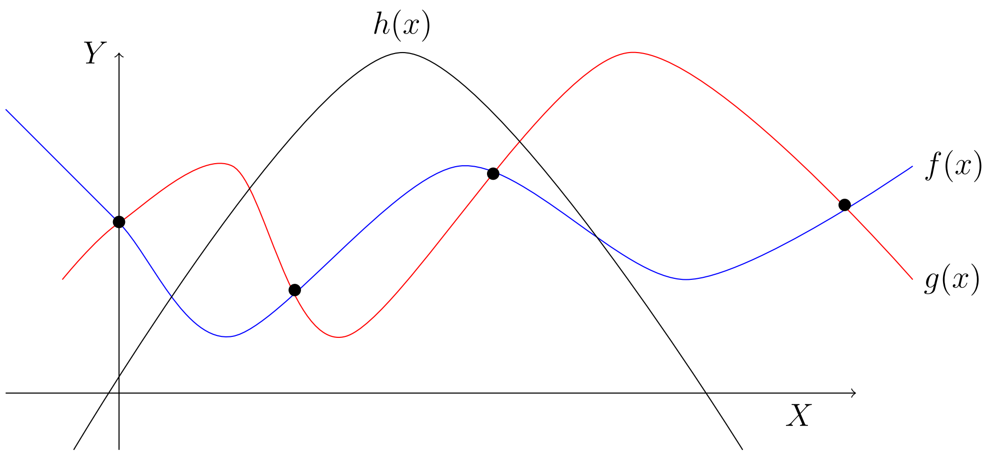
Week 4 Graded Assignment
Mathematics for Data Science - 1

1 Instructions:

- There are some questions which have functions with discrete valued domains (such as day, month, year etc). For simplicity, we treat them as continuous functions.
- **Notations:**
  - $\mathbb{R}$= Set of real numbers
  - $\mathbb{N}$= Set of natural numbers
- The set of natural numbers includes 0.

1. The figure shows the curves represented by polynomials $f(x)$, $g(x)$, and $h(x)$ of degrees 4, 4, and 2 respectively, on XY plane. Let $f(x) - g(x) = ax(x-2)(x-5)(x-9)$, $a \neq 0$. If b is a negative constant, then choose the most possible expression for $h(x)$ and other correct statements among the given options. (Note that figure is not according to scale .) [Marks 2]



Options:

1. $h(x) = b(x^2 + 8x - 7)$
2. $f(x) = g(x)$ at $x = 0, -2, -5, -9$
3. $h(x) = b(x^2 - 6x - 7)$
4. $h(x) = b(x^2 - 2x - 3)$
5. $h(x) = b(x^2 - 8x + 7)$
6. $h(x) = b(x^2 - 6x + 7)$
7. $f(x) = g(x)$ at $x = 0, 2, 5, 9$

(MSQ)

Answer: Option 3, option 7.

Solution:

As $f(x) - g(x) = ax(x-2)(x-5)(x-9)$, $f(x) - g(x)$ will be 0 when $x = 0, 2, 5, 9. f(x) = g(x)$ when $x \in \{0, 2, 5, 9\}$. Hence option 7 is correct but option 2 is incorrect.

$h(x)$ is a parabola with one intercept less than 0 and another x intercept between 5 and 9. Also its vertex will have x-coordinate between 2 and 5.

Now we can calculate the x-coordinate of the vertex of the parabolas given in the options. For option 1, the vertex is at $x = \frac{-(-8)}{2 \times -1} = -4$ (We consider the coefficient of x^2 to be-1 as b is a negative constant.) Therefore this option is not correct.

For the parabola option 3, the vertex is at $\frac{-6}{2 \times -1} = 3$. This is a correct option as it will have x-coordinates at $-1, 7$.

The parabola given at option 4 similarly, has the vertex at $x = 1$ which is not between 2 and 5.Hence option 4 is not correct.
The parabola at option 5 does not have any x-intercept less than 0 and the parabola given at option 6, will not have any x intercept between 5 and 9. Hence they are incorrect.

2. If the polynomials $x^3 + ax^2 + 3x + 5$ and $x^3 + 2x^2 + x + 2a$ leave the same remainder when divided by $(x - 1)$, then the value of a is:
   (NAT)
   Answer: 5
   Solution:
   Dividing $x^3 + ax^2 + 3x + 5$ by $(x - 1)$, we get the quotient $x^2 + (a + 1)x + (4 + a)$ and the remainder $(a + 9)$.
   Dividing $x^3 + 2x^2 + x + 2a$ by $(x - 1)$, we get the quotient $x^2 + 3x + 4$ and the remainder $(2a + 4)$.
   As the remainders should be equal in both cases, equating the remainders we get,
   $a + 9 = 2a + 4$
   $\implies a = 5$.

