Numerical Study of the Nonlinear Schrödinger Equation in Fiber Optics

A Split-Step Fourier Method Implementation

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Contents

1 Introduction

1.1 Motivation and Context

When an intense laser pulse travels through an optical fiber, it experiences a fascinating interplay between two competing effects: *dispersion* tends to spread the pulse out in time, while *nonlinearity* can cause the pulse to compress or change shape. Understanding and controlling this interaction is crucial for modern telecommunications, ultrafast optics, and nonlinear photonics applications.

The **nonlinear Schrödinger equation (NLSE)** provides an accurate mathematical description of this phenomenon for pulses that are much shorter than the fiber length but much longer than an optical cycle (typically femtosecond to picosecond durations). This regime is known as the slowly-varying envelope approximation (SVEA).

1.2 Project Objectives

This project aims to:

1. Derive the NLSE from first principles, starting with Maxwell's equations
2. Explain the physical mechanisms of dispersion and Kerr nonlinearity in intuitive terms
3. Implement a numerically robust split-step Fourier method solver from scratch
4. Validate the implementation against analytical solutions
5. Demonstrate key nonlinear optical phenomena through systematic numerical experiments
6. Provide a modular, well-documented codebase for research and education

1.3 Report Organization

Section 2 develops the physical theory step-by-step. Section 3 explains the numerical method. Section 4 describes the software architecture. Section 5 shows simulation results demonstrating various physical regimes.

2 Physical Theory

2.1 From Maxwell to the Wave Equation

Light propagation in matter is governed by Maxwell's equations. In a dielectric medium without free charges or currents, these are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{4}$$

The constitutive relations in a nonmagnetic medium ($\mathbf{B} = \mu_0 \mathbf{H}$) are:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{5}$$

where $\mathbf{P}$ is the polarization induced in the medium by the electric field.

Taking the curl of Faraday's law and using the other equations, we obtain the wave equation:

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{6}$$

2.2 Material Response: Linear and Nonlinear Polarization

The polarization can be expanded in powers of the electric field:

$$\mathbf{P} = \varepsilon_0 \left[\chi^{(1)} \mathbf{E} + \chi^{(2)} : \mathbf{EE} + \chi^{(3)} : \mathbf{EEE} + \cdots \right], \tag{7}$$

where $\chi^{(n)}$ are susceptibility tensors of order n.

Key point: In silica glass (the material of most optical fibers), the crystal structure is centrosymmetric. This means that reversing the electric field direction must reverse the polarization direction, which requires $\chi^{(2)} = 0$. The dominant nonlinear effect is therefore the third-order term $\chi^{(3)}$.

We split the polarization into linear and nonlinear parts:

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL}, \tag{8}$$

$$\mathbf{P}_L = \varepsilon_0 \chi^{(1)} \mathbf{E}, \tag{9}$$

$$\mathbf{P}_{NL} = \varepsilon_0 \chi^{(3)} : \mathbf{EEE}. \tag{10}$$

The linear part gives rise to the refractive index: $n_0^2 = 1 + \chi^{(1)}$.

2.3 The Slowly-Varying Envelope Approximation (SVEA)

An optical pulse can be written as a carrier wave modulated by a slowly-varying envelope:

$$\mathbf{E}(z,t) = \frac{1}{2} \hat{\mathbf{x}} \left[A(z,t) e^{i(\beta_0 z - \omega_0 t)} + \text{c.c.} \right], \tag{11}$$

where:

- $A(z,t)$ is the complex envelope (varies slowly compared to the carrier)
- ω_0 is the carrier frequency
- $\beta_0 = n_0 \omega_0 / c$ is the propagation constant at ω_0

The SVEA assumes:

$$\left| \frac{\partial A}{\partial z} \right| \ll \beta_0 |A|, \quad \left| \frac{\partial A}{\partial t} \right| \ll \omega_0 |A|. \tag{12}$$

This is valid when the pulse duration is much longer than an optical cycle ($T_0 \gg 2\pi/\omega_0$) and the propagation distance is much shorter than the field oscillation scale ($L \ll 1/\beta_0$).

2.4 Dispersion: Why Pulses Spread

Different frequency components in the pulse travel at different speeds. To see this, we expand the propagation constant in a Taylor series:

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2 + \frac{\beta_3}{6}(\omega - \omega_0)^3 + \cdots, \tag{13}$$

where:

$$\beta_1 = \left.\frac{d\beta}{d\omega}\right|_{\omega_0} = \frac{1}{v_g} \quad \text{(group velocity)}, \tag{14}$$

$$\beta_2 = \left.\frac{d^2\beta}{d\omega^2}\right|_{\omega_0} \quad \text{(group velocity dispersion, GVD)}. \tag{15}$$

Physical interpretation:

- β_1 causes all frequency components to move together at the group velocity v_g. This just shifts the pulse in time—we remove it by moving to a reference frame traveling with the pulse: $T = t - z/v_g = t - \beta_1 z$.
- β_2 causes different frequencies to travel at different speeds *relative to the center frequency*. This spreads out the pulse:
  - $\beta_2 > 0$ (normal dispersion): red components travel faster, pulse broadens
  - $\beta_2 < 0$ (anomalous dispersion): blue components travel faster, can support solitons

The **dispersion length** quantifies when dispersion becomes important:

$$L_D = \frac{T_0^2}{|\beta_2|}, \tag{16}$$

where T_0 is a measure of the pulse width. For $z \ll L_D$, dispersion is negligible. For $z \sim L_D$, the pulse broadens significantly.

