# A New Spiking Neuron Model

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*Abstract*— **A biologically realistic spiking neuron model has been proposed which contains a novel non linear spiking function. Proposed neuron model contains a lower order spike generating function in contrast to the spike generating function of Quadratic integrate fire neuron model. It is found that lower order terms of spike generating function is sufficient for generating biologically realistic spikes. It has been proved that proposed neuron model satisfies saddle node bifurcation. The spikes generated by the proposed neuron model has been compared with the spikes of existing spiking neuron models like Leaky Integrate fire neuron model and Quadratic Integrate and fire neuron model.**

## I. Introduction

VARIOUS spiking neuron models have been developed in the literature in the past. Hodgkin & Huxley model [7] deals with experimental studies of the giant axon of the squid in four differential equations. Hindmarsh-Rose model [8] is a three dimensional model that allows dynamic behavior for the membrane potential. Due to its nonlinearity and high dimensionality, various simpler models were proposed [1]. FitzHugh–Nagumo model [5] is a simplified version of the Hodgkin–Huxley model which models the activation and deactivation dynamics of a spiking neuron. Rinzel et al [11] explained phase plane analysis of stationary and oscillatory states of FitzHugh-Nagumo [5] model. Morris-Lucar model [10] combines the best aspects of Hodgkin-Huxley [7] and FitzHugh-Nagumo [5] models. It describes the relationship between membrane potential and the activation of ion channels within the membrane. Integrate-and-fire neuron [1] represents biological neuron as an RC-circuit. It consists of a capacitor in parallel with a resistor driven by a current. In the leaky integrate-and-fire neuron model [12], a leak term is added to the membrane potential in the Integrate and Fire neuron model. A general non linear integrate and fire neuron model was formulated by Abbott et al [2]. Leaky integrate and fire neuron model, quadratic model [4] and exponential integrate and fire neuron models [3] are derived as special instances of the spiking generating function of general non linear integrate and fire neuron model [6]. For a spike with very sharp intention, it is proved that exponential integrate and fire neuron model is equivalent to leaky integrate fire neuron model [6]

In this paper, a new spiking function has been designed for general non linear Integrate and Fire neuron model which generate spikes using lower order terms. The new spiking function is incorporated in the general non linear Integrate and Fire Neuron model. In contrast to Quadratic integrate fire neuron model, proposed neuron model contains a lower order spike generating function. It is shown that the proposed function satisfies voltage threshold and spike initiation conditions [6]. Analytical proof is given for non-hyperbolicity, non-degeneracy and transversality properties of the proposed spiking function using stable points to satisfy Saddle-node bifurcation [9]. It has been proved that the proposed neuron model is equivalent to quadratic integrate and fire neuron model at high frequencies. The spike generation mechanism and biological reliability of the proposed neuron model has been compared with existing spiking neuron models like Leaky Integrate fire neuron model and Quadratic Integrate and fire neuron model.

Rest of the paper is organized as follows. Section II describes existing Leaky Integrate fire neuron model, Quadratic and exponential integrate fire neuron models. Section III presents the proposed Spiking neuron Model. In Section IV describes the results obtained from proposed spiking neuron model when compared with LIF, QIF and EIF neuron models.

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## II. Nonlinear integrate-and-fire neuron models

Biological neurons commonly known as spiking neurons transmit information by spikes or action potentials. The behavior of spiking neurons is explained in terms of firing rate. In order to simulate a spiking neuron various spiking neuron models like Leaky Integrate Fire neuron, Quadratic Integrate fire neuron and Exponential Integrate and Fire Neuron are used. The family of non linear Integrate and Fire neuron models is given by

$$C\frac{dV}{dt} = -g_L(V - V_L) + \psi(V) + I_{syn}(t) \tag{1}$$

Where $V$ denotes membrane potential, $C$ is the membrane capacitance, $g_L$ is the leak conductance, $V_L$ is the leak potential, $I_{syn}$ is the external synaptic current and $\psi(V)$ is a function of voltage that describes the spike-generating currents.

