Probability and Statistics (MA6.101)

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Probability (contd.)

Distributions

Poisson Distribution (contd.)

Suppose we wish to send a bitstring of length $n = 10^4$. The probability that a bit can be corrupted is 10^{-6}. We need to know the probability that the message will arrive uncorrupted.

We can use the Poisson distribution by taking $\lambda = np = 10^4 10^{-6}$, *i.e.*, $X \sim$ Poisson(0.01) is the number of corrected bits. Therefore our answer is

$$\begin{aligned} P(X = 0) &= \left.\frac{\lambda^k}{k!} e^{-\lambda}\right|_{k=0} \\ &= \frac{1}{1} e^{-0.01} \\ &\approx 0.99 \end{aligned}$$

Expectations

Poisson Distribution

Let $X \sim$ Poisson, *i.e.*,

$$P_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & k \in R_X \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}
E[X] &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} \\
&= e^{-\lambda}\lambda e^{\lambda} \\
&= \lambda.
\end{aligned}$$

Binomial Distribution

Let $X \sim \text{Binomial}(n, p)$; then

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= \sum_{z=0}^{n-1} \binom{n-1}{z} p^{z+1} (1-p)^{n-z-1} \\
&= np.
\end{aligned}$$

Variance and Standard Deviation

The variance measures the divergence from the expected value of the actual values. It is defined by

$$\text{Var}(X) = E((X-\mu)^2) = \sum_x (x-\mu)^2 p(x).$$

Its positive square root is called the standard deviation:

$$\sigma = \sqrt{\text{Var(X)}}.$$

If X is a discrete random variable with mean μ, then

$$\text{Var}(X) = E[X^2] - \mu^2.$$

Higher Order Moments

Just as we have $E[(X-\mu)^2]$, we can define $E[(X-\mu)^3]$ and any other power. In other words, the n^{th} moment about the mean of a variable X is defined as

$$\mu_n = E[(X - E[X])^n].$$

The 0^{th} central moment μ_0 is 1.

The moment generating function $M_X(t)$ is the expectation value

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} p_X(x).$$

Clearly, $M_X(0) = 1$. Furthermore, $E[X] = M'_X(0)$, where the derivative is taken w.r.t t.