Assignment 2

MA3.101: Linear Algebra

Spring 2021

Answers

1. The given transformation is $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$. To show that it is not linear, it is sufficient to show that $T(c(x_1, x_2)) = cT(x_1, x_2)$ for some c.
   Consider $c = -1$. Then, $T(-(x_1, x_2)) = T(-x_1, -x_2) = (4(-x_1) - 2(-x_2), 3|-x_2|) = (2x_2 - 4x_1, 3|x_2|)$, which is not the same as $-T(x_1, x_2)$. Hence T is not linear.

2. Since T projects points from space onto the plane, it must be defined as $T(x_1, x_2, x_3) = (x_1, x_2)$. A basis of $\mathbb{R}^3$ is $[(1,0,0),(0,1,0),(0,0,1)]$, and a basis of $\mathbb{R}^2$ is $[(1,0),(0,1)]$. Therefore the matrix representation of T w.r.t these bases is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. (i) $D : P^3 \to P^2$, $D(p(x)) = \frac{d}{dx} p(x)$.
   The identity element of P^2 is the polynomial identically equal to 0, *i.e.*, $p(x) = 0$. Therefore the kernel is the set of antiderivatives of this polynomial, which is the set of all constant functions, *i.e.* $\ker f = \{p(x) | \exists c \in \mathbb{R}, \forall x p(x) = c\}$. The range of this function is $\mathbb{R}$ since all polynomials have an antiderivative, *i.e.*, are integrable.

   (ii) $S : P^1 \to \mathbb{R}$, $S(p(x)) = \int_0^1 p(x)dx$.
   Consider an arbitrary $p(x) \in P^1$, $p(x) = ax + b$. Then $S(p(x)) = S(ax + b) = [\frac{a}{2}x^2 + bx]_{x=0}^{x=1} = \frac{a}{2} + b$.
   The identity in $\mathbb{R}$ is 0; therefore $\ker f = \{ax + b | a + 2b = 0\}$.
   The range of this function is the range of $\frac{a}{2} + b$ as a, b range over $\mathbb{R}$. This is nothing but $\mathbb{R}$.

   (iii) $T : M_{22} \to M_{22}$, $T(A) = A^T$.
   The identity in M_{22} is the null matrix. Therefore the kernel is the transpose of this, which is also the null matrix. Hence,
$$\ker f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
   Since transposition is idempotent, the range is simply M_{22}.

4. We know that the scalar field has p^n elements and the dimension is k, and therefore the basis has k elements. All elements in the vector space can be formed by a scalar combination of the basis vectors; clearly, there are $(p^n)^k$ ways for this. Therefore the vector space has p^{nk} elements.

(a) A linear transformation is completely determined by the images of the basis vectors. For a general mapping, each basis vector can be mapped to any vector in the space. Therefore there are $(p^{nk})^k = p^{nk^2}$ different mappings.

(b) A linear transformation is invertible if it is one-one and onto. Since we are considering transformations from V to V, and V is a finite set, being one-one is equivalent to being onto*, so we only need to prove one of the properties.

The transformation is onto iff the basis vectors map to another basis. This can be proved as follows:

(i) forward implication: Let B be a basis and C, its image, also be a basis. Then an arbitrary $v \in V$ can be represented in the form $v = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k$. But this is nothing but $\alpha_1 T(b_1) + \alpha_2 T(b_2) + \cdots + \alpha_k T(b_k)$, which, by the linearity of T, is $T(\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k)$, where $w = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k$ is also a vector in V.

Hence, for all $v \in V$, there is a $w \in V$ such that $T(w) = v$, which means that T is onto, which makes it one-one, which makes it invertible.

(ii) backward implication: Let $T : V \to W$ be an onto transformation, and let C be the image of the basis B. Then, range(T) = $span(T(B)) = span(C)$. But $span(C) \subsetneq W$, since it is linearly dependent and has cardinality equal to the basis. This contradicts the fact that T is onto; therefore C must be LI and hence a basis.

