Probability and Statistics (MA6.101)

Monsoon 2021, IIIT Hyderabad
Assignment 4

Abhinav S Menon

Problem 1

We have a random variable $X \sim \text{Exponential}(\lambda)$, *i.e.*, $f_X(x) = \lambda e^{-\lambda x}$.

Part 1

We want to know the probability $P(X > 5)$. We can find it as follows:

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - F_X(5). \end{aligned}$$

Given $f_X(x)$, we know that $F_X(x) = 1 - e^{-\lambda x}$. This gives us

$$\begin{aligned} P(X > 5) &= 1 - F_X(5) \\ &= 1 - (1 - e^{-\lambda \cdot 5}) \\ &= e^{-5\lambda}. \end{aligned}$$

Part 2

Now, we want to calculate $P(X > 15 \mid X > 10)$.

We know that the exponential distribution has no memory, *i.e.*, that

$$P(X > x + a \mid X > x) = P(X > a).$$

Letting $x = 10, a = 5$ in this expression, we can see that $P(X > 15 \mid x > 10) = P(X > 5)$. We have found above that this is $e^{-5\lambda}$.

Problem 2

We have the PDF

$$f_R(r) = \begin{cases} 2e^{-2r} & r \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $R \sim \text{Exponential}(2)$.
We are given that the variable T is defined as $T = \frac{1}{R}$. We need to find its PDF.

Now, the function $g(x) = \frac{1}{x}$ is monotonically decreasing on the interval $[0, \infty)$. Thus we need to find the PDF of $T = g(R)$.

We can apply the usual rule for transformations, *i.e.*, for variables X, Y, such that $Y = g(X)$,

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{|g'(x_1)|} & g(x_1) = y \\ 0 & \text{otherwise,} \end{cases}$$

which means that

$$f_T(t) = \begin{cases} \frac{f_R(r)}{|-\frac{1}{r^2}|} & \frac{1}{r} = t \\ 0 & \text{otherwise.} \end{cases}$$

Since $\frac{1}{r} = t$, we know that $r = \frac{1}{t}$. Then we can substitute this along with f_R in the expression for f_T to find

$$f_T(t) = \begin{cases} \frac{2e^{-\frac{2}{t}}}{t^2} & t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3

Problem 4

We have a CRV $X \sim \text{Exponential}(\lambda)$, *i.e.*, $f_X(x) = \lambda e^{-\lambda x}$ for all $x \geq 0$.

Part 1

We want to find the CDF F_X of X. We can do this straightforwardly using integration:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(u) du \\ &= \int_{0}^{x} \lambda e^{-\lambda u} du \\ &= \lambda \cdot \frac{1}{-\lambda} [e^{-\lambda u}]_{u=0}^{u=x} \\ &= -[e^{-\lambda x} - e^{-\lambda \cdot 0}] \\ &= 1 - e^{-\lambda x}. \end{aligned}$$

Part 2

We are given that $Y = X^2$. We wish to find F_Y and f_Y.

Let $g(x) = x^2$. We can see that g is then a monotonically increasing function on $[0, \infty)$, and so we can apply the usual transformation. Thus, for F_Y, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) \\ &= 1 - e^{-\lambda\sqrt{y}}. \end{aligned}$$

We can find f_Y from this as

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= 0 - \left(e^{-\lambda\sqrt{y}} \frac{d}{dy}(-\sqrt{y}) \right) \\ &= \frac{e^{-\lambda\sqrt{y}}}{2\sqrt{y}}. \end{aligned}$$

for all $y \geq 0$, and 0 otherwise.

Problem 5

We are given a CRV X with range $[0, 3]$ and PDF $f_X(x) = kx^2$. We are also given a CRV $Y = X^3$.

Part 1

We want to find k and the CDF F_X of X.

To find k, we recall that $\int_{-\infty}^{\infty} f_X(u)du = 1$. We have

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(u)du &= \int_0^3 ku^2 du \\ &= k \left[\frac{u^3}{3} \right]_0^3 \\ &= 9k. \end{aligned}$$

Since this is equal to 1, we have $k = \frac{1}{9}$.