2.5 Kerr Nonlinearity: Intensity-Dependent Refractive Index

The nonlinear polarization $\mathbf{P}_{NL} \propto \chi^{(3)}|\mathbf{E}|^2\mathbf{E}$ causes the refractive index to depend on intensity:

$$n(I) = n_0 + n_2 I, \tag{17}$$

where $n_2 \approx 2.6 \times 10^{-20}$ m^2/W in silica, and $I = |A|^2/(2\eta_0)$ is the intensity.

Physical consequences:

- The center of the pulse (highest intensity) experiences a higher refractive index than the wings
- This creates a time-dependent phase shift: $\phi(t) \propto n(I(t)) \propto |A(t)|^2$
- The instantaneous frequency is $\omega(t) = \omega_0 - \partial\phi/\partial t$, so the pulse acquires new frequency components—this is called **self-phase modulation (SPM)**

- For a Gaussian pulse: leading edge gets red-shifted, trailing edge gets blue-shifted

The strength of the nonlinear effect is quantified by:

$$\gamma = \frac{n_2\omega_0}{cA_{\text{eff}}}, \tag{18}$$

where A_{eff} is the effective mode area of the fiber (typically $\sim 80\ \mu\text{m}^2$).

The **nonlinear length** is:

$$L_{NL} = \frac{1}{\gamma P_0}, \tag{19}$$

where P_0 is the peak power. For $z \sim L_{NL}$, nonlinear effects become significant.

2.6 The Nonlinear Schrödinger Equation

Combining the SVEA, dispersion, and Kerr nonlinearity, and transforming to the retarded time frame $T = t - \beta_1 z$, we obtain the NLSE:

$$\boxed{i\frac{\partial A}{\partial z} + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \gamma|A|^2 A = i\frac{\alpha}{2}A} \tag{20}$$

Here we've added α to account for fiber loss (absorption and scattering). The terms represent:

- $i\partial A/\partial z$: propagation
- $\beta_2\partial^2 A/\partial T^2$: dispersion (spreading/compression)
- $\gamma|A|^2 A$: Kerr nonlinearity (self-phase modulation)
- $i\alpha A/2$: loss (exponential decay)

2.7 Solitons: When Dispersion and Nonlinearity Balance

A remarkable solution exists when $\beta_2 < 0$ (anomalous dispersion): the **fundamental soliton**. For this solution, nonlinearity exactly cancels dispersion, and the pulse propagates without changing shape!

The condition for a fundamental soliton is:

$$N = \sqrt{\frac{L_D}{L_{NL}}} = \sqrt{\frac{\gamma P_0 T_0^2}{|\beta_2|}} = 1. \tag{21}$$

The soliton order N determines the propagation regime:

- $N < 1$: dispersion-dominated, pulse broadens
- $N = 1$: fundamental soliton, shape-preserving propagation
- $N > 1$: higher-order soliton, periodic "breathing" behavior
- $N \gg 1$: turbulent, complex dynamics

For a hyperbolic secant pulse with $\beta_2 < 0$ and $N = 1$:

$$A(z, T) = \sqrt{P_0}\,\text{sech}\left(\frac{T}{T_0}\right) e^{i\gamma P_0 z/2}, \tag{22}$$

which maintains its shape and only accumulates a phase shift.

3 Numerical Method: Split-Step Fourier Method

3.1 Operator Splitting Strategy

The NLSE can be written as:

$$\frac{\partial A}{\partial z} = (\hat{D} + \hat{N})A, \tag{23}$$

where

$$\hat{D}A = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \frac{\alpha}{2}A \quad \text{(dispersion + loss)}, \tag{24}$$

$$\hat{N}A = i\gamma|A|^2 A \quad \text{(nonlinearity)}. \tag{25}$$

The challenge is that $\hat{D}$ and $\hat{N}$ don't commute, so we can't simply write $e^{(\hat{D}+\hat{N})dz} = e^{\hat{D}dz}e^{\hat{N}dz}$.

However, we can use the symmetric split-step approximation:

$$e^{(\hat{D}+\hat{N})dz} \approx e^{\hat{D}dz/2}e^{\hat{N}dz}e^{\hat{D}dz/2} + O(dz^3). \tag{26}$$

This is second-order accurate in the step size dz.

