Probability and Statistics (MA6.101)

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Assignment 2

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Problem 1

10 shots are taken. Each has a probability 0.2 of hitting the target. The total number of hits is called X.

Part 1

We need to find the PMF of X. Consider the probability $P(X = n)$.

There are $\binom{10}{n}$ ways to choose n shots to have hit the target. Each shot then has a 0.2 chance of hitting the target. Furthermore, the remaining shots each have a 0.8 chance of *not* hitting the target. Therefore,

$$P(X = n) = \begin{cases} \binom{10}{n}(0.2)^n(0.8)^{10-n} & 0 \leq n \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

This is a binomial distribution.

Part 2

We wish to find the expectation and variance of X.

We know that since $X \sim \text{Binomial}(10, 0.2)$. This means that $E[X] = np = 2$.

For the variance, we use the fact that a binomial variable is the sum of n independent Bernoulli variables, and the fact that variance is linear for independent variables. Thus,
$\text{Var}(\text{Binomial}(n, p)) = n \cdot \text{Var}(\text{Bernoulli}(p))$.

Now, to find $\text{Var}(\text{Bernoulli}(p))$, let $B \sim \text{Bernoulli}(p)$ and $\mu = E[B]$. By definition, $\mu = 0 \cdot p + 1 \cdot (1 - p) = 1 - p$.

We know that $\mathrm{Var}(B) = E[(B-\mu)^2]$. Therefore,

$$\begin{aligned}
\mathrm{Var}(B) &= [0-(1-p)]^2 \cdot p + [1-(1-p)]^2 \cdot (1-p) \\
&= p(1-p)^2 + p^2(1-p) \\
&= p(1-p).
\end{aligned}$$

Therefore $\mathrm{Var}(\mathrm{Bernoulli}(n,p)) = n \cdot \mathrm{Var}(B) = np(1-p)$. This gives us $\mathrm{Var}(X) = 10 \cdot 0.2 \cdot 0.8 = 1.6$.

Part 3

We can see that $Y = 2X - 3$ (as he gains $2X$ dollars and loses 3). We know that expectation is linear; therefore $E[Y] = E[2X-3] = 2E[X] - 3$.

Hence, $E[Y] = 2(2) - 3 = 1$. Thus his expected profit is \$1.

Also, $Y^2 = 4X^2 - 12X + 9$, which means that $E[Y^2] = 4E[X^2] - 12E[X] + 9$. Moreover, $\mathrm{Var}(X) = E[X^2] - E[X]^2 \implies E[X^2] = 1.6 + 2^2 = 5.6$.

Substituting, $E[Y] = 4 \cdot 5.6 - 12 \cdot 2 + 9 = 7.4$. Therefore the variance of his profit is 7.4.

Part 4

In this case $Z = X^2$ is Tharun's profit. We need to find $E[Z] = E[X^2]$.

Consider $\mathrm{Var}(X)$. We know that $\mathrm{Var}(X) = E[(X-\mu)^2] = E[X^2 + \mu^2 - 2X\mu]$. From this we get

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] + E[\mu^2] - E[2X\mu] \\
&= E[X^2] + \mu^2 - 2\mu E[X] \\
&= E[X^2] - E[X]^2.
\end{aligned}$$

Therefore, $E[X^2] = \mathrm{Var}(X) + E[X]^2 = 1.6 + 2^2 = 5.6$.

Problem 2

X is a discrete random variable with the PMF

$$P_X(x) = \begin{cases} 0.1 & x = 0.2 \\ 0.2 & x = 0.4 \\ 0.2 & x = 0.5 \\ 0.3 & x = 0.8 \\ 0.2 & x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Part 1

The range of X is the set $\{0.2, 0.4, 0.5, 0.8, 1\}$.

