Advection of Inertial Particles in the Presence of the History Force: Higher Order Numerical Schemes
(Anton Daitche (ArXiv:1210.2576v2))

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CONTENTS

I. INTRODUCTION

The equation of motion for a small spherical particle in a viscous fluid is given by the Maxey-Riley equation

$$m_p \frac{d\mathbf{v}}{dt} = m_f \frac{D\mathbf{u}}{Dt} - \frac{m_f}{2}\left(\frac{d\mathbf{v}}{dt} - \frac{D\mathbf{u}}{Dt}\right) - 6\pi a \rho_f \nu (\mathbf{v}\text{-}\mathbf{u}) - \underbrace{6a^2 \rho_f \sqrt{\pi\nu} \int_{t_o}^{t} \frac{1}{\sqrt{t-\tau}} \left(\frac{d\mathbf{v}}{d\tau} - \frac{d\mathbf{u}}{d\tau}\right) d\tau}_{\text{Basset History Term}} \tag{1}$$

where $\mathbf{v}$, a and m_p are velocity, radius and the mass of particle respectively and $\mathbf{u}(r,t)$, m_f, ν and ρ_f are the velocity, mass, viscosity and the density of the fluid. The derivatives in equation 1 are given by

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} \ \ , \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{v}.\nabla\mathbf{u} \ \text{ and } \ \mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{2}$$

The equation 1 is valid only if particle Reynolds number is less and the particle size and its diffusive time scale are much less than the smallest length and time scale of fluid flow. Before solving, we rewrite equation 1 in terms of dimensionless parameters S and R as follows.

$$\begin{gathered} \frac{d\mathbf{w}}{dt} = (R-1)\frac{d\mathbf{u}}{dt} - R\mathbf{w}.\nabla\mathbf{u} - \frac{R}{S}\mathbf{w} - R\sqrt{\frac{3}{\pi S}} \frac{d}{dt}\int_{t_o}^{t} \frac{1}{\sqrt{t-\tau}}\mathbf{w}(\tau) d\tau \\ \frac{d\mathbf{r}}{dt} = \mathbf{v} = \mathbf{w} + \mathbf{u} \end{gathered} \tag{3}$$

where

$$R = \frac{3m_f}{m_f + 2m_p} \ \ , \ \ S = \frac{a^2}{3\nu T} \ \text{ and } \ \mathbf{w} = \mathbf{v} - \mathbf{u} \tag{4}$$

where T is characteristic time of the flow and a is the radius of spherical particle. Before solving equation 3, we would like to derive a numerical scheme for solving the history term. This is discussed in section II.

II. THE BASSET HISTORY

In order to apply the quadrature scheme to solve the integral in equation 1, we rewrite the history term as follows.

$$\int_{t_o}^{t} K(t-\tau)\frac{d}{d\tau}f(\tau)d\tau + K(t-t_o)f(t_o) = \frac{d}{dt}\int_{t_o}^{t} K(t-\tau)f(\tau)d\tau \tag{5}$$

where $K(t-\tau) = 1/\sqrt{t-\tau}$ and $f(\tau) = \mathbf{v} - \mathbf{u}$. Moreover, the equation 1 is derived assuming that $\mathbf{v}(t_o) = \mathbf{u}(t_o) \Rightarrow f(t_o) = 0$. Using this, equation 5 reduces to

$$\int_{t_o}^{t} K(t-\tau)\frac{d}{d\tau}f(\tau)d\tau = \frac{d}{dt}\int_{t_o}^{t} K(t-\tau)f(\tau)d\tau \tag{6}$$

Now, using 1st order quadrature scheme, we approximate the history term as follows.

$$\int_{t_o}^{t} K(t-\tau)f(\tau) = \sqrt{h}\sum_{j=0}^{n}\alpha_j^n\, f(\tau_{n-j}) + \mathcal{O}(h^2)\sqrt{t-t_o} \tag{7}$$

where

$$\alpha_j^n = \frac{4}{3}\begin{cases} 1, & j = 0. \\ (j-1)^{3/2} + (j+1)^{3/2} - 2j^{3/2}, & 0 < j < n. \\ (n-1)^{3/2} - n^{3/2} + \frac{6}{4}\sqrt{n}, & j = n. \end{cases} \tag{8}$$

The expressions of second and third order approximations of history term are given in reference 1.