For F_X, we can again integrate to find

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(u)du \\
&= \int_0^3 \frac{1}{9}u^2 du \\
&= \left[\frac{1}{9}\frac{u^3}{3}\right]_0^x \\
&= \frac{x^3}{27}.
\end{aligned}$$

Part 2

We wish to find the expected value $E[Y]$.

We can find it as

$$\begin{aligned}
E[Y] &= \int_0^3 x^3 f_X(x)dx \\
&= \int_0^3 \frac{x^5}{9} \\
&= \left[\frac{x^6}{54}\right]_0^3 \\
&= \frac{729}{54} = 13.5
\end{aligned}$$

Part 3

We need to find the variance $\mathrm{Var}(Y)$.

We can use the identity $\mathrm{Var}(Y) = E[Y^2] - E[Y]^2$. We have found $E[Y]$ already; we need to find $E[Y^2]$. We can do so by noting that $Y^2 = X^6$ and following the same method as above, to get

$$\begin{aligned}
E[Y^2] &= \int_0^3 x^6 f_X(x)dx \\
&= \int_0^3 \frac{x^8}{9} \\
&= \left[\frac{x^9}{81}\right]_0^3 \\
&= 3^{9-4} = 243.
\end{aligned}$$

Therefore, $\mathrm{Var}(Y) = 243 - (13.5)^2 = 60.75$.

Part 4

We wish to find the PDF f_Y of Y. We can proceed using the transformation as before:

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{3x_1^2} & x_1^3 = y \\ 0 & \text{otherwise.} \end{cases}$$

We can apply this method as the cube function is strictly increasing in $[0, 3]$.

Note that the range of Y is $[0, 27]$. Now we can substitute $x_1 = \sqrt[3]{y}$ and f_X to find the distribution of Y as

$$f_Y(y) = \begin{cases} \frac{\sqrt[3]{y^2}}{9} \frac{1}{3\sqrt[3]{y^2}} & 0 \leq y \leq 27 \\ 0 & \text{otherwise,} \end{cases}$$

and cancel to get

$$f_Y(y) = \begin{cases} \frac{1}{27} & 0 \leq y \leq 27 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6

X is given to be the result of rolling a 4-sided die, and Y that of a 6-sided die. Z is the average of X and Y. Thus we have

$$f_X(x) = \begin{cases} \frac{1}{4} & x \in \{1, 2, 3, 4\} \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{6} & x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $Z \in \{1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5\}$. We can find its distribution on a case-by-case basis:

$$
\begin{aligned}
f_Z(1) &= f_X(1)f_Y(1) = \frac{1}{24} \\
f_Z(1.5) &= f_X(1)f_Y(2) + f_X(2)f_Y(1) = \frac{1}{12} \\
f_Z(2) &= f_X(1)f_Y(3) + f_X(2)f_Y(2) + f_X(3)f_Y(1) = \frac{1}{8} \\
f_Z(2.5) &= f_X(1)f_Y(4) + f_X(2)f_Y(3) + f_X(3)f_Y(2) + f_X(4)f_Y(1) = \frac{1}{6} \\
f_Z(3) &= f_X(1)f_Y(5) + f_X(2)f_Y(4) + f_X(3)f_Y(3) + f_X(4)f_Y(2) = \frac{1}{6} \\
f_Z(3.5) &= f_X(1)f_Y(6) + f_X(2)f_Y(5) + f_X(3)f_Y(4) + f_X(4)f_Y(3) = \frac{1}{6} \\
f_Z(4) &= f_X(2)f_Y(6) + f_X(3)f_Y(5) + f_X(4)f_Y(4) = \frac{1}{8} \\
f_Z(4.5) &= f_X(3)f_Y(6) + f_X(4)f_Y(5) = \frac{1}{12} \\
f_Z(5) &= f_X(4)f_Y(6) = \frac{1}{24}
\end{aligned}
$$

Part 1

We want to find $\sigma_X, \sigma_Y, \sigma_Z$. First, we will find the variances of the variables.

For X, we know that

$$E[X] = \frac{1}{4}\sum_{i=1}^{4} i = 2.5,$$

and

$$E[X^2] = \frac{1}{4}\sum_{i=1}^{4} i^2 = \frac{15}{2}.$$

Therefore,

$$
\begin{aligned}
\sigma_X^2 &= \frac{15}{2} - \frac{25}{4} \\
&= \frac{5}{4} = 1.25
\end{aligned}
$$

Therefore $\sigma_X = \sqrt{1.25} \approx 1.118$.

