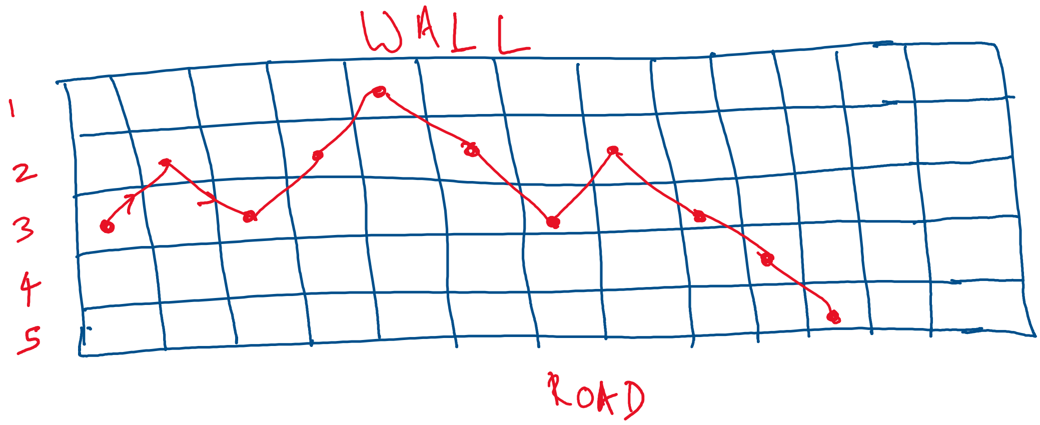
Markov Chains

04 November 2020 19:22

This is a type of stochastic / probabilistic process that is applicable to a huge class of problems / situations.

Example 1: A drunk Walker.

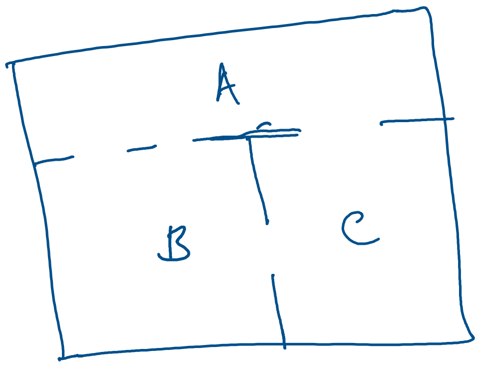


At every step, he moves forward one step and one step to the wall or road. If he enters the road, the process ends. How far can he go ahead?

Example 2: Gambler's ruin

Game between two players P1 and P2. P1 has l_1 rupees to begin with and P2 has l_2 rupees. At every step, they toss a coin. Whoever loses gives one rupee to the winner. Game continues till P1 or P2 is broke. What are the chances of P1 or P2?

<u>Example 3</u>: A mouse lives in this house. A bell rings at regular intervals and it randomly chooses a door to move to another room. What fraction of its life will it spend in each room?



<u>Example 4</u>: Mutating virus. Suppose a virus mutates itself into another strain every day. What is the prob that the n^{th} generation is a particular strain?

<u>Example 5</u>: Google Pagerank. They simulated the random surfer. Let N be the number of websites in all. For page i, let L_i be the set of outgoing links from it. $L_i \subseteq \{1, 2, \ldots N\}$.

Random surfer at page i.

With prob q, chooses a random outgoing link.

With prob $1-q$, chooses a new page from $\{1, 2, \ldots N\}$

$$\text{Prob}[\text{going from } i \text{ to } j] = \frac{q}{|L_i|} + \frac{1-q}{N} \quad (\text{if } j \in L_i)$$

$$\text{Prob}[\text{going from } i \text{ to } j] = \frac{1-q}{N} \quad (\text{if } j \notin L_i)$$

Total prob of next state (summed over all j)

$$= \sum_{j \in L_i} \left[\frac{q}{|L_i|} + \frac{1-q}{N} \right] + \sum_{j \notin L_i} \frac{1-q}{N}$$

$$= |L_i| \left[\frac{q}{|L_i|} + \frac{1-q}{N} \right] + (N - |L_i|) \frac{1-q}{N}$$

$$= q + (N + |L_i| - |L_i|) \frac{(1-q)}{N}$$

$= 1/1$

What is the asymptotic prob. of visiting a page i, after a large number of clicks?
This was one of the first definitions of page rank.