The slope of the I-V (current – voltage) curve at the threshold voltage is given by

$$\psi'(V_T) = g_L \tag{2}$$

The threshold current $(I_T)$ at the steady voltage $V_T$ is

$$I_T = -g_L(V_T - V_L) + \psi(V_T) \tag{3}$$

Input current greater than threshold current generates tonic spiking. The spike slope factor $\Delta_T$ is defined as

$$\Delta_T = \frac{g_L}{\psi''(V_T)} \tag{4}$$

The parameter $\Delta_T$ is inversely proportional to the curvature of the I–V curve at the threshold $V_T$ and measures the sharpness of spike initiation. Specific functions of spike-generating voltage functions $\psi(V)$ in "(1)" yield various spiking neuron models which are discussed in following subsections.

### A. *Leaky Integrate-and- Fire (LIF)*

Leaky Integrate Fire (LIF) neuron model [13] can be derived as a special instance of the family of neuron model by substituting $\psi(v) = 0$ in "(1)". In this model, a spike threshold $V_{th}$ is introduced to obtain spike generation [9]. The generated spike is flashed and the neuron is reset to a voltage $V_r$ after a spike. The Leaky Integrate Fire neuron is given as follows

$$C\frac{dV}{dt} = -g_L(V - V_L) + I_{syn}(t) \tag{5}$$

If $V \geq V_{th}$ then $V \leftarrow V_r$

Where C is the membrane capacitance, $g_L$ is the leak conductance, $V_L$ is the leak potential and $I_{syn}$ is the external synaptic current.

The solution of "(5)" under the initial condition $V(0) = V_r = 0$ is given by

$$V_L + \frac{I}{g_L} - \frac{(g_C V_L + I)}{g_L} * \exp\left(-\frac{R g_L t}{\tau}\right) = V(t) \tag{6}$$

Substitute $V = V_{th}$ yields the following equation

$$V_L + \frac{I}{g_L} - \frac{(g_C V_L + I)}{g_L} * \exp\left(-\frac{R g_L t^1}{\tau}\right) = V_{th} \tag{7}$$

The time of the first spike can be found by solving above equation and is given by

$$t = \frac{\tau}{R g_L}\log\left(\frac{g_L}{g_C V_L + I}\left(V_L + \frac{I}{g_L} - V_{th}\right)\right) \tag{8}$$

The mean firing rate of the neuron f with absolute refractory period $\Delta_{abs}$ is given by $1/t$

$$f = \left(\Delta_{abs} + \frac{\tau}{R g_L}\log\left(\frac{g_L}{g_C V_L + I}\left(V_L + \frac{I}{g_L} - V_{th}\right)\right)\right)^{-1}$$

(9)

### B. *Quadratic Integrate-and- Fire (QIF)*

The Quadratic Integrate Fire (QIF) neuron model is the simplest model of spiking neuron which is given as follows

$$C\frac{dV}{dt} = V^2 + I_{syn}(t) \tag{10}$$

if $V \geq V_{th}$ then $V \leftarrow V_r$

It is the topological normal form for the saddle node bifurcation [9] with the spike resetting condition. It is observed from "(10)" that equilibrium points are $\sqrt{|I|}$ and $-\sqrt{|I|}$ when $I < 0$. There are no equilibrium points when $I > 0$.If the injected current I changes from positive to negative value then membrane potential follows the resting state $-|I|$ in a quasi-static way till the bifurcation point $I = 0$ is reached. QIF model is derived by substituting the spike generating function $\psi(v)$ in "(1)". $\psi(v)$ is given by

$$\psi(v) = \frac{g_L}{2\Delta_T}\left((v - v_T)^2\right) + g_L\left((v - v_T)\right) - I_T \tag{11}$$

Where $I_T$ is the current threshold, $v_T$ and $\Delta_T$ are the spike threshold and sloping factor respectively. It can be observed that "(11)" satisfies properties defined in "(2)" and "(4)".

