DSA Assignment - 1

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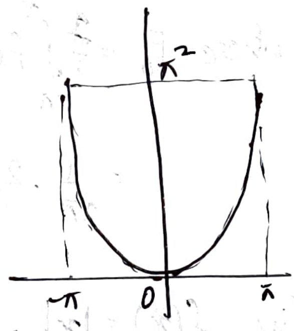
1. (i) $f(x) = x^2 \quad , -\pi < x < \pi$

[using the complex Fourier series]



$$F_n = \frac{1}{T} \int_{-\pi}^{\pi} f(t)\, e^{-in\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 \cdot e^{-int} \cdot dt = \frac{1}{2\pi} [I]_{-\pi}^{\pi}$$

$$I = \int t^2 e^{-int}\, dt = t^2 \int e^{-int}\, dt - \int 2t \left(\int e^{-int}\, dt\right) dt$$

$$= \frac{t^2}{-in} \cdot e^{-int} - \frac{2}{-in} \int t\, e^{-int}\, dt$$

$$= \frac{i}{n} t^2 e^{-int} - \frac{i}{n} 2 \left[t \int e^{-int}\, dt - \int 1 \left(\int e^{-int}\, dt \right) dt \right]$$

$$= \frac{i}{n} t^2 e^{-int} - \frac{\cancel{i}}{n} 2 \left[\frac{t}{-\cancel{i}n} e^{-int} - \frac{1}{(-in)^2} e^{-int} \right]$$

$$= \frac{i}{n} t^2 e^{-int} + \frac{2t}{n^2} e^{-int} + \frac{2}{in^3} e^{-int}$$

$$= \cancel{\frac{e^{-int}}{n^3} \left[\frac{it^2}{n} + \frac{2t}{n^2} \right]} \quad \frac{e^{-int}}{n^3} \left[n^2 i t^2 + n \cdot 2t - 2i \right]$$

$$= \frac{\cos(nt) \mp i \sin(nt)}{n^3} \left[n^2 i t^2 + n \cdot 2t - 2i \right]$$

$$F_n = \frac{1}{2\pi} \left[\frac{\cos(nt) - i\sin(nt)}{n^3} \{ n^2 i t^2 + n \cdot 2t - 2i \} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi \cdot n^3} \left[\{ ((-1)^n - 0) \cdot (n^2 i \cdot \pi^2 + n \cdot 2\pi - 2i) \} - \{ ((-1)^n - 0)(n^2 i \cdot \pi^2 - n \cdot 2\pi - 2i) \} \right]$$

$$= \frac{(-1)^n \cdot 4n\cancel{\pi}}{2\cancel{\pi} n^3} = (-1)^n \cdot \frac{2}{n^2}$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{int}$$

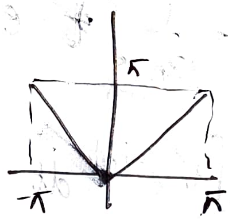
$$= \sum_{n=-\infty}^{\infty} (-1)^n \cdot \frac{2}{n^2} e^{int}$$

where $F_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{2\pi} \cdot \frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

(ii) $f(x) = |x|, \; -\pi < x < \pi$

[using the complex Fourier series]



$$F_n = \frac{1}{T}\int_{-\pi}^{\pi} f(t) \cdot e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|\, e^{-int}\,dt = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -t e^{-int}\,dt + \int_{0}^{\pi} t e^{-int}\,dt\right]$$

$$= \frac{1}{2\pi}\left[[-I_1]_{-\pi}^{0} + [I_1]_{0}^{\pi}\right]$$

$$I_1 = \int t e^{-int}\,dt = t\int e^{-int}\,dt - \int 1 \cdot \left(\int e^{-int}\,dt\right)dt$$

$$= \frac{t}{-in} e^{-int} - \frac{1}{(-in)^2} e^{-int}$$

$$= \frac{e^{-int}}{n^2}\left[nit + 1\right]$$

$$\therefore \cancel{F_n = \frac{1}{2\pi}} \; [-I_1]_{-\pi}^{0} = [I_1]_{0}^{-\pi} = \left[\frac{e^{-int}}{n^2}\{nit+1\}\right]_{0}^{-\pi}$$

$$= \frac{1}{n^2}\left[e^{in\pi}\{-ni\pi + 1\} - 1\right] = \cancel{\frac{1}{n^2}[\;]}$$

$$= \frac{1}{n^2}\left[(-1)^n(1 - ni\pi) - 1\right]$$

$$[I_1]_0^\pi = \left[\frac{e^{-int}}{n^2}\{nit+1\}\right]_0^\pi$$

$$= \frac{1}{n^2}\left[e^{in\pi}\{ni\pi+1\} - 1\right] = \frac{1}{n^2}\left[(-1)^n(ni\pi+1)-1\right]$$