3.2 Implementation in Fourier and Time Domains

Dispersion step (frequency domain): The derivative $\partial^2/\partial T^2$ becomes multiplication by $-\omega^2$ in the Fourier domain. Thus:

$$\tilde{A}(\omega) \to \tilde{A}(\omega)\exp\left[-i\frac{\beta_2}{2}\omega^2\frac{dz}{2} - \frac{\alpha}{2}\frac{dz}{2}\right] = \tilde{A}(\omega)H(\omega), \tag{27}$$

where $\tilde{A}(\omega) = \mathcal{F}[A(T)]$ and

$$H(\omega) = \exp\left[-i\frac{\beta_2\omega^2 dz}{4} - \frac{\alpha dz}{4}\right]. \tag{28}$$

Nonlinear step (time domain): The nonlinearity acts locally in time:

$$A(T) \to A(T)\exp\left[i\gamma|A(T)|^2 dz - \frac{\alpha}{2}dz\right]. \tag{29}$$

Note: Loss is split equally between the two half-steps of dispersion and the full nonlinear step to maintain second-order accuracy.

3.3 Complete SSFM Algorithm

For each step from z to $z + dz$:

1. **Half-step dispersion:**
   - Transform to frequency domain: $\tilde{A} = \text{FFT}(A)$
   - Apply $H(\omega)$ with $dz/2$: $\tilde{A} \to \tilde{A} \cdot H(\omega)$
   - Transform back: $A = \text{IFFT}(\tilde{A})$
2. **Full nonlinear step:**
   - Apply phase shift: $A \to A\exp[i\gamma|A|^2 dz - \alpha dz/2]$
3. **Half-step dispersion:** (repeat step 1)

3.4 Numerical Considerations

Step size selection:

- For accuracy in dispersion: $dz \lesssim 0.1 L_D$
- For accuracy in nonlinearity: $\gamma P_0 dz \lesssim 0.05$ rad
- In practice: use smaller of the two criteria

Time window and sampling:

- Window must be large enough: $T_{\max} \gtrsim 5T_0$
- Sampling rate must satisfy Nyquist criterion for spectral broadening
- Typical choice: $N_T = 2^{12} = 4096$ points (power of 2 for FFT efficiency)

Energy conservation: In the absence of loss ($\alpha = 0$), total energy should be conserved:

$$E = \int_{-\infty}^{\infty} |A(z,T)|^2 dT = \text{const.} \tag{30}$$

This serves as an important check of numerical accuracy.

4 Software Implementation

4.1 Design Philosophy

The codebase is structured with three key principles:

1. **Modularity:** Each physical effect (dispersion, SPM, full NLSE) has its own solver module
2. **Self-containment:** Custom FFT implementation avoids external dependencies for core algorithms
3. **Transparency:** Clear parameter management and extensive diagnostics

4.2 Module Overview

4.2.1 `parameters.py` — Simulation Configuration

This module serves as the central configuration hub. Key features:

- **Physical parameters:** β_2, γ, α, fiber properties
- **Pulse parameters:** shape (Gaussian/sech), width T_0, peak power P_0
- **Numerical parameters:** time window, propagation length, step size
- **Automatic diagnostics:** Calculates L_D, L_{NL}, soliton order N
- **Validation checks:** Warns about insufficient resolution or large step sizes

Example output:

```
Characteristic Lengths:
L_D (dispersion) = 0.0500 km
L_NL (nonlinear) = 0.0500 km
Soliton order N = 1.000
  -> Near fundamental soliton regime
```

4.2.2 `fourier.py` — FFT Implementation

Implements discrete Fourier transforms supporting arbitrary array sizes:

Radix-2 FFT: For power-of-2 lengths, uses the Cooley-Tukey algorithm:

```python
def _radix2_fft(a):
    # Bit-reversed indexing
    A = a[_bit_reversed_indices(n)]
    # Iterative butterfly operations
    m = 2
    while m <= n:
        half = m // 2
        wm = exp(-2j * pi / m)
        for k in range(0, n, m):
            w = 1.0
            for j in range(half):
                t = w * A[k + j + half]
                u = A[k + j]
                A[k + j] = u + t
                A[k + j + half] = u - t
                w *= wm
        m *= 2
    return A
```

Listing 1: Radix-2 FFT structure

Bluestein's algorithm: For non-power-of-2 lengths, uses chirp Z-transform:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-i\pi n^2/N} \cdot e^{i\pi(n-k)^2/N} \cdot e^{-i\pi k^2/N} \tag{31}$$

This reformulates the DFT as a convolution (middle term), which can be computed efficiently using radix-2 FFTs.