3. Let $f(x) = x^3 - 6x^2 + 3x + 10$, then choose the set of correct options regarding $f(x)$.
   (MSQ) [Marks: 3]

   Set of correct options:

   1. If $x \in [0, 1] \cup (5, \infty)$, then $f(x)$ is positive.
   2. If $x \in (-1, 2) \cup (5, \infty)$, then $f(x)$ is positive.
   3. If $x \in [0, 1] \cup (10, \infty)$, then $f(x)$ is positive.
   4. If $x \in (-\infty, -2] \cup (2, 5)$, then $f(x)$ is negative.
   5. If $x \in (-\infty, -1) \cup (3, 4)$, then $f(x)$ is negative.

   Set of incorrect options:

   1. If $x \in [0, 1] \cup (5, \infty)$, then $f(x)$ is negative.
   2. If $x \in [-2, 2] \cup (5, \infty)$, then $f(x)$ is positive.
   3. If $x \in (-\infty, 1] \cup (2, 5)$, then $f(x)$ is negative.
   4. If $x \in (-1, 4] \cup (5, \infty)$, then $f(x)$ is positive.
   5. If $x \in [2, 3] \cup (5, \infty)$, then $f(x)$ is positive.
   6. If $x \in (-\infty, -1] \cup (3, 6)$, then $f(x)$ is negative.

   Solution:
   Step 1:
   Draw the rough sketch of $f(x)$ and find its roots.

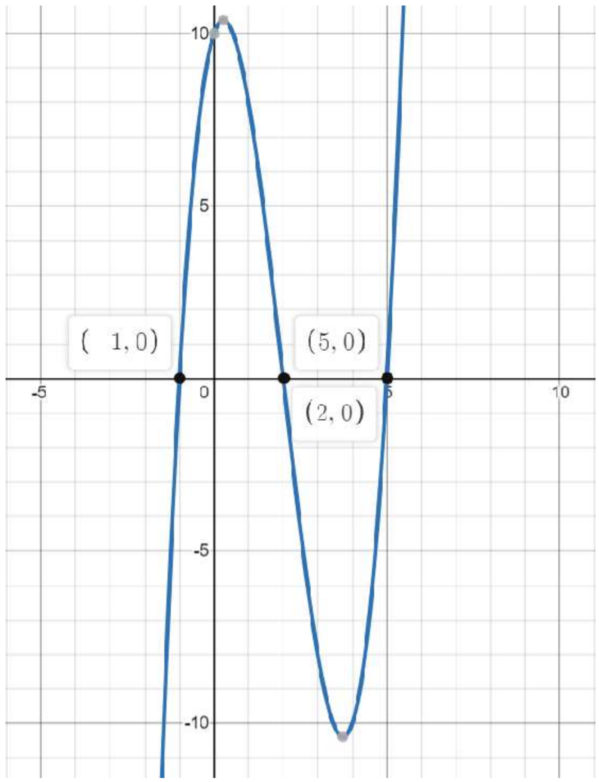


Figure for Question 3

Step 2:
From the graph, find the sets of correct and incorrect options.

$f(x) = x^3 - 6x^2 + 3x + 10 = x^3 - (1+5)x^2 + (5-2)x + 10$
$= x^3 - x^2 - 5x^2 - 2x + 5x + 10$
$= (x^2 - x - 2)(x - 5)$
$= (x^2 + x - 2x - 2)(x - 5)$
$\implies (x+1)(x-2)(x-5)$
Hence the roots of $f(x)$ are $-1, 2, 5$.
The graph of $f(x) = x^3 - 6x^2 + 3x + 10$ looks as shown above.

4. The volume of a box V, varies with some variable x as $V(x) = x^3 - 12x^2 + 44x - 48$ cubic metres. If $(x - a)$ metre is the measurement of one side of the box, then find the value for a.
(NAT)
(Answers: 2 OR 4 OR 6)
Solution:
$x^3 - 12x^2 + 44x - 48$
Volume of the box is the product of length(l), breadth(b) and height(h). So l, b, h are its factors.Likewise the factors of $V(x)$ can be obtained as follows.
$= x^3 - 6x^2 - 6x^2 + 36x + 8x - 48$
$= x^3 - 6x^2 + 8x - 6x^2 + 36x - 48$
$= (x^2 - 6x + 8)(x - 6)$
$= x^2 - 2x - 44x + 8)(x - 6)$
$= (x - 2)(x - 4)(x - 6)$
Hence, $a = 2$ or 4 or 6.