III. FULL SOLUTION OF MAXEY-RILEY EQUATION

We again rewrite equation 3 as follows

$$\frac{d\mathbf{w}}{dt} = \mathbf{G} + \mathbf{H} \tag{9}$$

with the following abbreviations.

$$\begin{aligned} \mathbf{G} &= (R-1)\frac{d\mathbf{u}}{dt} - R\mathbf{w}.\nabla\mathbf{u} - \frac{R}{S}\mathbf{w} \\ \mathbf{H} &= -R\sqrt{\frac{3}{\pi S}}\frac{d}{dt}\int_{t_o}^{t}\frac{1}{\sqrt{t-\tau}}\mathbf{w}(\tau)d\tau \end{aligned} \tag{10}$$

Then, the integration of equation 9 is

$$\mathbf{w}(t+h) = \mathbf{w}(t) + \int_{t}^{t+h}\mathbf{G}(\tau)d\tau + \mathbf{H}(t+h) - \mathbf{H}(t) \tag{11}$$

We use Adams-Bashforth multi-step method to approximate the integration of $\mathbf{G}(t)$ at different orders as follows.

$$\begin{aligned} &\text{1. First order: } \int_{t}^{t+h}\mathbf{G}(\tau)d\tau = h\mathbf{G}(t) + \mathcal{O}(h^2) \\ &\text{2. Second order: } \int_{t}^{t+h}\mathbf{G}(\tau)d\tau = \frac{h}{2}(3\mathbf{G}(t) - \mathbf{G}(t-h)) + \mathcal{O}(h^3) \\ &\text{3. Third order: } \int_{t}^{t+h}\mathbf{G}(\tau)d\tau = \frac{h}{12}(23\mathbf{G}(t) - 16\mathbf{G}(t-h) + 5\mathbf{G}(t-2h)) + \mathcal{O}(h^4) \end{aligned} \tag{12}$$

Using equations 11 and 12, we get the position $\mathbf{r}(t)$ and relative velocity $\mathbf{w}(t)$ of particle as follows.

1. First order solution:

$$\begin{aligned}\mathbf{r}_{n+1} &= \mathbf{r}_n + h(\mathbf{w}_n + \mathbf{u}_n) + \mathcal{O}(h^2) \\ (1+\xi\alpha_0^{n+1})\mathbf{w}_{n+1} &= \mathbf{w}_n + h\mathbf{G}_n - \xi\sum_{j=0}^{n}\left(\alpha_{j+1}^{n+1}\mathbf{w}_{n-j} - \alpha_j^n\mathbf{w}_{n-j}\right) + \mathcal{O}(h^2)\sqrt{t_n - t_0}\end{aligned} \tag{13}$$

2. Second order solution:

$$\begin{aligned}\mathbf{r}_{n+1} &= \mathbf{r}_n + \frac{h}{2}(3(\mathbf{w}_n + \mathbf{u}_n) - (\mathbf{w}_{n-1} + \mathbf{u}_{n-1})) + \mathcal{O}(h^3) \\ (1+\xi\beta_0^{n+1})\mathbf{w}_{n+1} &= \mathbf{w}_n + \frac{h}{2}(3\mathbf{G}_n - \mathbf{G}_{n-1}) - \xi\sum_{j=0}^{n}\left(\beta_{j+1}^{n+1}\mathbf{w}_{n-j} - \beta_j^n\mathbf{w}_{n-j}\right) + \mathcal{O}(h^3)\sqrt{t_n - t_0}\end{aligned} \tag{14}$$