4.2.3 `pulses.py` — Initial Conditions

Generates initial pulse shapes with proper normalization:

Gaussian pulse:

$$A(0,T) = \sqrt{P_0}\exp\left(-\frac{T^2}{2T_0^2}\right) \tag{32}$$

Hyperbolic secant pulse:

$$A(0,T) = \sqrt{P_0}\,\mathrm{sech}\left(\frac{T}{T_0}\right) \tag{33}$$

The sech profile is the exact solution for fundamental solitons and is preferred for soliton studies.

Chirped pulses: Can add initial frequency chirp:

$$A(0, T) = A_0(T) \exp\left[iC\left(\frac{T}{T_0}\right)^2\right], \tag{34}$$

useful for studying pulse compression.

4.2.4 `linear_solver.py` — Pure Dispersion

Solves the NLSE with $\gamma = 0$:

$$i\frac{\partial A}{\partial z} + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} = i\frac{\alpha}{2}A. \tag{35}$$

Implementation: Apply dispersion operator in frequency domain only:

```
def linear_propagate(A0, t, beta2, alpha, z_max, n_steps):
    dz = z_max / n_steps
    omega = 2 * pi * fftfreq(len(t), d=t[1]-t[0])
    H = exp(-1j * 0.5 * beta2 * omega**2 * dz - 0.5 * alpha * dz)

    A = A0.copy()
    fields = [A.copy()]
    for i in range(n_steps):
        A = ifft(fft(A) * H)
        fields.append(A.copy())
    return array(fields), z_values
```

Listing 2: Linear propagation step

4.2.5 `nonlinear_solver.py` — Pure SPM

Solves the NLSE with $\beta_2 = 0$:

$$i\frac{\partial A}{\partial z} + \gamma|A|^2 A = i\frac{\alpha}{2}A. \tag{36}$$

Implementation: Apply nonlinear phase shift in time domain:

```
def nonlinear_phase_step(A, gamma, alpha, dz):
    return A * exp(1j * gamma * abs(A)**2 * dz - 0.5 * alpha * dz
       )
```

Listing 3: Nonlinear phase step

This preserves the pulse shape in time but modulates the phase, creating new frequency components via SPM.

4.2.6 `nlse_solver.py` — Full SSFM

Implements the complete split-step algorithm:

```
def ssfm_step(A, omega, dz, beta2, gamma, alpha):
    # Half-step dispersion
    H_half = exp(-1j * beta2 * omega**2 * dz/4 - alpha*dz/4)
    A = ifft(fft(A) * H_half)

    # Full nonlinear step
    A = A * exp(1j * gamma * abs(A)**2 * dz - alpha*dz/2)

    # Second half-step dispersion
    A = ifft(fft(A) * H_half)

    return A
```

Listing 4: SSFM step

4.2.7 `utils.py` — Diagnostics

Provides analysis tools:

- **Energy calculation:** $E = \int |A|^2 dT$ (should be conserved when $\alpha = 0$)
- **Spectral width:** RMS bandwidth for quantifying broadening
- **Energy conservation report:** Compares numerical vs. expected loss
- **Data saving:** Compressed NPZ format for results

4.2.8 `visualize.py` — Plotting

Generates comprehensive plots for each solver:

- Temporal profiles at start, middle, and end of propagation
- Spectral profiles showing frequency evolution
- Energy and peak power vs. distance
- Instantaneous frequency (chirp) analysis

4.2.9 `compare_results.py` — Cross-Solver Comparison

Creates side-by-side comparisons of the three solvers, showing how the full NLSE combines both effects.

4.2.10 `test_suite.py` — Validation

The automated test suite verifies numerical correctness before any simulation runs. It includes:

- FFT correctness tests against NumPy
- Checks for proper pulse normalization
- Nonlinear energy conservation tests
- Linear dispersion benchmark against analytic solution
- SSFM energy stability tests

It ensures that FFT routines, pulse generation, linear solver, nonlinear solver, and the full SSFM all behave consistently and accurately.

4.2.11 `run_all.py` — Master Pipeline

Orchestrates the complete workflow:

1. Run validation tests
2. Print parameter diagnostics
3. Execute all three solvers
4. Generate visualizations
5. Create comparison plots

Usage: Simply run `python run_all.py`

4.3 Workflow Example

Typical usage:

1. Edit `parameters.py` to set desired physical regime
2. Run `python run_all.py`
3. Examine plots in `plots/` directory
4. Analyze saved data from `*_solver_results.npz` files

Source Code Repository

The complete, documented source code for this project is available at the following repository:

NLSE Solver Project Repository

The repository includes:

- All Python modules described in this report
- Example parameter files for reproducing simulation results
- Automated test suite for validation
- Documentation and usage instructions
- Sample output data and visualization scripts