5. A manufacturing company produces a product from one raw material using a process A. This process generates total solid wastes (W_A) (in tonne) represented as $W_A(r) = \frac{1}{15000}(-2r^3 + 10r^2 + 400r)$, where r is the amount of raw material used in tonne and $r \in (0, 10)$. If the company uses a different process B to produce the same product from same raw material, then the total solid waste generated is W_B (in tonne) represented as $W_B(r) = \frac{1}{10000}(-2.2r^3 + 11r^2 + 440r)$. The company spends ₹$5,000$ in waste treatment using the process A by consuming 1 tonne of raw material. How much extra amount will the company have to pay in waste treatment for consuming 1 tonne of raw material if it uses the process B? (Enter your answer till two decimal places).
(NAT)
Answer: 3250 (range: 3243.00 - 3255.00)
Solution:
The question is,
$W_A = \frac{1}{15{,}000}(-2r^3 + 10r^2 + 400r) \to$ ₹5000 / tonne
$W_B = \frac{1}{10{,}000}(-2.2r^3 + 11r^2 + 440r) \to$ ₹? / tonne

Dividing W_B by W_A we get,
$\frac{W_B}{W_A} = \frac{-2.2r^3+11r^2+440r}{10{,}000} \times \frac{15{,}000}{-2r^3+10r^2+400r} = \frac{3}{2} \times \frac{1.1r^3-5.5r^2-220r}{r^3-5r^2-200r} = 1.5 \times 1.1 = 1.65.$
Hence if ₹x is spent per tonne in process W_B and ₹5000 is spent per tonne in process W_A, then $\frac{x}{5000} = 1.65$ or $x = 8250$. Therefore process W_B requires ₹8250 / tonne.
Now the question has asked, "how much extra amount will the company have to pay in waste treatment for consuming 1 tonne", so, the extra amount spent in the process W_B will be $8250 - 5000 =$₹3250.

6. Liala and Vinay both have to travel to various locations for advertising their company's products. The company reimburses their expenses such as accommodation, food etc. The company also blacklists an employee whenever the employee's expenditure in a

given month exceeds ₹ 12000. The accounts department fits the data of monthly expenditure to the polynomial $E_l(x)$ and $E_v(x)$ (in ₹) for Liala and Vinay respectively, where x is the number of months since they joined the company (i.e., $x = 1$ represents the completion of one month). The polynomial fit is known to be applicable for a period of M months (i.e., $x \leq M$). If $E_l(x) - 12,000 = a(x-k)(x-\ell)(x-m),\ a > 0$ and $E_v(x) - 12,000 = a(x-n)^2(x-k)(x-p),\ a > 0$. If Vinay and Liala have been blacklisted together for atleast N times in M months, then find the value of N.

(NAT) Marks: 3

Solution:

Step 1:

Find the roots of both Vinay and Liala's expenses functions and plot a rough graph.

Step 2:

Note that, whenever the y-axis values are positive , Liala and Vinay have been warned. Calculate the number of months where the curve will be positive within the given number of months.

Example:

Liala and Vinay both have to travel to various locations for advertising their com-

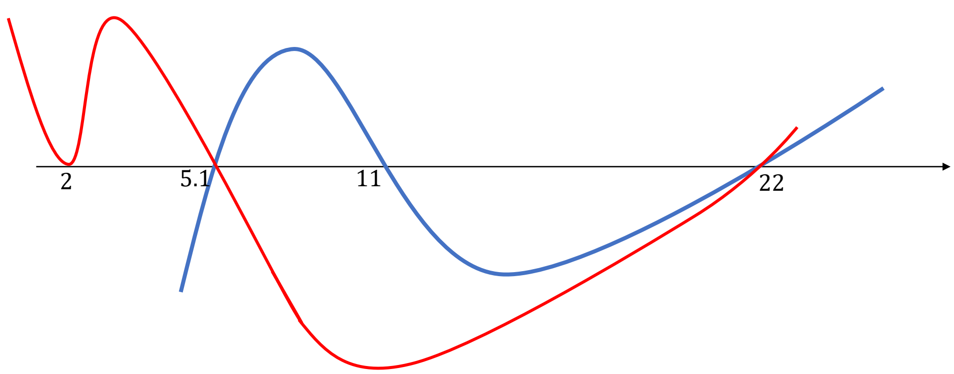


Figure for Question 6

pany's products. The company reimburses their expenses such as accommodation, food etc. The company also blacklists an employee whenever the employee's expenditure in

