Linear Algebra, IIIT Hyderabad

Spring 2021

Assignment 5

3.1 An Orthonormal Basis for Boolean Functions

Let $\mathcal{F}$ be the set of functions with domain $\{+1, -1\}^n$ and range $\mathbb{R}$.
An inner product is defined on this space as

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{+1,-1\}^n} f(x)g(x)$$

with the corresponding norm

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

The following set of functions is defined:

$$\{\chi_S\}_{S \subseteq \{1,\ldots,n\}},$$

where

$$\chi_S(x) = \prod_{i \in S} x_i.$$

If $S = \emptyset$, $\chi_S(x) = 1$ for all x.

(a) RTP: The functions χ_S form an orthonormal basis under the inner product defined.
Proof: We will first show that the χ_S are orthogonal to each other, and then that they all have unit norm.

(i) The χ_S are orthogonal.
Consider the inner product of two distinct functions, $\langle \chi_S, \chi_T \rangle$.
We will show by induction on n that out of the 2^n values of x, $\chi_S(x) = \chi_T(x)$ for exactly 2^{n-1} values, and $\chi_S(x) = -\chi_T(x)$ for exactly 2^{n-1} values.
Base case, $n = 1$: There are only two basis functions, $\chi_\emptyset$ and $\chi_{\{1\}}$. Clearly,

$$\chi_\emptyset(1) = \chi_{\{1\}}(1) = 1$$

and

$$\chi_\emptyset(-1) = -\chi_{\{1\}}(-1).$$

This proves the base case.
Induction step: Suppose $n > 1$, and the statement is true for all smaller values.
Let $\mathcal{P}(\{1, ..., n\}) \cap \mathcal{P}(\{1, ..., n-1\})$ [$\mathcal{P}$ = powerset] be called the *first group* of subsets of $\{1, ..., n\}$ (*i.e.*, those which do not include n). Let

all other subsets of $\{1, ..., n\}$ be called the *second group*. Further, let $x = (x_1, ..., x_n)$. Now, we distinguish three cases with respect to S and T:

1. S and T both belong the first group. In this case, x_n does not affect the values of $\chi_S(x)$ and $\chi_T(x)$; *i.e.*,

   $$\chi_S(x_1, x_2, ..., x_n) = \chi_S(x_1, x_2, ..., x_{n-1})$$

   and

   $$\chi_T(x_1, x_2, ..., x_n) = \chi_T(x_1, x_2, ..., x_{n-1}),$$

   both of which exist since S and T are subsets of $\{1, ..., n-1\}$. By the induction hypothesis, then, the statement holds.

2. S and T both belong the second group [$S' = S - \{n\}$ and $T' = T - \{n\}$]. In this case, x_n is present in the expression of $\chi_S(x)$ and $\chi_T(x)$; *i.e.*,

   $$\chi_S(x_1, x_2, ..., x_n) = x_n \cdot \chi_{S'}(x_1, x_2, ..., x_{n-1})$$

   and

   $$\chi_T(x_1, x_2, ..., x_n) = x_n \cdot \chi_{T'}(x_1, x_2, ..., x_{n-1}),$$

   both of which exist since S' and T' are subsets of $\{1, ..., n-1\}$. Therefore, if $\chi_{S'}(x_1, x_2, ..., x_{n-1}) = \chi_{T'}(x_1, x_2, ..., x_{n-1})$, then $\chi_S(x) = \chi_T(x)$; and if $\chi_{S'}(x1, x_2, ..., x_{n-1}) = -\chi_{T'}(x1, x_2, ..., x_{n-1})$, then $\chi_S(x) = -\chi_T(x)$, for all x. Therefore, by the induction hypothesis, the statement holds.