### C. *Exponential Integrate-and- Fire (EIF)*

Exponential Integrate Fire (EIF) neuron model is simulated by the following one dimensional differential equation which is given by

$$C\frac{dV}{dt} = -g_L(V - V_L) + g_L\Delta_T\exp\left(\frac{V - V_T}{\Delta_T}\right) + I_{syn}(t) \tag{12}$$

It is derived by substituting the following spike generating function $\psi(v)$ in "(1)"

$$\psi(v) = g_L\Delta_T\exp\left(\frac{V - V_T}{\Delta_T}\right) \tag{13}$$

$\psi(v)$ satisfies "(2)" and "(4)". $V_T$ and $\Delta_T$ are the spike threshold and sloping factor respectively. When $\Delta_T \to 0$ (denoting a spike with sharp initiation) the EIF model becomes equivalent to the LIF model with $V_{th} = V_T$

## III. Proposed Neuron model

An attempt has been made in this paper to generate spikes akin to QIF model with lower order terms. The spike generating function used in QIF is of order 2. Serpentine curves [14] have been used to design proposed spike generating function such that the rate of modulation of proposed spike generating function $\psi(V)$ decays faster than $\frac{1}{f}$ when the input to the model is a sinusoidal and weakly modulated current with frequency (f). The proposed spiking function is given by

$$\psi(V) = \frac{g_L}{2 v_T \Delta_T}\left(V - \frac{V}{(1 + (V - V_T)^2)}\right) + g_L V \tag{14}$$

Where $\Delta_T$ denotes spike sloping factor. The order of the new spiking function is of order 1 only.

Substituting the value of $\psi(V)$ in "(1)", the new neuron model is obtained as follows

$$C\frac{dV}{dt} = -g_L(V - V_L) + \frac{g_L}{2v_T\Delta_T}\left(V - \frac{V}{(1+(V-V_T)^2)}\right) + g_L V + I_{syn}(t) \quad (15)$$

The first derivative of $\psi(v)$ from "(14)" is given by

$$\psi'(v) = \left(\frac{-g_L}{2v_T\Delta_T}\right)\left(\frac{(1+(v-v_T)^2)-v(2v(v-v_T))}{1+(1+(v-v_T)^2)^2}\right) \quad (16)$$

From the above equation, it is seen that the proposed neuron model satisfies "(2)" of non linear spiking neuron models. Further it is observed that proposed neuron model also satisfies the condition related to spike sloping factor i.e.

$$\psi'(v_T) = \frac{g_L}{\Delta_T}.$$

The spike generating function in QIF model decays faster than $\frac{1}{f^2}$ at high frequencies [6]. The following lemma shows the possible behavior of proposed neuron model for the firing rate modulation at high frequencies.

***Lemma 3.1***

The rate modulation of proposed spike generating function $\psi(V)$ decays faster than $\frac{1}{f}$ when a sinusoidal and weakly modulated current $I_{syn}(t) = I_0 + I_1\cos(2\pi ft) + I_{noise}$ is input to the proposed model.

***Proof:***

TABLE I

PARAMETERS USED TO DRAW VOLTAGE CURVES

| Neuron Models | $g_L$ | $v_L$ | $\Delta_T$ | $v_T$ |
|---|---|---|---|---|
| Proposed model | 0.2 | 0.1 | 0.1 | 0.05 |
| QIF | 0.01 | 0.1 | 0.05 | 0.1 |

If v then the spike

$$\lim_{v\to\infty} \psi(v) \quad v\to\infty \frac{av}{(v-v_T)^2}+\frac{bv}{(v-v_T)^2}+bv \quad (17)$$

Hence it is proved that proposed spiking function decays faster than $\frac{1}{f}$ at high frequencies.

Let the right hand side of the proposed neuron model shown in Eq. (15) be denoted by $F(V, I)$ which is a non linear membrane potential function

$$C\frac{dV}{dt} = F(V, I) \quad (18)$$

The plot of the two dimensional function $F(V, I)$ at I =2 amp using $g_l = 0.8, v_T = 2, t = 1$ is shown in Fig-1. This shows that the function has a stable point at $v_T = 2$. The function $F(V, I)$ has to satisfy three properties viz. non-hyperbolicity, non-degeneracy and transversality for Saddle-node bifurcation [9]. It is proved in the appendix that $F(V, I)$ satisfies Saddle-node bifurcation.