Now, any set of k linearly independent vectors form a basis, and therefore, for T to be invertible, the image of B can be any set of k LI vectors. We therefore need to find the number of sets of LI vectors, which can be done as follows:

Let the first vector be $c_1 = x_1 b_1 + x_2 b_2 + \cdots + x_k b_k$. The x_i can take any value in F except all 0, so there are $((p^n)^k - 1)$ ways to select c_1. Then, c_2 can be any vector which is not a scalar multiple of c_1 (to preserve linear independence), which gives us $((p^n)^k - p^n)$ ways to select it. Then c_3 can be any vector which is not of the form $\mu c_1 + \nu c_2$, which means there are $((p^n)^k - (p^n)^2)$ ways to select it. Proceeding in this manner till c_k, we find that there are $(p^{nk}-1)(p^{nk}-p^n)(p^{nk}-p^{2n})\cdots(p^{nk}-p^{n(k-1)})$ possibilities for C (including ordering). Therefore, the number of ways to map B to another basis is $(p^{nk}-1)(p^{nk}-p^n)(p^{nk}-p^{2n})\cdots(p^{nk}-p^{n(k-1)})$, which is also the number of invertible linear transformations.

(c) We can find this by proving a bijection between the matrices of determinant Δ, where $\Delta \notin \{0, 1\}$. We arbitrarily select a row or column, say row 1.

Then we can say that given a matrix of determinant 1, we can multiply this row by Δ throughout to obtain a matrix of determinant Δ. Similarly, given a matrix of determinant Δ, we can multiply

the row throughout with Δ^{-1} (since $\Delta \neq 0$), to get a matrix of determinant 1.
Thus, there is a bijection between the matrices of determinant 1 and the matrices with determinant of any other nonzero value. Therefore, $\frac{1}{p^n-1}$ of all nonsingular matrices have determinant 1; hence, the total number of matrices is

$$\frac{(p^{nk}-1)(p^{nk}-p^n)(p^{nk}-p^{2n})\cdots(p^{nk}-p^{n(k-1)})}{p^n-1}.$$

$*$ If a transformation is one-one, all elements have a unique image. Therefore the cardinality of the range is the same as that of the domain. For a transformation from V to V, this means its range is the whole set V and it is therefore onto.
If a transformation from V to V is onto, all vectors in V have at least one pre-image; therefore the cardinality of the domain is at least the cardinality of the range. (It will be greater if some vector has more than one pre-image.) Since the cardinality of the domain is the same as the cardinality of the range, the transformation must be one-one.

5. We need to prove that if S is a subspace of V, then $span(S) = S$. We know that $span(S) = \{(a_1s_1 + a_2s_2 + \cdots + a_ns_n)|a_i \in F, s_i \in S\}$.

   (i) $span(S) \subseteq S$
   We will use induction on n to show that $(a_1s_1 + a_2s_2 + \cdots + a_ns_n) \in S$.
   Base case ($n = 1$): $\forall a_1 \in F,\ s_1 \in S \implies a_1s_1 \in S$. This follows from the property of subspaces that $(ax + by) \in S$ whenever $a, b \in F$ and $x, y \in S$, setting $a = a_1, x = s, b = 0$.
   Induction step: Suppose that $s = (a_1s_1 + a_2s_2 + \cdots + a_{n-1}s_{n-1}) \in S$ for all $a_i \in F$ and $s_i \in S$. Therefore, $s' = (a_1s_1 + a_2s_2 + \cdots + a_{n-1}s_{n-1} + a_ns_n) = s + a_ns_n \in S$. This follows from the property of subspaces that $(ax + by) \in S$ whenever $a, b \in F$ and $x, y \in S$, setting $a = 1, x = s, b = a_n, y = s_n$.
   Therefore $span(S) \subseteq S$.

   (ii) $S \subseteq span(S)$
   This is trivial; for any $s \in S$, setting $a_1 = 1, s_1 = s$ and all other a_i to 0, we get that $s \in span(S)$. Therefore $S \subseteq span(S)$.
   From (i) and (ii), we see that S is closed under linear combination, *i.e.* $S = span(S)$, QED.

6. $T : V \to V$, such that

$$T(x^2) = x + m$$

$$T(x) = (m-1)x$$

$$T(1) = x^2 + m$$

   (a) $B = \{x^2, x, 1\}$ is a basis.

(i) B is linearly independent. This can be proved by assuming that for some c_i, $c_2x^2 + c_1x + c_0 = 0$. Comparing the coefficients of x^i, we see that all $c_i = 0$, which means B is linearly independent.