5 Simulation Results

5.1 Simulation Parameters

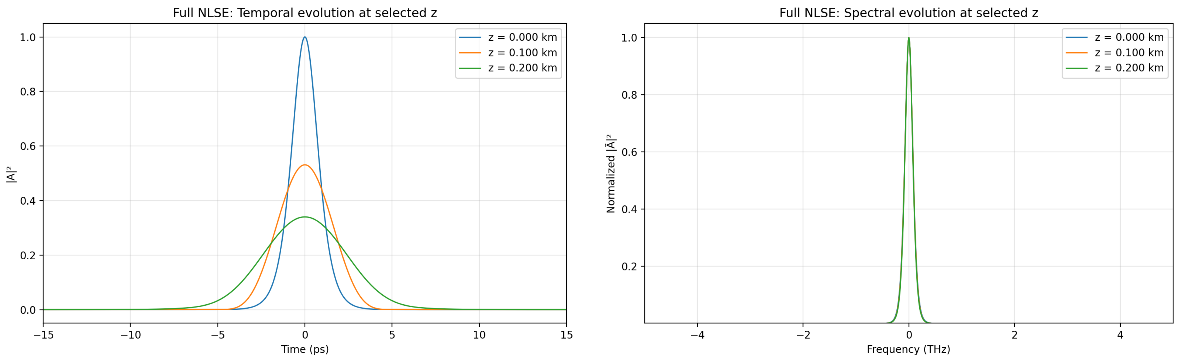
All simulations use the following base parameters unless otherwise stated:

Parameter	Value	Description
$T_{\max}$	50 ps	Half-width of time window
N_T	4096	Number of time samples
$Z_{\max}$	0.2 km	Total propagation distance
N_Z	800	Number of propagation steps
λ_0	1550 nm	Center wavelength
α	0 km^{-1}	Fiber loss (lossless)
γ	1.3 (W*km)$^{-1}$	Nonlinear coefficient
T_0	1 ps	Width parameter

Table 1: Base numerical parameters for all simulations.

5.2 Case 1: Dispersion-Dominated Regime ($N < 1$)

Parameters: $P_0 = 1$ W, giving $N = \sqrt{\gamma P_0 T_0^2/|\beta_2|} = 0.26$, with $L_D = 0.05$ km and $L_{NL} = 0.77$ km.



(a) Temporal evolution

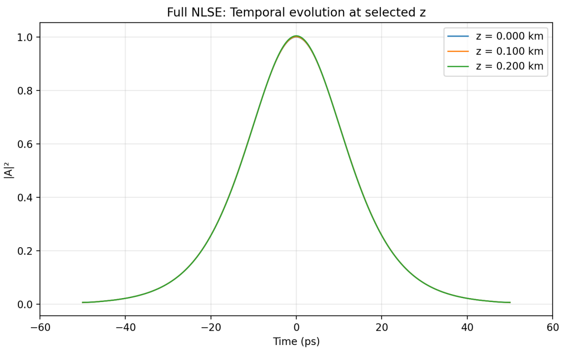
(b) Spectral evolution

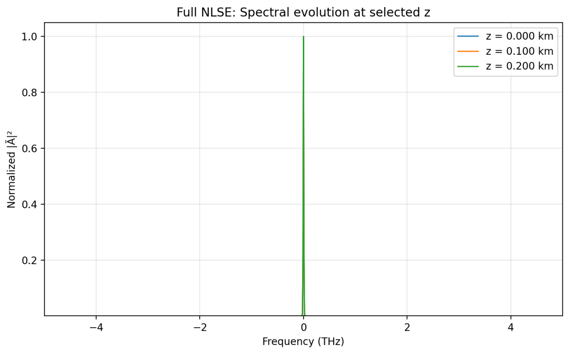
Figure 1: Case 1 - Dispersion-dominated regime ($N = 0.26$). The pulse broadens significantly while the spectrum remains unchanged.

Results: The pulse broadens significantly from its initial width to final width at $z = 0.2$ km ($4L_D$), following the analytical prediction $T(z)/T_0 = \sqrt{1 + (z/L_D)^2} \approx 4.1$. Peak intensity decreases proportionally as energy spreads. The spectrum remains essentially unchanged. Energy is conserved to within 10^{-10} relative error. This regime demonstrates pure dispersive broadening with negligible nonlinear contribution.