Voltage curves have also been drawn for LIF, QIF neuron models and compared with the proposed neuron model. For all the models, I=1amp; C=1 $\mu F/cm^2$; dt=0.01; V(1)=0.2 mv. Fig-2 and Fig-3 show the time versus voltage curves for proposed neuron model and quadratic Integrate and Fire neuron model for various parameter values as shown in Table 1. It is seen from Fig-2 and Fig-3 that the voltage curves of QIF and proposed neuron model are similar. The plot for time versus Voltage is shown in Fig-4 for Leaky Integrate fire neuron model. It is drawn by using gl = 0.2 $ms/cm^2$, vl = 0.1 mV.

## IV. COMPARISON OF SPIKE GENERATION USING DIFFERENT NEURON MODELS

This section presents the spike dynamics of the proposed neuron model, Leaky Integrate and fire model and Quadratic Integrate and Fire model. It is shown that proposed neuron model generates spikes equivalent to Quadratic Integrate and fire neuron model with lower order terms of spike generating function. Euler's method is used for generation of spikes using various models. Fig-5 shows the spikes generated using Leaky Integrate and fire neuron model. In general, Leaky Integrate and Fire neuron model do not generate any spikes [9]. In order to generate spikes, spike threshold condition has been incorporated. Fig-5 has been drawn using threshold = 4 and $I_0 = 1$Amp. Fig-6 shows the spiking generated using proposed neuron model. Fig-6 has been drawn using external current I = 1Amp with $g_l = 0.2$, $v_l = 0.1$ and $\Delta t = 0.5$. Fig-7 shows the spikes generated using proposed neuron model using $\Delta t = 0.01$ with other parameters being the same. Fig-8 shows the spikes generated using QIF neuron model for the values $g_l = 0.2$, $v_l = 0.1$ and I = 1 Amp.

It can be observed from Fig-6 and Fig-8 that the spikes generated using proposed model and QIF are analogous to each other. QIF model satisfies saddle-node bifurcation when V = 0 and I = 0 amp [9]. The saddle node bifurcation of proposed model is proved in the appendix.