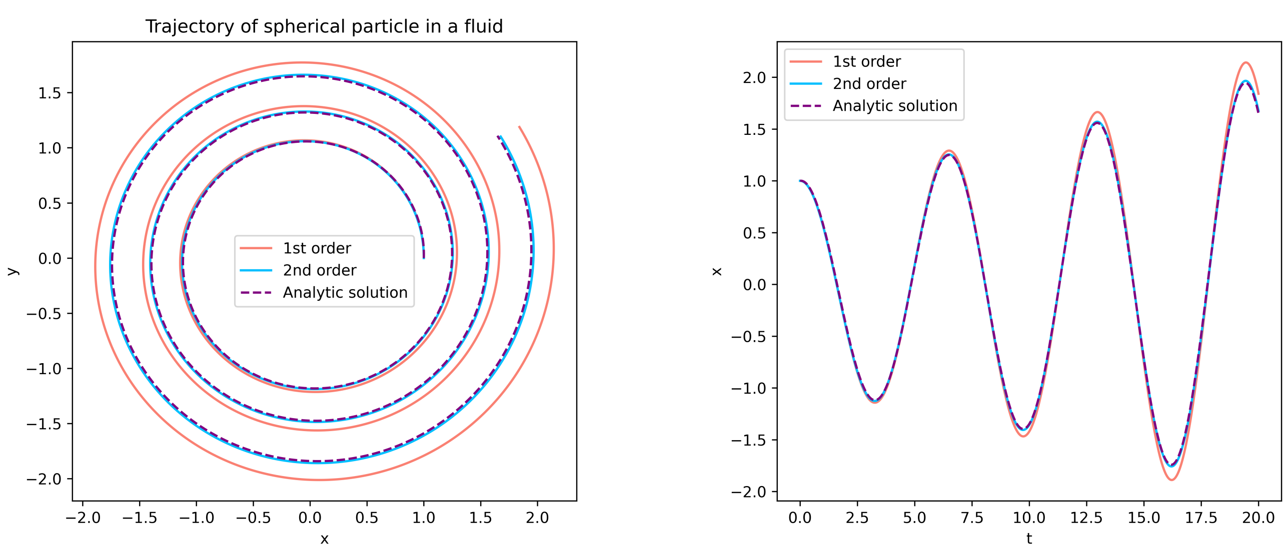
3. Third order solution:

$$\begin{aligned}\mathbf{r}_{n+1} &= \mathbf{r}_n + \frac{h}{12}(23(\mathbf{w}_n + \mathbf{u}_n) - 16(\mathbf{w}_{n-1} + \mathbf{u}_{n-1}) + 5(\mathbf{w}_{n-2} + \mathbf{u}_{n-2})) + \mathcal{O}(h^4) \\ (1+\xi\gamma_0^{n+1})\mathbf{w}_{n+1} &= \mathbf{w}_n + \frac{h}{12}(23\mathbf{G}_n - 16\mathbf{G}_{n-1} + 5\mathbf{G}_{n-2}) - \xi\sum_{j=0}^{n}\left(\gamma_{j+1}^{n+1}\mathbf{w}_{n-j} - \gamma_j^n\mathbf{w}_{n-j}\right) + \mathcal{O}(h^4)\sqrt{t_n - t_0}\end{aligned} \tag{15}$$

where $\xi = R\sqrt{3h/\pi S}$.

The trajectory of particle obtained using above equations is shown in Section IV. The expressions for β_j^n and γ_j^n can be found in reference 1.

IV. RESULTS AND CONCLUSION



(a) (b)

Figure 1: The trajectory of spherical particle starting at $\mathbf{r}_0 = (1,0)$, $\mathbf{w}_0 = (0,0)$ with parameters R = 0.75 and S = 0.3. Here, the velocity of fluid is $u = |r|\hat{e}_\phi$