3. One of S and T belongs to the first group, and the other to the second. WLOG, say S belongs to the first group and T to the second [$T' = T - \{n\}$]. Then,

   $$\chi_S(x_1, x_2, ..., x_n) = \chi_S(x_1, x_2, ..., x_{n-1})$$

   for all x.
   Now, if $x_n = 1$, then

   $$\chi_T(x_1, x_2, ..., x_n) = \chi_{T'}(x_1, x_2, ...x_{n-1}),$$

   which exists since T' is a subset of $\{1, ..., n-1\}$. Among all such values of x, then, half are such that $\chi_S(x) = \chi_T(x)$ and half such that $\chi_S(x) = -\chi_T(x)$.
   If $x_n = -1$, then

   $$\chi_T(x_1, x_2, ..., x_n) = -\chi_{T'}(x_1, x_2, ..., x_{n-1}).$$

   Therefore, if $\chi_S(x_1, x_2, ..., x_{n-1}) = \chi_{T'}(x_1, x_2, ..., x_{n-1})$, then $\chi_S(x) = -\chi_T(x)$; and if $\chi_S(x_1, x_2, ..., x_{n-1}) =$

$-\chi_{T'}(x_1, x_2, ..., x_{n-1})$, then $\chi_S(x) = \chi_T(x)$. Therefore, among all such values of x, half are such that $\chi_S(x) = \chi_T(x)$ and half such that $\chi_S(x) = -\chi_T(x)$. This proves the claim.

Therefore, for all distinct S and T, half of the values of x are such that $\chi_S(x) = \chi_T(x)$, and half are such that $\chi_S(x) = -\chi_T(x)$. Then, in the sum

$$\langle \chi_S, \chi_T \rangle = \frac{1}{2^n} \sum_{x \in \{+1,-1\}^n} \chi_S(x)\chi_T(x),$$

exactly half the terms (those for which $\chi_S(x) = \chi_T(x)$) have the value 1, and the remaining (those for which $\chi_S(x) = -\chi_T(x)$) have the value -1. Hence the sum is $2^{n-1} - 2^{n-1} = 0$, QED.

(ii) The χ_S have unit norm. Considering definition of the norm, we see that

$$\|\chi_S\| = \sqrt{\langle \chi_S, \chi_S \rangle} = \sqrt{\frac{1}{2^n} \sum_{x \in \{+1,-1\}^n} \chi_S^2(x)}.$$

Clearly, $\chi_S^2(x) = 1$ for all 2^n values of x. Therefore, the term under the square root is simply $\frac{2^n}{2^n} = 1$. Therefore $\|\chi_S\| = 1$, QED.

Now, we know that all χ_S are all orthogonal (and therefore linearly independent) and have unit norm. Since 2^n linearly independent vectors in a space of dimension 2^n must span it, the χ_S span $\mathcal{F}$. Therefore, they form an orthonormal basis, QED.

(b) Let $f \in \mathcal{F}$ be a function with range $\{+1, -1\}$, such that

$$f = \sum_{S \subseteq \{1,...,n\}} \hat{f}_S \chi_S,$$

where $\hat{f}_S \in \mathbb{R}$ for all $S \subseteq \{1, ..., n\}$.
RTP: $\sum_{S \subseteq \{1,...,n\}} (\hat{f}_S)^2 = 1$.
Proof: Consider the following enumeration of the values in $\{+1, -1\}^n$, which we will write as $x^{i|n}$ for $1 \leq i \leq 2^n$:
For $n = 1$,

$$x^{1|1} = (+1)$$
$$x^{2|1} = (-1).$$

For $n \geq 1$, $x^{1|n+1}$ to $x^{2^n|n+1}$ are respectively the values of $x^{1|n}$ to $x^{2^n|n}$, with $+1$ appended at the end of each. For all remaining $x^{i|n+1}$, we put the $x^{i|n+1}$'s in the same order as the $x^{i|n}$'s with -1 appended at the end of each. In other words, if we represent appending with a colon (:),

$$x^{i|n+1} = \begin{cases} x^{i|n} : (+1) & 1 \leq i \leq 2^n \\ x^{i-2^n|n} : (-1) & 2^n < i \leq 2^{n+1}. \end{cases}$$

For example,

$$x^{1|2} = (+1, +1),$$
$$x^{2|2} = (-1, +1),$$
$$x^{3|2} = (+1, -1),$$
$$x^{4|2} = (-1, -1).$$

Further, let us define an enumeration of the subsets of $\{1, ..., n\}$, which we will write as $S^{i|n}$ for $1 \leq i \leq 2^n$, as follows:
For $n = 1$,

$$S^{1|1} = \{\}$$
$$S^{2|1} = \{1\}.$$

For $n \geq 1$, $S^{i|n+1}$ to $S^{2^n|n+1}$ are respectively the exact same as $S^{i|n}$ to $S^{2^n|n}$. The remaining $S^{i|n+1}$ are the $S^{i|n}$ in the same order, with $n+1$ added to each. In other words,

$$S^{i|n+1} = \begin{cases} S^{i|n} & 1 \leq i \leq 2^n \\ S^{i|n} \cup \{n+1\} & 2^n < i \leq 2^{n+1}. \end{cases}$$