Part 2

$$\begin{aligned} P(X \le 0.5) &= P(0.2) + P(0.4) + P(0.5) \\ &= 0.1 + 0.2 + 0.2 \\ &= 0.5. \end{aligned}$$

Part 3

$$\begin{aligned} P(0.25 < X < 0.75) &= P(0.4) + P(0.5) \\ &= 0.2 + 0.2 \\ &= 0.4. \end{aligned}$$

Part 4

The expectation of X can be found as

$$\begin{aligned} E[X] &= \sum_x x P(X = x) \\ &= (0.2 \cdot 0.1) + (0.4 \cdot 0.2) + (0.5 \cdot 0.2) + (0.8 \cdot 0.3) + (1 \cdot 0.2) \\ &= 0.02 + 0.08 + 0.10 + 0.24 + 0.2 \\ &= 0.64. \end{aligned}$$

To find the variance of X, first we calculate

$$\begin{aligned} E[X^2] &= \sum_x x^2 P(X = x) \\ &= (0.2)^2 \cdot 0.1 + (0.4)^2 \cdot 0.2 + (0.5)^2 \cdot 0.2 + (0.8)^2 \cdot 0.3 + (1)^2 \cdot 0.2 \\ &= 0.004 + 0.032 + 0.05 + 0.192 + 0.2 \\ &= 0.433. \end{aligned}$$

Then we get $\mathrm{Var}(X) = E[X^2] - E[X]^2 = 0.433 - 0.4096 = 0.0234$.

Problem 3

We have the PMF of X as

$$P_X(x) = \begin{cases} 0.2 & x = 0 \\ 0.2 & x = 1 \\ 0.3 & x = 2 \\ 0.3 & x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

We also have $Y = X(X-1)(X-2)$. Therefore,
if X is 0, then Y is 0;
if X is 1, then Y is 0;
if X is 2, then Y is 0; and
if X is 3, then Y is 6.

Thus,

$$\begin{aligned}P(Y=0) &= P(X=0) + P(X=1) + P(X=2) \\ &= 0.2 + 0.2 + 0.3 \\ &= 0.7,\end{aligned}$$

and

$$P(Y=6) = P(X=3) = 0.3.$$

We can write Y's PMF as

$$P_Y(y) = \begin{cases} 0.7 & y = 0 \\ 0.3 & y = 6. \end{cases}$$

Problem 4

A player is dealt 13 cards from a standard 52-card deck.

Part 1

(assuming replacement)

The probability that any card dealt is a king is equal to

$$\frac{\text{number of kings}}{\text{total number of cards}},$$

which is $\frac{4}{52} = \frac{1}{13}$. Therefore this is also the probability that the 13th card is a king.

Part 2

(assuming replacement)

Each of the first 12 cards dealt must *not* be a king. The probability of *not* getting a king on any given card is $1 - \frac{1}{13} = \frac{12}{13}$. Once this happens, the probability that the 13th card is a king is (as found above) $\frac{1}{13}$.

Therefore the the probability that the 13th card is the first king dealt becomes $(\frac{12}{13})^{12}(\frac{1}{13})$.

Problem 5

An average of 20 customers arrive per hour. We have defined X as the number of customers from 1300h to 1500h; we know that $X \sim \text{Poisson}(40)$ (as the interval is of 2 hours).

We wish to find $P(15 < X < 25)$. Since X is a Poisson random variable,

$$P(X = k) = \frac{e^{-40}40^k}{k!},$$

which implies that

$$\begin{aligned} P(15 < X < 25) &= \sum_{i=16}^{24} P(X = i) \\ &= \sum_{i=16}^{24} \frac{e^{-40}40^i}{i!} \\ &\approx 0.00448. \end{aligned}$$

Problem 6

Each program has a probability p of being correct. X is defined as the number of tries until the program works correctly.
We can see that $P(X = k) = p(1-p)^{k-1}$ for all $k \geq 1$. This is a geometric distribution, *i.e.*, $X \sim \text{Geometric}(p)$.

Now, by definition,

$$E[X] = \sum_{k \geq 1} kp(1-p)^{k-1}.$$

Let $f(x) = x + x^2 + x^3 + \cdots$. From this we know that $f'(x) = 1 + 2x + 3x^2 + \cdots$, which means that $E[X] = pf'(1-p)$.
We can now find $f'(x)$ as

$$f'(x) = \left(\frac{1}{1-x} - 1\right)' = \frac{1}{(1-x)^2}.$$

Substituting, $E[X] = p\frac{1}{p^2} = \frac{1}{p}$.

For the variance, we have $\text{Var}(X) = E[X^2] - E[X]^2$. We need to find $E[X^2] = \sum_{k \geq 1} k^2 p(1-p)^{k-1}$.
With $f(x)$ as defined above, $xf'(x) = x + 2x^2 + 3x^3 + \cdots$, from which we get $(xf'(x))' = 1 + 2^2x + 3^2x^2 + \cdots$. Therefore $E[X^2] = p(xf'(x))' \mid_{x=1-p}$.