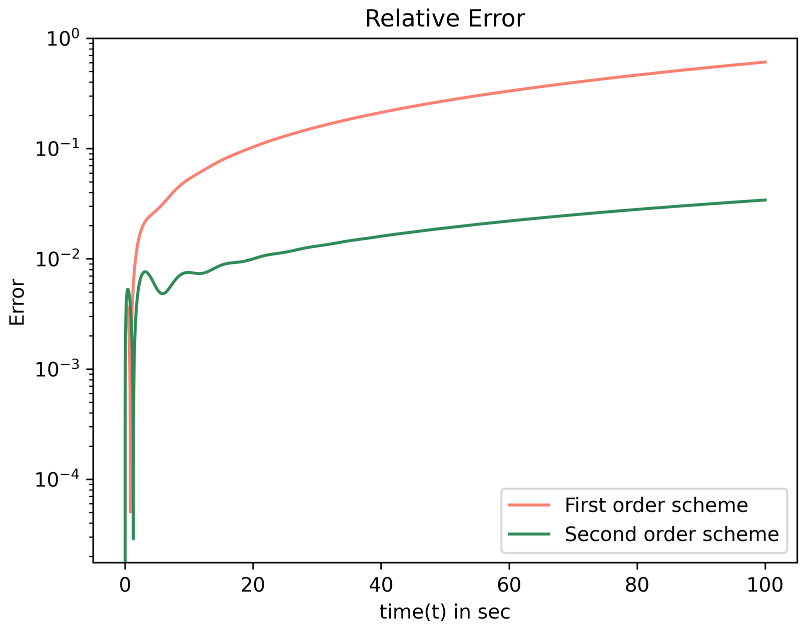
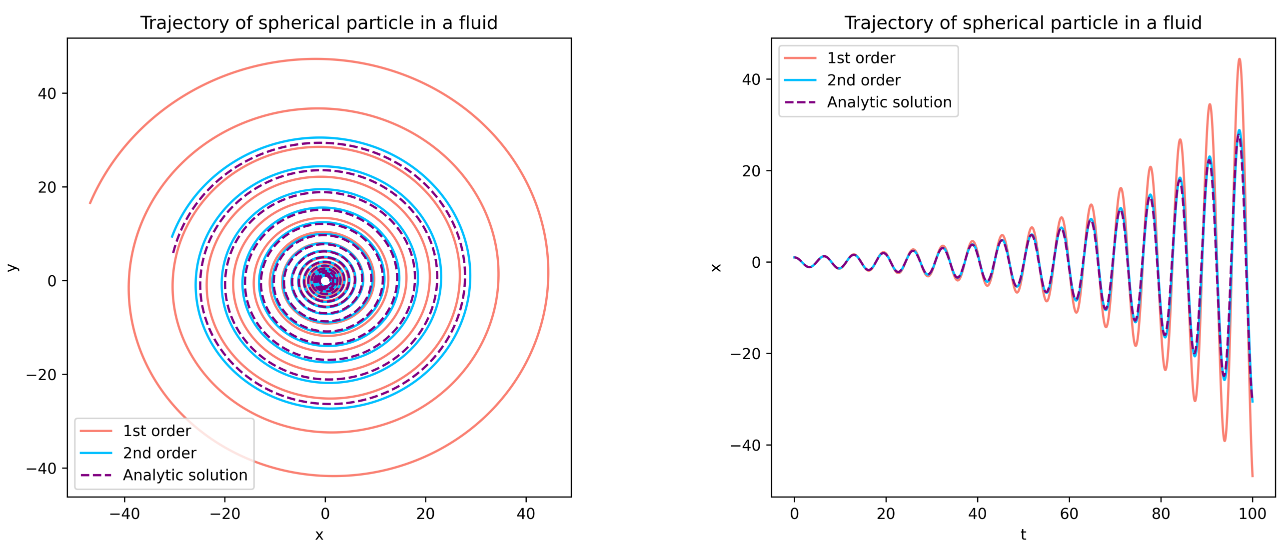
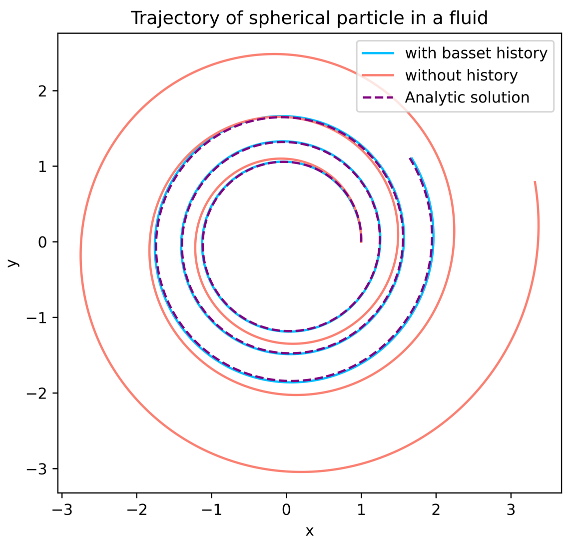


Figure 2: The relative error of numerical solution for first and second order approximations for $h = 10^{-2}$.

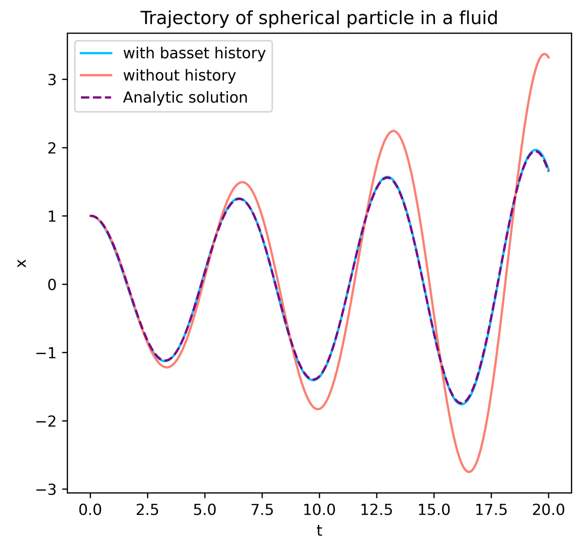


(a) (b)