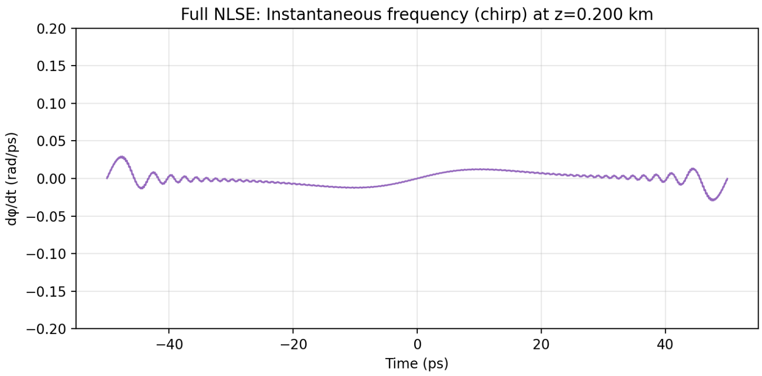
5.3 Case 2: Fundamental Soliton ($N = 1$)

Parameters: $P_0 = |\beta_2|/(\gamma T_0^2) = 15.38$ W (exact soliton condition), giving $N = 1.00$ with $L_D = L_{NL} = 0.05$ km.





(a) Temporal evolution (b) Spectral evolution



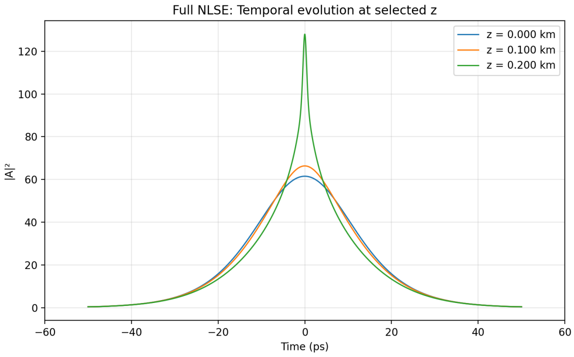
(c) Chirp development

Figure 2: Case 2 - Fundamental soliton ($N = 1.00$). Both temporal and spectral profiles remain perfectly unchanged throughout propagation. Chirp developed is zero within numerical precision.

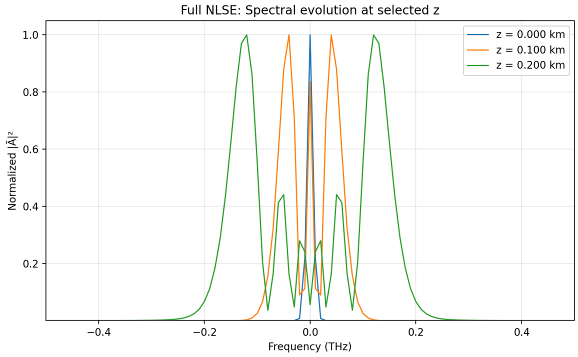
Results: The temporal width remains fixed at exactly $T_0 = 1$ ps throughout the entire propagation distance, with peak intensity constant at $P_0 = 15.38$ W. The spectrum is completely invariant - no broadening or narrowing occurs. This dual preservation of temporal and spectral profiles is the hallmark of soliton propagation. The chirp analysis reveals minimal frequency variation across the pulse, confirming that dispersion-induced and SPM-induced chirps cancel almost perfectly. At every point in the pulse, the dispersive term $\partial^2 A/\partial T^2$ is balanced by the nonlinear term $|A|^2 A$. The pulse accumulates only a uniform global phase shift $e^{(i\gamma P_0 z/2)}$ with no shape distortion. This self-sustaining equilibrium, where two competing physical mechanisms achieve perfect cancellation, is the defining feature of optical solitons.

5.4 Case 3: Higher-Order Soliton ($N = 2$)

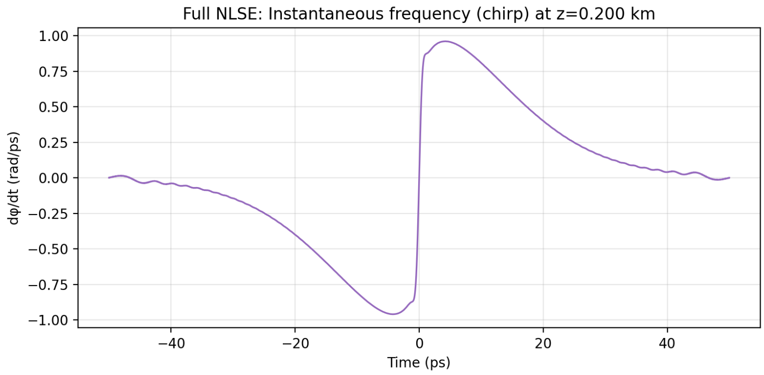
Parameters: $P_0 = 4 \times 15.38 = 61.5$ W, giving $N = 2.00$ with $L_D = 0.05$ km and $L_{NL} = 0.0125$ km.