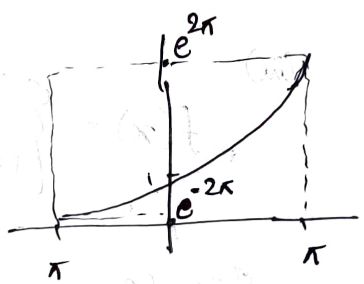
$$\therefore F_n = \frac{1}{\cancel{2}\pi}\cdot\frac{1}{n^2}\left((-1)^n(2) - 2\right) = \frac{(-1)^n 2 - 2}{\pi n^2}$$

$$F_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}|t|\,dt = \frac{\cancel{2}}{2\pi}\int_0^\pi t\,dt = \frac{\pi^2}{2}\cdot\frac{1}{\pi} = \frac{\pi}{2}$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{int} = \cancel{\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}\, i}{n} e^{int}} = \sum_{n=-\infty}^{\infty} 2\left(\frac{(-1)^n - 1}{\pi n^2}\right)e^{int}$$

where $F_0 = \pi/2$

(iii) $f(x) = e^{2x}$, $-\pi < x < \pi$



$$F_n = \frac{1}{T}\int_{-\pi}^{\pi} f(t)\cdot e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t}\cdot e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(2-in)t}\,dt$$

$$= \frac{1}{2\pi}\left[\frac{1}{(2-in)}e^{(2-in)t}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(2-in)}\left(e^{2t}\cdot(\cos(nt) - i\overset{0}{\cancel{\sin(nt)}})\right)_{-\pi}^{+\pi}$$

$$= \frac{(-1)^n}{2\pi(2-in)}\left(e^{2\pi} - e^{-2\pi}\right)$$

$$F_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t}\,dt = \frac{1}{4\pi}\left(e^{2\pi} - e^{-2\pi}\right)$$

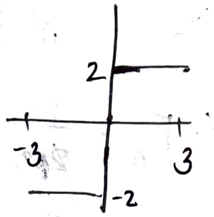
$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{int} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2\pi(2-in)}\left(e^{2\pi} - e^{-2\pi}\right)$$

$$= \left(\frac{e^{2\pi} - e^{-2\pi}}{2\pi}\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(2-in)} \quad \forall n \in \mathbb{Z}$$

2. (i)

$$f(x) = \begin{cases} 2, & 0 < x < 3 \\ -2, & -3 < x < 0 \end{cases} \quad ; T = 6$$

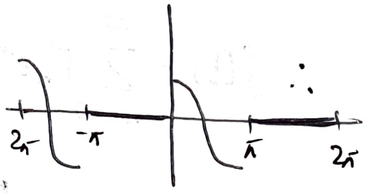
This f^n is <u>odd</u>.



(ii)

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases} \quad ; T = 2\pi.$$

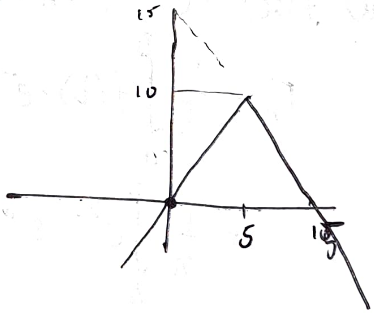
This f^n is neither even nor odd.



(iii)

$$f(x) = \begin{cases} 2x, & x < 5 \\ 15 - x, & x \geq 5 \end{cases}$$

This f^n is neither even nor odd.



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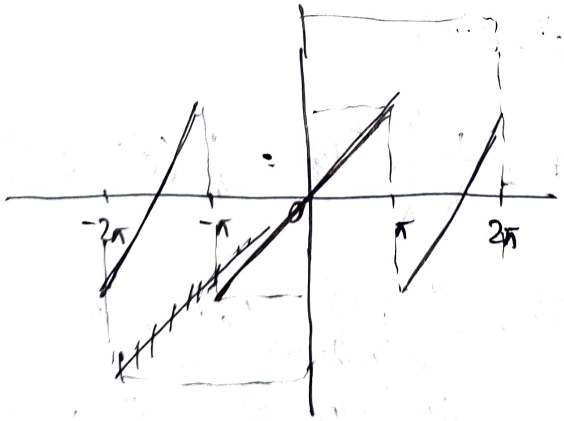
3. (i) $f(x) = x \quad ; -\pi < x < \pi$

