Does interaction increase chaos?

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1 Introduction

Chaos theory explores the unpredictable behavior of deterministic systems, where small variations in initial conditions can lead to vastly different outcomes over time. One of the classic examples illustrating chaos is the double pendulum system. Unlike a single pendulum, which follows predictable oscillations, a double pendulum comprises two connected pendulums, introducing a higher degree of complexity. Due to its non-linearity and sensitivity to initial conditions, the motion of a double pendulum can become chaotic, exhibiting seemingly random behavior. Even small differences in the initial positions or velocities of the pendulums can lead to dramatically different trajectories over time. This sensitivity to initial conditions makes the double pendulum a fascinating example of chaotic dynamics, demonstrating how deterministic systems can give rise to unpredictable behavior.

In this article we first explore the chaos nature of double pendulum. Then we move on to coupled double pendulum and see the affect to coupling on the nature of chaos.

In this article we have used only SI units, although we do not write any units for our convenience.

2 Lyapunov exponent

The Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. Quantitatively, two trajectories in phase space with initial separation vector $\delta \mathbf{x}_0$ diverge (provided that the divergence can be treated within the linearized approximation) at a rate given by

$$|\delta \mathbf{x}(t)| \approx e^{\lambda t} |\delta \mathbf{x}_0| \tag{1}$$

where λ is the Lyapunov exponent [1].

3 Double pendulum

A typical double pendulum of looks like as shown in fig 1.

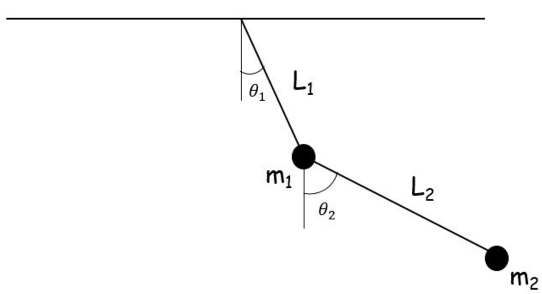


Figure 1: Double pendulum.

To solve the equation of motion of the double pendulum we make use of Lagrangian formalism. First we setup the coordinate system at the top of the double pendulum, where it is attached to the wall. The Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + m_1 g y_1 + m_2 g y_2 \tag{2}$$

where everything can be written in terms of θ_1 and θ_2. Then using the Euler-Lagrange equation we can solve for the equation of motion

$$\begin{aligned} \frac{\partial \mathcal{L}}{d\theta_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{d\dot{\theta}_1} &= 0 \\ \frac{\partial \mathcal{L}}{d\theta_2} - \frac{d}{dt}\frac{\partial \mathcal{L}}{d\dot{\theta}_2} &= 0 \end{aligned} \tag{3}$$

3.1 Numerical results

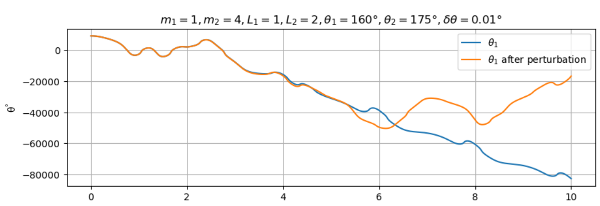
The goal of this article is to conclude the effect of the coupling on the Lyapunov exponent for coupled pendulum but we first begin with measuring the Lyapunov exponent of uncoupled double pendulum. For a double pendulum the dynamical variables are θ_1 and θ_2. We perturb the initial condition and measure the Lyapunov exponent corresponding to following quantity

$$\theta = \sqrt{\theta_1^2 + \theta_2^2} \tag{4}$$

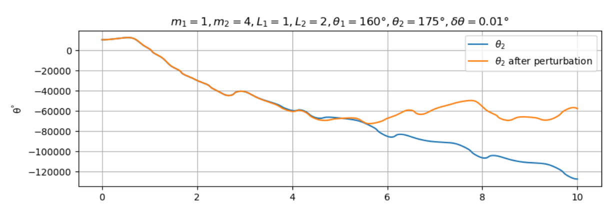
We choose the values of parameters to be $m_1 = 1, m_2 = 4, L_1 = 1, L_2 = 2$ and the initial conditions $\theta_1 = 160°, \theta_2 = 175°, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0$ and finally the perturbation $\delta\theta_0 = \delta\theta_2 = 0.01°$.

3.1.1 Typical separation of θ_1 and θ_2

First we look at typical separation of θ_1 and θ_2 as time grows.