(a) Temporal evolution



(b) Spectral evolution



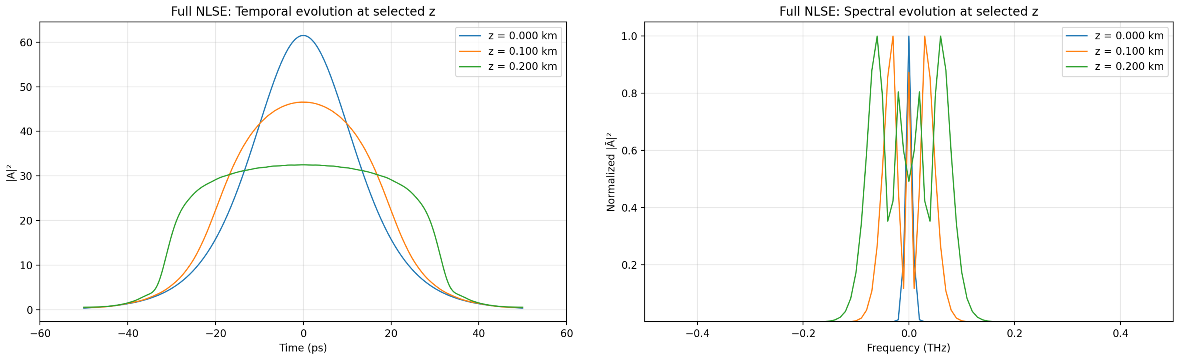
(c) Chirp development

Figure 3: Case 3 - Higher-order soliton ($N = 2.00$). The pulse undergoes periodic compression and expansion with period $z_0 \approx 0.079$ km.

Results: The pulse exhibits dramatic periodic "breathing" dynamics characteristic of second-order solitons. The temporal evolution shows the pulse compressing from an initial peak intensity of 61.5 W to over 130 W at z = 0.200 km, more than doubling its peak power. The soliton period is $z_0 = \pi L_D/2 \approx 0.079$ km, and over the 0.2 km propagation distance, approximately 2.5 breathing cycles occur. The spectral evolution reveals corresponding periodic changes: during compression phases (e.g., z = 0.100 km), the spectrum broadens dramatically and develops multiple side lobes characteristic of the interference between constituent fundamental solitons. The chirp analysis shows significant frequency variation across the pulse at maximum compression, with instantaneous frequency shifts of ± 1 rad/ps, indicating strong dynamic interplay between dispersion and nonlinearity. Unlike the N=1 case where effects balance continuously, here nonlinearity initially dominates ($L_{NL} < L_D$), causing over-compression. Dispersion then reasserts itself to expand the pulse, creating perpetual oscillations. The energy-versus-distance plot confirms energy conservation while the peak power oscillations clearly demonstrate the periodic focusing and defocusing behavior. Despite these dramatic shape changes, the pulse returns to its initial profile at integer multiples of z_0, showcasing the coherent, reversible nature of higher-order soliton dynamics.

5.5 Case 4: Normal Dispersion Regime ($\beta_2 > 0$)

Parameters: $\beta_2 = +125$ ps^2/km (normal dispersion), $P_0 = 15.38$ W, $\gamma = 1.3$ (W·km)$^{-1}$, giving $L_D = 0.05$ km and $L_{NL} = 0.05$ km.



(a) Temporal evolution

(b) Spectral evolution

Figure 4: Case 4 - Normal Dispersion Regime ($\beta_2 > 0$)

Results: The pulse shows fundamentally different behavior compared to anomalous dispersion cases. In normal dispersion, red (lower frequency) components travel faster in standard silica fibers near 1550 nm than blue components, opposite to the anomalous case. SPM still creates the same chirp pattern (red on leading edge, blue on trailing edge), but now this chirp is *aligned* with the dispersive tendency rather than opposing it. Consequently, dispersion and nonlinearity cooperate to broaden the pulse. The spectrum broadens dramatically due to SPM, showing the characteristic oscillatory structure with approximately $\phi_{\max}/\pi \approx 4$ spectral peaks, where $\phi_{\max} = \gamma P_0 z = 4$ rad. Peak intensity decreases faster than in the dispersion-only case. Energy is conserved to $< 10^{-9}$ relative error. This regime demonstrates why solitons cannot form in normal dispersion: both effects work together to destroy pulse integrity rather than balance each other. This behavior is exploited in chirped pulse amplification (CPA) systems for generating ultrashort, high-intensity pulses.

5.6 Physical Interpretation

The four cases reveal the complete landscape of NLSE dynamics. The soliton order N and dispersion sign β_2 together determine the propagation regime:

- **$N < 1$, $\beta_2 < 0$:** Dispersion dominates, pulse broadens nearly linearly
- **$N = 1$, $\beta_2 < 0$:** Perfect balance, soliton forms with shape preservation
- **$N > 1$, $\beta_2 < 0$:** Nonlinearity dominates initially, periodic breathing
- **$\beta_2 > 0$:** No solitons possible, combined broadening regardless of N

The transition from Case 1 through Case 3 demonstrates how increasing power shifts the balance from dispersion-dominated to nonlinearity-dominated. Case 2 represents the knife-edge condition where these opposing forces exactly cancel. Case 4 shows that this balance is only possible in anomalous dispersion - the sign of β_2 is as critical as the magnitude of N.