(ii) B spans V. We know that an arbitrary polynomial in V is of the form $ax^2 + bx + c$, which is clearly a linear combination of the elements of B.

From (i) and (ii), we see that B forms a basis of V, QED.

(b) [skipped]

(c) The columns of the matrix are the coordinate vectors of the images of each of the basis vectors. Therefore, the columns are

$$[T(x^2)]_B = [x + m]_B = \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$$

$$[T(x)]_B = [(m-1)x]_B = \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$$

$$[T(1)]_B = [x^2 + m]_B = \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}.$$

This makes the transformation matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}.$$

(d) We need to find all a, b, c such that $T(ax^2 + bx + c) = 0$. From this equation, we get $aT(x^2) + bT(x) + cT(1) = a(x + m) + b((m-1)x) + c(x^2 + m) = cx^2 + (a + b(m-1))x + (a + c)m = 0$.

The trivial solution to this is $a = b = c = 0$, *i.e.*, $0 \in \ker T$ for all values of m.

We see that regardless of the value of m, $c = 0$. For the values of a and b, we distinguish three cases:

- If $m = 0$, then $a + c = a$ can take any value. However, from the coefficient of x, $a - b = 0 \implies a = b$. Therefore $\ker T = \{a(x^2 + x) | a \in F\}$, which includes 0.

- If $m = 1$, then from the constant term, $a + c = 0 \implies a = -c = 0$. This makes the coefficient of x 0 in all cases and is therefore sufficient; hence the kernel in this case is $\{bx | b \in F\}$, which also includes 0.

- For any other value of m, we see that $am = 0 \implies a = 0$, and $b(m-1) = 0 \implies b = 0$. Therefore the kernel consists of 0 alone.

(e) We have seen above that $T(ax^2 + bx + c) = cx^2 + (a + b(m-1))x + (a+c)m$. Here again, we distinguish three cases based on the value of m:

- If $m = 0$, the image reduces to $cx^2 + (a-b)x$. This covers all polynomials with no constant term; therefore the image of T is $\{ax^2 + bx\}$.

- If $m = 1$, the image reduces to $cx^2 + ax + (a+c)$. This covers all polynomials whose constant term is the sum of their other two coefficients; therefore the image of T is $\{ax^2 + bx + (a+b)\}$.

- For any other value of m, let $cx^2 + (a + b(m-1))x + (a+c)m = px^2 + qx + r$, an arbitrary polynomial. Then, we see that

$$a = \frac{r}{m} - p, b = \frac{1}{m-1}(q - \frac{r}{m} + p), c = p,$$

  which are well-defined for all p, q, r. Therefore the image of T in this case is V itself.

7. Let $S : V \to W$ and $T : U \to V$ be two linear transformations that are onto. We need to prove that $S \circ T$ is onto.
   Consider an arbitrary $w \in W$. Now, since S is onto, there exists some $v \in V$ such that $S(v) = w$. Also, since T is onto, there exists some $u \in U$ such that $T(u) = v$. Substituting the latter equation in the former, we see that for any $w \in W$, there exists $u \in U$, such that $S(T(u)) = (S \circ T)(u) = w$. This proves that $S \circ T$ is onto.

8. Let $S : V \to W$ and $T : U \to V$ be two linear transformations that are one-one. We need to prove that $S \circ T$ is one-one.
   We know that for all $u_1, u_2 \in U$, $u_1 \neq u_2 \implies T(u_1) \neq T(u_2)$, by injectivity of T. We also know that for all $v_1, v_2 \in V$, $v_1 \neq v_2 \implies S(v_1) \neq S(v_2)$, by injectivity of S. Letting $v_1 = T(u_1)$ and $v_2 = T(u_2)$, we see that $u_1 \neq u_2 \implies T(u_1) \neq T(u_2) \implies S(T(u_1)) \neq S(T(u_2))$. This means that $u_1 \neq u_2 \implies (S \circ T)(u_1) \neq (S \circ T)(u_2)$, *i.e.* $S \circ T$ is one-one.

