Distillation and Bound Entanglement

Abhinav Sapra

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Abstract

This is a report on the paper "Distillation and Bound Entanglement"[2] by Pawel and Ryszard Horodecki, written for the course "Physics of Quantum Information" offered at CMI. In this report we will focus on Distillation and Bound Entanglement in *Bipartite Systems*.

1 Prologue

The phenomenon of entanglement in quantum mechanics plays a central role in quantum information theory and quantum computation. However, for a general bipartite mixed quantum state, it is still not known how to determine whether such a state is entangled or separable. Since the results of many experiments are not pure but in fact mixed states, this is a problem that is of both fundamental and practical importance within quantum mechanics and quantum information theory.
Quantum entanglement, discovered in 1935 by EPR and Schrödinger, led to Bell's inequalities, unveiling the strange properties of entangled states. Today, it underpins quantum communication (like cryptography and teleportation) and quantum computing. However, realizing these applications is impeded by sensitivity to quantum noise. This lead to invention of *Distillation of Noisy Entanglement*.
The aim of this report is to present an overview of that part of the quantum information theory which concerns entanglement distillation together with its limits symbolized by so called bound entanglement, but we will mostly deal with bipartite systems.

2 Entanglement in Bipartite Systems and Positive Maps

One of the most studied mathematical tools for determining whether a quantum state is entangled or not is the theory of positive maps of operators on a Hilbert space. In particular, the construction of indecomposable positive maps is a topic of particular importance, because they can be used to form strong necessary criteria for a quantum state to be separable.

2.1 Entangled and Separable states

We begin by defining entanglement for bipartite quantum systems. Let $\mathcal{H}_i$ be two finite dimensional Hilbert Spaces for $i \in \{1, 2\}$. Define $\mathcal{B}(\mathcal{H})$ to be the set of bounded operators on $\mathcal{H}$. A bipartite quantum state is $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is said to *separable* if and only if we can write ρ in the form

$$\rho = \sum_{i=1}^{k} p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$$

where $p_i > 0, \sum p_i = 1$ and $\rho_i^{(j)} \in \mathcal{B}(\mathcal{H}_j)$ are density matrices on individual parts of the bipartite system. Hence, a bipartite system that is not separable is know as *entangled*. In general it is hard to check separability of a given state, but if however the $m \otimes n$ is pure then it is separable iff the reduced density matrix is pure. We just use the Schmidt decomposition to get,

$$|\psi\rangle = \sum_i a_i |e_i\rangle \otimes |f_i\rangle, \;\; a_i < 0, \;\; a_i < a_{i+1}$$

Thus, $|\psi\rangle$ is separable $\iff$ only one of a_i-s is non-zero.

2.2 Positive maps and Entanglement

Let $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a Linear map. Then we say that Λ is a *Positive* map if for all hermitian $\sigma \in \mathcal{H}$, if $\sigma \geqslant 0$ then $\Lambda(\sigma) \geqslant 0$.

Let $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a Linear map. Then we say that Λ is a *Completely-Positive* map if for all Hilbert spaces $\mathcal{K}$ the map $\mathcal{I} \otimes \Lambda : \mathcal{B}(\mathcal{K} \otimes \mathcal{H}) \to \mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ is positive.

We say that Λ is *Completely Co-positive* map if $\Lambda \otimes \mathcal{T}$ is completely positive, where $\mathcal{T}$ is Transpose map.

Due to Choi and Kraus, we know that a completely-positive map $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has a decomposition of the form

$$\Lambda(\rho) = \sum_i V_i \rho V_i^\dagger$$

where $\{V_i\} \in \mathcal{B}(\mathcal{H})$ is the set of Kraus operators. Moreover, Λ describes a Quantum Channel iff $\sum V_i^\dagger V_i = \mathcal{I}$.

Existence of positive maps that are not completely-positive helps us produce sufficient conditions for bipartite systems to be *Separable*.

If ρ is separable, i.e., $\rho = \sum p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$. Then for a positive map Λ, we have

$$(\mathcal{I} \otimes \Lambda)(\rho) = \sum_i p_i \rho_i^{(1)} \otimes \Lambda(\rho_i^{(2)}) \geqslant 0$$

But if $\mathcal{I} \otimes \Lambda(\rho) \ngeqslant 0$ then we can be sure that ρ is Entangled.

