Existence of Repulsive Induced Charge and Solution of 2D Laplace's Equation.

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1 Introduction

When a charged particle is brought near a neutral conducting plate, charges appear on the surface of the conductor such that charges having the opposite sign to that of the particle appear on the side of the plate facing the particle. The "induced charges", with their opposite sign face the charge and cause an attractive force $\boldsymbol{F} < 0$. Given how general and applicable this phenomenon is in a wide variety of cases including when dealing with grounded conductors, dielectrics etc., one might conclude that induced charges are always result in an attractive force, but in this paper we discuss a simple counterexample, primarily following the discussion in [1], and subsequently solve 2D Laplace's Equation for the charge-conductor system for all space using appropriate Numerical Methods.

2 Charge-Conductor System

We consider unit charged particle moving along the $\boldsymbol{z}$ axis in the negative direction and a thin semicircular conducting annulus centred at $z = 0$ (Fig 1a). As we bring the particle closer, induced charges tend to appear on the surface of the conductor which results in the conductor having an electrostatic potential energy, $U(z)$, that varies with the particle distance z. The magnitude of force between the particle and the conductor then is nothing but

$$F_z = -\frac{dU(z)}{dz}. \tag{1}$$

Consider the situation when the charged particle is situated infinitely far away from the conductor, here the electric fields of the particle are radially outward, symmetric and are unaffected by the presence of the conductor which results in the conductor having an electrostatic energy $U(\infty) = 0$ which is expected. This exact same situation also occurs when the particle is placed at $z = 0$ (Fig 1b) i.e. $U(0) = U(\infty) = 0$. The electrostatic energy of the system is a thus a non-monotonically changing function which necessarily guarantees that the negative gradient, force, will be positive for some small positive value of z. A positive force implies repulsion which shows that for some small positive value of z the induced charge facing the particle must be positive (Fig 1c).

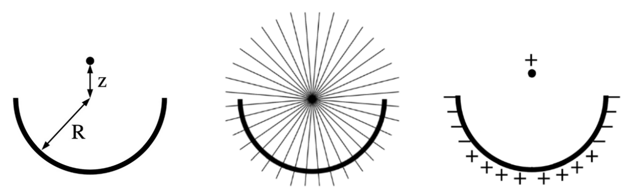


Figure 1: a) System, b) Field lines of particle at z=0, c) Charge density on conductor.

The boundary values for the system in terms of electrostatic potential, $\phi(z)$, are:

$$\phi(z) \to 0 \text{ for } z \to \infty, \tag{2}$$

$$\phi = \text{ constant, on the surface of conductor.} \tag{3}$$

Coupled with the fact that conductor is overall electrically neutral the electrostatic potential as well as electrostatic energy for the conductor can be obtained analytically ([1]).

$$U(z) = \frac{1}{2} log \left(\frac{2 - \frac{4R^2}{(R+z+\sqrt{R^2+z^2})^2}}{1 + \frac{z}{\sqrt{R^2+z^2}}} \right). \tag{4}$$

Armed with the potential for the conductor and the potential of the charged particle this we can solve for the potential for the entire 2D space using the 2D Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{5}$$

3 Numerical Analysis

3.1 Non-Linear Curve Fitting

For many such electrostatic boundary problems we can deduce the asymptotic behaviour of the electric potential and just like how we showed the existence of a repulsive force without knowing the form of the solution we would like to know how well we can make conclusive statements just from knowing an approximate (asymptotic) solution. We have our approximate solution to the potential problem

$$\tilde{U}(z) = \frac{a}{z} + b\ log\left(\frac{z}{c+z}\right). \tag{6}$$

Here a,b,c are free parameters that must be varied for curve fitting.

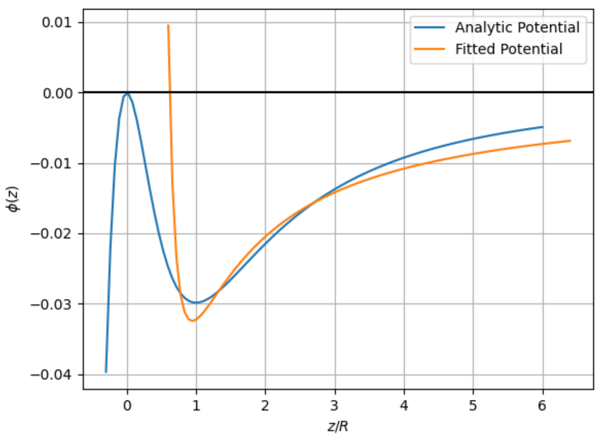


Figure 2: Best fit a=0.049, b= 0.19, c=0.5

The approximate solution, although saturating at $z \to \infty$ as required, diverges very quickly for $z \to 0$ but we our only concerned $z \geq 0$ anyways.

