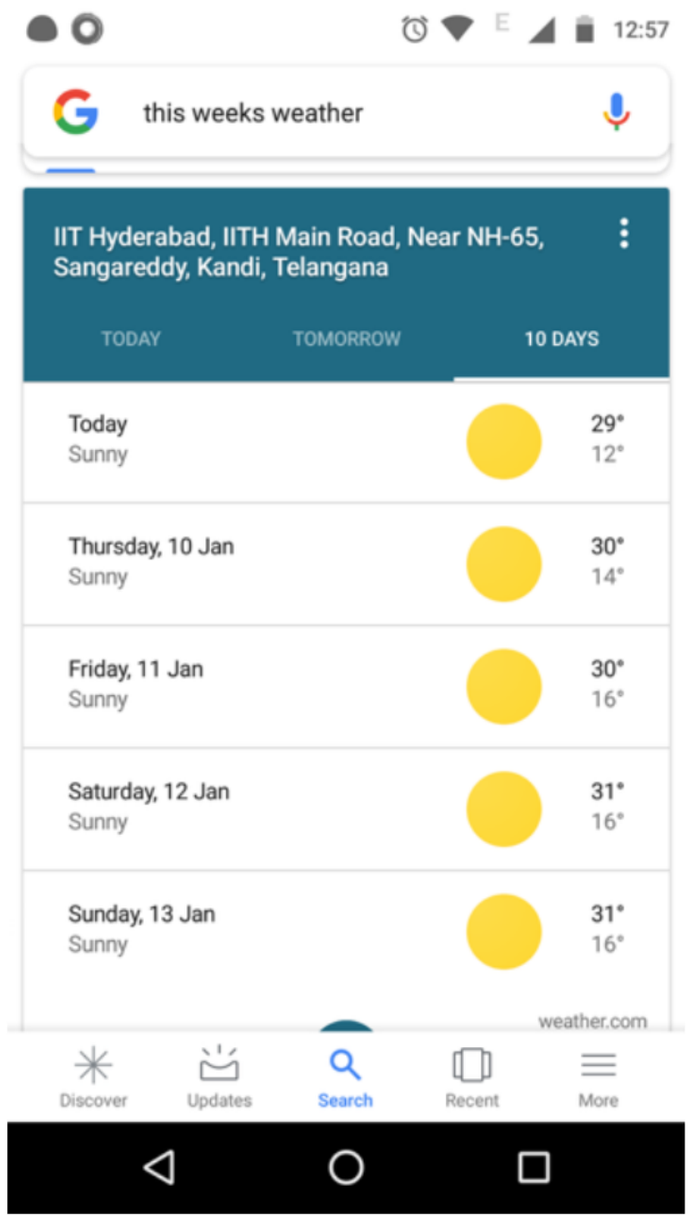
Rain prediction : ML estimation

X = [1,0,0, 0, 1, 0,….]

Model ?
Likelihood : p(X I model)
Learn model parameters : maximum likelihood (ML) estimation

$$\widehat{\Theta}_{ML} = \underset{\Theta}{\operatorname{argmax}}\ \mathcal{L}$$

$$\mathcal{L} = \sum_{\mathbf{x}_i \in \mathcal{X}} log\ prob(\mathbf{x}_i|\Theta)$$