Thus this gives a way to check if a bipartite system is entangled, i.e.,
if $\mathcal{I} \otimes \Lambda \ngeqslant 0 \implies$ Bipartite state ρ must be entangled.

Let us look at an example of a map that is positive but *not* completely-positive:

Consider the *Transpose* map. Clearly, it is positive, to see this consider a positive matrix A in $\mathcal{B}(\mathcal{H})$, then we know that for any $v \in \mathcal{H}$, $\langle v|A|v\rangle = v^* A v \geqslant 0$ and on taking the transpose on both sides we get, $v^T A^T v^{*T} = v^T A^T \bar{v} = \langle \bar{v}|A^T|\bar{v}\rangle \geqslant 0$, where $\bar{v}$ is conjugate of v. Thus, we get A^T is also positive, therefore, Transpose map is positive.

To see that Transpose is not a completely-positive map, just consider the matrix

$$A = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}$$

and now consider the image of this matrix under the map $\mathcal{I} \otimes \mathcal{T}$, we get,

$$
\begin{aligned}
(\mathcal{I} \otimes \mathcal{T})(A) &= \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T \end{bmatrix} \\
&= \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}
\end{aligned}
$$

which is not positive as it's eigenvalues are $\{1, 1, 1, -1\}$.

This leads to the *Partial Transpose* condition for separability, $(\mathcal{I} \otimes \mathcal{T})(\rho) \geqslant 0$.

For $2 \otimes 2$ and $2 \otimes 3$ bipartite systems, this condition is sufficient as well. More generally, there exists entangled states that do not violate this condition. Those are known as *Bound Entangle* state.

2.3 Decomposable Maps

A map $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called *Decomposable* if it can be written as the sum of a completely positive map and a completely co-positive map. We care for such maps because for decomposable map Λ,

$$(\mathcal{I} \otimes \mathcal{T})(\rho) \geqslant 0 \implies (\mathcal{I} \otimes \Lambda)(\rho) \geqslant 0$$

Thus, partial transpose conditions can detect any entanglement that the condition formed from a decomposable map.

A positve map is called *Indecomposable* if it is not decomposable. Separability conditions formed from *indecomposable* maps are important as they are the only conditions that can detect Bound Entanglement.

2.4 Entanglement Witnesses

An operator $W \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called *Entanglement Witness* if $Tr(W\rho_s) \geqslant 0$ for all separable states ρ_s and there exists atleast one entangled state ρ_e such that $Tr(W\rho_e) < 0$

Thus, if a bipartite state $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is such that $Tr(W\rho) < 0$ for some E.W. $W \implies \rho$ must be entangled.

Entanglement witnesses have a clear geometric meaning, the expectation value of an observable depends linearly on the state, thus the set of states satisfying $Tr(W\rho) = 0$ is a hyperplane in the set of states, separating the set in two halves, one containing the set of separable states and the other half being the set of states detected by W.

For each entangled state ρ_e, there exists an entanglement witness W detecting it and idea of the proof is the set of separable states is convex and closed, thus we can find a line separating the set of separable states and a point outside (the entangled state).

Example of E.W. :

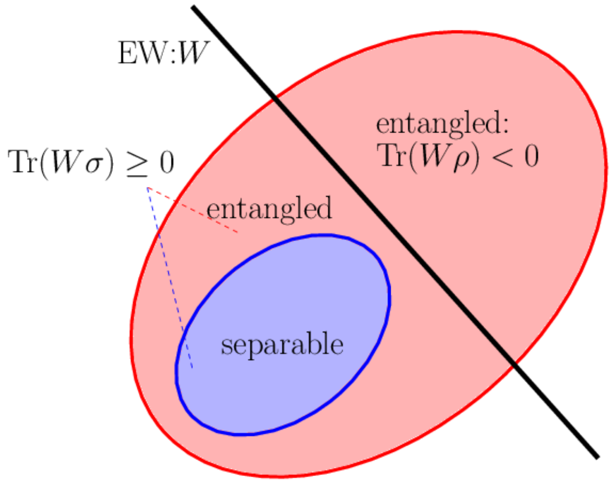

Figure 1: Visualising Entanglement Witness

Let ρ_e be a state that violates the PPT criterion. Thus, there exists a negtive eigenvalue λ_- of $\rho_e^{T_B}$, where T_B is the partial transpose map. Let the corresponding eigenvector be $|\nu\rangle$.
Consider the operator $W = |\nu\rangle\langle\nu|^{T_B}$, then we get

