Q→! If $\dim(V) = \dim(W) = n$ then

To prove V is isomorphic to W.

i.e V is one to one to W

& V is onto to W.

(i) Let V be not one-one to W.

i.e Let p, q be $\in V$ such that $p \neq q$

but $L(p) = L(q)$.

for a Linear map 'L' from $V :\longrightarrow W$.

We also assume

$v_1\ v_2\ ---\ v_n$ be basis of V

$w_1\ w_2\ ---\ w_n$ be " " W.

~~Now~~ & we define a linear map $L : V \to W$ as

$L(v_1) = w_1$

$L(v_2) = w_2$

|

|

$L(v_n) = w_n$.

Then p & q can be defined as a linear combination of basis vectors $v_1 ---- v_n$ as

$p = \alpha_1 v_1 + \alpha_2 v_2 --- \alpha_n v_n$

& $q = \beta_1 v_1 + \beta_2 v_2 --- \beta_n v_n$.

where $\alpha_1 ---- \alpha_n$ &

$\beta_1 --- \beta_n$ are coefficients

& there is an $i \in \{1 ---- n\}$

such that $\alpha_i \neq \beta_i$.

Now

$$L(p) = L(\alpha_1 v_1 + \alpha_2 v_2 \text{ --- } \alpha_n v_n)$$

$$= \alpha_1 L(v_1) + \alpha_2 L(v_2) \text{ --- } \alpha_n L(v_n)$$

$$= \alpha_1 w_1 + \alpha_2 w_2 \text{ --- } \alpha_n w_n$$

$$L(q) = L(\beta_1 v_1 + \beta_2 v_2 \text{ --- } \beta_n v_n)$$

$$= \beta_1 L(v_1) + \beta_2 L(v_2) \text{ ----- } \beta_n L(v_n)$$

$$= \beta_1 w_1 + \beta_2 w_2 \text{ --- } \beta_n w_n.$$

As per assumption

$$L(p) = L(q)$$

$$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 \text{ --- } \alpha_n w_n = \beta_1 w_1 + \beta_2 w_2 \text{ -- } \beta_n w_n$$

$$\Rightarrow \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \vdots \\ \alpha_n - \beta_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\Rightarrow \alpha_i = \beta_i \text{ for } i = \{1 \text{ --- } n\}.$$

$$\Rightarrow v = v$$

which contradicts our assumption

hence linear map b/w V 4 W is one one.

Let q be any vector in W
with basis $w_1, w_2 \cdots w_n$

Then q can be written as a linear combination of
$w_1 \cdots w_2 \cdots w_n$ as

$$q = \alpha_1 w_1 + \alpha_2 w_2 \cdots \alpha_n w_n$$

$$\Rightarrow q = \alpha_1 L(v_1) + \alpha_2 L(v_2) \cdots \alpha_n L(v_n)$$ {from the Linear map defined earlier}.

$$\Rightarrow q = L\{\alpha_1 v_1 + \alpha_2 v_2 \cdots \alpha_n v_n\}$$, {from the property of linear map $L(u+v) = L(u) + L(v)$}

$$= L(p)$$

i.e There exist a vector p in V
for every vector q in W
hence we prove that
$L : V \to W$ is onto.

Therefore as linear map $L : V \to W$ is one-one
as well as onto, it is ~~isomorphism~~.

(c) To prove $L: V \to W$ is linear

Let p & q be linear combination of basis vectors $v_1 \cdots v_n$ as

$$p = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$$

$$q = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_n v_n$$

where $\alpha_1 \cdots \alpha_n$ & $\beta_1 \cdots \beta_n$ are coefficients

Then $p+q = (\alpha_1+\beta_1)v_1 + (\alpha_2+\beta_2)v_2 + \cdots (\alpha_n+\beta_n)v_n$

$$= L\left(\sum_{i=1}^{n} (\alpha_i+\beta_i)v_i\right).$$

ie $L\left(\sum_{i=1}^{n} \alpha_i v_i\right) + L\left(\sum_{i=1}^{n} \beta_i v_i\right) = L\left(\sum_{i=1}^{n} (\alpha_i+\beta_i)v_i\right)$

Also

$$L(\alpha v) = L\left(\alpha \sum_{i=1}^{n} \alpha_i v_i\right)$$

$$= \sum_{i=1}^{n} \alpha \alpha_i v_i \quad \forall\, i \in \{1 \cdots n\}$$

$$= \alpha L(v)$$

There we also prove that Linear map $L: V \to W$ is a linear transformation.

Hence as $L: V \to W$ is
- one one
- onto
- linear

Hence it is proved that V is isomorphic to W.

(b) Let q be any vector in W.
with basis $w_1, w_2 - - - w_n$

Then q can be written as a linear combination of
$w_1 - - w_2 - - - w_n$ as

$q = \alpha_1 w_1 + \alpha_2 w_2 - - - \alpha_n w_n$

$\Rightarrow q = \alpha_1 L(v_1) + \alpha_2 L(v_2) - - - \alpha_n L(v_n)$ {from the Linear map defined earlier}.

$\Rightarrow q = L(\alpha_1 v_1 + \alpha_2 v_2 - - - \alpha_n v_n)\}$ → {from the property of linear map $L(U+V) = L(U) + L(V)$}.

