Probability and Statistics (MA6.101)

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Probability

Distributions (contd.)

Standard Gaussian Distribution

This distribution is also known as the standard normal distribution. A CRV Z is said to be a standard Guassian random variable ($Z \sim N(0,1)$) if its PDF is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, z \in \mathbb{R}.$$

The Central Limit Theorem states (roughly) that if we add a large number of random variables, the distirbution of the sum is normal.

We can check that $E[Z] = 0$, as the integral is of an odd function. Also, $\text{Var}(Z) = 1$.

The CDF of the standard Gaussian distribution is denoted by Φ, and defined as

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} \, du$$

This integral does not have a closed solution, but values of it have been tabulated. The CDF of any normal distribution can be written in terms of Φ.

Some properties of this function are:

- $\lim_{x\to\infty} \Phi(x) = 1, \lim_{x\to-\infty} \Phi(x) = 0.$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

Further, for all $x \geq 0$, it satisfies the following bound:

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{\frac{-x^2}{2}} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{\frac{-x^2}{2}}.$$

We can prove this in the following way.

$$
\begin{aligned}
1-\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}} \\
&\le \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{u}{x}e^{-\frac{u^2}{2}}du \text{ [as } u \ge x \text{]} \\
&= \frac{1}{x\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}d\left(\frac{u^2}{2}\right) \\
&= \frac{1}{x\sqrt{2\pi}}\left[-e^{-\frac{u^2}{2}}\right]_x^\infty \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}.
\end{aligned}
$$

General Gaussian Distribution

The general Gaussian distribution is obtained from the standard Gaussian by shifting and scaling. If $X = \sigma Z + \mu$, where $\sigma > 0$, then $X \sim N(\mu, \sigma^2)$.

We can see that $E[X] = \sigma E[Z] + \mu$, which is μ, and $\mathrm{Var}(X) = \sigma^2 \mathrm{Var}(Z)$, which is σ^2.

Conversely, if $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$.

The CDF of X is given by

$$
\begin{aligned}
F_X(x) = P(X \le x) &= P(\sigma Z + \mu \le x) \\
&= P\left(Z \le \frac{x-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{x-\mu}{\sigma}\right).
\end{aligned}
$$

We can then easily find the PDF of X as

$$
\begin{aligned}
f_X(x) &= \frac{d}{dx}F_X(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) \\
&= \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}f_Z\left(\frac{x-\mu}{\sigma}\right) \\
&= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
\end{aligned}
$$

Thus, if X is a normal RV with mean μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$.

We can see, then, that a linear transformation of a Gaussian RV is another Gaussian RV. More concretely, if $X \sim N(\mu_X, \sigma_X^2)$ and $Y = aX + b$, then $Y \sim N(a\mu_X + b, a^2\sigma_X^2)$.

Gamma Distribution

The gamma function $\Gamma(x)$ is an extension of the factorial function to real numbers.