Probability and Statistics (MA6.101)

Monsoon 2021, IIIT Hyderabad
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Probability

Distributions

Gamma Distribution

The relation between the gamma function and the factorial function is $\Gamma(n) = (n-1)!$. It is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \alpha > 0.$$

Some properties of the gamma function are:

- $\int_0^\infty x^{\alpha-1}e^{-\lambda x}dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Now, a CRV X is said to have a gamma distribution with parameters $\alpha > 0, \lambda > 0$, written $X \sim \text{Gamma}(\alpha, \lambda)$, if

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha = 1$, we get

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $\text{Gamma}(1, \lambda) = \text{Exponential}(\lambda)$.

Further, the sum of n independent $\text{Exponential}(\lambda)$ CRVs is $\text{Gamma}(n, \lambda)$.

Some properties of the gamma distribution are:

- $E[X] = \frac{\alpha}{\lambda}$
- $\text{Var}(X) = \frac{\alpha}{\lambda^2}$

Mixed Random Variables

Suppose X is a CRV with the PDF

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and let Y be an RV such that

$$Y = g(X) = \begin{cases} X & 0 \leq X \leq \frac{1}{2} \\ \frac{1}{2} & X \geq \frac{1}{2}. \end{cases}$$

Then Y is a mixed random variable. Its CDF is not continuous, but not in the staircase form either. CDFs of mixed RVs can be written as the sum of the CDFs of a continuous and a discrete RV. In the case of Y, this is

$$F_Y(y) = C(y) + D(y),$$

where

$$C(y) = \begin{cases} \frac{1}{4} & y \leq \frac{1}{2} \\ y^2 & 0 \leq y \leq \frac{1}{2} \\ 0 & y < 0. \end{cases}$$

and

$$D(y) = \begin{cases} \frac{3}{4} & y \geq \frac{1}{2} \\ 0 & y < \frac{1}{2}. \end{cases}$$

We can differentiate the continuous part to get $c(y) = \frac{dC(y)}{dy}$, wherever it is differentiable. If $\{y_1, y_2, ...\}$ is the set of jump points of $D(y)$, then

$$\int_{-\infty}^{\infty} c(y) + \Sigma_{y_k} P(Y = y_k) = 1.$$

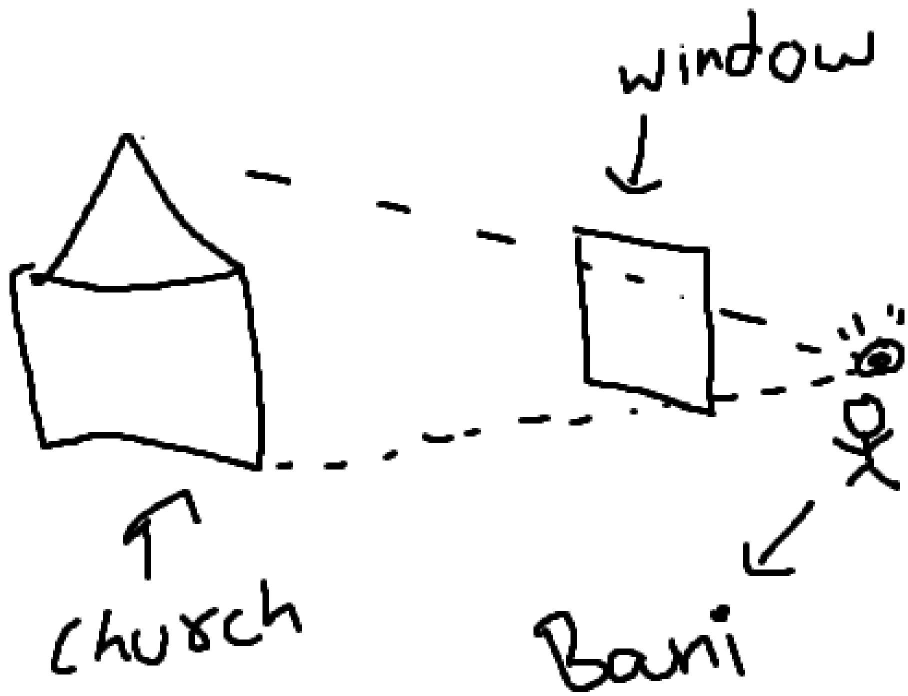

Figure 1: Amal's Art

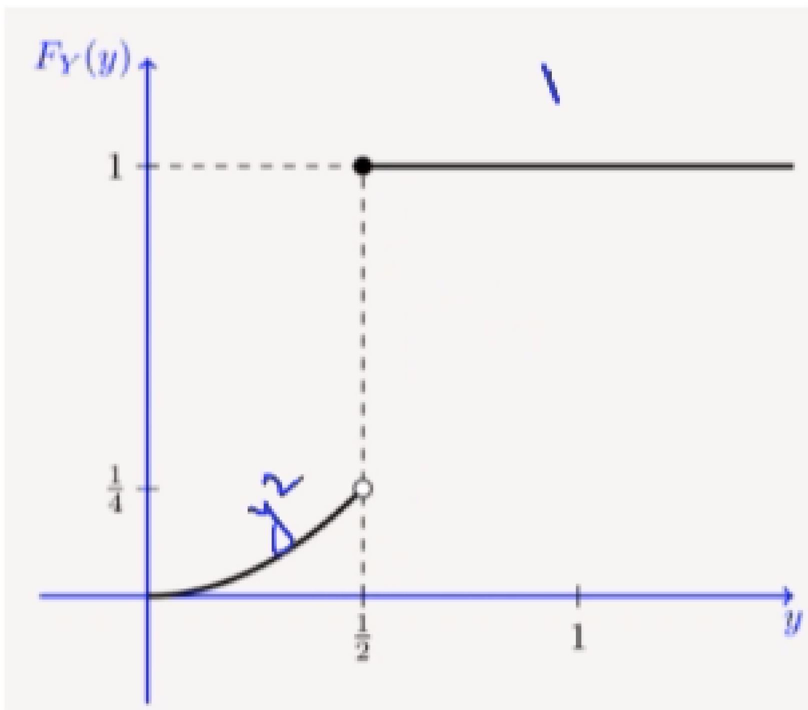

Figure 2: CDF of $Y