Probability and Statistics (MA6.101)

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Probability

Transforming PDFs

Suppose we wish to find the PDF of a function of a random variable, say $Y = g(X)$. We can use the following formula in the cases where g is differentiable and strictly increasing:

$$f_Y(x) = \begin{cases} \frac{f_X(x_1)}{g'(x_1)} & g(x_1) = y \\ 0 & \nexists x : g(x) = y \end{cases}$$

To prove this, note that since g is strictly increasing, g^{-1} is well-defined. For each y, there exists x_1 such that $g(x_1) = y$.
Now, we can try to find the CDF of Y as follows.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(x) \leq y) \\ &= P(x \leq g^{-1}(y)) \\ &= F_X(g^{-1}(y)) \end{aligned}$$

We can then find the PDF of Y by differentiating and applying the chain rule.

$$\begin{aligned} \frac{d}{dy} F_Y(y) &= \frac{d}{dy} F_X(g^{-1}(y)) \\ &= F'_X(x_1) \frac{dx_1}{dy} f_Y(y) \\ &= \frac{F'_X(x_1)}{g'(x_1)} \end{aligned}$$

In the case where g is strictly decreasing, the formula is simply the negative of what we already have. The modification to the proof occurs in the steps

$$\begin{aligned}F_Y(y) &= P(Y \leq y)\\ &= P(g(x) \leq y)\\ &= P(x \geq g^{-1}(y))\\ &= 1 - P(x \leq g^{-1}(y))\end{aligned}$$

Now, when we differentiate, the constant 1 disappears and only the negative sign remains, which proves our claim.

We can then write a general formula for any monotonic function.

$$f_Y(x) = \begin{cases} \frac{f_X(x_1)}{|g'(x_1)|} & g(x_1) = y \\ 0 & \nexists x : g(x) = y \end{cases}$$

We can also generalise further to the case where g is piecewise strictly monotonic and differentiable, to

$$F_Y(y) = \sum_{i=1}^{n} \frac{F_X(x_i)}{|g'(x_1)|}$$

where x_i is the inverse of y in the interval i.

Continuous Distributions

Uniform Distribution

The PDF of $X \sim \text{Uniform}(a, b)$ is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

The expectation of this distribution is given by $E[X] = \frac{a+b}{2}$ and its variance by $\text{Var}(X) = \frac{(a-b)^2}{12}$.

Exponential Distribution

Let X be a CRV, having an exponential distribution with parameter $\lambda > 0$, denoted $X \sim \text{Exponential}(\lambda)$. Its PDF is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The CDF for this distribution is

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda t}$$

For this distribution $E[X] = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

An important property of the exponential distribution is that it is memoryless, *i.e.*,

$$P(X > x + a \mid X > a) = P(X > x),$$

for all $a, x \geq 0$. This can be easily shown to be true:

$$\begin{aligned}
P(X > x + a \mid X > a) &= \frac{P(X > x + a, X > a)}{P(X > a)} \\
&= \frac{P(X > x + a)}{P(X > a)} = \frac{1 - F_X(x + a)}{1 - F_X(a)} \\
&= \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} \\
&= P(X > x)
\end{aligned}$$