Random Numbers

SADINENI ABHINAY - CS21BTECH11055

Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/1.1.c
```

Now compile and exceute

```
$ gcc -o 1.1 1.1.c
$ ./1.1
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/1.2.py
$ python3 1.2.py
```

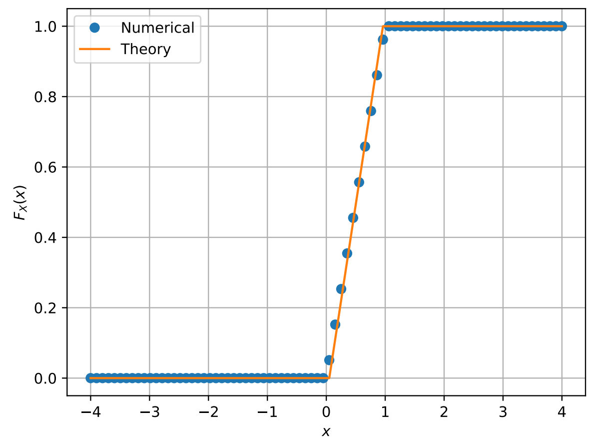

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since it is uniform distribution then pdf is given by:

$$p_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{1.2}$$

for $0 < x < 1$:

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{1.3}$$
$$= \int_{0}^{x} 1.dx \tag{1.4}$$
$$= x \tag{1.5}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & 1 < x \end{cases} \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution: Download and excecute the code

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/1.3-1.4.c
```

Now compile and excecute

```
$ gcc -o 1.3-1.4 1.3-1.4.c
$ ./1.3-1.4
```

Experimental values:

$$\text{Mean} = 0.499772 \tag{1.9}$$
$$\text{Variance} = 0.083368 \tag{1.10}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.11}$$

Solution: Mean and variance are given by :

$$E[U] = \int_0^1 x.dx = 0.5 \tag{1.12}$$
$$Var[U] = E\left[U^2\right] - E[U]^2 = \int_0^1 x^2.dx - 0 \tag{1.13}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.14}$$
$$= \frac{1}{12} \tag{1.15}$$

By comparing these values obatined with results (1.9) and (1.10) we can conclude that simulation varies with theoritical analysis.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Soltion: Download

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.1.c
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.1.py
```

Now compile the c-code to generate the data

```
$ gcc -o 2.1 2.1.c
$ ./2.1
```

Now excecute the python code which take the data from output of c code:

```
$ python3 2.1.py
```

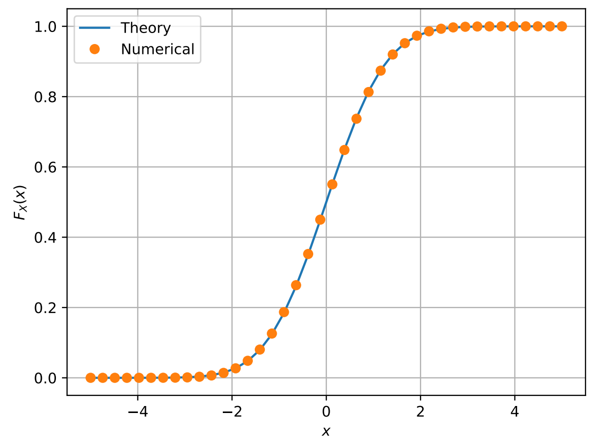

Fig. 2.2: The CDF of X

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2
Properties:
a) it is symmetric about $(0, 0.5)$
b) its tends to one as we approch $x = 6$

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.2.py
$ python3 2.2.py
```

$$Q(x) = \int_x^{\infty} exp\left(\frac{-x^2}{2}\right) dx \tag{2.2}$$
$$erf(x) = \int_0^x exp\left(\frac{-x^2}{2}\right) dx \tag{2.3}$$
$$F_X(x) = \int_{-\infty}^x exp\left(\frac{-x^2}{2}\right) dx \tag{2.4}$$
$$F_X(x) = 1 - Q(x) \tag{2.5}$$
$$Q(x) = \frac{1}{2}\left(1 - erf\left(\frac{x}{\sqrt{2}}\right)\right) \tag{2.6}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.7}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below
Properties:

