1. $M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $v_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$; $Mv_t = v_{t+1}$

(i) To find: all v_0 s.t. $v_0 = v_1 = v_2 = v_3 = \cdots$

By definition, then, $Mv_0 = v_0$ is the only solⁿ.

$$\Rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\Rightarrow 3x_0 + 2y_0 = x_0 \Rightarrow 2x_0 + 2y_0 = 0$$

$$\& \; 2x_0 + 3y_0 = y_0 \quad \& \; 2x_0 + 2y_0 = 0.$$

The only solⁿ to these eqⁿs is $x_0 = -y_0 = s$ (say), i.e.,

$$v_0 = \cancel{\begin{bmatrix} t \\ -t \end{bmatrix}} \; \underline{\underline{\begin{bmatrix} s \\ -s \end{bmatrix}}} \quad \exists s \in \mathbb{R}$$

(ii) To find all v_0 s.t. $v_0, v_1, v_2 \cdots \in$ a 1-D subspace.

If the subspace containing all the v_t is 1-D, its basis has only one vector. This can be any vector v_t; say v_0.

Then, the subspace is simply $\{cv_0 \mid c \in \mathbb{R}\}$,

i.e. $Mv_0 = \lambda v_0$ for some $\lambda \in \mathbb{R}$.

$\therefore v_0$ is an eigenvector of M.

$$M - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$\det(M - \lambda I) = (3-\lambda)^2 - 4$ is the characteristic polynomial

$$(3-\lambda)^2 - 4 = 0 \Rightarrow 3 - \lambda = \pm 2$$

$$\Rightarrow \lambda = +1, -5.$$

$\cancel{\lambda=-1 : \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = -\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}$

$\cancel{\Rightarrow 4x_0 + 2y_0 = -x_0 \Rightarrow 5x_0 + 2y_0 = 0}$

$\cancel{\& \ 2x_0 + 4y_0 = -y_0 \Rightarrow 2x_0 + 5y_0 = 0}$

of which the only sol^n is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\cancel{\lambda=-3}$ $\lambda = 1$: done in part (i).

$\exists s \in \mathbb{R} : v_0 = \begin{bmatrix} s \\ -s \end{bmatrix}$

$\lambda = -5 : \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = -5\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$8x_0 + 2y_0 = -5x_0 \Rightarrow 13x_0 + 2y_0 = 0$

$\& \ 2x_0 + 8y_0 = -5y_0 \Rightarrow 2x_0 + 13y_0 = 0$

which has the solution $v_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only.

$\therefore$ The set v_0 is $\left\{ \begin{bmatrix} s \\ -s \end{bmatrix} \mid s \in \mathbb{R} \right\}$ //.