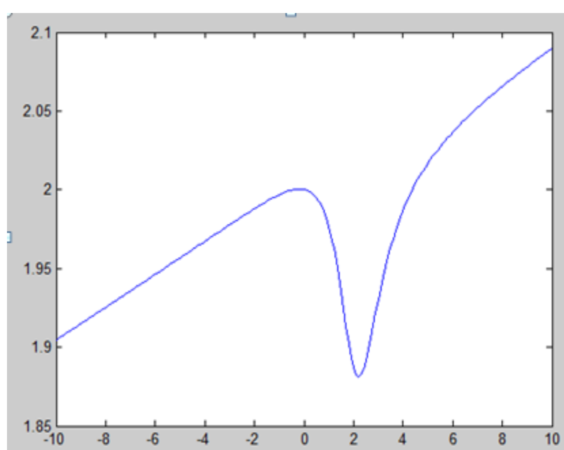


Fig. 1. Stable point of F(V,I) at voltage threshold = 2mV

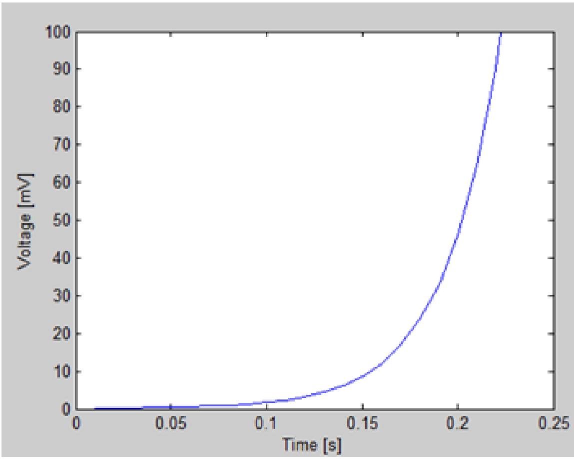

Fig. 2. Time versus Voltage plot for proposed neuron model

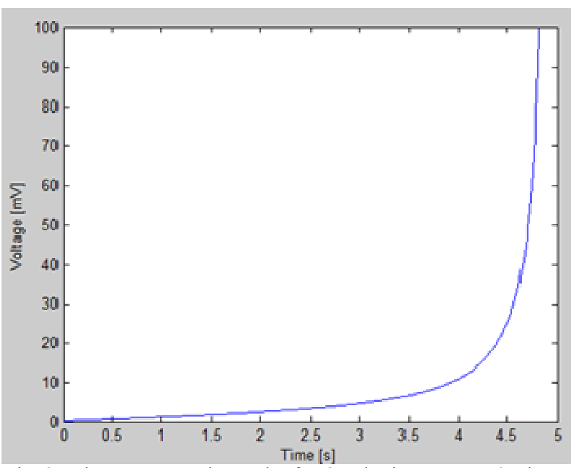

Fig. 3. Time versus Voltage plot for Quadratic Integrate & Fire neuron

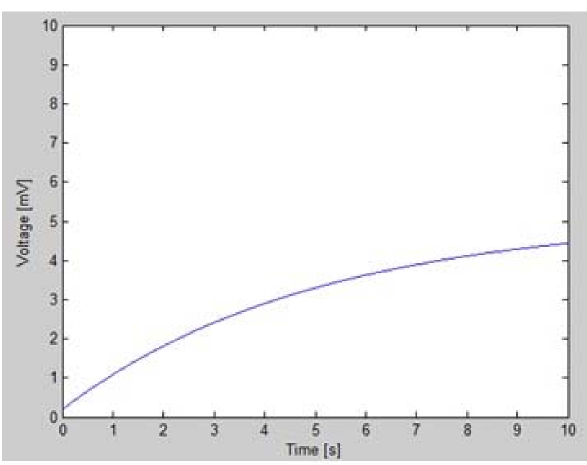

Fig. 4. Time versus Voltage plot for Leaky Integrate & Fire neuron model

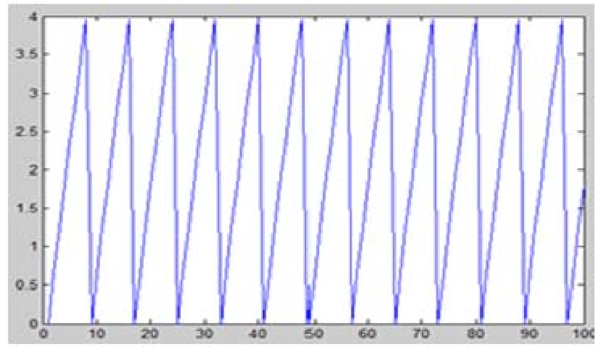

Fig. 5. Time versus Voltage plot for Leaky Integrate & Fire neuron model

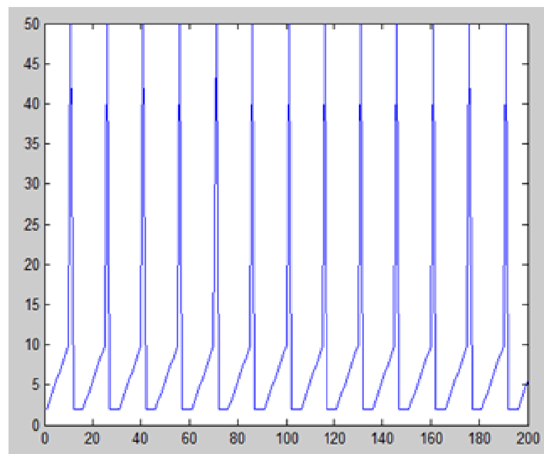

Fig. 6. Time versus Voltage plot for proposed neuron model.