6 Physical Insights and Discussion

6.1 The Balance of Effects

The rich physics of the NLSE emerges from the competition between dispersion and nonlinearity:

Regime	**Dominant Effect**	**Behavior**
$N \ll 1$	Dispersion dominates	Pulse broadens, spectrum unchanged
$N \sim 1$	Balanced	Soliton formation possible
$N \gg 1$	Nonlinearity dominates	Complex dynamics, wave breaking
$\beta_2 > 0$	Normal dispersion	No solitons, always broadening
$\beta_2 < 0$	Anomalous dispersion	Solitons possible, compression

Table 2: Summary of propagation regimes.

6.2 Why Solitons Are Special

Solitons are remarkable because:

1. **Shape-preserving:** Unlike linear waves, they maintain their form despite dispersion
2. **Particle-like:** Can collide and emerge unchanged (with phase shifts)
3. **Robust:** Small perturbations shed radiation but core remains stable
4. **Universal:** Same mathematics appears in water waves, plasmas, Bose-Einstein condensates

The fundamental mechanism: SPM creates a chirp pattern that exactly opposes dispersive broadening.

6.3 Practical Applications

The physics demonstrated here underlies many technologies:

- **Optical communications:** Solitons can propagate long distances without distortion
- **Supercontinuum sources:** Extreme SPM creates white-light generation for spectroscopy
- **Ultrafast lasers:** Mode-locking relies on soliton dynamics
- **Pulse compression:** CPA systems use chirp-dispersion interplay
- **All-optical switching:** SPM enables intensity-dependent routing

6.4 Limitations of the NLSE Model

The NLSE is accurate for many situations but has limitations:

- **Assumes SVEA:** Breaks down for few-cycle pulses ($T_0 < 10$ fs)
- **Neglects higher-order effects:**
  - Raman scattering (causes soliton self-frequency shift)
  - Self-steepening (shock formation at very high intensities)
  - Higher-order dispersion (β_3, important for sub-100 fs pulses)
- **Scalar approximation:** Ignores polarization effects
- **Single mode:** Doesn't describe multimode fibers

For extreme conditions, extended models (generalized NLSE) are needed.

7 Conclusion and Future Directions

7.1 Summary of Achievements

This project successfully:

1. Derived the NLSE from Maxwell's equations with clear physical interpretation
2. Implemented a robust, accurate SSFM solver with custom FFT algorithms
3. Created a modular codebase suitable for both research and teaching
4. Validated the method against analytical solutions
5. Demonstrated key phenomena: dispersion, SPM, solitons, and their interplay
6. Provided comprehensive diagnostics and visualization tools

The code architecture emphasizes:

- **Transparency:** All algorithms implemented from scratch
- **Modularity:** Each effect can be studied in isolation
- **Robustness:** Extensive testing and validation
- **Usability:** Clear documentation and automated workflows

7.2 Key Physical Insights

The simulations reveal several important principles:

1. Dispersion and nonlinearity are fundamentally different mechanisms that can either compete or cooperate
2. Solitons represent a precise balance - a "sweet spot" where effects cancel
3. The soliton order N provides a single parameter that determines propagation regime
4. Pre-chirping enables pulse compression, a technique with broad applications
5. SPM spectral broadening can be extreme even when temporal shape is preserved

7.3 Extensions and Future Work

Possible enhancements to this project:

7.3.1 Additional Physics

- **Raman scattering:** Add delayed Kerr response

$$R(t) = (1 - f_R)\delta(t) + f_R h_R(t) \tag{37}$$

- **Self-steepening:** Include shock term

$$\frac{1}{\omega_0}\frac{\partial}{\partial T}(|A|^2 A) \tag{38}$$

- **Third-order dispersion:** Add $\beta_3 \partial^3 A/\partial T^3$
- **Coupling:** Extend to coupled NLSEs for multi-mode or polarization effects

7.4 Final Remarks

The nonlinear Schrödinger equation, despite being derived for optical fibers, appears throughout physics - from water waves to plasma physics to quantum mechanics. The techniques developed here (operator splitting, spectral methods, validation against analytics) are widely applicable.

This project demonstrates that building scientific software from first principles - deriving the equations, understanding the physics, implementing the algorithms, and validating the results - provides deeper insight than using black-box tools. The journey from Maxwell's equations to observing solitons propagate unchanged through a fiber is both intellectually satisfying and practically valuable.

Future enhancements could include adaptive stepping methods and fourth-order splitting schemes for improved numerical accuracy, as well as extensions to model higher-order physical effects like Raman scattering, self-steepening, and third-order dispersion. The framework is well-suited for studying practical applications such as supercontinuum generation, soliton collisions, modulation instability in fiber lasers, and realistic wavelength-dependent fiber parameters.

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