$T = 2\pi$



(ii) $F_n = \frac{1}{T} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} t \cdot e^{-int}\, dt$

$$= \frac{1}{2\pi}\left[I\right]_{-\pi}^{\pi}$$

$$I = \int t e^{-int}\,dt = t\int e^{-int}\,dt - \int 1.\left(\int e^{-int}\,dt\right)dt$$

$$= \frac{t}{(-in)} e^{-int} - \frac{1}{(-in)^2} \cdot e^{-int}$$

$$= \frac{e^{-int}}{n^2}(itn + 1)$$

$$\therefore F_n = \frac{1}{2\pi}\left[\frac{e^{-int}}{n^2}(nit+1)\right]_{-\pi}^{\pi} = \frac{1}{2\pi} \cdot \frac{1}{n^2}\left[\begin{matrix}(e^{-in\pi})(n\pi i+1) \\ -(e^{ni\pi})(-n\pi i+1)\end{matrix}\right]$$

$$= \frac{1}{2\pi n^2}\left[\begin{matrix}(\cos n\pi - i\cancel{\sin n\pi}^{\,0})(n\pi i + 1) \\ -(\cos n\pi + i\cancel{\sin n\pi}^{\,0})(-n\pi i + 1)\end{matrix}\right]$$

$$= \frac{(-1)^n \cdot \cancel{2}n\cancel{\pi} i}{\cancel{2\pi} n^2} = (-1)^n \frac{i}{n}$$

$$F_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{i}{n} e^{int}, \quad F_0 = 0.$$

(iii) $t = \pi/2 \Rightarrow f(\pi/2) = \pi/2 = \sum_{n=-\infty}^{\infty} (-1)^n \cdot \frac{i}{n}\left[\left(\cos n\cdot\frac{\pi}{2}\right) + i\sin\left(n\frac{\pi}{2}\right)\right]$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \cdot \frac{1}{n}\left[i\cos\left(n\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right)\right]$$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}\left[i\cos\left(n\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right)\right] + (-1)^n \cdot \frac{1}{(-n)}\left[i\cos\left(-n\frac{\pi}{2}\right) - \sin\left(-n\frac{\pi}{2}\right)\right]$$

$$= \sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n}\right)\left[\cancel{i\cos\frac{n\pi}{2}} - \sin\frac{n\pi}{2} - \cancel{i\cos\frac{n\pi}{2}} - \sin\frac{n\pi}{2}\right]$$

$$= -2\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}\left[\sin\frac{n\pi}{2}\right]$$

$$= -2\sum_{k=1}^{\infty} (-1)^{2k-1}\cdot\frac{1}{2k-1}\left[\sin\frac{2k-1}{2}\pi\right] + (-1)^{2k}\cdot\frac{1}{2k-1}\left[\cancel{\sin\frac{2k}{2}\pi}^{\,0}\right]$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{2k-1}(-1)^{k+1} = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}(-1)^{n}$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5}\cdots \quad \underline{\underline{\text{QED}}}$$

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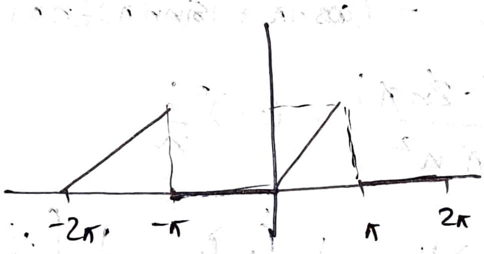
4. (i) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$; $T = 2\pi$



(ii) $$F_n = \frac{1}{T}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot e^{-int}\,dt + \int_{0}^{\pi} t\cdot e^{-int}\,dt\right]$$

$$= \frac{1}{2\pi}\left[0 + \left[\frac{e^{-int}}{n^2}(nit+1)\right]_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n^2}\right]\left[(e^{-in\pi})(ni\pi+1) - (1)\right]$$

$$= \frac{1}{2\pi n^2}\left\{\left[\cos(n\pi) - i\cancel{\sin(n\pi)}^{\,0}\right](ni\pi+1) - 1\right\}$$

$$= \frac{(-1)^n(ni\pi+1) - 1}{2\pi n^2}$$

$$F_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot dt + \int_{0}^{\pi} t\,dt\right] = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{4}.$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n(ni\pi+1)-1}{2\pi n^2}\right] e^{int} \;;\; F_0 = \pi/4$$

(iii) $t=0 \Rightarrow f(0) = 0 = \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n(ni\pi+1)-1}{2\pi n^2}\right]\cdot \cancel{e^{0}}\,1 + \frac{\pi}{4}$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{2\pi n^2}\left[-1-1+(-1)^n\begin{pmatrix} ni\pi+1 \\ +(-n)i\pi+1\end{pmatrix}\right].$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{2}{2\pi n^2}\left[-1+(-1)^n\right] = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left((-1)^n-1\right)$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left((-1)^{2k-1}-1\right) + \frac{1}{(2k)^2}\cancelto{0}{\left((-1)^{2k}-1\right)}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi}{4} - \frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

