Digital Signal Analysis (CS7.303)

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Fourier Series and Transform

Fourier Transform

Properties of FT (contd.)

Convolution is a mathematical operator on two functions, that produces a third function which expresses how one affects the shape of the other. It is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau.$$

The convolution property of the Fourier transform states that if

$$\begin{aligned} x(t) &\to X(\omega) \\ h(t) &\to H(\omega), \end{aligned}$$

then

$$\begin{aligned} x(t) * h(t) &\to X(\omega) \cdot H(\omega) \\ x(t) \cdot h(t) &\to X(\omega) * H(\omega) \end{aligned}$$

This can be easily proved:

$$\begin{aligned} F\left[f_1(t) * f_2(t)\right] &= \int_{-\infty}^{\infty} \left[f_1(t) * f_2(t)\right] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) e^{-j\omega t} d\tau dt \\ &= \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t} dt\right] d\tau \\ &= \int_{-\infty}^{\infty} f_1(\tau) F_2(\omega) e^{-j\omega\tau} d\tau \\ &= F_1(\omega) F_2(\omega). \end{aligned}$$

Discrete-Time Fourier Transform

As the Fourier transform takes as input and gives as output continuous real-valued functions, it is not possible for computers to compute it exactly. The discrete-time fourier transform overcomes this difficulty by simply taking the value of the input $x(t)$ at discrete values n:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$x(n)$ is defined as $x(t)\cdot\delta(t)$, where $\delta(t)$ is the *impulse train* (its value is zero everywhere except at discrete inputs separated by T_s, where it is infinite). Writing the input as $X\left(e^{j\omega}\right)$ makes it periodic (introducing discretisation in one domain introduces periodicity in the other).

More Details.
Say we have a continuous function $m(t)$ and an impulse train $\delta(t)$, with an infinite value at inputs separated by T_s. Note that the integral across each spike is 1.

We can find the Fourier series of $\delta(t)$:

$$\delta(t) = \frac{1}{T_s}\left[1 + \sum_{n=1}^{\infty} 2\cos(n\omega_0 t)\right].$$

Now, we multiply $m(t)$ with $\delta(t)$ to sample it:

$$g(t) = \frac{1}{T_s}\left[m(t) + 2\cos(\omega_0 t)m(t) + 2\cos(2\omega_0 t)m(t) + \cdots\right].$$

We can express this in complex form

$$g(t) = \frac{1}{T_s}\left[m(t) + e^{-j\omega_0 t}m(t) + e^{j\omega_0 t}m(t) + e^{-2j\omega_0 t}m(t) + e^{2j\omega_0 t}m(t) + \cdots\right],$$

which makes it easy to calculate the Fourier transform:

$$G(\omega) = \frac{1}{T_s}\left[M(\omega) + M(\omega+\omega_0) + M(\omega-\omega_0) + M(\omega+2\omega_0) + M(\omega-2\omega_0) + \cdots\right]$$

If $M(\omega)$ has some cutoff frequency f_m (*i.e.* for values $> f_m$ or $< -f_m$, it is zero), then $G(\omega)$ will have a number of possibly overlapping regions. This intereference is called aliasing.
The Nyquist Sampling Theorem tells us the minimum sampling rate $f_s = \frac{1}{T_s}$ to avoid aliasing:

$$f_s \geq 2f_m$$

The discrete-time Fourier transform can be inverted:

$$x(n) = \frac{1}{2\pi}\int_T X\left(e^{j\omega}\right)e^{j\omega n}d\omega.$$

Discrete Fourier Transform

The DTFT, however, outputs a continuous real-valued function, like the FT. The discrete Fourier transform samples the output function as well, at discrete values k, in order to avoid this.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{2\pi jnk}{N}}, k \in \{0, \dots, N-1\}.$$

This is called the N-point DFT. Here, we require $x(t)$ to be periodic with period N; thus we take $x_p(n)$, a periodic version of $x(t)$.