For Y, we know that

$$E[Y] = \frac{1}{6}\sum_{i=1}^{6} i = 3.5,$$

and

$$E[Y^2] = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{91}{6}$$

Therefore,

$$\begin{aligned}
\sigma_Y^2 &= \frac{91}{6} - \frac{49}{4} \\
&= \frac{35}{12} \approx 2.917
\end{aligned}$$

Therefore $\sigma_Y = \sqrt{2.917} \approx 1.708$.

For Z, we know that

$$E[Z] = E\left[\frac{1}{2}(X+Y)\right] = \frac{1}{2}(E[X] + E[Y]) = 3,$$

and

$$\begin{aligned}
E[Z^2] = E\left[\frac{1}{4}(X+Y)^2\right] &= \frac{1}{4}(E[X^2] + E[Y^2] + 2E[XY]) \\
&= \frac{1}{4}\left(\frac{45+91+105}{6}\right) \\
&= \frac{241}{24}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma_Z^2 &= \frac{241}{24} - 9 \\
&\approx 1.0417
\end{aligned}$$

Therefore $\sigma_Z = \sqrt{1.0417} \approx 1.02$.

Part 2

Let W be a random variable indicating the amount won. We can find its probability distribution as follows.

$$\begin{aligned}
f_W(2) &= f_X(1)P(Y < 1) = 0 \\
f_W(4) &= f_X(2)P(Y < 2) = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24} \\
f_W(6) &= f_X(3)P(Y < 3) = \frac{1}{4} \cdot \frac{2}{6} = \frac{1}{12} \\
f_W(8) &= f_X(4)P(Y < 4) = \frac{1}{4} \cdot \frac{3}{6} = \frac{1}{8} \\
f_W(-1) &= \frac{3}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned} E[W] &= (2)(0) + \frac{4}{24} + \frac{6}{12} + \frac{8}{8} + (-1)\left(\frac{3}{4}\right) \\ &= \frac{11}{12}. \end{aligned}$$

Hence, after playing 60 games, we have an expected winning of $60E[W] = 55$ dollars.

Problem 7

We are given

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

We wish to find $E[X^n]$ for any $n \in \mathbb{N}$. We can do so by integrating.

$$\begin{aligned} E[X^n] &= \int_0^1 x^n f_X(x) dx \\ &= \int_0^1 x^n \left(x + \frac{1}{2}\right) dx \\ &= \int_0^1 \left(x^{n+1} + \frac{1}{2}x^n\right) dx \\ &= \left[\frac{x^{n+2}}{n+2} + \frac{1}{2}\frac{x^{n+1}}{n+1}\right]_0^1 \\ &= \frac{3n+4}{(n+1)(n+2)}. \end{aligned}$$

Problem 8

We have a CRV X with PDF

$$f_X(x) = \begin{cases} x^2\left(2x + \frac{3}{2}\right) & 0 < x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

We are given that

$$Y = \frac{2}{X} + 3$$

and we need to find $\text{Var}(Y)$.

We can see that $g(x) = \frac{2}{x} + 3$ is strictly decreasing in the domain $(0, 1]$. Thus, we can apply the transformation, using $g'(x) = -\frac{2}{x^2}$.

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{|-\frac{2}{x_1^2}|} & \frac{2}{x_1} + 3 = y \\ 0 & \text{otherwise.} \end{cases}$$

Given g, we see that if $g(x_1) = y$, then $x_1 = \frac{2}{y-3}$. Thus, for $y \in [5, \infty)$, we have

$$\begin{aligned} f_Y(y) &= x_1^2 \left(2x_1 + \frac{3}{2}\right) \cdot \frac{x_1^2}{2} \\ &= \frac{1}{2} \left(\frac{2}{y-3}\right)^4 \left(\frac{4}{y-3} + \frac{3}{2}\right) \\ &= \frac{32}{(y-3)^5} + \frac{12}{(y-3)^4}. \end{aligned}$$