(a) θ_1 vs t



(b) θ_2 vs t

Figure 2: Typical separation of θ_1 and θ_2

3.1.2 Lyapunov exponent

We now measure the Lyapunov exponent by looking at $\delta\theta(t)$. To measure the Lyapunov exponent we make a exponential fit to the plot of $\delta\theta(t)$. We evolve the system for very small times so that the mean square error between the fit and the actual value to be very small.

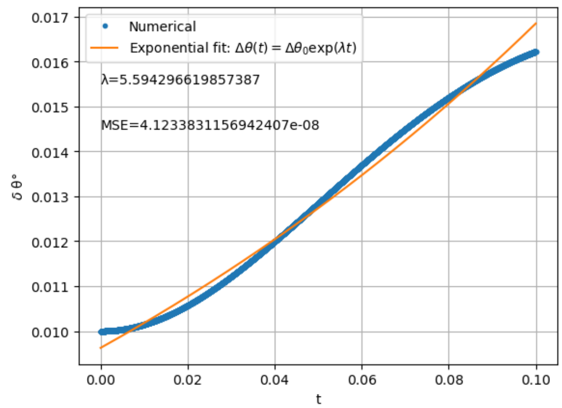


Figure 3: $\delta(t)$ vs t

From the above plot we conclude that the Lyapunov exponent for the chosen values of parameters to be

$$\lambda = 5.59 \tag{5}$$

And this value is very reliable since the mean square error is of the order $\sim 10^{-8}$.

3.1.3 Hamiltonian

To show that there is no numerical error in the calculation of the Lyapunov exponent, we plot the Hamiltonian of the system.

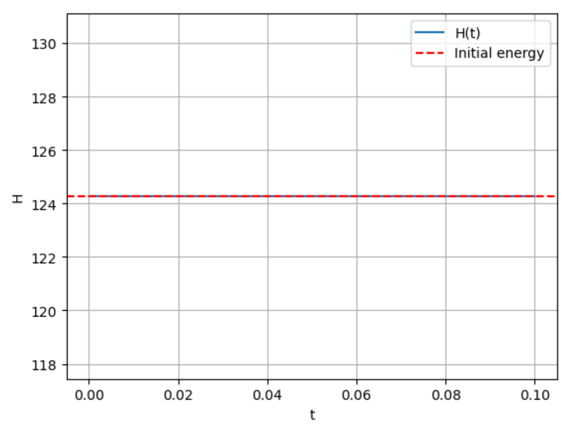


Figure 4: $H(t)$ vs t

We have plotted the Hamiltonian till the same time we used to calculate the Lyapunov exponent. We see that the Hamiltonian remains constant, as it should be since out system is conservative. Hence we conclude that the error in our numerical simulation is negligible.

4 Coupled Double Pendulum

Now we take two double pendulums and couple them with a spring, with spring constant k,in following way

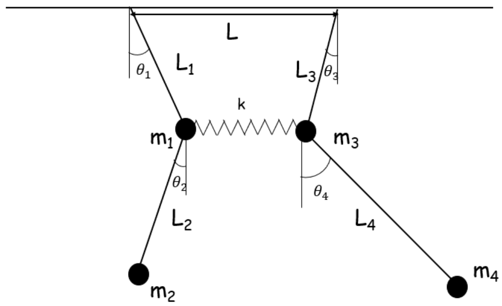


Figure 5: Coupled double pendulum.

where the natural length of the spring is L. The coordinate system is again put at the point where the pendulum of mass m_1 is attached to the ceiling. The Lagrangian of the coupled double pendulum is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_1^3 + \frac{1}{2}m_4v_4^2 + m_1gy_1 + m_2gy_2 + +m_3gy_3 + m_3gy_3 \\ & \frac{1}{2}k\left(\sqrt{(x_1-x_3)^2+(y_1-y_3)^2} - L\right)^2\end{aligned} \tag{6}$$

where everything can be written in terms of θ_1, θ_2, θ_3, and θ_4. To get the equation of motion we solve the corresponding Euler-lagrange equation similar to (3).

4.1 Numerical results

From fig 5, it is obvious that we measure the Lyapunov exponent for the quantity

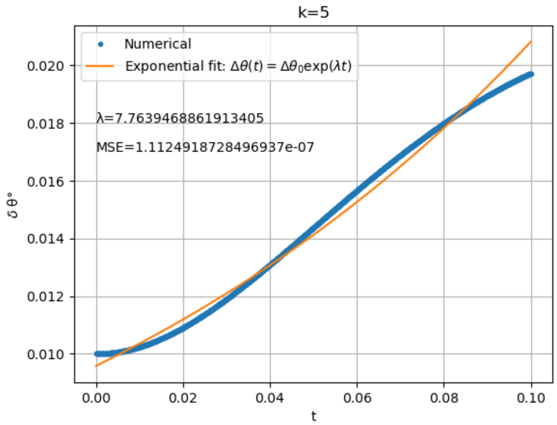
$$\theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2} \tag{7}$$

In this case, we choose the values of parameters to be $m_1 = 1, m_2 = 4, m_3 = 2, m_4 = 5, L_1 = 1, L_2 = 2, L_3 = 1, L_4 = 3$ and the initial conditions $\theta_1 = 160°, \theta_2 = 175°, \theta_3 = -150°, \theta_4 = 140°, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0, \dot{\theta}_3 = 0, \dot{\theta}_4 = 0$ and finally the perturbation $\delta\theta_0 = \delta\theta_2 = 0.01°$.