A discrete time Markov Chain (DTMC) has a state space (could be countably infinite), and transition probabilities P_{ij}.

A discrete time stochastic process is a Markov chain if

$$P(X_t = a_t \mid X_{t-1} = a_{t-1}, X_{t-2} = a_{t-2}, \dots X_1 = a_1, X_0 = a_0)$$

$$= P(X_t = a_t \mid X_{t-1} = a_{t-1})$$

$$= P_{a_{t-1}, a_t}.$$

The transition probability from state i does not depend on the history of how state i was reached.

Markov = Memory less.

$$P_{i,j} = P_r(X_t = j \mid X_{t-1} = i)$$

The transition prob is independent of $X_{t-2}, X_{t-3} \cdots$

In this lecture, we will only see DTMC (Discrete Time Markov Chain).

Transition Probability Matrix

Possibly infinitely many rows and columns.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ & \vdots & & \\ P_{i1} & P_{i2} & P_{i3} & \cdots \\ & \vdots & & \end{bmatrix}$$

$(i,j)^{th}$ entry of $P = P_{ij} = P_r[X_t = j \mid X_{t-1} = i]$

$$\sum_{j=1}^{\infty} P_{ij} = 1, \quad \forall i.$$

Applications of Markov Chains

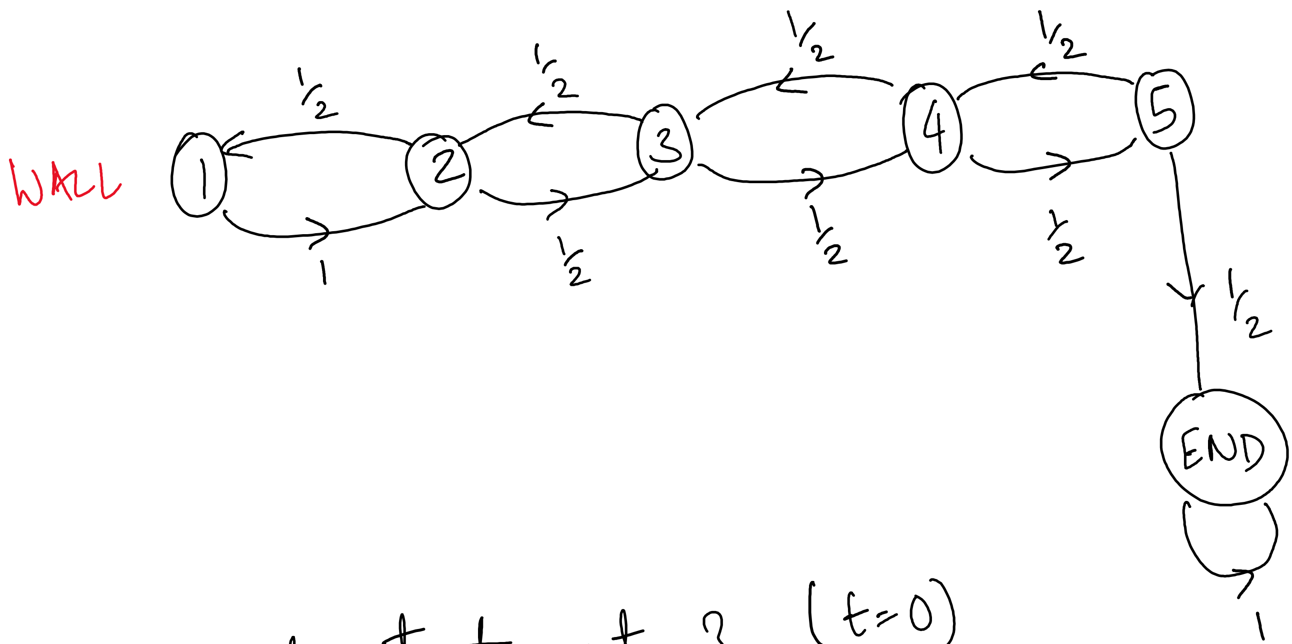
- **Many many other real-world processes ...**

  Dynamical systems with stochastic (partially or fully random) dynamics.
  Some are really fundamentally random, others are 'practically' random.
  E.g.