For example,

$$S^{1|2} = \{\}$$
$$S^{2|2} = \{1\}$$
$$S^{3|2} = \{\} \cup 2 = \{2\}$$
$$S^{4|2} = \{1\} \cup 2 = \{1, 2\}.$$

This enumeration of $S^{i|n}$ is equivalent to an enumeration of the basis functions χ. Here, we note an important property of this enumeration: if $j > 2^{n-1}$, then $\chi_{S^{j|n}}(x^{i|n-1} : (\xi)) = \xi \cdot \chi_{S^{j-2^{n-1}}|n}(x^{i|n-1})$. This is clear from the fact that $S^{j|n} = S^{j-2^{n-1}|n} \cup n$, and that the n^{th} entry of $x^{i|n-1} : (\xi)$ is ξ.

Now, we define a matrix X_n as

$$X_n = \begin{bmatrix} \chi_{S^{1|n}}(x^{1|n}) & \chi_{S^{2|n}}(x^{1|n}) & \cdots & \chi_{S^{2^n|n}}(x^{1|n}) \\ \chi_{S^{1|n}}(x^{2|n}) & \chi_{S^{2|n}}(x^{2|n}) & \cdots & \chi_{S^{2^n|n}}(x^{2|n}) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{S^{1|n}}(x^{2^n|n}) & \chi_{S^{2|n}}(x_{2^n|n}) & \cdots & \chi_{S^{2^n|n}}(x^{2^n|n}) \end{bmatrix},$$

i.e., $(X_n)_{ij} = \chi_{S^{j|n}}(x^{i|n})$, and two vectors $\hat{f}$ and v_f as

$$\hat{f} = \begin{bmatrix} \hat{f}_{S^{1|n}} \\ \hat{f}_{S^{2|n}} \\ \vdots \\ \hat{f}_{S^{2^n|n}} \end{bmatrix}, v_f = \begin{bmatrix} f(x^{1|n}) \\ f(x^{2|n}) \\ \vdots \\ f(x^{2^n|n}) \end{bmatrix}.$$

From the definition of the $\hat{f}_S$, it is clear that $v_f = X_n \hat{f}$, *i.e.*, $\hat{f} = X_n^{-1} v_f$.

X_n has two important properties, both of which we will prove by induction:
(i) X_n is symmetric. For the base case $n = 1$,

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

which is clearly symmetric.
Now, we will try to build X_{n+1} from X_n (which we can assume to be symmetric) for $n \geq 1$. Consider an arbitrary element $(X_{n+1})_{ij}$; we distinguish four cases:
1. $1 \leq i, j \leq 2^n$: This element is, by definition, equal to $\chi_{S^{j|n+1}}(x^{i|n+1})$. From the constraint on i and j, it is clear that this is the same as $\chi_{S^{j|n}}(x^{i|n} : (+1)) = \chi_{S^{j|n}}(x^{i|n})$, which is nothing but $(X_n)_{ij}$. Therefore, the top-left quarter of X_{n+1} is identical to X_n.
2. $1 \leq i \leq 2^n < j \leq 2^{n+1}$: This element is equal to $\chi_{S^{j|n+1}}(x^{i|n+1})$. From the constraint on i and j, this is the same as $\chi_{S^{j-2^n|n} \cup \{n+1\}}(x^{i|n} : (+1))$. By the property of the χ_S stated above, this is simply $(+1) \cdot \chi_{S^{j-2^n|n}}(x^{i|n}) = \chi_{S^{j-2^n|n}}(x^{i|n}) = (X_n)_{i(j-2^n)}$. Therefore, the top-right quarter of X_{n+1} is also identical to X_n.
3. $1 \leq j \leq 2^n < i \leq 2^{n+1}$: This element is equal to $\chi_{S^{j|n+1}}(x^{i|n+1})$. From the constraint on i and j, this is the same as $\chi_{S^{j|n}}(x^{i-2^n|n} : (-1))$. However, $S^{j|n}$ doesn't include $n + 1$, and so the (-1) is not part of its expansion. Hence this is the same as $\chi_{S^{j|n}}(x^{i-2^n|n})$, which is $(X_n)_{(i-2^n)j}$. Therefore, the bottom-left quarter of X_{n+1} is also identical to X_n.
4. $2^n < i, j \leq 2^{n+1}$: This element is equal to $\chi_{S^{j|n+1}}(x^{i|n+1})$. From the constraint on i and j, this is the same as $\chi_{S^{j-2^n|n} \cup \{n+1\}}(x^{i-2^n|n} : (-1))$. By the property of the χ_S stated above, this is simply $(-1) \cdot \chi_{S^{j-2^n|n}}(x^{i-2^n|n}) = -(X_n)_{(i-2^n)(j-2^n)}$. Therefore, the bottom-right quarter of X_{n+1} is identical to $-X_n$.
From the above, we can say that