$$Tr(W\rho_e) = Tr(|\nu\rangle\langle\nu|^{T_B}\rho_e) = Tr(|\nu\rangle\langle\nu|\rho_e^{T_B}) = \langle\nu|\rho_e^{T_B}|\nu\rangle = \lambda_- < 0$$

and for any sperable state (which means it's partial transpose is a postive operator) we get

$$Tr(W\rho_s) = Tr(|\nu\rangle\langle\nu|^{T_B}\rho_s) = Tr(|\nu\rangle\langle\nu|\rho_s^{T_B}) = \langle\nu|\rho_s^{T_B}|\nu\rangle \geqslant 0$$

Thus W is a valid E.W. that detects entanglement in ρ_e.

2.5 Positive Map $\leftrightarrow$ Entanglement Witness

Choi and Jamiołkowski discovered that there is a one-one correspondence between Positive maps and Entanglement Witnesses.This can be seen in the following way:

For any linear map $\Lambda : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$, we can define a 4-index array Λ_{ijkl} by

$$\Lambda(|k\rangle\langle l|) = \sum_{i,j} \Lambda_{ijkl}|i\rangle\langle j|. \tag{1}$$

This 4-index array is essentially what defines an entanglement witness when Λ is positive. The Jamiołkowski form of this correspondence can be written in the form

$$W_\Lambda = (\Lambda \otimes I)\left(\sum_{k,l}|k\rangle\langle l| \otimes |k\rangle\langle l|\right) = \sum_{ijkl}\Lambda_{ijkl}|i\rangle\langle j| \otimes |k\rangle\langle l| \tag{2}$$

which when Λ is positive defines an entanglement witness $W_\Lambda \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

3 Quantum Teleportation

The resources required for quantum teleportation are a communication channel capable of transmitting two classical bits, a means of generating an entangled Bell state of qubits and distributing to two different locations, performing a Bell measurement on one of the Bell state qubits, and manipulating the quantum state of the other qubit from the pair. Of course, there must also be some input qubit (in the quantum state $|\phi\rangle$) to be teleported. The protocol is then as follows:

- A Bell state is generated with one qubit sent to location A and the other sent to location B.
- A Bell measurement of the Bell state qubit and the qubit to be teleported ($|\phi\rangle$) is performed at location A. This yields one of four measurement outcomes which can be encoded in two classical bits of information. Both qubits at location A are then discarded.
- Using the classical channel, the two bits are sent from A to B. (This is the only potentially time-consuming step after step 1 since information transfer is limited by the speed of light.)
- As a result of the measurement performed at location A, the Bell state qubit at location B is in one of four possible states. Of these four possible states, one is identical to the original quantum state $|\phi\rangle$, and the other three are closely related. The identity of the state actually obtained is encoded in two classical bits and sent to location B. The Bell state qubit at location B is then modified in one of three ways, or not at all, which results in a qubit identical to $|\phi\rangle$, the state of the qubit that was chosen for teleportation.

4 Distillation

Entanglement distillation (also called entanglement purification) is the transformation of N copies of an arbitrary entangled state ρ into some number of approximately pure Bell pairs, using only local operations and classical communication.
Quantum entanglement distillation can in this way overcome the degenerative influence of noisy quantum channels by transforming previously shared less entangled pairs into a smaller number of maximally entangled pairs.

4.1 Motivation

Consider two people, say Alice and Bob, who want to communicate quantum information, as we saw in the previous section. According to the Quantum Teleportation technique they would require to share a Bell State in the beginning, that would assist them to communicate a qubit using classical communication. But presently Alice and Bob only share n copies of a state ρ_{AB}.