  – physics: quantum mechanics, solids/liquids/gases at nonzero temperature, diffusion
  – biology: interacting molecules, cell motion, predator-prey models,
  – medicine: epidemiology, gene transmission, population dynamics,
  – commerce: stock markets & exchange rates, insurance risk, derivative pricing,
  – sociology: herding behaviour, traffic, opinion dynamics,
  – computer science: internet traffic, search algorithms,
  – leisure: gambling, betting,

(Image Source: Ton Coolen)

Drunk Walker.



Suppose he starts at 3. $(t=0)$

In step $t=1$ he is likely to be at 2 or 4
w.p. $\frac{1}{2}$ each.

In step $t=2$, Prob distribution as below

	1	2	3	4	5	END
$t=0$	0	0	1	0	0	0
$t=1$	0	$1/2$	0	$1/2$	0	0
$t=2$	$1/4$	0	$1/2$	0	$1/4$	0
$t=3$	0	$1/2$	0	$3/8$	0	$1/8$
$t=4$	$1/4$	0	$7/16$	0	$3/16$	$1/8$

Even when we have a fixed starting point

The matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ END \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & END \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Let say the starting distribution is $p(0)$. → row vector

The distribution at $t=1$ is

$$p(0) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$$

$$p_j(t) = \sum_i p_i(t-1)\, P_{ij}$$

So $\overline{p(t)} = \overline{p(t-1)}\, P.$

Row vectors

$$\overline{p(1)} = \overline{p(0)} \cdot P.$$

$$\overline{p(2)} = \overline{p(1)} \cdot P = \overline{p(0)} . P . P$$

$$= \overline{p(0)}\, P^2$$

Similarly $\overline{p(3)} = \overline{p(0)}\, P^3$

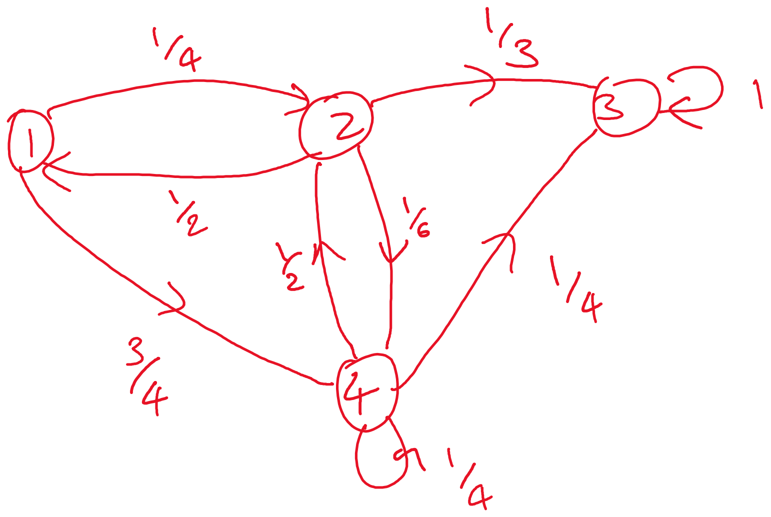
$$\overline{p(t)} = \overline{p(t-m)}\, P^m$$

The transition probability matrix over m steps will be the m^{th} power of the one-step transition probability matrix P.

$$P_{ij}^{(m)} = Pr\left(X_{t+m} = j \mid X_t = i \right)$$

$$= (i,j)^{th} \text{ entry of } P^m$$

Another Example



$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{array}{l} \longrightarrow \text{Sum}=1 \\ \longrightarrow \text{Sum}=1 \\ \\ \\ \end{array}$$

