Conditioning on random variables: Summary

$$p_{X|A}(x) := \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}.$$

$$E[X/A] = \sum_x x p_{X|A}(x).$$

$$p_X(x) = \sum_{i=1}^n \mathbb{P}(A_i) p_{X|A_i}(x)$$

$$E[X] = \sum_{i=1}^n \mathbb{P}(A_i) E[X|A_i]$$

$$p_{X,Y}(x,y) = p_{X|Y}(x|y) p_Y(y)$$

$$p_X(x) = \sum_y p_{X|Y}(x|y) p_Y(y)$$

$$E[X|Y=y] := \sum_x x p_{X|Y}(x|y)$$

$$E[X] = \sum_y p_Y(y) E[X|Y=y]$$

How about all this for continuous X & Y?

$$\int_{x \in B} f_{X|A}(x) = \mathbb{P}(X \in B|A).$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

$$E[X/A] = \int_{-\infty}^{\infty} x f_{X|A}(x) dx$$

$$f_X(x) = \int_y f X|Y(x|y) f_Y(y) dy$$

$$f_X(x) = \sum_{i=1}^{n} \mathbb{P}(A_i) f_{X|A_i}(x)$$

$$E[X|Y = y] = \int_x x f_{X|Y}(x|y) dx$$

$$E[X] = \sum_{i=1}^{n} \mathbb{P}(A_i) E[X|A_i]$$

$$E[X] = \int_y E[X|Y = y] f_Y(y) dy$$

Conditional expectation $E[X|Y]$

Recall that

$$E[X|Y=y] := \sum_x x p_{X|Y}(x|y)$$

- $E[X|Y=y]$ is a function of y.
- Now consider $E[X|Y]$.
- $E[X|Y]$ is a function of Y, say $g(Y)$.
- When Y takes the value y,(this happens with probability $p_Y(y)$)$E[X|Y]$ takes the value $E[X|Y=y]$.
- What is the expectation of $E[X|Y]$?

Conditional expectation $E[X|Y]$

- $g(Y) = E[X|Y]$.
- What is $E[g(Y)] = E[E[X|Y]]$?
- $E[g(Y)] = \sum_y g(y)p_Y(y) = \sum_y E[X|Y = y]p_Y(y)$.
- This implies $E[g(Y)] = E[E[X|Y]] = E[X]$. This is the law of iterated expectation.

$$E[E[X|Y]] = E[X]$$

Sampling from continuous random variables

Lemma

Let U be uniform random variable over $[0,1]$. Consider any continuous cdf $F(.)$.Consider a random variable X defined as follows

$$X := F^{-1}(U)$$

Then the cdf of X is $F(.)$.

Proof:

- Let $F_X(x)$ be the cdf of X,i.e., $F_X(x) := \mathbb{P}[X \leq x]$.Then

$$F_X(x) = \mathbb{P}[F^{-1}(U) \leq x]$$

$$= \mathbb{P}[U \leq F(x)]$$

$$= F(x)$$

Sampling from continuous random variables

Lemma

Let U be uniform random variable over $[0,1]$. Consider any continuous cdf $F(.)$. Consider a random variable X defined as follows

$$X := F^{-1}(U)$$

Then the cdf of X is $F(.)$.

- Using this lemma, how to generate samples of a continuous random variable X using samples of U?
- **Answer:** Draw $u \sim U$ and obtain $F^{-1}(u)$. This is a sample from X.

Convergence of Random Variables

Modes of Convergence ($X_n \to X$)

Pointwise or Sure convergence

> $\{X_n, n \geq 0\}$ converges to X pointwise or surely if for all $\omega \in \Omega$ we have $\lim_{n\to\infty} X_n(\omega) = X(\omega)$

- Consider $\Omega = \{H, T\}$.
- Further, $X_n = \begin{cases} \frac{1}{n} & \text{if } \omega = H \\ 1 + \frac{1}{n} & \text{if } \omega = T. \end{cases}$ and $X = \begin{cases} 0 & \text{if } \omega = H \\ 1 & \text{if } \omega = T. \end{cases}$

