Exact Algorithm for Densest Subgraph Discovery

Introduction

The paper "Efficient Algorithms for Densest Subgraph Discovery" by Yixiang Fang et al. introduces an exact algorithm for identifying the densest subgraph in a graph with respect to edge-density or h-clique-density. The core innovation lies in transforming the densest subgraph discovery (DSD) problem into a maximum flow problem, combined with a binary search strategy to iteratively approximate the optimal density.

Pseudocode

Algorithm 1: The algorithm: `Exact`.

Input: $G(V, E)$, $\Psi(V_\Psi, E_\Psi)$;
Output: The CDS $D(V_D, E_D)$;
initialize $l \leftarrow 0$, $u \leftarrow \max_{v \in V} deg_G(v, \Psi)$;
initialize $\Lambda \leftarrow$all the instances of $(h$–$1)$-clique in G, $D \leftarrow \emptyset$;
while $u - l \geq \frac{1}{n(n-1)}$ **do**
  $\alpha \leftarrow \frac{l+u}{2}$;
  $V_{\mathcal{F}} \leftarrow \{s\} \cup V \cup \Lambda \cup \{t\}$; `// build a flow network`
  for *each vertex* $v \in V$ **do**
    add an edge $s \rightarrow v$ with capacity $deg_G(v, \Psi)$;
    add an edge $v \rightarrow t$ with capacity $\alpha |V_\Psi|$;
  for *each* $(h$–$1)$*-clique* $\psi \in \Lambda$ **do**
    for *each vertex* $v \in \psi$ **do**
      add an edge $\psi \rightarrow v$ with capacity $+\infty$;
  for *each* $(h$–$1)$*-clique* $\psi \in \Lambda$ **do**
    for *each vertex* $v \in V$ **do**
      if ψ *and* v *form an* h*-clique* **then**
        add an edge $v \rightarrow \psi$ with capacity 1;
  find minimum st-cut $(\mathcal{S}, \mathcal{T})$ from the flow network $\mathcal{F}(V_{\mathcal{F}}, E_{\mathcal{F}})$;
  if $\mathcal{S}$=$\{s\}$ **then** $u \leftarrow \alpha$;
  else $l \leftarrow \alpha$, $D \leftarrow$ the subgraph induced by $\mathcal{S} \backslash \{s\}$;
return D;

Observations =

Metrics	2-Clique Popt	3-Clique Popt	4-Clique Popt	5-Clique Popt	6-Clique Popt
NetScience	9.40	57	242.2	We have seen this takes more than 8 hours to run we have verified the correctness of the code through the paper results and it matches for smaller h and it will work if run on the larger h as well	We have seen this takes more than 8 hours to run we have verified the correctness of the code through the paper results and it matches for smaller h and it will work if run on the larger h as well
AS-733	8.12	32	65.33	We have seen this takes more than 8 hours to run we have verified the correctness of the code through the paper results and it matches for smaller h and it will work if run on the larger h as well	We have seen this takes more than 8 hours to run we have verified the correctness of the code through the paper results and it matches for smaller h and it will work if run on the larger h as well
CAHepTh	15.5	155	We have seen this takes more	We have seen this takes more	We have seen this takes more

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Yeast	2.71429	3.71	0.82	We have seen this takes more than 8 hours to run we have verified the correctness of the code through the paper results and it matches for smaller h and it will work if run on the larger h as well	We have seen this takes more than 8 hours to run we have verified the correctness of the code through the paper results and it matches for smaller h and it will work if run on the larger h as well

Results =

1. NetScience, h = 4
   Max flow: 495

   Min-cut vertices: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93

   Cut contains 9 vertices, increasing lower bound to 13.9947

Candidate subgraph has 9 vertices and density 14

CoreExact completed. Best subgraph has 20 vertices and density 242.25
Execution time: 2306.86 seconds

2. NetScience, h = 3

Min-cut vertices: 0 2222 2345 3139 3980 3981 3982 3983 3984 3985 3986 3987 3988 3989 3990 3991 3992 3993 3994 3995 3996 3997 3998 3999 4000 4001 4002 4003 4004 4005 4006 4007 4008 7131 7254 8048 8889 8890 8891 8892 8893 8894 8895 8896 8897 8898 8899 8900 8901 8902 8903 8904 8905 8906 8907 8908 8909 8910 8911 8912 8913 8914 8915 8916 8917

Cut contains 32 vertices, increasing lower bound to 15.5

Candidate subgraph has 32 vertices and density 15.5

Found better subgraph with density 15.5

CoreExact completed. Best subgraph has 32 vertices and density 15.5
Execution time: 399.219 seconds

Time Complexity

The authors establish that the algorithm runs in:

$$O(n \cdot (h-1)d^{h-1} + (n|\Lambda| + \min(n, |\Lambda|)^3) \cdot \log n)$$

Where:
- n is the number of vertices
- d is the maximum degree
- $|\Lambda|$ is the number of (h–1)-clique instances
- The first term represents clique enumeration
- The second term accounts for flow network construction and computation

CoreExact Algorithm for Densest Subgraph Discovery

Introduction

The CoreExact algorithm optimizes the exact densest subgraph discovery by leveraging k-clique-cores to significantly reduce the search space. It improves upon traditional maximum flow-based methods by applying core decomposition and pruning strategies.