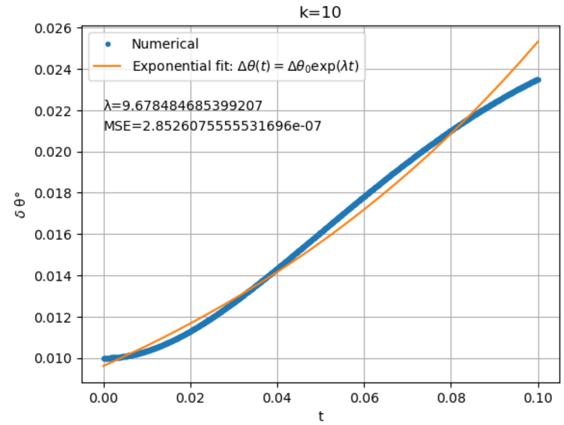
The typical separation in θ's is very similar to fig.2. So we now move on to the calculation of Lyapunov exponents

4.1.1 Lyapunov exponents

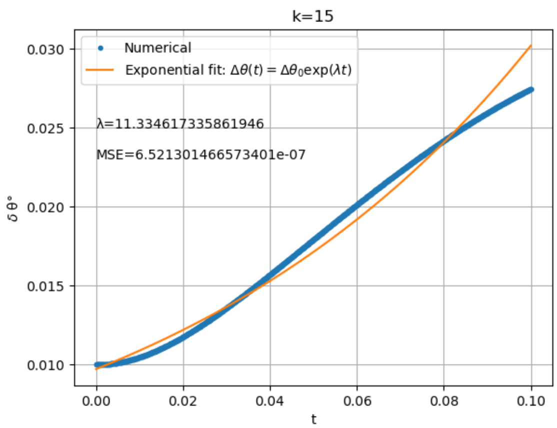
In this subsection we will plot $\delta\theta(t)$ for various values of k, and calculate the corresponding Lyapunov exponent to conclude the dependence of chaos on the coupling (k).



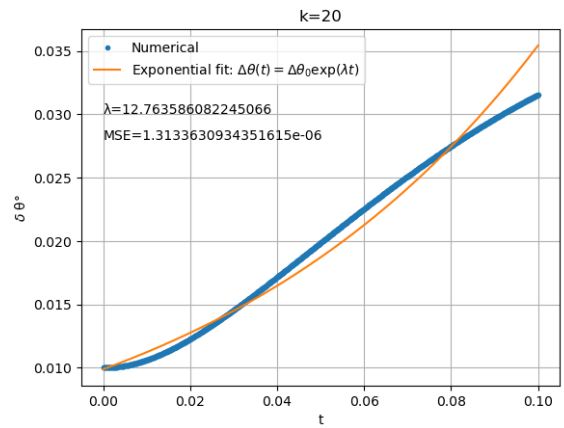
(a) $\delta(t)$ vs t for $k = 5$



(b) $\delta(t)$ vs t for $k = 10$



(c) $\delta(t)$ vs t for $k = 15$



(d) $\delta(t)$ vs t for $k = 20$

Figure 6: $\delta(t)$ vs t for different k's.

From the above plots we draw the following table Since, the mean square error is negligible

k	λ	MSE
5	7.76	$\sim 10^{-7}$
10	9.68	$\sim 10^{-7}$
15	11.33	$\sim 10^{-7}$
20	12.76	$\sim 10^{-6}$

Table 1: Lyapunov exponent λ and MSE for different values of k's

compared to the Lyapunov exponent, all the values of λ is very reliable. And we conclude that the chaos increases as we increase the coupling.

5 Conclusion

First, we showed that the double pendulum is chaotic. Then we tried to answer the question that if we bring two chaotic systems together, how does the chaos of the whole system changes? From our analysis we conclude that the chaos increases as we increase the coupling between two chaotic systems.

References

[1] Wikipedia. Lyapunov exponent. `https://en.wikipedia.org/wiki/Lyapunov_exponent`.