Almost sure convergence

X_n converges to X almost surely if

$$P\left(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1.$$

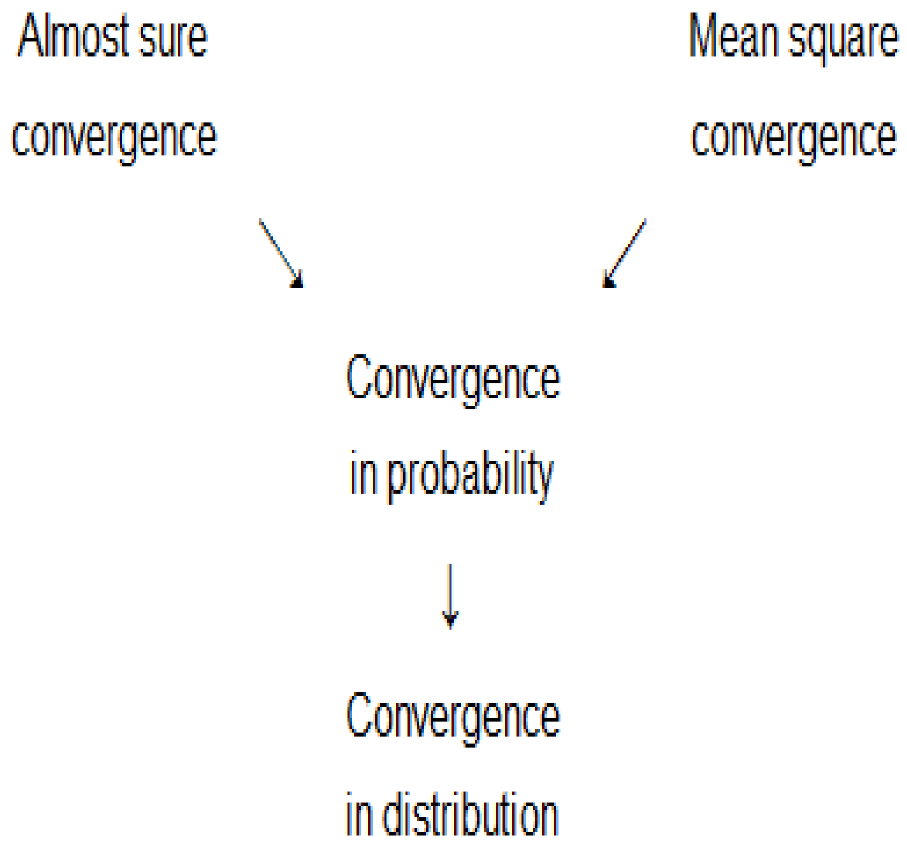
- The set of outcomes where the convergence does not happen has measure 0. $P\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 0.$
- Consider $\Omega = [0,1]$ where you pick a number uniformly in $[0,1]$. Let $X_n(\omega) = \omega^n$ for all $\omega \in \Omega$ and $X(\omega) = 0$ for all ω.
- $X_n(\omega) \to X(\omega)$ for $\omega \in [0,1)$.
- $X_n(\omega) \nrightarrow X(\omega)$ for $\omega = 1$ and $\mathbb{P}\{\omega = 1\}$.
- This is almost sure convergence as $\mathbb{P}\{[0,1)\} = 1$.

Example 2 (SLLN): Let $\{X_n, n \geq 0\}$ denote a sequence of i.i.d random variables with mean μ and denote $S_n = \sum_{i=1}^{n} X_i$. Then $\frac{S_n}{n} \to \mu$ a.s.

Summary

Pointwise convergence	$\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for every ω
Almost sure convergence	$\lim_{n\to\infty} X_n(\omega) = X(\omega)$ almost surely
Convergence in probability	$\lim_{n\to\infty} \mathrm{P}(\lvert X_n - X\rvert > \epsilon) = 0$ for any $\varepsilon > 0$
Mean-square convergence	$\lim_{n\to\infty} \mathrm{E}[(X_n - X)^2] = 0$
Convergence in distribution	$\lim_{n\to\infty} F_n(x) = F(x)$ for any continuity point x

Relation between modes of convergence (no proofs)



https://en.wikipedia.org/wiki/Proofs_of_convergence_of_random_variables

Interchanging limits and expectation

- Suppose $X_n \to X$ a.s. Then when is $\lim_{n\to\infty} E[X_n]$ equal to $E[\lim_{n\to\infty} X_n] = E[X]$?

- A counterexample where the exchange is not possible?

- $U \sim U(0,1)$ and $X_n = n1_{\{U<\frac{1}{n}\}}$.

- In this example, $X_n \to 0$ but $E[X_n] = 1$ and hence the interchange is not possible.

Monotone Convergence Theorem

Theorem

Suppose X_n is an increasing sequence of non-negative random variables, i.e., $X_n(\omega) \leq X_{n+1}(\omega)$ for all n and $\omega \in \Omega$. Then $X = \lim_{n\to\infty} X_n$ exists and $E[X_n] \uparrow E[X]$ as $n \to \infty$.

Corollary

If $Y_i \geq 0$, then $E[\sum_{i=1}^{\infty} Y_i] = \sum_{i=1}^{\infty} E[Y_i]$.

Hint: Set $X_n = \sum_{i=1}^{n} Y_n$.

Dominated Convergence Theorem

Theorem

Suppose $X_n \to X$ a.s. and there exists a random variable Y with $E[Y] < \infty$ such that $|X_n| \leq Y$ for all n. Then $E[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} E[X_n]$.

- Example 1: $X \sim N(0,1)$ and $X_n = min(X, n)$.
- Example 2: $U \sim U(0,1)$ and $X_n = U/n$. The limit $X = 0$.
- If Y is a constant, we often call it the Bounded convergence theorem.