Instructions

- There are a total of 9 questions and need to attempt all the questions.
- Each question is worth of 4 points. (Total -> 4 x 9 = 36)

Problem 1

The Moment Generating Function (MGF) for a random variable X, $mgf_X(t)$ is given. Find the probability :

$$P(|X| <= 2)$$

$$mgf_X(t) = \frac{1}{10}e^{-20t} + \frac{1}{5}e^{-3t} + \frac{3}{10}e^{4t} + \frac{2}{5}e^{5t}$$

Problem 2

Given a random variable X with Moment Generating function $M(t)$, compute the Moment generating functions for the following random variables (in terms of M). In each subpart, k is a scalar.

1. kX
2. $X + k$
3. A random variable Y with $PDF(y) = PDF(x + k)$, both PDF defined over R.
4. A random variable Y with $PDF(y) = PDF(2x)$

Problem 3

Suppose that n people throw their hats in a box and then each pick one hat at random. (Each hat can be picked by only one person, and each assignment of hats to persons is equally likely.) What is the expected value of X, the number of people that get back their own hat?

Problem 4

The lifetime of a bulb is modeled as a Poisson variable. You have two bulbs types A and B with expected lifetime 0.25 years and 0.5 years, respectively. When a bulb's life ends, it stops working. You start with new bulb of type A at the start of the year. When it stops working, you replace it with a bulb of type B. When it breaks, you replace with a type A bulb, then a type B bulb, and so on.

1. Find the expected total illumination time (in years), given you do exactly 3 bulb replacements
2. Your replacements are now probabilistic. If your current bulb breaks, you replace it with a bulb of type A with probability p, and with type B with probability $(1 - p)$. Find the expected total illumination time (in years), given you do exactly n bulb replacements, and start with bulb of type A

Answer for part 2 exists in closed form in terms of n and p.

Problem 5

Vinay and Mahesh alternate playing at the casino table. (Vinay starts and plays at odd times $i = 1, 3, \ldots$; Mahesh plays at even times $i = 2, 4, \ldots$) At each time i, the net gain of whoever is playing is a random variable G_i with the following PMF:

$$P_G(g) = \begin{cases} 1/3, & \text{for } g = -2 \\ 1/2, & \text{for } g = 1 \\ 1/6, & \text{for } g = 3 \\ 0, & \text{otherwise} \end{cases}$$

Assume that the net gains at different times are independent. We refer to an outcome of -2 as a "loss."

1. They keep gambling until the first time where a loss by Mahesh immediately follows a loss by Vinay. Write down the PMF of the total number of rounds played. (A round consists of two plays, one by Vinay and then one by Mahesh.)
2. Write down the PMF for Z (number of trials), defined as the time at which Mahesh has his third loss.
3. Let N be the number of rounds until each one of them has won at least once. Find $E[N]$.

Problem 6

One of 225 different colors is assigned to a ball where each color is equally likely. For a group of n balls,

1. Find the expected number of colors which are assigned to exactly k balls.
2. Find the expected number of colors which have been assigned to more than one ball.

Problem 7

A book of 500 pages contains 500 misprints on an average. Estimate the chances that a given page contains at least three misprints. Consider occurrence of errors as a Poisson process.

Problem 8

Given an undirected graph $G = (V, E)$, and a 3-color assignment colors R, G, B to the vertices of the graph a.

$$a : V \to \{R, G, B\}$$

Given an assignment a, the set of monochromatic edges

$$E(a) := \{(u, v) \in E : a(u) = a(v)\}$$

is the set of edges that has same colors for endpoints. Let a be randomly chosen, i.e. for every $v \in V$, it is chosen to be R, G, B uniformly and independent of the other vertices.

1. For any edge $e \in E$, let X_e be the random variable which is 1 when e is monochromatic and 0 otherwise. Even though the set of random variables $\{X_e\}_{e \in E}$ are pairwise independent, show that they are not independent of all other X_e(you are not required to prove the pairwise independence part).
2. Let Y be the random variable corresponding to the number of non-monochromatic edges. That is $Y := |E/E(a)|$. Find $E[Y]$.

Problem 9

Shashank performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of Z fair coins.

1. Given that Shashank has just had a trial with Z tails, what is the probability that next two trials will also have this result?

2. Sandeep conducts an experiment like Shashank's, except that he uses M coins for the first trial, and then he obeys the following rule: Whenever all the coins land on the same side in a trial, Sandeep permanently removes one coin from the experiment and continues with the trials. He follows this rule until the $(M-1)^{th}$ time he removes a coin, at which point the experiment ceases. Find $E[X]$, where X is the number of trials in Sandeep's experiment.