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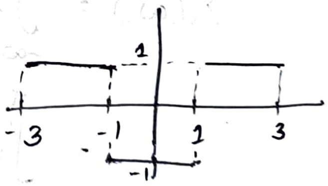
5. $$x(t) = \begin{cases} 1, & 1 \leq |t| \leq 3 \\ -1, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt = \int_{-3}^{3} x(t)e^{-i\omega t}\,dt$$

$$= \int_{-3}^{-1}(1)\cdot e^{-i\omega t}\,dt + \int_{-1}^{1}(-1)\cdot e^{-i\omega t}\,dt + \int_{1}^{3}(1)\cdot e^{-i\omega t}\,dt$$

$$= [I]_{-3}^{-1} + [-I]_{-1}^{1} + [I]_{1}^{3}$$

$$I = \int e^{-i\omega t}\,dt = \frac{1}{-i\omega}e^{-i\omega t} = \frac{i}{\omega}e^{-i\omega t}$$

$$\therefore X(\omega) = \frac{i}{\omega}\left\{\left[e^{-i\omega t}\right]_{-3}^{-1} - \left[e^{-i\omega t}\right]_{-1}^{1} + \left[e^{-i\omega t}\right]_{1}^{3}\right\}$$

$$= \frac{i}{\omega}\left\{\cancel{\frac{\cos(-\omega)}{-i\sin(-3\omega)}} - \cancel{\frac{\cos(\omega)}{-i\sin(-\omega)}} + \cancel{\cos(3\omega)}\right\}$$

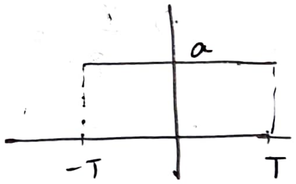
$$= \frac{i}{\omega}\left\{\begin{array}{l}(\cos(-\omega) - i\sin(-\omega)) - (\cos(\omega) - i\sin(\omega)) \\ - (\cos(-3\omega) - i\sin(-3\omega)) + (\cos(-\omega) - i\sin(-\omega)) \\ + (\cos(3\omega) - i\sin(3\omega)) \\ - (\cos(\omega) - i\sin(\omega))\end{array}\right\} = 0$$

$$= \frac{i}{\omega}\left[2i\sin(\omega) + 2i\sin(\omega) - 2i\sin(3\omega)\right]$$

$$X(\omega) = \frac{2}{\omega}\left[\sin(3\omega) - 2\sin(\omega)\right]$$

6. (a)

$$x(t) = \begin{cases} a, & -T \leq t \leq T \\ 0, & \text{otherwise.}\end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$$

$$= \int_{-T}^{T} a\cdot e^{-i\omega t}\,dt = \frac{a}{(-i\omega)}\left[e^{-i\omega t}\right]_{-T}^{T}$$

$$= \frac{ai}{\omega}\left[\begin{array}{l}\cos(\omega T) - i\sin(\omega T) \\ -\cos(-\omega T) + i\sin(\omega T)\end{array}\right]$$

$$= -ai\cdot\frac{2i\sin(\omega t)}{\omega} = \frac{2a\sin(\omega t)}{\omega}$$

$$\Rightarrow |X(\omega)| = \left|\frac{2a\sin(\omega t)}{\omega}\right| \rightarrow \text{magnitude spectrum}$$

$$\phi(X(\omega)) = \tan^{-1}\left(\frac{0}{\frac{2a\sin(\omega t)}{\omega}}\right) = 0 \rightarrow \text{phase spectrum}$$

(b) $x(t) = \delta(t-a)$; $a \in \mathbb{R}$

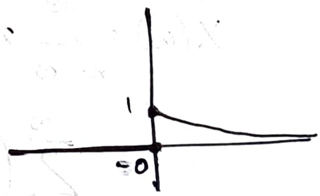
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

$$= e^{-i\omega a} = e^{i(-\omega a)}$$

$\Rightarrow |X(\omega)| = 1$ → magnitude ~~function~~ spectrum

$\varphi(X(\omega)) = -\omega a$ → phase spectrum.