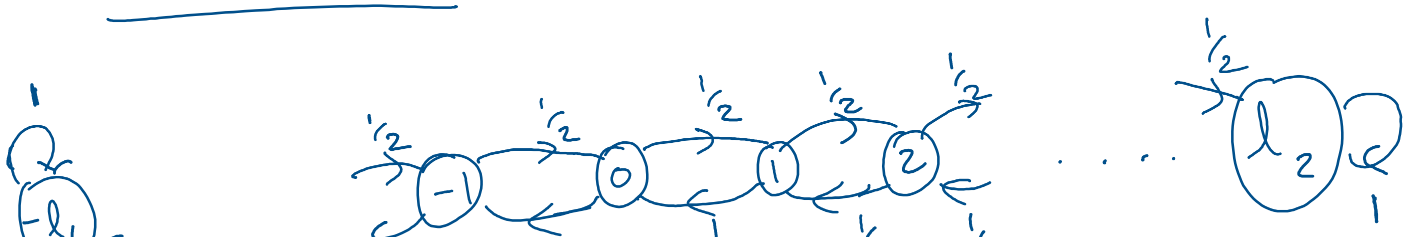
Two step matrix:

$$P^2 = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{13}{48} & \frac{11}{48} \\ 0 & \frac{5}{24} & \frac{3}{8} & \frac{5}{12} \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{23}{48} & \frac{7}{48} \end{bmatrix}$$

$2 \to 1 \to 4$
$2 \to 4 \to 4$

$4 \to 2 \to 1$

Gambler's Ruin



The entry in each state denotes P1's winnings till now.

What is the probability that P1 wins l_2 before he loses l_1? (Starting from 0).

If $l_1 = l_2$, by symmetry this prob $= \frac{1}{2}$

Let P_i^t be the probability that chain is in state i after t steps. If $-l_1 < i < l_2$, we have

$$\lim_{t \to \infty} P_i^t = 0$$

(i is a transient state)

Let q = Prob that P1 wins $l_2 = \lim_{t \to \infty} P_{l_2}^t$

Let q_j be the prob that P1 wins from j.

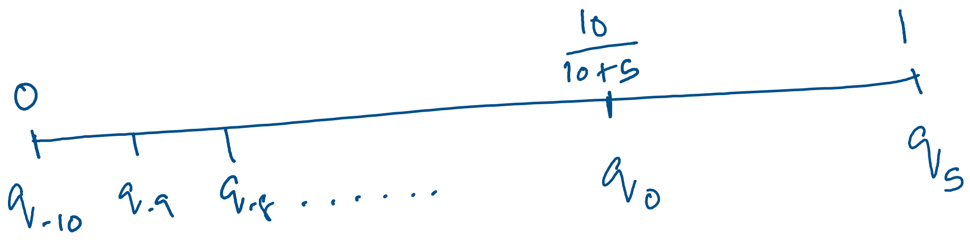
So $q_0 = q$. $q_{-l_1} = 0$ $q_{l_2} = 1$.

$q_j = q_{j-1} + q_{j+1}$

For $-l_1 < j < l_2$, $q_j = \frac{\cdots}{2}$

So the q_j's are in arithmetic progression.

Say $l_1 = 10$ and $l_2 = 5$



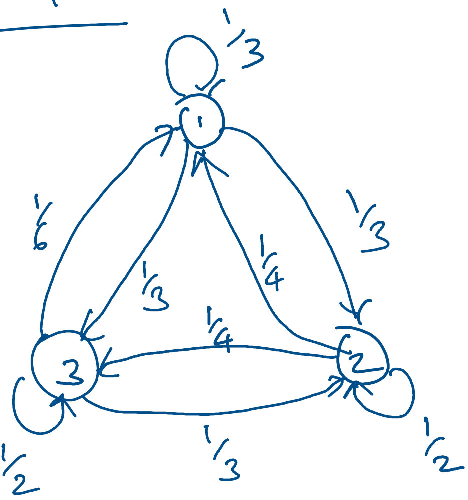
So $q = q_0 = \frac{l_1}{l_1 + l_2}$.