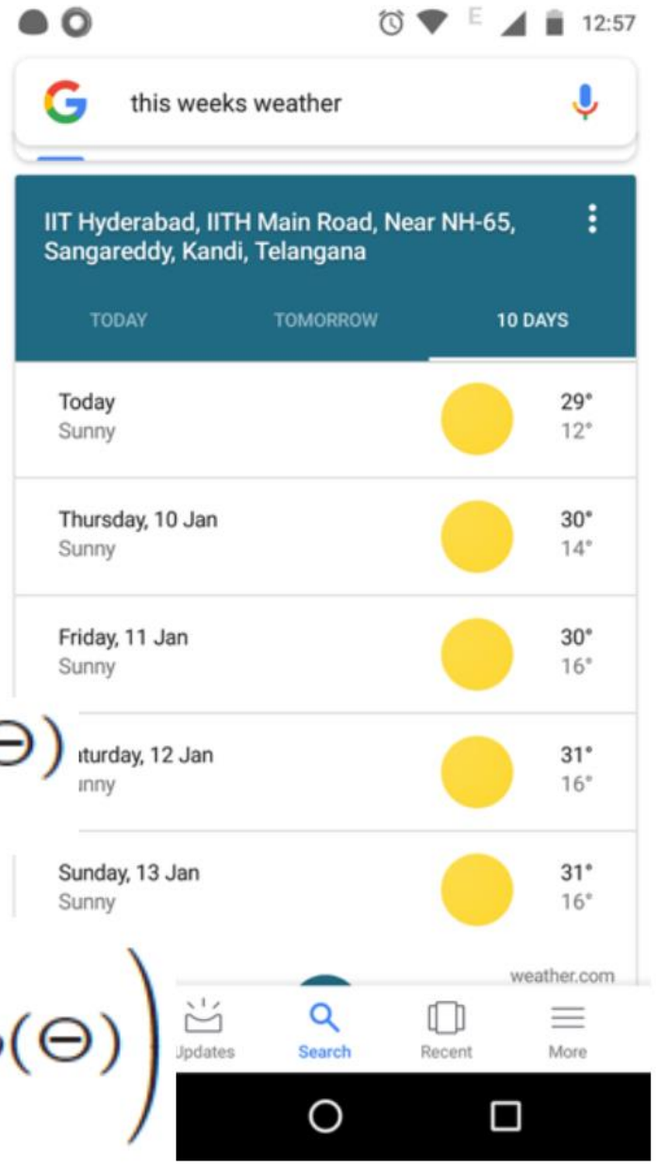
Rain prediction : MAP estimation

Maximum A-Posteriori (MAP) estimation

$$prob(\Theta|\mathcal{X}) = \frac{prob(\mathcal{X}|\Theta) \cdot prob(\Theta)}{prob(\mathcal{X})}$$

$$\widehat{\Theta}_{MAP} = \underset{\Theta}{\text{argmax}} \prod_{\mathbf{x}_i \in \mathcal{X}} prob(\mathbf{x}_i|\Theta) \cdot prob(\Theta)$$

$$\widehat{\Theta}_{MAP} = \underset{\Theta}{\text{argmax}} \left(\sum_{\mathbf{x}_i \in \mathcal{X}} log\ prob(\mathbf{x}_i|\Theta) + log\ prob(\Theta) \right)$$



Seek that value for θ which maximizes the posterior prob(θ|X).

What Does the MAP Estimate Get Us That the ML Estimate Does NOT

- MAP estimate allows us to inject into the estimation calculation our prior beliefs regarding the parameters values in θ

- MAP estimation “pulls” the estimate toward the prior. The more focused our prior belief, the larger the pull toward the prior.

- “smoothing” role (Laplace smoothing) for parameter estimation.

MAP Estimation Example [Election]

- Consider a survey where voters were asked if they will vote for NDA or UPA in the next election. Let p be the probability that an individual will vote NDA.

- xi is either NDA or UPA, and p is the probability some body votes for NDA

- nd is the number of individuals who are planning to vote NDA

$$\hat{p}_{ML} = \frac{n_d}{N}$$

N= 20 and if 12 out of 20 said that they were going to vote NDA, we get the following the ML estimate for p: ˆpML = 0.6.

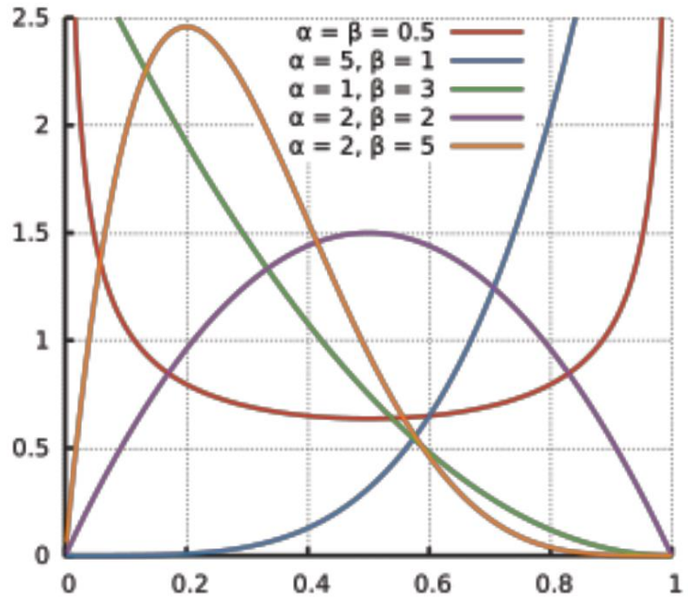
MAP Estimate : Prior Belief on p

- The prior for p must be zero outside the [0, 1] interval.
- In most cases, we would want to choose a distribution for the prior beliefs that peaks somewhere in the [0, 1] interval.
- beta distribution that is parameterized by two “shape” constants α and β does the job nicely for expressing our prior beliefs concerning p:

$$prob(p|\alpha,\beta) = \frac{1}{B(\alpha,\beta)}p^{\alpha-1}(1-p)^{\beta-1}$$

$$B = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\text{mode} \quad \frac{\alpha-1}{\alpha+\beta-2}$$



Example [Indian Election]

- Due to some emerging situations, lets assume that voters are equally likely to vote for UPA and NDA.
- Consider a prior distribution for p that has a peak at 0.5. Setting α = β=5 gives us a distribution for p that has a peak in the middle of the [0, 1] interval.

$$\hat{p}_{MAP} = \underset{p}{\text{argmax}} \left(\sum_{x \in \mathcal{X}} log\ prob(x|p) \ + \ log\ prob(p) \right)$$