We now need to find $E[Y]$ and $E[Y^2]$. For $E[Y]$,

$$\begin{aligned} E[Y] &= \int_5^\infty \left[\frac{32y}{(y-3)^5} + \frac{12y}{(y-3)^4}\right] dy \\ &= \int_5^\infty \left[32\left\{\frac{y-3}{(y-3)^5}\right\} + \frac{96}{(y-3)^5} + 12\left\{\frac{y-3}{(y-3)^4}\right\} + \frac{36}{(y-3)^4}\right] dy \\ &= \int_2^\infty \left(\frac{96}{x^5} + \frac{68}{x^4} + \frac{12}{x^3}\right) dx \\ &= \frac{96}{-4}\left[\frac{1}{x^5}\right]_2^\infty + \frac{68}{-3}\left[\frac{1}{x^3}\right]_2^\infty + \frac{12}{-2}\left[\frac{1}{x^2}\right]_2^\infty \\ &= (-24)\left(-\frac{1}{16}\right) + \left(-\frac{68}{3}\right)\left(-\frac{1}{8}\right) + (-6)\left(-\frac{1}{4}\right) \\ &= \frac{3}{2} + \frac{17}{6} + \frac{3}{2} = \frac{35}{6}. \end{aligned}$$

For $E[Y^2]$,

$$\begin{aligned} E[Y^2] &= \int_5^\infty \left[\frac{32y^2}{(y-3)^5} + \frac{12y^2}{(y-3)^4}\right] dy \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_5^\infty \frac{32y^2}{(y-3)^5} dy \\ &= \int_5^\infty \left[32\frac{(y-3)^2}{(y-3)^5} - \frac{32 \cdot 9}{(y-3)^5} + \frac{32 \cdot 6y}{(y-3)^5}\right] dy \\ &= \int_5^\infty \left[\frac{32}{(y-3)^2} - \frac{288}{(y-3)^5} + 192\left\{\frac{y-3}{(y-3)^5}\right\} + \frac{192 \cdot 3}{(y-3)^5}\right] dy, \end{aligned}$$

and

$$
\begin{aligned}
I_2 &= \int_5^\infty \frac{12y^2}{(y-3)^4} dy \\
&= \int_5^\infty \left[12\frac{(y-3)^2}{(y-3)^4} - \frac{12 \cdot 9}{(y-3)^4} + \frac{12 \cdot 6y}{(y-3)^4} \right] dy \\
&= \int_5^\infty \left[\frac{12}{(y-3)^2} - \frac{108}{(y-3)^4} + 72 \left\{ \frac{y-3}{(y-3)^4} \right\} + \frac{72 \cdot 3}{(y-3)^4} \right] dy.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
E[Y^2] &= I_1 + I_2 \\
&= \int_5^\infty \left[\frac{12}{(y-3)^2} + \frac{32+72}{(y-3)^3} + \frac{192+216-108}{(y-3)^4} + \frac{576-288}{(y-3)^5} \right] dy \\
&= \int_2^\infty \left(\frac{12}{x^2} + \frac{104}{x^3} + \frac{300}{x^4} + \frac{288}{x^5} \right) dx \\
&= \frac{12}{-1} \left[-\frac{1}{x} \right]_2^\infty + \frac{104}{-2} \left[-\frac{1}{x^2} \right]_2^\infty + \frac{300}{-3} \left[-\frac{1}{x^3} \right]_2^\infty + \frac{288}{-4} \left[\frac{1}{x^4} \right]_2^\infty \\
&= (-12) \left(-\frac{1}{2} \right) + (-52) \left(-\frac{1}{4} \right) + (-100) \left(-\frac{1}{8} \right) + (-72) \left(-\frac{1}{16} \right) \\
&= 6 + 13 + \frac{25}{2} + \frac{9}{2} = 36.
\end{aligned}
$$

Now we can find $\text{Var}(Y)$ as

$$
\begin{aligned}
\text{Var}(Y) &= E[Y^2] - E[Y] \\
&= 36 - \left(\frac{35}{6} \right)^2 \\
&= 6^2 - \left(6 - \frac{1}{6} \right)^2 = \left(\frac{1}{6} \right) \left(\frac{71}{6} \right) = \frac{71}{36}.
\end{aligned}
$$