9. Let $S : V \to W$ and $T : U \to V$ be two linear transformations such that $S \circ T$ is onto. We need to prove that S is onto. We will prove this by contradiction; suppose that $S \circ T$ is onto, but S is not.
   Since $S \circ T$ is onto, for all $w \in W$, there exists some $u \in U$ such that $S(T(u)) = w$. However, since S is not onto, there is some $w' \in W$, for

which there is no $v \in V$ such that $S(v) = w'$.
Setting $w = w'$ in the first equation, we see that there must be some $u \in U$ such that $S(T(u)) = w'$. But $v = T(u) \in V$ is such that $S(v) \in w'$, which is a contradiction. Therefore S must be onto, QED.

10. Let A be the matrix representation of T in with respect to the bases B and C. Since $[T(b_i)]_C$ can take only one value, there is only one A for T. If another linear transformation S has the same matrix representation A with respect to bases B and C, then for all $v \in V$, $[S(v)]_C = A[v]_B = [T(v)]_C \implies S(v) = T(v)$, which means $S = T$. Therefore, given a matrix representation, it can correspond to only one transformation.

11. $T : V \to W$ is a linear transformation and $A = [T]_{C \leftarrow B}$ is its matrix representation. We need to show that nullity(T) = nullity(A).
Consider a vector $v \in \ker T$; we know that $T(v) = 0$, which means that $[T(v)]_C = A[v]_B = 0$. Therefore $[v]_B \in N(A)$, where $N(A)$ is the null space of A.
Similarly, let $x \in V$ be a vector such that $A[x]_B = 0$, *i.e.* $[x]_B$ is in $N(A)$. Then $[T(x)]_C = 0$, which means that $T(x) = 0$. Therefore $x \in \ker T$.
The above shows that $N(A) = \{[v]_B | v \in \ker T\}$.
Now, consider a basis of $\ker T$, $K = \{k_1, k_2, ..., k_n\}$, and the corresponding set $K_B = \{[k_1]_B, [k_2]_B, ..., [k_n]_B\}$ in $N(A)$.
We know that K is linearly independent, and that the coordinate vectors of a set of LI vectors is also LI. This allows us to conclude that K_B is linearly independent.
Further, $span(K) = \ker T$. Given an arbitrary vector $k \in \ker T$, we can write $k = c_1k_1 + c_2k_2 + \cdots c_nk_n$. Taking the coordinate vectors on both sides, we get $[k]_B = c_1[k_1]_B + c_2[k_2]_B + \cdots + c_n[k_n]_B$. Therefore, all vectors $[k]_B \in N(A)$ belong to $span(K_B)$.
The above two properties of K_B show that it is a basis of $N(A)$.
Now, $|K| = |K_B|$ by definition. But $|K| = \dim(\ker T)$, and $|K_B| = \dim N(A)$. Therefore $\dim(\ker T) = \dim N(A)$, *i.e.* the nullity of T is equal to the nullity of A, QED.

12. (a) The equations are

$$x + 3y + 5z = 14$$
$$2x - y - 3z = 3$$
$$4x + 5y - z = 5$$

Therefore the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ and $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & 0 & -8 & -26 \end{bmatrix}$$

$R_3 \to -\frac{1}{2}R_3$ and $R_2 \to -R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 0 & 4 & 13 \end{bmatrix}$$

This is the row echelon form. Converting back to linear equations, we have

$$x + 3y + 5z = 14$$

$$7y + 13z = 25$$

$$4z = 13$$

We can solve these using backsubstition, getting the answer

$$x = \frac{36}{7}, y = -\frac{69}{28}, z = \frac{13}{4}.$$

(b) The equations are

$$y + z = 4$$

$$3x + 6y - 3z = 3$$

$$-2x - 3y + 7z = 10$$

Therefore the augmented matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$R_2 \to \frac{1}{3}R_2$ and $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$R_3 \to \frac{1}{4}R_3$ and $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1 + 3R_3$ and $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This is the reduced row echelon form. Converting back to linear equations, we directly get the answer

$$x = -1, y = 2, z = 2.$$

(c) The equations are

$$\begin{aligned} \sqrt{2}x + y + 2z &= 1 \\ \sqrt{2}y - 3z &= -\sqrt{2} \\ -y + \sqrt{2}z &= 1 \end{aligned}$$