$$X_{n+1} = \begin{bmatrix} X_n & X_n \\ X_n & -X_n \end{bmatrix}.$$

Therefore,

$$(X_{n+1})^T = \begin{bmatrix} (X_n)^T & (X_n)^T \\ (X_n)^T & (-X_n)^T \end{bmatrix} = \begin{bmatrix} X_n & X_n \\ X_n & -X_n \end{bmatrix} = X_{n+1},$$

and X_{n+1} is also symmetric.
(ii) $(X_n)^{-1} = -\frac{1}{2^n} X_n$: For this, let us consider the product matrix $P = (X_n)^2$. Now, by the definition of matrix multiplication, each P_{ij} is the dot product of the i^{th} row of X_n and the j^{th} column of X_n. Since the i^{th} row is the same as the i^{th} column (by symmetry), we can see that

$$P_{ij} = \chi_{S^{i|n}}(x^{1|n})\chi_{S^{j|n}}(x^{1|n}) + \chi_{S^{i|n}}(x^{2|n})\chi_{S^{j|n}}(x^{2|n}) + \cdots + \chi_{S^{i|n}}(x^{2^n|n})\chi_{S^{j|n}}(x^{2^n|n}) = 2^n \cdot \langle \chi_{S^{i|n}}, \chi_{S^{j|n}} \rangle.$$

From part (a), we know that the inner product is 1 if $i = j$ and 0 otherwise. Therefore, P is a diagonal matrix, all of whose diagonal elements are 2^n, *i.e.*,

$$P = (X_n)^2 = 2^n I.$$

Therefore, we can premultiply by $(X_n)^{-1}$ on both sides to get $(X_n)^{-1} = \frac{1}{2^n} X_n$, QED.

Now, we wish to find the value of

$$\sum_{S \subseteq \{1,\ldots,n\}} (\hat{f}_S)^2.$$

This is nothing but the dot product of the vector $\hat{f}$ (defined above) with itself, *i.e.*, $\hat{f}^T \cdot \hat{f}$. Therefore,

$$\begin{aligned}
\hat{f}^T \cdot \hat{f} &= (X_n^{-1} v_f)^T \cdot (X_n^{-1} v_f) \\
&= (v_f^T (X_n^{-1})^T) \cdot (X_n^{-1} v_f) \\
&= v_f^T \cdot (\frac{1}{2^n} X_n)^T \cdot \frac{1}{2^n} X_n \cdot v_f \\
&= \frac{1}{2^{2n}} (v_f^T \cdot (X_n)^2 \cdot v_f) \\
&= \frac{1}{2^{2n}} (v_f^T \cdot 2^n I \cdot v_f) \\
&= \frac{1}{2^n} (v_f^T \cdot v_f) \\
&= \frac{1}{2^n} \sum_{i=1}^{2^n} (v_f)_i^2.
\end{aligned}$$

However, all entries of v_f are either $+1$ or -1. Therefore, all terms of the sum are 1, which means the sum itself is 2^n. Hence,

$$\sum_{S \subseteq \{1,\ldots,n\}} (\hat{f}_S)^2 = \hat{f}^T \cdot \hat{f} = 1,$$

QED.

