Probability and Statistics (MA6.101)

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Quantum Computing

Bits and Qubits

A classical bit is either

$$0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \text{ or } 1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

Even if we don't know the value, we can identify probabilities p_0 and p_1 s.t. $p_0 + p_1 = 1$.

In contrast, a qubit can be in a superposition of the states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers s.t. $|\alpha|^2 + |\beta|^2 = 1$.
Any attempt to measure the state results in $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$. After measurement, the system will be in the measured state

$$|\psi'\rangle = 0 \text{ or } 1.$$

Any subsequent measurement will yield the same value.

The state of a qubit is more than a probability mass function. For example,

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

$$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle),$$

$$\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle),$$

are all equally likely to be $|0\rangle$ or $|1\rangle$, but correspond to different superpositions.

The Hadamard gate H behaves in the following way:

$$H|+\rangle = H\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \to |0\rangle$$

$$H|-\rangle = H\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \to |1\rangle$$

The two-qubit Bell state $|\phi^+\rangle$ is defined as

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle.$$

Quantum phenomena are described in linear algebra terms. We need to define a new form of multiplication on matrices called tensor multiplication:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}.$$

We also define the conjugate transpose $A^\dagger = (A^*)^T$.

If we have two vectors $|u\rangle, |v\rangle$, we define the inner product $\langle u|v\rangle$ as $\langle u|\times|v\rangle = u^\dagger v$. Similarly, the outer product is $|u\rangle\langle v|$, which is a matrix. If $|u\rangle$ is a unit vector then $|u\rangle\langle u|$ is known as a projector; it projects an arbitrary vector $|v\rangle$ onto the subspace $|u\rangle$.

$$(|u\rangle\langle u|)|v\rangle = |u\rangle(\langle u|v\rangle) = (\langle u|v\rangle)|u\rangle$$

A matrix A is normal if $AA^\dagger = A^\dagger A$. It is hermitian if $A = A^\dagger$. It is unitary if $AA^\dagger = A^\dagger A = I$.

Postulates of QM

State space

Associated to any isolated physical system is a copmlex vector space with an inner product (a Hilbert space). The state is completely described by its state vector (which is a unit vector).

Qubits are quantum states in the space $\mathbb{C}^2$. An example of a physical realisations of qubits is electron spin.

Evolution

The time evolution of the state of a closed system is described by the Schrodinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle,$$

where H is a fixed Hermitian operator called the Hamiltonian of the system.

We can say that

$$|\psi_{t_1}\rangle = \exp\left(\frac{iH(t_1 - t_0)}{\hbar}\right)|\psi_{t_0}\rangle.$$

The Pauli matrices are important single-qubit unitary matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

and the Hadamard matrix is

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We can check that $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$.

Measurement

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. They ar on the state space of the system. The index m refers to the outcomes that may occur.

If the state is $|\psi\rangle$ before measurement, the probability of the m^{th} outcome is

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}.$$

The measurement operators must satisfy

$$\sum_m M_m^\dagger M_m = I.$$

This property is called completeness.

We can write a one-qubit state as

$$|\psi\rangle = e^{i\theta}(\alpha|0\rangle + \beta e^{i\phi}|1\rangle) = e^{i\theta}|\psi'\rangle,$$

where α, β are positive real numbers. θ is known as the global phase and has no observable consequences.

Composition

The state space of a composite system is the tensor product of the state space of the component physical systems. More generally, if the systems are numbered 1 to n, where system i is in state $|\psi_i\rangle$, then the state of the composite system is $|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$.