Digital Signal Analysis (CS7.303)

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Z-Transform

We have seen that the discrete-time Fourier transform is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}.$$

However, this expression may not always converge. The Z-transform, therefore, takes care of this by adding another parameter:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n},$$

where $z = re^{j\omega}$. By choosing z appropriately (within the region of convergence or RoC), we can ensure that $X(z)$ converges.

Examples

We will consider five examples of the Z-transform.

First, let $x(n) = \delta(n)$. Then we have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} = 1.$$

The RoC is the entire z-plane in this case.

Next, consider $x(n) = \delta(n-k)$. Then,

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n-k)z^{-n} = z^{-k}.$$

Here, the RoC is $\mathbb{C}$ if $k = 0$, $\mathbb{C} - \{0\}$ if $k > 0$, and $\mathbb{C} - \{\infty\}$ if $k < 0$.

Third, we have $x(n) = \{4, 3, 0, 1\}$, with $x(0) = 3$. This gives us

$$X(z) = 4z + 3 + \frac{1}{z^2},$$

and so the RoC is $\mathbb{C} - \{0, \infty\}$.

Fourth, let $x(n) = p^n u(n)$. Then we get

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} p^n u(n) z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{p}{z}\right)^n \\ &= \frac{z}{z-p}. \end{aligned}$$

For the RoC,

$$\left|\frac{p}{z}\right| < 1,$$

which means that

$$|z| = r > |p|.$$

Lastly, we take $x(n) = -p^n u(-n-1)$. Then,

$$\begin{aligned} X(k) &= \sum_{n=-\infty}^{-1} -\left(\frac{p}{z}\right)^n \\ &= -\sum_{k=1}^{\infty} \left(\frac{p}{z}\right)^k \\ &= -\frac{p}{z} \cdot \frac{1}{1 - \frac{p}{z}} \\ &= \frac{z}{z-p}. \end{aligned}$$

The RoC here is

$$|z| = r < |p|.$$

Note that the expression for $X(z)$ is the same as in example 4, but the RoC is different. Example 4 is a right-side signal, while example 5 is a left-side signal.

If $x(n)$ is expressed as $x_1(n) + x_2(n) + \cdots$, then its RoC is given by $R_1 \cap R_2 \cap \cdots$. Also, the Z-transform is linear.

Properties

By the time-shifting property of the Z-transform, if $x(n)$ transforms to $X(z)$ with RoC R, then $x(n-k)$ transforms to $z^{-k} X(z)$.
The RoC for this is $R - \{0\}$ if $k > 0$, and $R - \{\infty\}$ if $k < 0$.

Scaling in the z-domain is related to exponential rise in the time domain; thus $a^n x(n)$ transforms to $X\left(\frac{z}{a}\right)$.
The RoC changes from $z \in R$ to $\frac{z}{a} \in R$.

Similarly, the time inversion property states that $x(-n)$ transforms to $X\left(\frac{1}{z}\right)$.
The RoC, here too changes to $\frac{1}{z} \in R$.

Furthermore, $nx(n)$ transforms to $-z\frac{d}{dz}X(z)$.

Let $x_1(n)$ and $x_2(n)$ transform to $X_1(z)$ and $X_2(z)$, with RoCs R_1 and R_2 respectively. Then the convolution $x_1(n) * x_2(n)$ transforms to $X_1(z) \cdot X_2(n)$.
This property can be used to solve LTI systems. We know that $y(n) = x(n) * h(n)$; thus we can find $Y(z) = X(z) * H(z)$ and find the inverse Z-transform to obtain $y(n)$.

Furthermore, the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z),$$

and the final value theorem states that

$$\lim_{n \to \infty} x(n) = \lim_{z \to 1} (z-1)X(z).$$

Inverse Z-Transform

To take the inverse Z-transform, we can employ several methods, like:

- contour integration
- table lookup
- using properties
- series expansion (by long division or other methods)
- partial fractions

Note that if the RoC is not given, we assume $X(z)$ to be a right-side signal, *i.e.*, take the rightmost possible RoC.

The RoC may change the signal we obtain. For example, consider the signal

$$X(z) = \frac{a}{1 - \frac{b}{z}}.$$

If the RoC is given to be $|z| > |b|$, then the inverse is

$$x(n) = ab^n u(n),$$

which is called the right-side signal. However, if the RoC is $|z| < |b|$, then the inverse is

$$x(n) = -ab^n u(-n-1),$$

i.e., the left-side signal.

Application

Consider a digital system given by

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1),$$

and we need to find $h(n)$. We can do this by converting to the z-domain:

$$Y(z) + 0.1z^{-1}Y(z) - 0.2z^{-2}Y(z) = X(z) + z^{-1}X(z).$$

Now we know that $H(z) = \frac{Y(z)}{X(z)}$, which gives us

$$H(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}.$$

We can then calculate the IZT of $H(z)$ to find $h(n)$, which will come out to be

$$h(n) = 1.556(0.4)^n u(n) - 0.556(-0.5)^n u(n).$$