3.2

The inner product on the space V is defined as

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

We are given the basis $\{1, x, x^2, x^3\}$. Let these vectors be v_1, v_2, v_3, v_4 respectively. We will first use the Gram-Schmidt process to generate an orthogonal basis w_1, w_2, w_3, w_4, and then convert it into an orthonormal basis w'_1, w'_2, w'_3, w'_4. For the orthogonal basis,

$$w_1 = v_1,$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1,$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2,$$

$$w_4 = v_4 - \frac{\langle v_4, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_4, w_2\rangle}{\langle w_2, 2_2\rangle} w_2 - \frac{\langle v_4, w_3\rangle}{\langle w_3, w_3\rangle} w_3.$$

First,

$$w_1 = v_1 = 1,$$

from which we get

$$\langle w_1, w_1\rangle = \int_0^1 1 \cdot dx = [x]_0^1 = 1,$$

$$\langle v_2, w_1\rangle = \int_0^1 x dx = [\frac{1}{2}x^2]_0^1 = \frac{1}{2},$$

$$\langle v_3, w_1\rangle = \int_0^1 x^2 dx = [\frac{1}{3}x^3]_0^1 = \frac{1}{3},$$

$$]langlev_4, w_1\rangle = \int_0^1 x^3 dx = [\frac{1}{4}x^4]_0^1 = \frac{1}{4}.$$

Therefore,

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = x - \frac{\frac{1}{2}}{1}(1) = x - \frac{1}{2}.$$

From this we get

$$\langle w_2, w_2\rangle = \int_0^1 (x - \frac{1}{2})^2 dx = [\frac{1}{3}(x - \frac{1}{2})^3]_0^1 = \frac{1}{12},$$

$$\langle v_3, w_2\rangle = \int_0^1 x^2(x - \frac{1}{2}) dx = [\frac{1}{4}x^4 - \frac{1}{6}x^3]_0^1 = \frac{1}{12},$$

$$\langle v_4, w_2\rangle = \int_0^1 x^3(x - \frac{1}{2}) dx = [\frac{1}{5}x^5 - \frac{1}{8}x^4]_0^1 = \frac{3}{40}.$$

Therefore,

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}(x - \frac{1}{2}) = x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}.$$

From this we get

$$\langle w_3, w_3\rangle = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx$$

$$= [\frac{1}{5}x^5 - \frac{1}{2}x^4 - \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}]_0^1 = \frac{1}{5} - \frac{1}{2} - \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{1}{180},$$

$$\langle v_4, w_3\rangle = \int_0^1 x^3(x^2 - x + \frac{1}{6})dx = [\frac{1}{6}x^6 - \frac{1}{5}x^5 + \frac{1}{24}x^4]_0^1$$

$$= \frac{1}{6} - \frac{1}{5} + \frac{1}{24} = \frac{1}{120}.$$

Therefore,

$$w_4 = v_4 - \frac{\langle v_4, w_1\rangle}{\langle w_1, w_1\rangle}w_1 - \frac{\langle v_4, w_2\rangle}{\langle w_2, 2_2\rangle}w_2 - \frac{\langle v_4, w_3\rangle}{\langle w_3, w_3\rangle}w_3$$

$$= x^3 - \frac{\frac{1}{4}}{1}(1) - \frac{\frac{3}{40}}{\frac{1}{12}}(x - \frac{1}{2}) - \frac{\frac{1}{120}}{\frac{1}{180}}(x^2 - x + \frac{1}{6})$$

$$= x^3 - \frac{1}{4} - \frac{9}{10}(x - \frac{1}{2}) - \frac{3}{2}(x^2 - x + \frac{1}{6}) = x^3 - \frac{1}{4} - \frac{9}{10}x + \frac{9}{20} - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{1}{4}$$

$$= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$$

From this we get

$$\langle w_4, w_4\rangle = \int_0^1 (x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20})^2 dx$$

$$= [\frac{1}{7}x^7 - \frac{1}{2}x^6 + (\frac{6}{5} + \frac{9}{4})\frac{1}{5}x^5 + (-\frac{1}{10} - \frac{9}{5})\frac{1}{4}x^4 + (\frac{3}{20} + \frac{9}{25})\frac{1}{3}x^3 - \frac{3}{50}\frac{1}{2}x^2 + \frac{1}{40}x]_0^1$$

$$= \frac{1}{7} - \frac{1}{2} + \frac{69}{100} - \frac{19}{40} + \frac{51}{300} - \frac{3}{100} + \frac{1}{40} = \frac{4}{175}.$$

