Assignment 2
MA3.101: Linear Algebra
Spring 2021

March 14, 2021

1. Show that the transformation $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

2. Let $T : R^3 \to R^2$ projects each point (x_1, x_2, x_3) onto the x_1x_2 plane. Find the matrix representation of the linear transformation T

3. Find the kernel and range of the following linear transformations:
   (i) Differential operator $D : P^3 \to P^2$ defined by $D(p(x)) = \frac{d}{dx}p(x)$, P^i is a polynomial of degree i.
   (ii) $S : P^1 \to R$, $S(p(x)) = \int_0^1 p(x)dx$
   (iii) $T : M_{22} \to M_{22}$, $T(A) = A^T$, M_{22} is a $2 \otimes 2$ matrix and T is the transpose of the matrix.

4. Suppose F is a finite field with p^n elements (such that p is a prime.) Let V be a k-dimensional vector space over F. Then count the cardinality of the following:

   (a) The number of linear transformations $T : V \to V$.

   (b) The number of invertable linear transformations $T : V \to V$.

   (c) The number of linear transformations $T : V \to V$ with determinant 1.

5. Prove that if S is a subspace of a vector space V, then $span(S) = S$.

6. Define V as the vector space of all polynomials in x of degree < 3 over $\mathbb{R}$. Define a set $B = \{x^2, x, 1\}$. Define a transformation T as

$$\begin{aligned} T(x^2) &= x + m \\ T(x) &= (m-1)x \\ T(1) &= x^2 + m \end{aligned}$$

   Answer the following:

   (a) Prove that B is a basis.

(b) Show that T is a linear transformation.

(c) Find the matrix representation of T relative to the given basis.

(d) Find kernel(T) for all values of m.

(e) Find the image of T for all values of m.

7. Let $S : V \to W$ and $T : U \to V$ be linear transformation. Show that if S and T are onto, so is $S \circ T$.

8. Let $S : V \to W$ and $T : U \to V$ be linear transformation. Show that if $S \circ T$ are one to one, so is T.

9. Let $S : V \to W$ and $T : U \to V$ be linear transformation. Show that if $S \circ T$ are onto , so is S

10. Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces and let B and C be bases for V and W respectively. Show that the matrix T with respect to the bases B and C be unique.

11. Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces and let B and C be bases for V and W respectively. Let $A = [T]_{C \leftarrow B}$. Show that nullity(T)= nullity(A).

12. (a) Using Gaussian elimination, solve for x, y and z in

$$x + 3y + 5z = 14, 2x - y - 3z = 3, 4x + 5y - z = 7$$

(b) Using Gauss-Jordan elimination, solve for x, y and z in

$$y + z = 4, 3x + 6y - 3z = 3, -2x - 3y + 7z = 10$$

(c) Using Gauss-Jordan elimination, solve for x, y and z in

$$\sqrt{2}x + y + 2z = 1, \sqrt{2}y - 3z = -\sqrt{2}, -y + \sqrt{2}z = 1$$

13. Show $< u + v, u - v >= ||u||^2 - ||v||^2$

14. Show $||u||^2 + ||v||^2 + 2 < u, v >= ||u + v||^2$

15. Prove that $||u + v|| = ||u - v||$ if and only if u and v are orthogonal.

16. Let $T : P_2 \to P_2$ be the linear transformation defined by $T(p(x)) = p(2x - 1)$. Find the matrix of T with respect to the basis $B = [1, x, x^2]$ and compute $T(3 + 2x - x^2)$