Assignment 2

(CS1.502) Information-Theoretic Methods in Computer Science, Spring 2023

Due on 24^{th} February (Friday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. (a) Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean μ and variance σ^2. Let $\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ be the sample mean. Show that $\bar{Z}_n$ converges to μ in probability, i.e., for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathrm{P}(|\bar{Z}_n - \mu| > \epsilon) = 0.$$

This is known as the weak law of large numbers (WLLN).

(3 marks)

(b) For a sequence $x^n := (x_1, x_2, \ldots, x_n)$ with x_i, for $i \in [1:n]$, taking values in a finite alphabet $\mathcal{X}$, the empirical PMF is defined as $\pi_{x^n}(x) = \frac{|\{i : x_i = x\}|}{n}$. Let $X^n = (X_1, X_2, \ldots, X_n)$ be a sequence of i.i.d. random variables with X_i, for each $i \in [1:n]$, distributed according to P_X over $\mathcal{X}$. Using WLLN, show that $\pi_{X^n}(x)$ converges to $P_X(x)$ in probability, i.e., $\lim_{n\to\infty} \mathrm{P}(|\pi_{X^n}(x) - P_X(x)| > \epsilon) = 0$, for all $x \in \mathcal{X}$.

(2 marks)

Problem 2. A tripartite graph is a graph where the vertex set can be partitioned into three disjoint sets A, B, and C such that no two vertices within the same set have an edge between them. Let n_1 be the number of edges between vertices in A and vertices in B, n_2 be the number of edges between B and C, and n_3 be the number of edges between A and C. Suppose n denotes the maximum number of triangles in such a graph. Prove that $n^2 \leq n_1 n_2 n_3$.

[*Hint.* A triangle can be represented by three vertices, one each from A, B, and C. Pick a triangle uniformly at random from the set of all triangles in the graph.]

(5 marks)

Problem 3. Let S be a random variable distributed according to P_S over the set of all subsets of $[1:n]$ and μ be such that $\mathrm{P}(i \in S) \geq \mu$, for all $i \in [1:n]$. Then for jointly distributed random variables $X_1, X_2, \ldots, X_n$,

$$\mu H(X_1, X_2, \ldots, X_n) \leq \mathbb{E}_{S\sim P_S}[H(X_S)].$$

(5 marks)