7. (i) $x(t) = e^{-|a|t} \cdot u(t)$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-|a|t} \cdot 1 \cdot e^{-i\omega t}\, dt$$

$$= \frac{1}{-(|a|+i\omega)} \left[e^{-(|a|+i\omega)t} \right]_0^{\infty} = \frac{1}{|a|+i\omega} = \frac{|a|-i\omega}{(a^2+\omega^2)}$$

∴ (a) $|X(\omega)| = \dfrac{1}{\sqrt{a^2+\omega^2}}$

(b) $\angle X(\omega) = \tan^{-1}\left(-\dfrac{\omega}{|a|}\right)$

(c) $\operatorname{Re}\{X(\omega)\} = \dfrac{|a|}{a^2+\omega^2}$

(d) $\operatorname{Im}\{X(\omega)\} = -\dfrac{\omega}{a^2+\omega^2}$

(ii) $x(t) = e^{(-1+2j)t} \cdot u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} \cdot dt = \int_{0}^{\infty} e^{(-1+2j-j\omega)t} \cdot dt$$

$$= \frac{1}{-1+(2-\omega)i} \left[e^{(-1+2j-j\omega)t} \right]_0^{\infty} = \frac{-1-(2-\omega)i}{1+(2-\omega)^2}$$

(a) $|X(\omega)| = \sqrt{1+(2-\omega)^2}$

(b) $\angle X(\omega) = \tan^{-1}(2-\omega)$

(c) $\text{Re}\{X(\omega)\} = \dfrac{-1}{1+(2-\omega)^2}$

(d) $\text{Im}\{X(\omega)\} = \dfrac{-(2-\omega)}{1+(2-\omega)^2}$

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8. (i) $x[n] = \left(\frac{1}{5}\right)^n u(n+1)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u(n+1) e^{-i\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{5} e^{-i\omega}\right)^n$$

[assuming $u(0) = 1$]

$$X(\omega) = \frac{1}{1 - \frac{1}{5} e^{-i\omega}}$$

(ii) $x[n] = u(n)$ [assuming $u(0) = 1$]

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} u(n) \cdot e^{-i\omega n} = \sum_{n=0}^{\infty} e^{-i\omega n}$$

$$X(\omega) = \frac{1}{1 - e^{-i\omega}}$$

(iii) $x[n] = \left(\frac{1}{2}\right)^{n+2} u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n+2} u(n) \cdot e^{-i\omega n}$$

$$= \left(\frac{1}{2}\right)^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-i\omega n}$$ [assuming $u(0) = 1$]

$$X(\omega) = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{1 - \frac{1}{2} e^{i\omega}}$$

(iv) $x[n] = \left(\frac{1}{2}\right)^n u(n-4)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n-4) \cdot e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{\cancel{n}+4} \cdot u(n) \cdot e^{-i\omega(n+4)} = \left(\frac{1}{2}\right)^4 \cdot e^{-4i\omega} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cdot u(n) \, e^{-i\omega n}$$

$$= \frac{e^{-4i\omega}}{2^4} \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-i\omega}\right)^n = \frac{e^{-4i\omega}}{2^4\left(1 - \frac{1}{2} e^{-i\omega}\right)}$$

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9. (i) $\cancel{X(e^{i\omega}) = \frac{e^{-i\omega} - (1/4)}{1 - (\frac{1}{4}) e^{-i\omega}}}$ $x[n] = \frac{1}{4}\delta(n)$

$$X(k) = \sum_{n=0}^{7} x(n) \cdot e^{-i \cdot 2\pi \frac{n}{8} \cdot k} = \cancel{\sum_{n=0}^{3} x(n)} \frac{1}{4}\delta(0) \cdot 1 = \frac{1}{4}\delta(0)$$

(ii) $x[n] = \{1, -1, 1, -1, 1, \cancel{\ldots}\}$

[assuming $x[i] = (-1)^i$ for $0 \le i \le 4$
$= 0$ for $5 \le i \le 7$

$$\therefore X(k) = \sum_{n=0}^{7} x[n] \cdot e^{-i 2\pi \cdot \frac{n}{8} k} = \cancel{e^{-2\pi \cdot \frac{k}{8}}} - \cancel{e^{+i2\pi \frac{k}{8}}} + \cancel{e^{-2\pi}}$$

$$= 1 \cdot e^0 - 1 \cdot e^{-i \cdot 2\pi \frac{1}{8} k} + 1 \cdot e^{-i \cdot 2\pi \cdot \frac{2}{8} k} - 1 \cdot e^{-i \cdot 2\pi \cdot \frac{3}{8} k} + 1 \cdot e^{-i 2\pi \frac{4}{8} k}$$

$$= 1 - \left(\cos\left(\frac{\pi}{4}k\right) - i\sin\left(\frac{\pi}{4}k\right)\right) + \left(\cos\left(\frac{\pi}{2}k\right) - i\sin\left(\frac{\pi}{2}k\right)\right)$$

$$- \left(\cos\left(\frac{3\pi}{4}k\right) - i\sin\left(\frac{3\pi}{4}k\right)\right) + \left(\cos(\pi k) - \cancel{i\sin(\pi k)}^{0}\right)$$