<u>Defn</u>: A stationary distribution (equilibrium distribution) of a Markov chain is a distribution $\bar{\pi}$ such that

$$\bar{\pi} = \bar{\pi} \cdot P$$

* If a chain starts with distribution $\bar{\pi}$, then it maintains the distribution in the future.
* Stationary distribution is like steady state.
* Useful in analyzing Markov Chains

Example



$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/6 & 1/3 & 1/2 \end{bmatrix}$$

$$\pi = \begin{bmatrix} 6/25 & 10/25 & 9/25 \end{bmatrix}$$

Existence: For any finite, irreducible, aperiodic Markov chain, there is a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots \pi_n)$.

Also, $\pi_i = \lim_{t \to \infty} P^t_{j,i}$ (for all i, j).

Irreducible: Underlying graph is strongly connected.

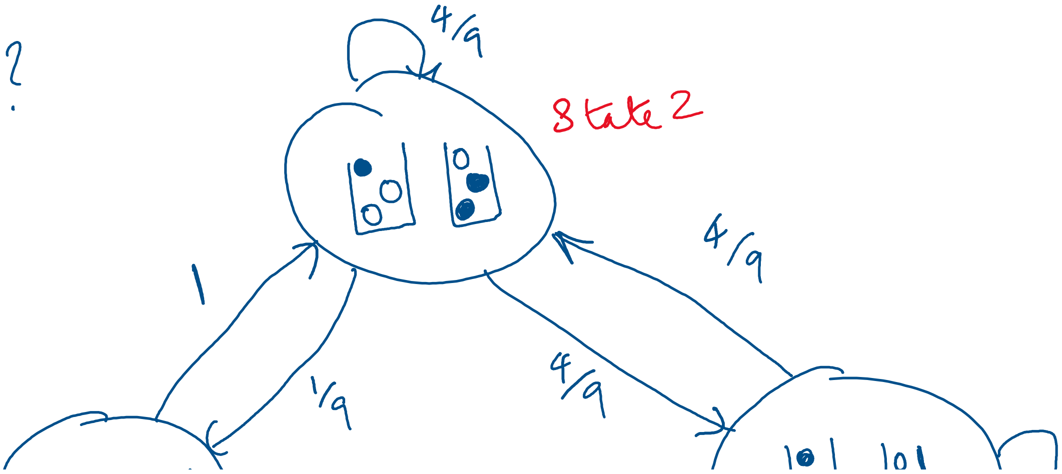
Aperiodic: There is no period (Δ) such that $P(X_{t+s} = j \mid X_t = j) > 0$ only for s multiples of Δ.

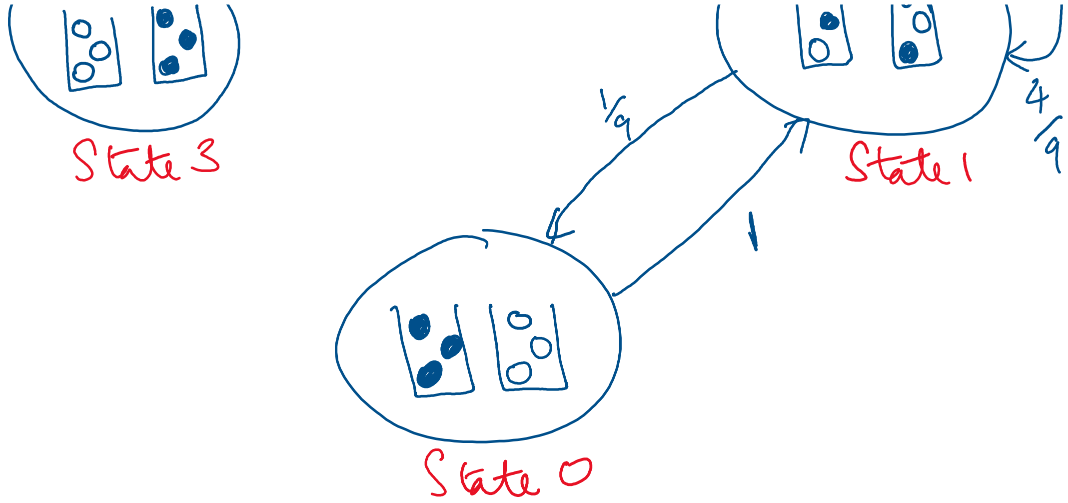
Some more definitions

Recurrent state: State i is recurrent if the probability that MC will return to state i is 1.