Therefore the augmented matrix is

$$\begin{bmatrix} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{\sqrt{2}}R_2$

$$\begin{bmatrix} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$R_3 \to -\sqrt{2}R_3$ and $R_1 \to R_1 - \frac{1}{\sqrt{2}}R_2$

$$\begin{bmatrix} \sqrt{2} & 0 & (2 + \frac{3}{2}\sqrt{2}) & 2 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - (2 + \frac{3}{2}\sqrt{2})R_1$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 & 2 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{\sqrt{2}} R_1$ and $R_2 \to R_2 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{\sqrt{2}} R_2$

$$\begin{bmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is the reduced row echelon form. Converting back to linear equations, we directly get the answer

$$x = \sqrt{2}, y = -1, z = 0.$$

13. We need to show that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$.
Applying the distributive law twice to the LHS, we see that $\langle u+v, u-v\rangle = \langle u, u-v\rangle + \langle v, u-v\rangle = \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle$.
From here, we apply the commutative law and the definition of norm to get $\|u\|^2 - \langle u, v\rangle + \langle u, v\rangle - \|v\|^2$.
Cancelling, we see that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$, QED.

14. We need to show that $\|u\|^2 + \|v\|^2 + 2\langle u, v\rangle = \|u+v\|^2$.
Applying the definition of norm and the commutative law to the LHS, we see that it is equal to $\langle u, u\rangle + \langle u, v\rangle + \langle v, v\rangle + \langle v, u\rangle$.
From here, we apply the distributive law twice to get $\langle u, u+v\rangle + \langle v, v+u\rangle = \langle u+v, u+v\rangle$.
By the definition of norm, this is the same as the RHS. Therefore, $\|u\|^2 + \|v\|^2 + 2\langle u, v\rangle = \|u+v\|^2$, QED.

15. We need to show both directions of implication for this. We will use the definition of orthogonality: $\langle u, v\rangle = 0$.

    (i) $\|u+v\| = \|u-v\| \implies \langle u, v\rangle = 0$.
    First, we square both sides and apply the identity proved in 14 (above), to get $\|u+v\| = \|u-v\| \implies \|u+v\|^2 = \|u-v\|^2 \implies \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle = \|u\|^2 + \|-v\|^2 + 2\langle u, -v\rangle$.
    However, we know that $\langle u, -v\rangle = -\langle u, v\rangle$. Applying this to $\|-v\|^2 = \langle -v, -v\rangle$ and to $\langle u, -v\rangle$, we get the above equality to be the same as $\langle u, v\rangle = -\langle u, v\rangle$.
    This means that $\langle u, v\rangle = 0$, *i.e.* u and v are orthogonal.

(ii) $\langle u, v\rangle = 0 \implies \|u+v\| = \|u-v\|$.
Since $\langle u, v\rangle = 0$, we can say that $2\langle u, v\rangle = -2\langle u, v\rangle = 2\langle u, -v\rangle$.
We add $\|u\|^2$ on both sides. Then, we add $\|v\|^2$ on the LHS and $\|-v\|^2$ on the RHS to get $\|u\|^2 + \|v\|^2 + 2\langle u, v\rangle = \|u\|^2 + \|-v\|^2 + 2\langle u, -v\rangle$.
From the identities proved in 14 (above), we see that this is equivalent to $\|u+v\|^2 = \|u-v\|^2$. We can take the positive root on both sides as the inner product always takes nonnegative values; hence we conclude that $\|u+v\| = \|u-v\|$.
From (i) and (ii), we can say that $\|u+v\| = \|u-v\|$ iff u and v are orthogonal.

16. $T : P_2 \to P_2$, $T(p(x)) = p(2x-1)$. The basis of P_2 is $B = [1, x, x^2]$. Therefore,

$$T(1) = 1$$
$$T(x) = 2x - 1$$
$$T(x^2) = (2x-1)^2 = 4x^2 - 4x + 1.$$

Expressing these in the same basis, we get the coordinate vectors to be

$$[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$[T(x)]_B = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$
$$[T(x^2)]_B = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix},$$

and therefore the matrix representation of T is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}.$$

To calculate $T(3 + 2x - x^2)$, we multiply A with the coordinate vector of $3 + 2x - x^2$, which is $(3, 2, -1)$.

$$T(3 + 2x - x^2) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}.$$

This is the coordinate representation of the image. Converting it back to a polynomial, we see that $T(3 + 2x - x^2) = 8x - 4x^2$.