MAP Estimation [Indian Election]

$$\hat{p}_{MAP} = \underset{p}{\text{argmax}} \Big(n_d \cdot log\ p + (N - n_d) \cdot log\ (1 - p) + log\ prob(p) \Big)$$

$$\frac{n_d}{p} - \frac{(N - n_d)}{(1 - p)} + \frac{\alpha - 1}{p} - \frac{\beta - 1}{1 - p} = 0$$

$$\hat{p}_{MAP} = \frac{n_d + \alpha - 1}{N + \alpha + \beta - 2}$$

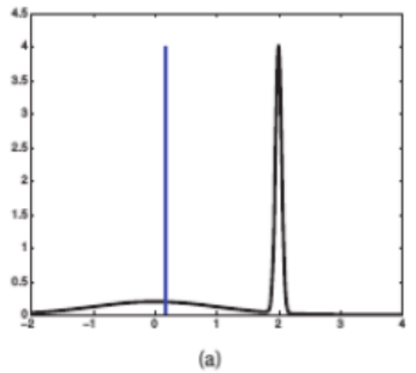
$$= \frac{n_d + 4}{N + 8}$$

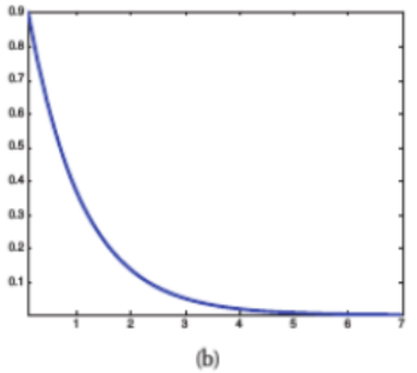
With N = 20 and with 12 of the 20 previously saying they would vote NDA, the MAP estimate forp is 0.571 with α and β both set to 5.

Rain prediction : Bayesian estimation

Both ML and MAP return only single and specific values for the parameter Θ.

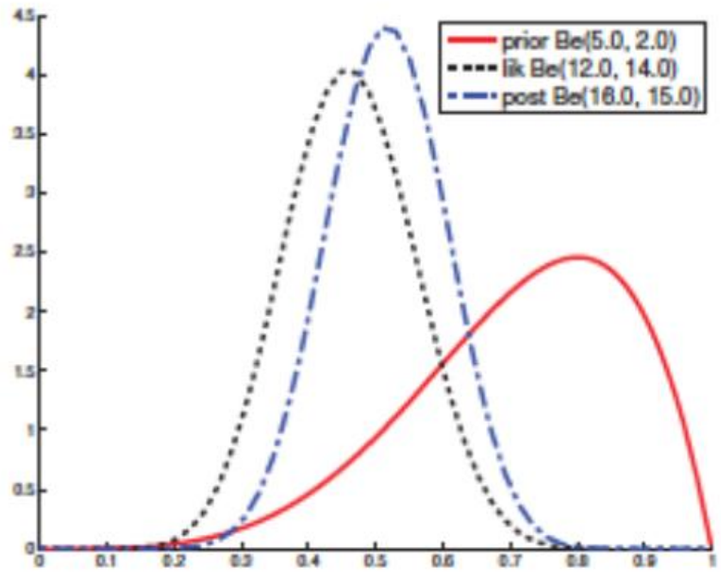
$$prob(\Theta|\mathcal{X}) = \frac{prob(\mathcal{X}|\Theta) \cdot prob(\Theta)}{prob(\mathcal{X})}$$





$$prob(\mathcal{X}) = \int_{\Theta} prob(\mathcal{X}|\Theta) \cdot prob(\Theta)\ d\Theta$$

$$prob(\tilde{\mathbf{x}}|\mathcal{X}) = \int_{\Theta} prob(\tilde{\mathbf{x}}|\Theta) \cdot prob(\Theta|\mathcal{X})\ d\Theta$$



Posterior is a “compromise” between the prior and likelihood.
posterior mean is convex combination of the prior mean and the MLE
: $\lambda m_1 + (1-\lambda)\hat{\theta}_{MLE}$


ధన్యవాదాలు धन्यवाद
ధన్యవాదగళు ನಂತಿ धन्यवाद ಧನ್ಯವಾದ ధన్యవాదాలు આભાર நன்றி धन्यवाद
ನಂತಿ ધન્યવાદ ધન્યવાદગళು ನಂತಿ ధన్యవాదాలు நன்றி
न्यवाद ధన్యవాదాలు நன்றி આભાર धन्यवाद ધન્યવાદ
ಂತಿ நன்றி THANK YOU ధన్యవాదగళ
ಯವಾದ ధన్యవాదాలు ధన్యవాదగళు ನಂತಿ ધન્યવાદ ನಂತಿ धन्यवाद ધન્યવ
್ಯವಾದಾಲು धन्यवाद આભાર धन्यवाद ధన్యవాదగళు ನ
ವಾದಗಳು ನಂತಿ ધન્યવાદ ನಂತಿ ధన్యవాదాలు આભાર நன்றி