Now, we have an orthogonal basis. For the orthonormal basis,

$$w_1' = \frac{w_1}{\|w_1\|} = \frac{w_1}{\sqrt{\langle w_1, w_1\rangle}} = \frac{1}{1}(1) = 1,$$

$$w_2' = \frac{w_2}{\|w_2\|} = \frac{w_2}{\sqrt{\langle w_2, w_2\rangle}} = \frac{1}{\frac{1}{2\sqrt{3}}}(x - \frac{1}{2}) = \sqrt{3}(2x - 1),$$

$$w_3' = \frac{w_3}{\|w_3\|} = \frac{w_3}{\sqrt{\langle w_3, w_3\rangle}} = \frac{1}{\frac{1}{6\sqrt{5}}}(x^2 - x + \frac{1}{6}) = \sqrt{5}(6x^2 - 6x + 1),$$

$$w_4' = \frac{w_4}{\|w_4\|} = \frac{w_4}{\sqrt{\langle w_4, w_4\rangle}} = \frac{1}{\frac{2}{5\sqrt{7}}}(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20})$$

$$= \frac{\sqrt{7}}{2}(5x^3 - \frac{15}{2}x^2 + 3x - \frac{1}{4}).$$

3.3

The vector space V is spanned by $\{\sin t, \sin 2t, \sin 3t, \sin 4t\}$ and has the inner product defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

We have a subspace $U = \text{span}\{\sin t + \sin 2t\}$. We need to find its orthogonal complement $U^\perp$.
Let $u = c\sin t + c\sin 2t \in U$ be an arbitrary vector in U, and let $v = (\alpha \sin t + \beta \sin 2t + \gamma \sin 3t + \delta \sin 4t)$ be one in $U^\perp$. This can only happen if

$$\int_0^{2\pi} c(\sin t + \sin 2t)(\alpha \sin t + \beta \sin 2t + \gamma \sin 3t + \delta \sin 4t)dt = 0$$

$$\implies \int_0^{2\pi} (c\alpha \sin^2 t + c\beta \sin t \sin 2t + c\gamma \sin t \sin 3t + c\delta \sin t \sin 4t + c\alpha \sin t \sin 2t$$

$$+c\beta \sin^2 2t + c\gamma \sin 2t \sin 3t + c\delta \sin 2t \sin 4t)dt = 0$$

Now, consider the integral

$$\int_0^{2\pi} sin(mx)sin(nx)dx$$

for $m \neq n$. Applying trigonometric identities, this is equal to

$$\int_0^{2\pi} \frac{1}{2}[\cos{(m+n)x} - \cos{(m-n)x}] = [\frac{1}{2}(\frac{1}{m+n}\sin{(m+n)x} - \frac{1}{m-n}\sin{(m-n)x})]_0^{2\pi} = \frac{1}{2}(0-0) = 0.$$

If $m = n$, we get

$$\int_0^{2\pi} \sin^2(mx)dx = \int_0^{2\pi} \frac{1}{2}(1 - \cos 2mx)dx = \frac{1}{2}[x - \frac{1}{2m}\sin 2mx]_0^{2\pi} = \frac{1}{2}[2\pi - 0] = \pi.$$

Applying these results to the above expansion, we see that it reduces to

$$c\alpha\pi + c\beta\pi = 0 \implies \alpha = -\beta.$$

Therefore, an arbitrary vector in $U^\perp$ is of the form $\alpha \sin t - \alpha \sin 2t + \gamma \sin 3t + \delta \sin 4t$ for any α, γ, δ. We can also write

$$U^\perp = \text{span}\{\sin t - \sin 2t, \sin 3t, \sin 4t\}.$$

From this, we can see that $\sin t - \sin 2t$ is already in $U^\perp$; therefore, if we wish to write it as a sum of vectors from U and $U^\perp$, it is simply $0 + (\sin t - \sin 2t)$. We can verify this fact by evaluating the integral using the above results:

$$\int_0^{2\pi} (\sin t + \sin 2t)(\sin t - \sin 2t)dt = \int_0^{2\pi} (\sin^2 t - \sin^2 2t)dt = \pi - \pi = 0.$$