Transient state: (Not recurrent): There is a non zero probability that MC will not return to state i.

Exercise: Three white and three black balls are in two urns, with three balls per urn. At each step, we draw a random ball from each urn and swap them. Model this as a Markov chain. What is the stationary distribution?





$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 0 & 0 \\
1 & 1/9 & 4/9 & 4/9 & 0 \\
2 & 0 & 4/9 & 4/9 & 1/9 \\
3 & 0 & 0 & 1 & 0
\end{array} = P
$$

Stationary distribution : $\bar{\pi} P = \bar{\pi}$

let $\bar{\pi} = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]$

$$\pi_0 = \tfrac{1}{9}\pi_1$$

$$\pi_1 = \pi_0 + \tfrac{4}{9}\pi_1 + \tfrac{4}{9}\pi_2$$

$$\pi_2 = \tfrac{4}{9}\pi_1 + \tfrac{4}{9}\pi_2 + \pi_3$$

$$\pi_3 = \tfrac{1}{9}\pi_2$$

$$\pi_1 = \tfrac{1}{9}\pi_1 + \tfrac{4}{9}\pi_1 + \tfrac{4}{9}\pi_2$$

$$\tfrac{4}{9}\pi_1 = \tfrac{4}{9}\pi_2 \Rightarrow \pi_1 = \pi_2$$

$$\bar{\pi} = [\tfrac{1}{9}\pi_1, \pi_1, \pi_1, \tfrac{1}{9}\pi_1]$$

Since it's a prob. distribution, it sums to 1.

$$\Rightarrow \tfrac{2}{9}\pi_1 + 2\pi_1 = 1$$

$$\Rightarrow \pi_1 = \tfrac{9}{20}$$

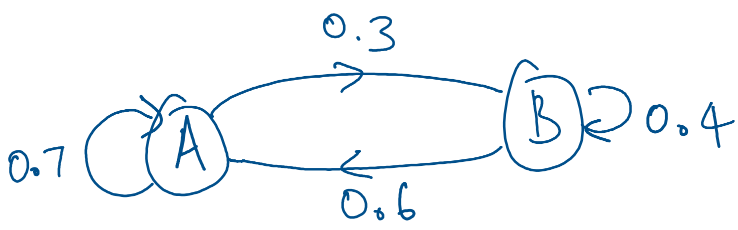
So $\bar{\pi} = \left[\tfrac{1}{20}, \tfrac{9}{20}, \tfrac{9}{20}, \tfrac{1}{20}\right]$

If you start from state 0, what is the prob distribution after 10 steps?

$$= [1 \quad 0 \quad 0 \quad 0] \cdot P^{9}$$

Example : You have two coins, coin A and coin B. Coin A falls heads w.p. 0.7. Coin B falls heads with prob 0.6. In day 0, we choose A or B at random and toss. If you get heads, you toss coin A the next day. If you get tails, you toss coin B the next day.

State is the coin tossed on the day.



$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

What is the prob that coin A is flipped on day 3 ?

$$= [0.5 \quad 0.5]\, P^3$$

Stationary distribution : $\pi = [\pi_A \; \pi_B]$

$$\pi_A = 0.7\,\pi_A + 0.6\,\pi_B \Rightarrow 0.3\,\pi_A = 0.6\,\pi_B$$

$$\Rightarrow \pi_A = 2\pi_B$$

$\pi_A + \pi_B = 1$. So $\underline{\pi_A = 2/3 \quad \pi_B = 1/3}$.