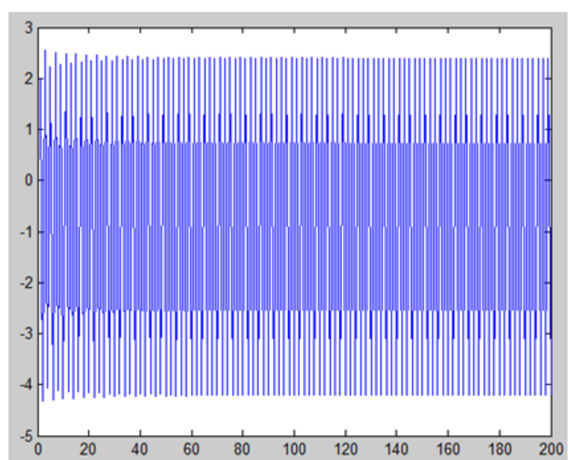

Fig. 7. Time versus Voltage plot for proposed neuron model.

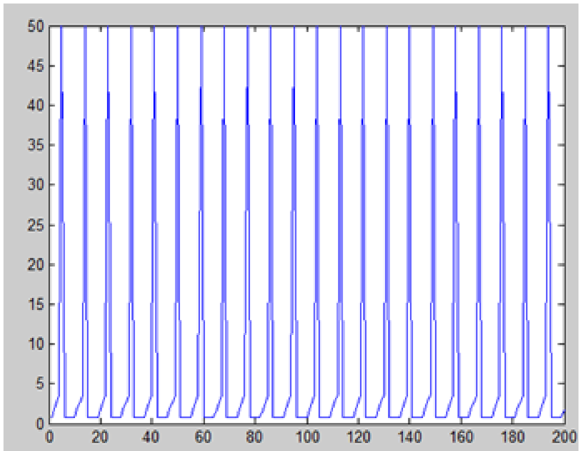

Fig. 8. Time versus Voltage plot for Quadratic Integrate and Fire neuron model.

## V. CONCLUSION

A biological realistic spiking neuron model has been evolved in this paper using a novel spike generating function. The new spike generating function satisfies saddle node bifurcation property. The proposed spike generating function decays faster than reciprocal of the frequency when the input to the model is sinusoidal and weakly modulated current. The new neuron model generates spikes equivalent to Quadratic Integrate and fire neuron model with lower order terms of spike generating function.

## APPENDIX

The explicit equation for $F(V, I)$ for the proposed model is given by

$$F(V,I) = g_L V_L + \frac{g_L}{2v_T \Delta_T}\left(V - \frac{V}{(1+(V-V_T)^2)}\right) + I_{syn}(t) \qquad (19)$$

$$F_V(V,I) = \frac{g_L}{2V_T\Delta_T}\left[1 - \frac{1+[V-V_T]^2-[2V[V-V_T]^2]}{[1+[V-V_T]^2]^2}\right] \qquad (20)$$

Where $F_V$ denotes the first derivative of F with respect to V.

The eigan value $\lambda$ is obtained by $\lambda = F_V(V,I) = 0$ at $V = V_{sn}$. It is observed that $F_V(V,I) = 0$ at $V = V_T$. This shows the proposed model satisfies Non-hyperbolicity property. Let the second order derivative for F is denoted by $F_{VV}$.

$$F_{VV}(V,I) = \frac{g_L}{2V_T\Delta_T}\frac{d}{dV}\left[\frac{V^2-{V_T}^2-1}{[1+[V-V_T]^2]^2}\right] \qquad (21)$$

Expanding R.H.S of "(21)",

$$F_{VV}(V,I) = \left[\frac{\left[1+[V-V_T]^2\right]^2(2V)-[V^2-V_T^2-1][1+[V-V_T]^2][V-V_T]}{[1+[V-V_T]^2]^4}\right] \qquad (22)$$

It is observed that $F_{VV}(V,I) \neq 0$ at $V = V_T$.

This shows that the proposed model satisfies Non-degeneracy property. It is observed that $F_I(V, I_{sn}) \neq 0$ for $V = V_T$.Hence the proposed model satisfies Transversality property. Hence $F(V_{sn}, I_{sn})$ results at saddle node bifurcation.

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