$= L(p)$

i.e There exist a vector p in V
for every vector q in W
hence we prove that
$L : V \to W$ is onto.

Therefore as linear map $L : V \to W$ is one-one as well as onto, it is V is isomorphic to W.

Q2 ~~To prove~~ $\langle v, w\rangle = v_1w_1 - (v_1w_2 + v_2w_1) + 2v_2w_2$
To prove
is a valid inner product where $v = \langle v_1, v_2\rangle$, $w = \langle w_1, w_2\rangle$.

$$\langle v, w\rangle = v_1w_1 - (v_1w_2 + v_2w_1) + 2v_2w_2$$

$$= w_1v_1 - (w_2v_1 + w_1v_2) + 2w_2v_2$$

$$= \langle w, v\rangle$$

$\Rightarrow$ $\langle v, w\rangle$ is symmetric. $\rightarrow$ (1)

$$\langle v, v\rangle = v_1^2 - (v_1v_2 + v_1v_2) + 2v_2^2$$

$$= (v_1 - v_2)^2 + v_2^2$$

$\Rightarrow$ $\langle v, v\rangle \geq 0$ hence it is positive definite. $\rightarrow$ (2).

Now to prove $\langle v, w\rangle$ is bilinear we need to prove that

(1) $\langle v+u, w\rangle = \langle v, w\rangle + \langle u, w\rangle$

&

(2) $\langle \lambda v, w\rangle = \lambda \langle v, w\rangle$

$$\langle v+u, w\rangle = (v_1+u_1)w_1 - ((v_1+u_1)w_2 + (v_2+u_2)w_1) + 2(v_2+u_2)w_2$$

$$= (v_1w_1 - (v_1w_2 + v_2w_1) + 2v_2w_2) + (u_1w_1 - (u_1w_2 + u_2w_1) + 2u_2w_2)$$

$$= \langle v, w\rangle + \langle u, w\rangle.$$

Also

$$\langle \lambda v, w \rangle = \lambda v_1 w_1 - (\lambda v_1 w_2 + \lambda v_2 w_1) + 2\lambda(v_2 w_2)$$

$$= \lambda(v_1 w_1 - (v_1 w_2 + v_2 w_1) + 2 v_2 w_2)$$

$$= \lambda \langle v, w \rangle.$$

Hence $\langle v, w \rangle$ is bilinear

Therefore as $\langle v, w \rangle$ is, symmetric,

→ positive definite

→ bilinear

hence $\langle v, w \rangle$ is a valid inner product.

Q→3

E_1 = infected with covid.

E_2 = Tested +ve

$P(E_1) = 0.1 \qquad P(E_1') = 0.9.$

$P(E_2/E_1) = 1 - 0.2 = 0.8$

$P(E_2/E_1') = 0.1.$

$$P(E_1/E_2) = \frac{P(E_2/E_1)\,P(E_1)}{P(E_2/E_1)\cdot P(E_1) + P(E_2/E_1')\,P(E_1')}$$

$$= \frac{(0.8)(0.1)}{(0.8)(0.1) + (0.1)(0.9)}.$$

$= 0.4705.$

Q5 4

(a). $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$= \int_{-\infty}^{c} 0\,dx + \int_{c}^{e} \frac{1}{x}\,dx + \int_{e}^{\infty} 0\,dx = 1$

2, 3, 4 ✓

$= \log_e x \Big]_c^e = 1$

$\Rightarrow 1 - \log_e c = 1$

$\Rightarrow \boxed{c = 1}$

(b). $E(X) = \int_{1}^{e} \frac{1}{x} \cdot x\,dx = (e-1)$.

(4) (c). $P\{X>2\} = 1 - P\{X \le 2\}$

$$P\{X \le 2\} = \int_{-\infty}^{2} f_x \, dx$$

$$= \int_{c}^{2} \frac{1}{x} \, dx = \log_e 2 - \log_e 1$$

$$= \log_e 2$$

$$\Rightarrow P\{X>2\} = 1 - \log_e 2$$

R15 (a)

By Makov inequality

$$P\{X \geq a\} \leq \frac{EX}{a}$$

$EX = 10^6$

$a = 2 \times 10^6$

$$\Rightarrow \quad P\{X \geq 2\times 10^6\} \leq \frac{10^6}{2\times 10^6} =$$

$$\Rightarrow \quad P\{X \geq 2\times 10^6\} \leq \frac{1}{2}$$

Hence $1/2$ is the upper bound according to Markov's inequality.

(b) By chebyshev's inequality

$$P\{|X-\mu| \geq b\} \leq \frac{\operatorname{Var} X}{b^2}.$$

~~$\mu = 10^6$~~ $\mu = 10^6$

$\operatorname{Var}(X) = 10^{10}$.

$b = 2\times 10^6$

$\Rightarrow$

$$P\{X \geq 2mn\}$$

$$= P\{(X - 1mn) \geq 1mn) \leq \frac{\operatorname{Var} X}{b^2}$$

$$= P\{(X-10^6) \geq 10^6\} \leq \frac{10^{10}}{10^{12}}.$$

$$= P\{(X-10^6) \geq 10^6\} \leq \frac{1}{100}.$$

Hence $\frac{1}{100}$ or 0.01 is the upper bound according to chebyshev's inequality.