a given month exceeds ₹ 12000. The accounts department fits the data of monthly expenditure to the polynomial $E_l(x)$ and $E_v(x)$ (in ₹) for Liala and Vinay respectively, where x is the number of months since they joined the company (i.e., $x = 1$ represents the completion of one month). The polynomial fit is known to be applicable for a period of 24 months (i.e., $x \leq 24$). If $E_l(x) - 12,000 = a(x-5.1)(x-11)(x-22),\ a > 0$ and $E_v(x) - 12,000 = a(x-2)^2(x-5.1)(x-22),\ a > 0$. If Vinay and Liala have been blacklisted together for atleast N times in 24 months, then find the value of N.
Answer: 2

$(E_l(x) - 12,000)$ has 3 roots $5.1, 11$ and 22. All roots are of odd power. $(E_v(x) - 12,000)$ has 3 roots, $+2, +5.1, +22$. At $+2$, the graph will bounce off because of the even power. In the graph drawn above, the red line is $E_v(x)$ and the blue line is $E_l(x)$. Whenever the y-axis values are positive , Liala and Vinay have been warned. So Vinay have been warned at $x = 1, 3, 4, 5, 23, 24$ i.e. for 6 times. Similarly Liala is warned at $x = 6, 7, 8, 9, 10, 23, 24$. Hence Liala is warned for 7 times. After $x = 22$, both graphs are positive, So at $x = 23, 24$ both of them will be warned. Hence they will be warned 2 times together.

7. Polynomial fit for the data given in the table recorded by a student is $y = f(x) = x^3 + 2x^2 + nx$. Find the value of n (till one decimal place), so that SSE (sum squared error) will be minimum?
(NAT) [Marks: 3]

x	-2	-1	0	1	2
$y = f(x)$	5	4	1	9	15

Answer: -0.9
Solution:
The SSE can be calculated as (fitted value - obtained exact value)^2.
The fitted value for the given equation will be-
$f(-2) = -8 + 8 - 2n = -2n$
$f(-1) = 1 + 2 - n = 1 - n$
$f(0) = 0$
$f(1) = 1 + 2 + n = 3 + n$
$f(2) = 8 + 8 + 2n = 16 + 2n$
Hence SSE is
$(-2n-5)^2 + (1-n-4)^2 + (0-1)^2 + (3+n-9)^2 + (16+2n-15)^2$
$= (2n+5)^2 + (3+n)^2 + 1 + (n-6)^2 + (2n+1)^2$
$= 10n^2 + 18n + 72$
Now we need the SSE to be minimum.
Minimum value of the quadratic function $10n^2 + 18n + 72$ occur at the vertex of the

parabola, i.e. at $\frac{-18}{2\times10} = -0.9$
Hence at $n = -0.9$, the SSE will be minimum.

8. If m, n and p are the roots of the polynomial $x^3 + 5x - 8$ and sum of the roots is 0, then find the value of $m^3 + n^3 + p^3$.
(NAT) [Marks: 2]
Answers: 24
Solution:
If m, n, p are the roots of the polynomial $x^3 + 5x - 8$.
Hence, we can write,
$m^3 + 5m - 8 = 0$
$n^3 + 5n - 8 = 0$
$p^3 + 5p - 8 = 0$
Adding these three, we get,
$m^3 + n^3 + p^3 + 5(m + n + p) - 24 = 0$
Again, $(m + n + p) = 0$, therefore,
$m^3 + n^3 + p^3 = 24$.

9. Let $h(x) = \frac{f(x)}{g(x)}$, where $h(x)$ is polynomial of x. If $f(x) = x^3-2x^2-3x$ and $g(x) = (x-3)$, then find the number of real solutions of the equation $h(x) = 0$ in the interval $[-2.0, 2.0]$.
(NAT) [Marks: 3]
Answer: 2.
Solution:
$h(x) = \frac{f(x)}{g(x)} = \frac{x^3-2x^2-3x}{x-3} = x^2 + x = x(x + 1)$
Therefore $h(x)$ has two roots $x = 0$ and $x = -1$ in the interval $[-2.0, 2.0]$.