Linear Algebra, IIIT Hyderabad

Spring 2021

Assignment 6

5.1 An Equivalence

$M \in \mathbb{R}^{n \times n}$ is a symmetric matrix. We need to show that
1. $\forall v \in \mathbb{R}^n, v^T M v \geq 0$,
2. All eigenvalues of M are nonnegative,
3. $\exists B \in \mathbb{R}^{n \times n}, M = B^T B$,

are all equivalent. We will do this by showing that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).
(i) (1) $\implies$ (2)
By assumption, $\forall v \in \mathbb{R}^n$, $v^T M v \geq 0$. We will prove from this by contradiction that all eigenvalues of M are nonnegative.
Let some eigenvalue λ_i be negative, with the corresponding eigenvector u_i, *i.e.*,

$$Mu_i = \lambda_i u_i,$$

where $\lambda_i < 0$. Then, consider the product $u_i^T M u_i$. Using the associative property, we see that

$$u_i^T M u_i = u_i^T (M u_i) = u_i^T (\lambda_i u_i) = \lambda_i (u_i^T u_i).$$

But we know that $\langle u, v \rangle = u^T v$, and that $\langle v, v \rangle = \|v\|^2$. Applying these properties,

$$u_i^T M u_i = \lambda_i \|u_i\|^2 \leq 0,$$

But we know that $u_i^T M u_i \geq 0$; this means that $\lambda_i \|u_i\|^2 = 0$. Since $\lambda_i < 0$, we see that $\|u_i\|^2 = 0 \implies u_i = 0$, which is a contradiction as all eigenvectors are nonzero. Therefore no λ_i can be negative, QED.
(ii) (2) $\implies$ (1)
Assuming that all eigenvalues of M are nonnegative, we need to show that $\exists B \in \mathbb{R}^{n \times n}$ such that $M = B^T B$. We will show this by constructing B and proving that it satisfies the given property.
Let

$$B = \begin{bmatrix} \sqrt{\lambda_1} u_{11} & \sqrt{\lambda_1} u_{12} & \cdots & \sqrt{\lambda_1} u_{1n} \\ \sqrt{\lambda_2} u_{21} & \sqrt{\lambda_2} u_{21} & \cdots & \sqrt{\lambda_2} u_{21} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_n} u_{n1} & \sqrt{\lambda_n} u_{n1} & \cdots & \sqrt{\lambda_n} u_{n1} \end{bmatrix},$$

where u_{ij} denotes the j^{th} entry of the i^{th} eigenvector u_i. B is real and well-defined since all $\lambda_i \geq 0$, by assumption. Now, let us consider the product $B^T B$.

$$B^T B = \begin{bmatrix} \sqrt{\lambda_1}u_{11} & \sqrt{\lambda_2}u_{21} & \cdots & \sqrt{\lambda_n}u_{n1} \\ \sqrt{\lambda_1}u_{12} & \sqrt{\lambda_2}u_{22} & \cdots & \sqrt{\lambda_n}u_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_1}u_{1n} & \sqrt{\lambda_2}u_{2n} & \cdots & \sqrt{\lambda_n}u_{nn} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1}u_{11} & \sqrt{\lambda_1}u_{12} & \cdots & \sqrt{\lambda_1}u_{1n} \\ \sqrt{\lambda_2}u_{21} & \sqrt{\lambda_2}u_{21} & \cdots & \sqrt{\lambda_2}u_{21} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_n}u_{n1} & \sqrt{\lambda_n}u_{n1} & \cdots & \sqrt{\lambda_n}u_{n1} \end{bmatrix},$$

which we can also write as

$$\begin{bmatrix} \sqrt{\lambda_1}u_1 & \sqrt{\lambda_2}u_2 & \cdots & \sqrt{\lambda_n}u_n \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1}u_1^T \\ \sqrt{\lambda_2}u_2^T \\ \vdots \\ \sqrt{\lambda_n}u_n^T \end{bmatrix}.$$

Evaluating this, we see that

$$B^T B = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T = M,$$

QED.
(iii) (3) $\implies$ (1)
Let there exist a matrix B such that $M = B^T B$. We have to show that for all $v \in \mathbb{R}^n$, $v^T M v \geq 0$.
Substituting $B^T B$ for M in the expression, we see that

$$v^T M v = v^T B^T B v = (Bv)^T (Bv).$$

But we know that $\langle u, v \rangle = u^T v$, and that $\langle v, v \rangle = \|v\|^2$. Therefore,

$$v^T M v = \langle Bv, Bv \rangle = \|Bv\|^2 \geq 0,$$

QED.
This completes the proof.

5.2

We are given that $v \in V$ is a nonzero vector, that $\dim V = n$, and that $L : V \to V$ is a linear transformation.
(a) RTP: $\{v, Lv, \ldots, L^n v\}$ is a linearly dependent set.
We know that L is a linear transformation from V to V. Therefore, $L^i v \in V$ for all $0 \leq i \leq n$. This means that the given set contains $(n+1)$ vectors, all of which belong to an n-dimensional space; they must therefore be linearly dependent, QED.
(b) RTP: There exist scalars α_i, not all zero, such that $\alpha_0 v + \alpha_1 Lv + \cdots + \alpha_n L^n V = 0$.
Given that $\{v, Lv, \ldots, L^n v\}$ are linearly dependent, this follows from the definition of linear dependence.

(c) RTP: If m is the largest integer such that $\alpha_m \neq 0$, and $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$, then we can write $p(z)$ as

$$\alpha_m(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_m).$$

Since $p(z)$ is a polynomial having real coefficients and degree m, by the Fundamental Theorem of Algebra, it has m roots (real as well as complex).
(d) RTP: $(L - \lambda_1 I)(L - \lambda_2 I)\cdots(L - \lambda_m I)v = 0$.
We can rewrite the LHS of the equation in (b) as

$$(\alpha_0 + \alpha_1 L + \cdots + \alpha_m L^m)v,$$

(omitting terms with higher powers of L since the corresponding coefficients are all 0) and then factorise the transformation in brackets the same way as $p(z)$, *i.e.*,

$$\alpha_m(L - \lambda_1 I)(L - \lambda_2 I)\cdots(L - \lambda_m I)v = 0.$$

However, $\alpha_m \neq 0$ by assumption. Since there are no zero-divisors in a field, we can cancel α_m, giving us

$$(L - \lambda_1 I)(L - \lambda_2 I)\cdots(L - \lambda_m I)v = 0,$$

QED.
(e) RTP: One of the numbers λ_i, $1 \leq i \leq m$ must be an eigenvalue of L.
Consider the following sequence of vectors:
$w_m = (L - \lambda_m I)v$,
$w_{m-1} = (L - \lambda_{m-1} I)(L - \lambda_m I)v$,
$\vdots$
$w_1 = (L - \lambda_1 I)(L - \lambda_2 I)\cdots(L - \lambda_m I)v$.
In the above sequence, the last entry is given to be zero. Let k be the largest number such that $(L - \lambda_k I)(L - \lambda_{k+1} I)\cdots(L - \lambda_m I)v = 0$, or in other words, let k be such that w_k is the first zero entry of the above sequence. k must exist because the last entry w_1 of the sequence is given to be zero.
Clearly, $w_k = (L - \lambda_k I)w_{k+1} = 0$, and $w_{k+1} \neq 0$. This means that $Lw_{k+1} = \lambda_k w_{k+1}$ (if $k < m$), or that $Lv = \lambda_m v$ (if $k = m$). In both cases, λ_k is an eigenvalue of L, QED.