31/Oct/2020

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Concentration Bounds: Random variables are "concentrated" around the mean/expectation.

Markov Inequality: Let X be a r.v such that $X \geq 0$.

For all $a \geq 0$, $P_r(X \geq a) \leq \frac{EX}{a}$

Proof: Let I be a r.v that indicates if $X \geq a$

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{if } X < a \end{cases}$$

$E(I) = P_r(X \geq a)$.

But $I \leq X/a$. So $E(I) \leq \frac{E(X)}{a}$.

Chebyshev's Inequality: For any $a > 0$,

$$P_r(|X - EX| \geq a) \leq \frac{Var(X)}{a^2}$$

Proof: $P_r(|X-EX| \geq a) = P_r((X-EX)^2 \geq a^2)$

[illegible]

Proof: $\Pr(|X - EX| \geq a) = \Pr($ ...

Markov $\longrightarrow$ $$\leq \frac{E\left((X-EX)^2\right)}{a^2} = \frac{Var(X)}{a^2}$$

Example: A coin is flipped n times. What is the probability of getting 90% heads?

Markov: $$\Pr(X \geq 0.9n) \leq \frac{n/2}{0.9n} = \frac{1/2}{0.9} = \frac{5}{9}.$$

Chebyshev: Consider r.v. $X_1, X_2, \dots X_n$. Each $X_i = 1$ if the ith toss is heads and $X_i = 0$ if ith toss is tails.

$$E(X_i) = 1/2.$$

$$Var(X_i) = E(X_i^2) - E(X_i)^2$$
$$= 1/2 - 1/4 = 1/4.$$

Let X = total no. of heads $= \sum X_i$.

Since the tosses are indep., $Var(X) = \sum Var(X_i)$
$$= n/4$$

$$\Pr(X \geq 0.9n) \leq \Pr(|X - EX| \geq 0.4n)$$

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$$\leq \frac{Var(X)}{(0.4n)^2} = \frac{1/4\, n}{0.16 n^2} = \frac{25}{16n}$$

$25/16n$ becomes smaller when n increases.

Chernoff Bounds: let $X_1, X_2 \ldots X_n$ be independent 0/1 Poisson trials such that $\Pr(X_i = 1) = p_i$. let $X = \sum X_i$ and $\mu = E(X)$. For $0 < \delta < 1$.

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}$$

Coin Flip Example:

$$\Pr[X \geq 0.9n] \leq \Pr[|X - EX| \geq 0.4n]$$

$$= \Pr[|X - EX| \geq 0.8\mu]$$

$$\leq 2e^{-\frac{0.5n \cdot 0.8^2}{3}}$$

$$= 2e^{-\frac{0.32}{3}n}$$

$$\leq 2e^{-0.1n} = \frac{2}{e^{0.1n}}$$

As n increases, this prob drops very fast.

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Suppose there is a coin. It falls heads w.p. 0.6, and tails w.p. 0.4. You win if it is heads.

Three possibilities

(1) We toss 100 times, for each toss, the stake is Re. 1.

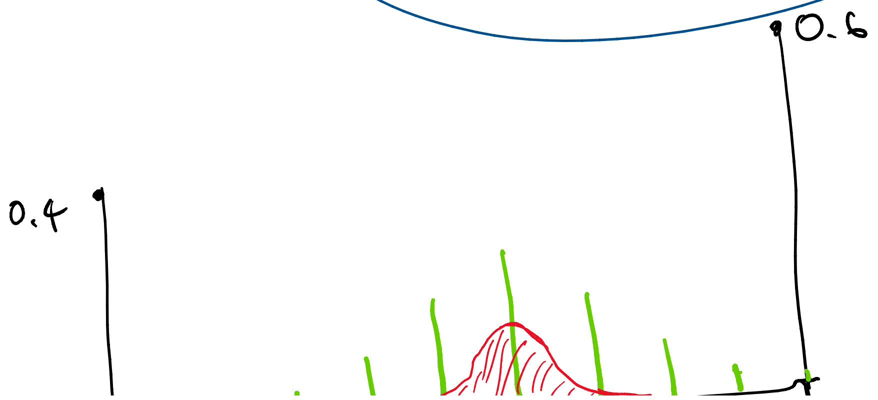
(2) We toss 10 times, for each toss, the stake is Rs. 10.

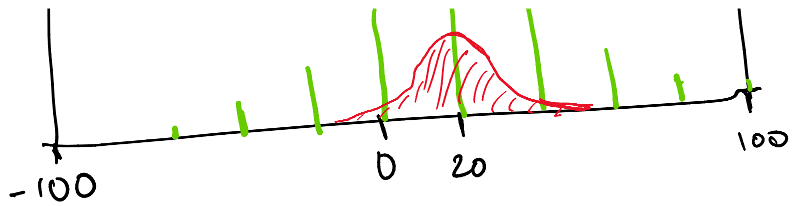
(3) One toss, with stake Rs. 100.

Total payout is Rs. 100/- in all 3 cases.

Expectation?

This idea is used in random forests.





Prob of you winning money is highest in the first case, and least in the third case.

Beta Distribution

$$P.D.F = \frac{1}{B(\alpha,\beta)} \, x^{\alpha-1} (1-x)^{\beta-1} \qquad x \in [0,1].$$

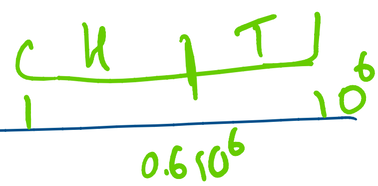
where $\alpha, \beta > 0$.

$$E(X) = \alpha / \alpha+\beta .$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

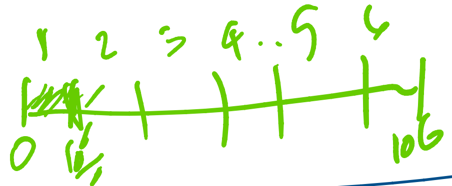
Diff cases based on relation b/w α and β

Used in statistics, estimation etc.



Given you have access to a uniform random number generator. How can you create a given distribution, say like a die from this? A coin with 0.6 prob Heads and

0.4 prob tails?



X = time taken for bus to arrive after last bus has passed.

If X follows exp. dist,

You go to bus stop

Maybe 30 mins has already passed when you arrive.

P(your waiting time $\geq$ 5 mins)

$$= Pr(X \geq 35 \mid X \geq 30)$$

Maybe bus has passed just 1 min before you reached.

Pr(your waiting time $\geq$ 5 mins)

$$= Pr(X \geq 6 \mid X \geq 1)$$

Both of these are same in exp. dist!

$$E(XY) = \sum_{X=x,\, Y=y} xy \Pr(X=x, Y=y)$$

$$EX = \sum_{X=x} x \Pr(X=x)$$