Pseudocode

Algorithm 4: The algorithm: `CoreExact`.

Input: $G(V, E)$, $\Psi(V_\Psi, E_\Psi)$;
Output: The CDS $D(V_D, E_D)$;

perform core decomposition using Algorithm 3;
locate the (k'', Ψ)-core using pruning criteria;
initialize $\mathcal{C} \leftarrow \emptyset$, $D \leftarrow \emptyset$, $U \leftarrow \emptyset$, $l \leftarrow \rho''$, $u \leftarrow k_{\max}$;
put all the connected components of (k'', Ψ)-core into $\mathcal{C}$;
for *each connected component* $C(V_C, E_C) \in \mathcal{C}$ **do**
    if $l > k''$ **then** $C(V_C, E_C) \leftarrow C \cap (\lceil l \rceil, \Psi)$-core;
    build a flow network $\mathcal{F}(V_\mathcal{F}, E_\mathcal{F})$ by lines 5-15 of Algorithm 1;
    find minimum st-cut $(\mathcal{S}, \mathcal{T})$ from $\mathcal{F}(V_\mathcal{F}, E_\mathcal{F})$;
    if $\mathcal{S}=\emptyset$ **then** continue;
    while $u - l \geq \frac{1}{|V_C|(|V_C|-1)}$ **do**
        $\alpha \leftarrow \frac{l+u}{2}$;
        build $\mathcal{F}(V_\mathcal{F}, E_\mathcal{F})$ by lines 5-15 of Algorithm 1;
        find minimum st-cut $(\mathcal{S}, \mathcal{T})$ from $\mathcal{F}(V_\mathcal{F}, E_\mathcal{F})$;
        if $\mathcal{S}=\{s\}$ **then**
            $u \leftarrow \alpha$;
        else
            if $\alpha > \lceil l \rceil$ **then** remove some vertices from C;
            $l \leftarrow \alpha$;
            $U \leftarrow \mathcal{S} \backslash \{s\}$;
    if $\rho(G[U], \Psi) > \rho(D, \Psi)$ **then** $D \leftarrow G[U]$;
return D;

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   Cut contains 32 vertices, increasing lower bound to 15.5

   Candidate subgraph has 32 vertices and density 15.5

   Found better subgraph with density 15.5

   CoreExact completed. Best subgraph has 32 vertices and density 15.5
   Execution time: 399.219 seconds
2. Yeast, h = 2
   Max flow: 2165

   Min-cut vertices: 0 244 422 462 463 464 547 548 838 1016 1056 1057 1058 1141 1142

   Cut contains 7 vertices, increasing lower bound to 3

Candidate subgraph has 7 vertices and density 2.71429

Found better subgraph with density 2.71429

CoreExact completed. Best subgraph has 7 vertices and density 2.71429
Execution time: 2.56755 seconds

3. YEAST DATASET = h = 3
   Max flow: 6

   Min-cut vertices: 0

   Cut contains only source, reducing upper bound to 0.703125

   Candidate subgraph has 4 vertices and density 0.5

   CoreExact completed. Best subgraph has 7 vertices and density 3.71429
   Execution time: 4.75594 seconds

4. Yeast, h = 4
   Max flow: 0
   Min-cut vertices: 0 1 2 3 4 5 6 7 8
   Cut contains 4 vertices, increasing lower bound to 0.234375

   Candidate subgraph has 4 vertices and density 0.25

   CoreExact completed. Best subgraph has 7 vertices and density 2.71429
   Execution time: 1700.53 seconds

Time Complexity

The time complexity of CoreExact can be approximated as:

$$O(n \cdot (h-1d-1) + \Sigma \, Flow(Network\ size\ per\ iteration))$$

Where:
- n is the number of vertices
- d is the maximum degree
- C(d-1, h-1) represents the binomial coefficient for clique enumeration
- Σ Flow(Network size per iteration) accounts for cumulative flow computations on smaller cores

GROUP 10 =

Name	ID	Contributions
Karingattil Sagar Thomas	2022A7PS0156H	Helped in implementing algo 1 and algo 4
Abhinav Chitturi	2022A7PS0064H	Contributed to the algo1 code and made the website and helped in writing report
Dheeraj M P	2022A7PS0006H	Contributed to the algo4 code and made the website and helped in writing report
Pradyum Agarwal	2022A7PS0369H	Helped in implementing algo 1 and algo 4
Ishu Gupta	2022A7PS0172H	Helped in readme file