3.2 Force

To calculate the force we use simple finite difference method for different grid-sizes and compare the result along with the derivative of our approximate solution. we also calculate the order of convergence which is second order as it should be.

$$\text{Forward: } F_i = -\frac{(U_{i+1} - U_i)}{\Delta z} \tag{7}$$

$$\text{Central: } F_i = -\frac{(U_{i+1} - U_{i-1})}{2\Delta z} \tag{8}$$

$$\text{Backward: } F_i = -\frac{(U_i - U_{i-1})}{\Delta z} \tag{9}$$

$$n(z) = log\left|\frac{F_{4h} - F_{2h}}{F_{2h} - F_h}\right| \tag{10}$$

We can clearly observe that for the region $0 \leq z \leq 1$ the force is positive hence demonstrating the existence of repulsive induced charge. Although our assumed solution

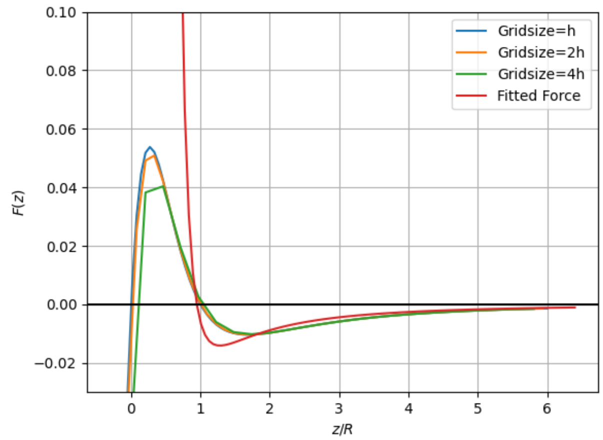


Figure 3: Force for different grid-sizes

for the problem does go into the positive regime, we cannot conclude anything reliably as it diverges very quickly for $z \to 0$

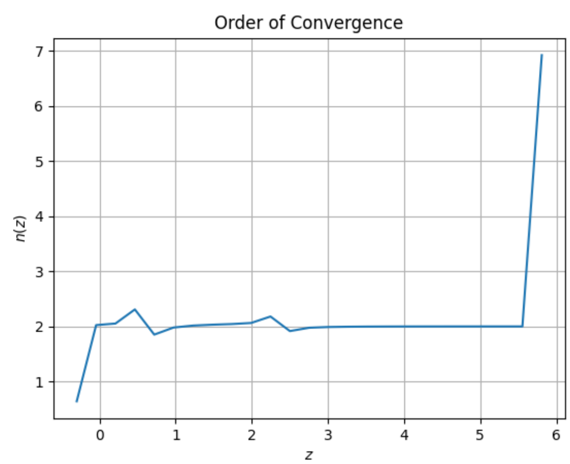


Figure 4: Order of convergence

3.3 Laplace's Equation

With the potential of both particle and conductor known, we can solve the boundary value problem for all space using $\nabla^2 V = 0$. To do this we once again use second order finite difference,

$$\frac{V(x_{i+1}, y_j) - 2V(x_i, y_j) + V(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{V(x_i, y_{j+1}) - 2V(x_i, y_j) + V(x_i, y_{j-1})}{\Delta y^2} = 0. \tag{11}$$

A simple rearrangement gives us,

$$V(x_i, y_j) = \frac{1}{4}(V(x_{i+1}, y_j) + V(x_{i-1}, y_j) + V(x_i, y_{j+1}) + V(x_i, y_{j-1})). \quad (12)$$

This is nothing but the "method of relaxation" [2], which is a special property of Laplace's Equation in 2D, where the potential at point i, j is nothing but the average of the potential at neighbouring points. Solving Eq. 12 for a large number of iterations gives us an accurate value of V.

We can observe the result for different value of z for all space (Fig 5), as we expect the contribution of annulus vanishes for z=0 and we recover Coulomb potential.

Although Eq. 12 turns out to be independent of grid-size, h, it is certainly dependent on the number of grid points. We repeat our calculation for increasing number of grid-points to see if our solution actually converges or not.

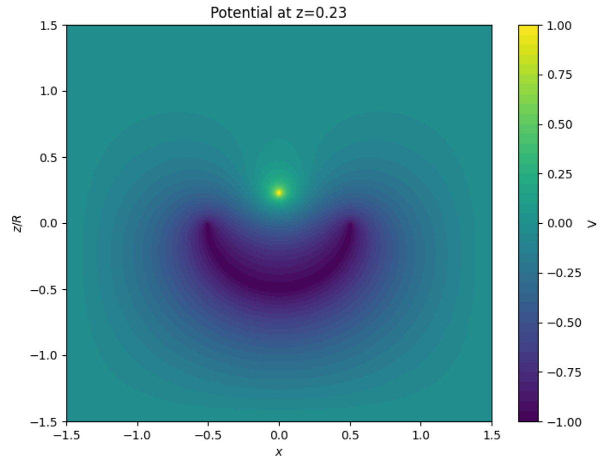
To do this we calculate the absolute error, E_{i-j}, which is the difference in the value of potential calculated for a particular 1D slice of space (x=0 slice for z=0 case), between total grid points, $i * 100$ and $j * 100$ (e.g. between 400x400 grid and 200x200 grid).