Figure 3: The trajectory of spherical particle starting at $\mathbf{r}_0 = (1, 0)$, $\mathbf{w}_0 = (0, 0)$ with parameters R = 0.75 and S = 0.3 for longer time. The first and second order approximations don't hold at large times. Here, the velocity of fluid is $u = |r|\hat{e}_\phi$



(a)



(b)

Figure 4: The trajectory of spherical particle starting at $\mathbf{r}_0 = (1, 0)$, $\mathbf{w}_0 = (0, 0)$ with parameters R = 0.75 and S = 0.3. These figure show the difference between the numerical solutions of Maxey-Riley equation with and without Basset history.

We can draw following conclusions from the plots.

1. The trajectory of particle evaluated using first and second order approximations match with the analytical solution for small time. As shown in the figure 1, only the second order numerical solution matches with the analytical solution as time increases. But this approximation doesn't work for much larger time. In figure 3, it is shown that at $t = 100$ sec, even the second order solution doesn't match well with the analytic solution and we may have to use the higher order terms in order to get the desired accuracy.

2. Figure 2 shows the evolution of relative error of numerical solution for first and second order solution. It can be seen that the second order scheme has error much less than the first order scheme. Hence, it matches well with analytic solution. But the error increases with time, which suggests that the numerical solution would have values different than exact solution at large times.

3. The figure 4 shows the trajectory of particle evaluated with and without basset history. We see that the particle is spiraling away much faster in the case where no basset history is taken into account than the particle with basset history.

V. FEEDBACK

I was advised to do a convergence test to check validity of numerical solution. In figure 5, I plot the analytic solution and first order solution for step size $h = 10^{-2}$ and $h = 2 \times 10^{-3}$. We see that the the first order solution approaches the analytical solution for smaller step size, hence the relative error reduces. This shows that the solution converges to exact solution as we reduce the step size.

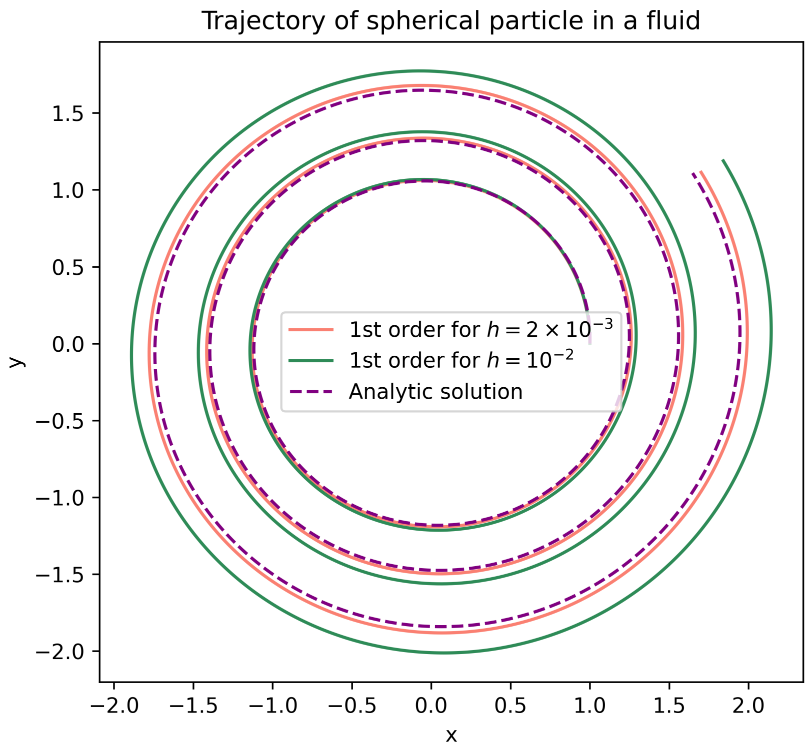


Figure 5: First order numerical solution for step size $h = 10^{-2}$ and $h = 2 \times 10^{-3}$.

VI. REFERENCES

1. Advection of Inertial Particles in the Presence of the History Force: Higher Order Numerical Schemes

2. On the effect of the Boussinesq–Basset force on the radial migration of a Stokes particle in a vortex (Candelier et al.)