$$= 1 - \left(\cos\left(\pi \frac{k}{4}\right) + \cos\left(k\pi - \frac{k}{4}\pi\right)\right)$$

$$+ i\left(\sin\left(\pi \frac{k}{4}\right) + \left(k\pi - \frac{k}{4}\pi\right)\right)$$

$$+ \cos\left(\frac{\pi}{2}k\right) - i\sin\left(\frac{\pi}{2}k\right) + \cos(\pi k)$$

odd $k \Rightarrow \; = \cancel{1} - 0 \qquad [\cos(\pi - x) = -\cos x]$

$+ 2i\sin\left(\frac{\pi k}{4}\right) \qquad [\sin(\pi - x) = \sin(x)]$

$+ \cos\left(\frac{\pi}{2}k\right) + i\sin\left(\frac{\pi}{2}k\right) - \cancel{1}$

even $k \Rightarrow \; = 1 - \cancel{2}\cos\left(\frac{\pi k}{4}\right) \qquad [\cos(2\pi - x) = \cos x]$

$+ 0 \qquad [\sin(2\pi - x) = -\sin x]$

$+ \cos\left(\frac{\pi}{2}k\right) + i\sin\left(\frac{\pi}{2}k\right) + 1$

$= 2 - 2\cos\frac{\pi k}{4} + \cos\left(\frac{\pi}{2}k\right) + i\sin\left(\frac{\pi}{2}k\right)$

(iii) $x[n] = \cos\left(\frac{\pi n}{4}\right) = \left\{1, \frac{1}{\sqrt{2}}; 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$

$X(k) = \sum_{n=0}^{7} x[n] \cdot e^{-i \cdot 2\pi \cdot \frac{n}{8} k} = \sum_{n=0}^{7} \cos\left(\frac{\pi n}{4}\right)\left[\cos\left(\frac{\pi n}{4}k\right) - i\sin\left(\frac{\pi n}{4}k\right)\right]$

$= \sum_{n=1}^{3} \left\{ \cos\left(\frac{\pi n}{4}\right)\left[\cos\left(\frac{\pi k}{4}n\right) - i\sin\left(\frac{\pi k}{4}n\right)\right]\right.$

$\left. + \cos\left(\frac{\pi}{4}(8-n)\right)\left[\cos\left(\frac{\pi k}{4}(8-n)\right) - i\sin\left(\frac{\pi k}{4}(8-n)\right)\right]\right\}$

$+ \underbrace{1[1-0]}_{n=0} + \underbrace{(-1)[(-1)^k - 0]}_{n=4}$

$= \sum_{n=1}^{3} \cos\left(\frac{\pi n}{4}\right)\left[2\cos\left(\frac{\pi n}{4}k\right)\right] + 1 + (-1)^{k+1}$

$= 2\left[\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}k\right) + 0 - \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}3k\right)\right] + 1 + (-1)^{k+1}$

$= \sqrt{2}\left[\cos\left(\frac{\pi k}{4}\right) - \cos\left(\frac{3\pi k}{4}\right)\right] + 1 - (-1)^k$

(iv) $x[n] = \{i, 0, -i, 1\}$ [assuming $x[i] = 0$ for $4 \le i < 7$]

$X(k) = \sum_{n=0}^{7} x[n] \cdot e^{-i \cdot 2\pi \frac{n}{8} k}$

$= i \cdot e^{-i 2\pi \cdot 0 \cdot k} + 0 \cdot e^{-2\pi \cdot \frac{1}{8} k}$

$- i \cdot e^{-i 2\pi \cdot \frac{2}{8} k} + 1 \cdot e^{-i2\pi \cdot \frac{3}{8} k}$

$= i + \cancel{0} - i\left(\cos\left(\frac{\pi}{2}k\right) - i\sin\left(\frac{\pi}{2}k\right)\right) + \left(\cos\left(\frac{3\pi}{4}k\right) - i\sin\left(\frac{3\pi}{4}k\right)\right)$

$$= i - i\left(\cos\left(\frac{\pi}{2}k\right) + \sin\left(\frac{3\pi}{4}k\right)\right) - \left(\sin\left(\frac{\pi}{2}k\right) - \cos\left(\frac{3\pi}{4}k\right)\right)$$

