1. Given : $u_1, \dots, u_n \in \mathbb{R}^n$ are ON vectors

RTP :

$$\sum u_i u_i^T = I$$

Proof :

Let $A = \sum u_i u_i^T$.

Since the $u_i$ are ON, they are linearly indept. and therefore form a basis of $\mathbb{R}^n$.

Consider the product $Au_j$ for any $j \in \{1, \dots, n\}$

$$Au_j = \sum u_i u_i^T u_j$$

$$= \sum u_i \langle u_i, u_j \rangle = \sum u_i \delta_{ij}$$

[orthonormality]

$$= u_j \qquad (1)$$

$$= I u_j$$

$\therefore Au_j = Iu_j \;\; \forall j \in \{1, \dots, n\}$.

Further, let $v \in \mathbb{R}^n$ be an arbitrary vector.
~~It~~ Since the $u_i$ form a basis, we can say

$$v = a_1 u_1 + \cdots + a_n u_n$$

for some scalars $a_i$.

Then, $$\begin{aligned} Av &= A(a_1 u_1 + \cdots + a_n u_n) \\ &= a_1(Au_1) + \cdots + a_n(Au_n) \quad \text{[distributivity]} \\ &= a_1 u_1 + \cdots + a_n u_n \quad \text{[from (1)]} \\ &= v \\ &= Iv \end{aligned}$$

$\therefore$ $Av = Iv \quad \forall v \in \mathbb{R}^n$.

If we let $v = e_k$ [the k'th standard basis vector] we see that the $k^{th}$ column of $A$ [$= Ae_k$] is equal to the $k^{th}$ column of $I$ [$= Ie_k$], for all $k \in \{1, \ldots, n\}$.

Therefore $A = I$, i.e.,

$$\sum u_i u_i^T = I,$$

QED.