Now $xf'(x) = \frac{x}{(1-x)^2}$. Therefore,

$$(xf'(x))' = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

This gives us

$$E[X^2] = p\left(\frac{2-p}{p^3}\right) = \frac{2-p}{p^2},$$

from which we can conclude that

$$\text{Var}(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

Problem 7

We have $X \sim \text{Poisson}(\alpha)$, $Y \sim \text{Poisson}(\beta)$ and $Z = X + Y$. We wish to find Z's PMF.

$$\begin{aligned}
P(Z = n) &= \sum_{k=1}^{n-1} P(X = k)P(Y = n-k) \\
&= \sum_{k=1}^{n-1} \left(\frac{e^{-\alpha}\alpha^k}{k!}\right)\left(\frac{e^{-\beta}\beta^{n-k}}{(n-k)!}\right) \\
&= e^{-(\alpha+\beta)} \sum_{k=1}^{n-1} \binom{n}{k} \alpha^k \beta^{n-k} \\
&= \frac{e^{-(\alpha+\beta)}(\alpha+\beta)^n}{n!}.
\end{aligned}$$

Thus, we have $Z \sim \text{Poisson}(\alpha + \beta)$.

Problem 8

Each paper receives a grade from the set $G = \{A, A-, B, B-, C, C-\}$ with equal probability. Let X be the number of papers handed in before each possible grade is received at least once.

Consider $P(X = n)$. For X to have the value n, the first $n-1$ papers must have grades from some subset of G containing only 5 grades; and the n^{th} paper must have the remaining grade. There are 6 ways to pick the grade that the n^{th} paper is the first to get; the probability of each paper from the first $n-1$ receiving a grade from the other five is $\frac{5}{6}$; and the probability that the last paper gets the chosen grade is $\frac{1}{6}$. Therefore,

$$P(X = n) = 6 \cdot \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right).$$

Hence $X = 6Y$ where $Y \sim \text{Geometric}(\frac{1}{6})$. We wish to find $E[X] = 6E[Y]$.

It has been shown (Problem 6) that the expected value of a variable having a geometric distribution with parameter p is $\frac{1}{p}$. Therefore $E[Y] = \frac{1}{\frac{1}{6}} = 6$, which gives us $E[X] = 36$.

Problem 9

A mosquito lands with probability 0.5 each second. Therefore the number of mosquitoes (say X) landing per second follows a Poisson distribution, *i.e.*, $X \sim \text{Poisson}(0.5)$.

Further, the expected value for a Poisson distribution is $\lambda = 0.5$ itself. Therefore an average of $E[X] = 0.5$ mosquitoes lands every second, which implies that one mosquito lands every $\frac{1}{E[X]} = 2$ seconds. Thus the expected time between two mosquito *landings* is 2 seconds.

It is given that with a probability of 0.2, the mosquito (having landed) will bite. This means that one-fifth of the mosquito landings result in a bite, which tells us that the expected interval between two mosquito bites is 10 seconds.

We have shown that the time between successive bites $T = \frac{5}{X}$. Therefore, to find $\text{Var}(X)$, we must calculate $\text{Var}\left(\frac{5}{X}\right)$; but the variance of the reciprocal of a random variabl has no relation to its own variance. Therefore the variance of the time cannot be found.

Problem 10

X has mean μ_X and variance σ_X^2.
Y has mean μ_Y and variance σ_Y^2.

Since $Z = 3X + 4Y$, we know that $E[Z] = \mu_Z = 3E[X] + 4E[Y]$ (linearity of expectation).
Substituting, $\mu_Z = 3\mu_X + 4\mu_Y$.

To find $\text{Var}(Z)$, we can use the fact that X and Y are independent, allowing us to directly sum their variances. This gives us $\text{Var}(Z) = \text{Var}(3X) + \text{Var}(4Y)$, from which we get $\text{Var}(Z) = 9\text{Var}(X) + 16\text{Var}(Y)$.
Again, we can immediately substitute to get $\sigma_Z^2 = 9\sigma_X^2 + 16\sigma_Y^2$.