Firstly, using our knowledge of entanglement detection, we will check if the shared pair of qubits is entangled or not.

If the state is entangled and the fidelity $F(\rho_{AB}, |\psi^-)\rangle > \frac{1}{2}$, then consider the following protocol for distilling out a maximally entangled state out of these copies of ρ_{AB} :

4.2 BBPSSW Reccurence Protocol

We take 2 copies of the entangled state shared by Alice and Bob, call them $\rho_{A_1B_1}$ and $\rho_{A_2B_2}$

Before		After	(n.c. = no change)
Source	Target	Source	Target
$\Phi^{\pm}$	Φ^{+}	n.c.	n.c.
$\Psi^{\pm}$	Φ^{+}	n.c.	Ψ^{+}
$\Psi^{\pm}$	Ψ^{+}	n.c.	Φ^{+}
$\Phi^{\pm}$	Ψ^{+}	n.c.	n.c.
$\Phi^{\pm}$	Φ^{-}	$\Phi^{\mp}$	n.c.
$\Psi^{\pm}$	Φ^{-}	$\Psi^{\mp}$	Ψ^{-}
$\Psi^{\pm}$	Ψ^{-}	$\Psi^{\mp}$	Φ^{-}
$\Phi^{\pm}$	Ψ^{-}	$\Phi^{\mp}$	n.c.

Figure 2: Bilateral XOR,[1]

- *Twirling Operation*: Alice picks an operator at random from the set of Unitary matrices acting on a single qubit and tells Bob her pick and then Bob applies it to his half.
  Say she picked U, then both locally apply U to their part of the entangled pair.

  We get $\sigma_{A_iB_i} = (U \otimes U)\rho_{A_iB_i}(U^\dagger \otimes U^\dagger)$.

  This random local application of Unitary matrices is the so called *Twirling Map*, $\Phi : \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Note that this map is not unitary, even though each operator is unitary in itself. This is due to the randomly choosing and applying the unitary operation. Instead this belong to a general class of operators, i.e., CPTP.

  Now the output of the twirling map is a Werner State(which is obvious because output of twirling map is invariant to application of a unitary matrix) and hence it is diagonalizable in Bell basis. Thus we get,

$$\Phi(\rho_{A_iB_i}) = x|\psi^-\rangle\langle\psi^-| + \frac{1-x}{4}\mathcal{I}$$

  substitute $x + \frac{1-x}{4} = F$

$$\Phi(\rho_{A_iB_i}) = F|\psi^-\rangle\langle\psi^-| + \frac{1-F}{3}|\psi^+\rangle\langle\psi^+| + \frac{1-F}{3}|\phi^-\rangle\langle\phi^-| + \frac{1-F}{3}|\phi^+\rangle\langle\phi^+|$$

- *Convert from ψ^- to ϕ^+* : Alice applies $\mathcal{Y}$ to her half the sates.

$$\mathcal{Y} \otimes \mathcal{I}(\Phi(\rho_{A_iB_i}) = \frac{1-F}{3}|\psi^-\rangle\langle\psi^-| + \frac{1-F}{3}|\psi^+\rangle\langle\psi^+| + \frac{1-F}{3}|\phi^-\rangle\langle\phi^-| + F|\phi^+\rangle\langle\phi^+|$$

- *Apply CNOT (Bilateral XOR) Locally*: Alice and Bob, both apply XOR operation at their ends to the pair of particles they posses, thus we get

$$(U_{XOR})_{Alice} \otimes (U_{XOR})_{Bob}(\rho_{A_1B_1}\rho_{A_2B_2})(U_{XOR})^\dagger_{Alice} \otimes (U_{XOR})^\dagger_{Bob}$$

- *Local Measurements* : Alice and Bob local measure the second copy in standard basis and obtain $a, b \in \{|0\rangle, |1\rangle\}$ and they communicate this via classical channel
  If $a = b$, they keep the pair. But if $a \neq b$ they discard the pair left with them and start over with another set of qubits from the available copies.