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10 (i) $X(e^{j\omega}) = \dfrac{e^{-j\omega} - \frac{1}{4}}{1 - \frac{1}{4}e^{-j\omega}}$

$$\cancel{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cdot e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{j\omega(n-1)} - \frac{1}{4}e^{j\omega n}}{1 - \frac{1}{4}e^{-j\omega}}\, d\omega}$$

$$= \frac{1}{\left(1 - \frac{1}{4}e^{-j\omega}\right)}\left(e^{-j\omega} - \frac{1}{4}\right) = \left(e^{-j\omega} - \frac{1}{4}\right)\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$= \left(e^{-j\omega} - \frac{1}{4}\right)\sum_{n=0}^{\infty}\frac{1}{4^n}e^{-j\omega n} = \sum_{n=0}^{\infty}\frac{1}{4^n}e^{-j\omega(n+1)} - \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty}\frac{4}{4^n}e^{-j\omega n} - \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{4^{n-1}}\cdot u(n-1) - \frac{1}{4^{n+1}}\cdot u(n)\right)\cdot e^{-j\omega n}$$

$$\Rightarrow x[n] = \frac{u(n-1)}{4^{n-1}} - \frac{u(n)}{4^{n+1}}$$

(ii) $X(e^{j\omega}) = \dfrac{1 - \left(\frac{1}{2}\right)^4 e^{-4j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$

$$= \frac{1 - \left(\frac{1}{2}e^{-j\omega}\right)^4}{1 - \frac{1}{2}e^{-j\omega}}$$

$$= 1 + \left(\frac{1}{2}e^{-j\omega}\right) + \left(\frac{1}{2}e^{-j\omega}\right)^2 + \left(\frac{1}{2}e^{-j\omega}\right)^3$$

$$= \sum_{n=0}^{3}\left(\frac{1}{2}e^{-j\omega}\right)^n = \sum_{n=0}^{3}\frac{1}{2^n}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2^n}u(n)u(3-n)\cdot e^{-j\omega n} \quad \therefore x[n] = \frac{u(n)u(3-n)}{2^n}$$

(iii) $X(e^{j\omega}) = \cos^2 3\omega + \sin^2\omega = \cos^2 3\omega + 1 - \cos^2\omega$

$$= \frac{\cancel{1} + \cos 6\omega}{2} + 1 - \frac{\cancel{1} + \cos 2\omega}{2} = \frac{1}{2}(\cos 6\omega)$$

$$- \frac{1}{2}(\cos 2\omega) + 1$$

$$\Rightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\cos 6\omega - \frac{1}{2}\cos 2\omega + 1\right) \cancel{d\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi} \cos 6\omega \cdot e^{j\omega n}\, d\omega - \frac{1}{2}\int_{-\pi}^{\pi} \cos 2\omega\, e^{j\omega n}\, d\omega + \int_{-\pi}^{\pi} e^{j\omega n}\, d\omega\right]$$

~~for n ≠ 0~~ for $n \neq 0$:

let $I(a) = \int \cos(a\omega) \cdot e^{j\omega n} d\omega = \int \cos(a\omega)\left[\cos(\omega n) + i\sin(\omega n)\right] d\omega$

$$[I(a)]_{-\pi}^{\pi} = \int_{-\pi}^{\pi} \cos(a\omega) \cdot \cos(\omega n) d\omega + i \cancel{\int_{-\pi}^{\pi} \cos(a\omega) \cdot \sin(\omega n) d\omega}^{\,0}$$

$$= \begin{cases} 0, & a \neq n \\ \pi, & a = n. \end{cases} = \delta(a-n) \quad [\delta(0) = 1]$$

$$\therefore x[n] = \frac{1}{2\pi}\left[\frac{1}{2}\delta(a-6) + \delta(a-2) + \int_{-\pi}^{\pi} e^{j\omega n} d\omega\right] \text{ for } n \neq 0$$

$$\int_{-\pi}^{\pi} e^{j\omega n} d\omega = \cancel{\frac{1}{jn}\left[e^{j\omega n}\right]_{-\pi}^{\pi}} = \cancel{\frac{1}{jn}\left[\cos(\omega n) + i\sin(\omega n)\right]_{-\pi}^{\pi}}$$

$$= \cancel{\frac{(-1)^n}{jn}} \cdot \int_{-\pi}^{\pi} \cos(\omega n) + i\sin(\omega n)\, d\omega = \frac{(-1)^n}{jn}$$

for $n = 0$, $I(a) = \int_{-\pi}^{\pi} \cos(a\omega) d\omega = -\frac{1}{a}\left[\sin(a\omega)\right]_{-\pi}^{\pi} = 0$

since $a \neq 0$, this is is taken care of by the $n \neq 0$ case.

$$\int_{-\pi}^{\pi} e^{j\omega n} d\omega = \int_{-\pi}^{\pi} d\omega = 2\pi$$

$$\therefore x[n] = \frac{1}{2\pi}\left[\frac{1}{2}\delta(a-6) + \frac{1}{2}\delta(a-2)\right] + \begin{cases} \frac{(-1)^n}{2\pi jn}, & n \neq 0 \\ 1, & n = 0. \end{cases}$$

