Assignment 3

(CS1.502) Information-Theoretic Methods in Computer Science, Spring 2023

Due on 9th April (Sunday).

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Consider a 4×4 matrix in which all the entries are equal to 1. Note that this has an equivalent representation in the form of a complete bipartite graph and the permutation $\sigma = (1\ 2\ 3\ 4)$ (i.e., $\sigma(i) = i$, for all $i \in [1:4]$) corresponds to a perfect matching of that graph. For an arbitrary permutation τ, define $k = \tau^{-1}(1)$ and $N_1(\sigma, \tau) = |\{1,2,3,4\} \setminus \{\sigma(\tau(1)), \sigma(\tau(2)), \ldots, \sigma(\tau(k-1))\}|$. For $j \in [1:4]$, argue that for all permutations τ such that $\tau(j) = 1$, we have $N_1(\sigma, \tau) = 5 - j$. Finally, conclude that the number of permutations τ for which $N_1(\sigma, \tau) = j$ is equal to 6, for all $j \in [1:4]$.

(5 marks)

Problem 2. For a continuous random variable X with mean zero and variance σ^2, show that

$$h(X) \leq \frac{1}{2} \log \left(2\pi e \sigma^2\right)$$

with equality if and only if X is a Gaussian random variable.

(4 marks)

Problem 3. For any two probability distributions P_X and Q_X, suppose $D(P_X \| Q_X) \geq \frac{k}{\ln 2} TV(P_X, Q_X)^2$ holds, for a constant k. Show that $k \leq 2$.

[*Hint.* Consider $P_X(1) = \frac{1}{2} = 1 - P_X(0)$ and $Q_X(1) = \frac{1}{2} + \epsilon = 1 - Q_X(0)$, for $\epsilon \in (0, \frac{1}{4})$.]

(3 marks)

Problem 4. Let P_X and Q_X be two probability distributions on $\mathcal{X}$ and $f : \mathcal{X} \to [0, B]$, where B is a constant. Prove that $|E_{X \sim P_X}[f(X)] - E_{X \sim Q_X}[f(X)]| \leq B.TV(P_X, Q_X)$.

(3 marks)