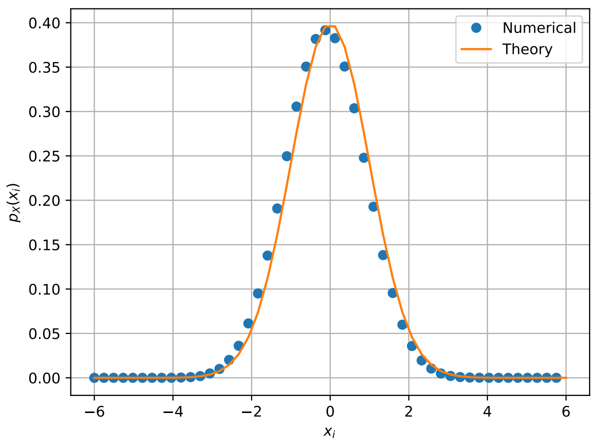

Fig. 2.3: The PDF of X

a) it is symmetric about $x = 0$
b) its tends to zero as we approch $x = 6$

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/2.3.py
$ python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/karna−rash/A11110−
    Assignment/blob/master/codes/gau_mean.
    c
```

compile and execute:

```
gcc −o gau_mean gau_mean.c −lm
./gau_mean
```

Experimental values

$$\text{Mean} = -0.000354 \tag{2.8}$$
$$\text{Variance} = 1.001179 \tag{2.9}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.10}$$

repeat the above exercise theoretically.
Solution:

a) Mean :

$$E(X) = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.11}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\left(exp\left(-\frac{x^2}{2}\right)\right) \tag{2.13}$$
$$= 0 \tag{2.14}$$

b) variance:

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.15}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 exp\left(-\frac{x^2}{2}\right) dx \tag{2.16}$$
$$= \frac{1}{\sqrt{2\pi}}\left(x \int x exp\left(-\frac{x^2}{2}\right) dx\right) \tag{2.17}$$
$$- \frac{1}{\sqrt{2\pi}} \int \int \left(x exp\left(-\frac{x^2}{2}\right)\right) dx.dx \tag{2.18}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2}\right) dx \tag{2.19}$$
$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \tag{2.20}$$
$$= 1 \tag{2.21}$$

By comparing these values obatined with results (2.8) and (2.9) we can conclude that simulation varies with theoritical analysis.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Download the codes 3.1

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/3.1.c
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/3.1.py
```

Compile and execute:

```
$gcc −o 3.1 3.1.c
$./3.1
$python 3.1.py
```

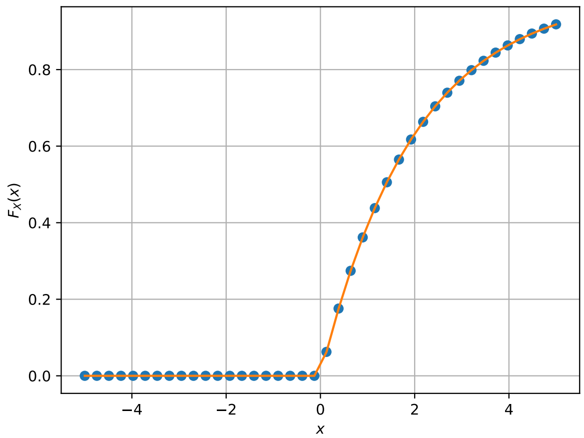

Fig. 3.1: The PDF of X

3.2 Find a theoretical expression for $F_V(x)$.
Soltion: Let $V = G(U)$ then

$$F_V(x) = F_U(G^{-1}(x)) \tag{3.2}$$
$$= F_U(1 - e^{-\frac{x}{2}}) \tag{3.3}$$
$$\tag{3.4}$$

$$F_V(x) = \begin{cases} G^{-1}(x) & 0 \le 1 - e^{-\frac{x}{2}} \le 1 \\ 0 & 1 - e^{-\frac{x}{2}} < 0 \end{cases} \tag{3.5}$$

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & 0 \le x \le \infty \\ 0 & x < 0 \end{cases} \tag{3.6}$$