- *Converting back to* $|\psi^-\rangle$: Alice locally apply $\mathcal{Y}$ to her half, this is counter the change of basis we did earlier for our convenience.
- We twirl the pair left with us and repeat the process until we get a maximally entangled pair.

This process leads to some survived pairs $\tilde{\sigma}_{A_iB_i}$ with $F(\tilde{\sigma}_{A_iB_i}, |\psi^-\rangle) = \frac{F^2+\frac{1}{9}(1-F)^2}{F^2+\frac{2}{3}F(1-F)+\frac{5}{9}(1-F)^2}$, this is increasing for $F > \frac{1}{2}$, thus after each iteration the fidelity increases.

4.3 General Distillation

Theorem 1. *PPT states cannot be distilled. [3]*

We prove that PPT states are closed under the action on LOCC. We know that LOCC operation is a completely positive map such that $\rho \xrightarrow{LOCC} \rho' = \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger$ for $\sum_i A_i^\dagger A_i \leqslant \mathcal{I}, \sum_i B_i^\dagger B_i \leqslant \mathcal{I}$.

If we start with a PPT state (i.e., $\rho^{T_B} \geqslant 0$), we get $\rho' = \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger$.
Taking partial transpose, we get

$$\rho'^{T_B} = \sum_i (A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger)^{T_B} = \sum_i A_i \otimes B_i^{\dagger T} \rho^{T_B} A_i \otimes B_i^T$$

where T is transpose map and T_B is parital transpose map. This implies $\rho'^{T_B} \geqslant 0$, therefore ρ' is also a PPT state. Thus, we have proved that PPT states are closed under LOCC and hence cannot to distilled. This shows that we need some minimum entanglement in the begining to perform distillation via LOCC.

Theorem 2. *Any entangled two qubit states are distillable [3]*

Formally in the previous protocol, Alice and Bob act on each copy with an operation given by $\{A \otimes B, \sqrt{I - A^\dagger A} \otimes B, A \otimes \sqrt{I - B^\dagger B}, \sqrt{I - A^\dagger A} \otimes \sqrt{I - B^\dagger B}\}$ and the event corresponding to Kraus operator $A \otimes B$ is the desired one and the complement corresponds the failure in improving properties. In the desired case the transformed state is

$$\sigma = \frac{A \otimes B \rho_{AB} A^\dagger \otimes B^\dagger}{\mathrm{Tr}(A \otimes B \rho_{AB} A^\dagger \otimes B^\dagger)}$$

with probability $p = \mathrm{Tr}(A \otimes B \rho_{AB} A^\dagger \otimes B^\dagger)$.

Now suppose that Alice and Bob are given n copies of entangled state with $F(\rho, |\psi^-\rangle) < \frac{1}{2}$. Then there exists a state μ such that $\langle\mu|\mathcal{I} \otimes \mathcal{T}\rho|\mu\rangle < 0$, where $\mu = \sum_{ij} a_{ij}|ij\rangle$. It can be shown that using a filter $M_\mu \otimes \mathcal{I}$, where $M_\mu = [a_{ij}]$, we can increase fidelity. This will result in a state

$$\sigma = \frac{M_\mu \otimes \mathcal{I} \rho_{AB} {M_\mu}^\dagger \otimes \mathcal{I}}{\mathrm{Tr}(M_\mu \otimes \mathcal{I} \rho_{AB} {M_\mu}^\dagger \otimes \mathcal{I}})$$

with fidelity greater than $\frac{1}{2}$.

Thus, the general distillation protocol is that Alice applies to each state a POVM defined by Kraus operators $\{M_\mu \otimes \mathcal{I}, \sqrt{\mathcal{I} - M_\mu^\dagger M_\mu \otimes \mathcal{I}}\}$. She then tells Bob when she succeeds. On average they select np pairs with probability $p = \mathrm{Tr}(M_\mu^\dagger M_\mu \otimes \mathcal{I} \rho_{AB})$. They then launch the previous protocol (BBPSSW) to distill entanglement.

Note, in the above protocol we assume that Alice and Bob know the state of the shared qubit before starting, but this can be done in general as well, where they sacrifice $\sqrt{n}$ copies of the qubit for calulating the state of the qubit.

References

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