We can observer that as we increase the total number of grid points the absolute error in general goes down, the difference in potential for 400 dimensional grid and 200 dimensional grid is much larger than the difference in value between a 1600 dimensional grid and 800 dimensional grid (Fig 6).

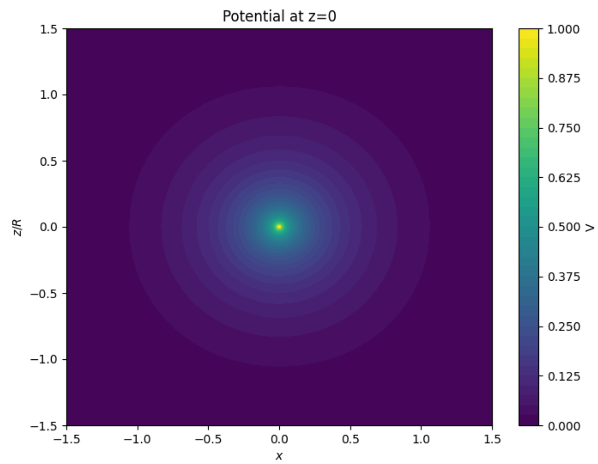
4 Further Improvements

There are many areas of improvement that are present in my numerical analysis of this problem:

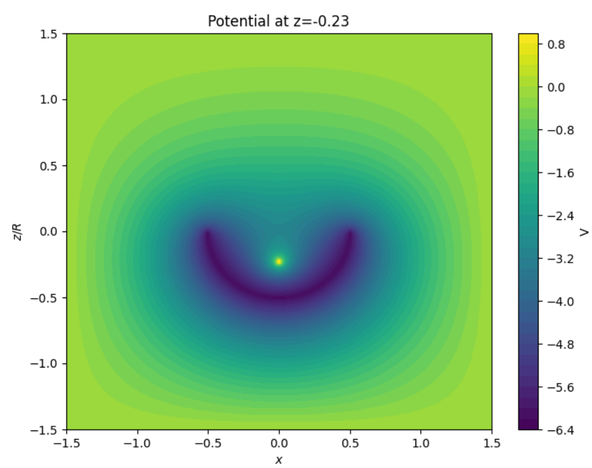
- $\tilde{U}(z)$ (Eq. 6) cannot possibly be even an approximate solution to the problem because it does not satisfy Laplace's Equation due to the presence of the $(c+z)$ term in the logarithm. Hence, more careful approximation can be made to the asymptotic solution for the problem.
- The authors of [1] solved for the exact potential both analytically and numerically, using "boundary-element method for multiply connected domains"[3], which I was unable to implement.
- In reality for the point charge, we do not have $\nabla^2 V = 0$ but rather Poisson's Equation $\nabla^2 V = -4\pi\delta(r)$. To deal with this I took a charged particle of a small but finite size but despite that the value of potential generated by the particle alone dwarfs those made by the conductor. Hence, I had to exaggerate the effects of the field made by the conductor for it to even be visible.



(a)



(b)



(c)

Figure 5: a) z=0.23, b) z=0, c) z=-0.23 .

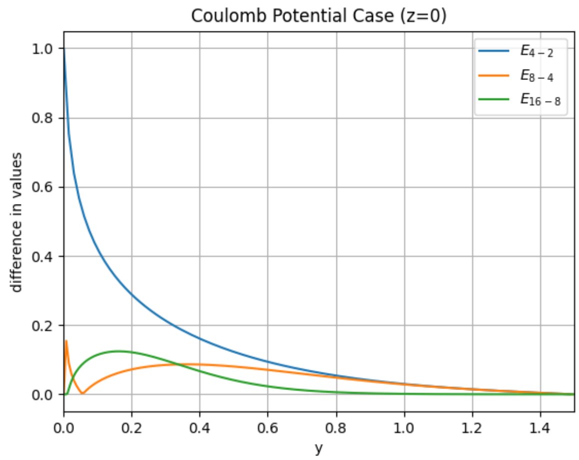


Figure 6: Absolute Error

5 Code

The Python code for the project can be accessed via Github.

6 References

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2. D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed.

3. A. Greenbaum, L. Greengard, and G. B. McFadden, *Laplace's equation and the Dirichlet-Neumann map in multiply connected domains*, J. Comp. Phys. **105**, 267–278 (1993).

4. W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd ed.