11. $$x[n] = (n-1)\left(\frac{1}{a}\right)^{|n|}$$

let $$z[n] = \left(\frac{1}{a}\right)^{|n|}$$

$$\Rightarrow Z(\omega) = \sum_{n=-\infty}^{\infty} z[n]\cdot e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{a}\right)^{|n|}\cdot e^{-i\omega n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{a}\right)^{n} e^{-i\omega n} + \sum_{n=1}^{\infty}\left(\frac{1}{a}\right)^{n}\cdot e^{+i\omega n} + 1$$

$$= \frac{1}{1-\frac{1}{a}e^{-i\omega}} + \frac{1}{1-\frac{1}{a}e^{i\omega}} + \cancel{1} - \cancel{1} - 1$$

now, the transform of $nz[n]$ is $i\dfrac{dZ(\omega)}{d\omega}$.

and $x[n] = nz[n] - z[n] = (n-1)z[n]$

$\therefore$ by linearity,

$$X(\omega) = i\,\frac{dZ(\omega)}{d\omega} - Z(\omega)$$

$$= \left[-i\,\frac{\left(-\frac{1}{a}\right)(-i)e^{-i\omega}}{\left(1-\frac{1}{a}e^{-i\omega}\right)^2} - i\,\frac{\left(-\frac{1}{a}\right)(i)\,e^{i\omega}}{\left(1-\frac{1}{a}e^{i\omega}\right)^2} - 0 \right]$$

$$- \frac{1}{\left(1-\frac{1}{a}e^{-i\omega}\right)} - \frac{1}{\left(1-\frac{1}{a}e^{i\omega}\right)} + 1$$

$$= \frac{-\frac{1}{a}e^{-i\omega}}{\left(1-\frac{1}{a}e^{-i\omega}\right)^2} + \frac{-\frac{1}{a}e^{i\omega}}{\left(1-\frac{1}{a}e^{i\omega}\right)^2}$$

$$- \frac{1-\frac{1}{a}e^{-i\omega}}{\left(1-\frac{1}{a}e^{-i\omega}\right)^2} - \frac{1-\frac{1}{a}e^{i\omega}}{\left(1-\frac{1}{a}e^{i\omega}\right)^2} + 1$$

$$= -\frac{1}{\left(1-\frac{1}{a}e^{-i\omega}\right)^2} - \frac{1}{\left(1-\frac{1}{a}e^{i\omega}\right)^2} + 1$$

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12. $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\omega n}$$

$$= (1)\,e^{2j\omega} + (-2)\cdot e^{j\omega} + (7)\cdot + (8)\cdot e^{-2j\omega} + (-10)\,e^{-3j\omega}$$

(a) $$|X(e^{j\omega})| = \left| e^{2j\omega} - 2e^{j\omega} + 7 + 8e^{-2j\omega} - 10e^{-3j\omega} \right|$$

$$= \Big| \big(\cos(2\omega) - 2\cos(\omega) + 7 + 8\cos(-2\omega) - 10\cos(-3\omega)\big) + i\big(\sin(2\omega) - 2\sin\omega + 8\sin(-2\omega) - 10\sin(-3\omega)\big)\Big|$$

When we take the magnitude, we obtain an expression w/ 4 types of terms:

(a) $\cos^2(a\omega)$ (b) $\cos(a\omega)\cos(b\omega)$

(c) $\sin^2(a\omega)$ (d) $\sin(a\omega)\cos(b\omega)$

(e) $a\cos(b\omega)$ (f) a^2

When we integrate from $-\pi$ to π, only (a), (c) and (f) yield nonzero values $\rightarrow$ π π $2\pi a^2$

$$\therefore \int_{-\pi}^{\pi} |X(e^{j\omega})|\, d\omega = \pi + 4\pi + 64\pi + 100\pi + 49\pi\cdot 2 + \pi + 4\pi + 64\pi + 100\pi$$

$$= 636\pi$$

(b) $$\frac{d}{d\omega}(X e^{j\omega}) = 2je^{2j\omega} - 2je^{j\omega} + 0 - 16je^{-2j\omega} + 30je^{-3j\omega}$$

$$= (-2\sin 2\omega + 2\sin\omega + 16\sin(-2\omega) - 30\sin(-3\omega)) + i(2\cos(2\omega) - 2\cos\omega - 16\cos(-2\omega) + 30\cos(-3\omega))$$

As in part (a),

$$\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right| d\omega = 4\pi + 4\pi + 256\pi + 900\pi + 4\pi + 4\pi + 256\pi + 900\pi$$

$$= 2328\pi$$