Random Numbers

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/1.1.c
```

Now compile and exceute

```
$ gcc −o 1.1 1.1.c
$ ./1.1
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/1.2.py
$ python3 1.2.py
```

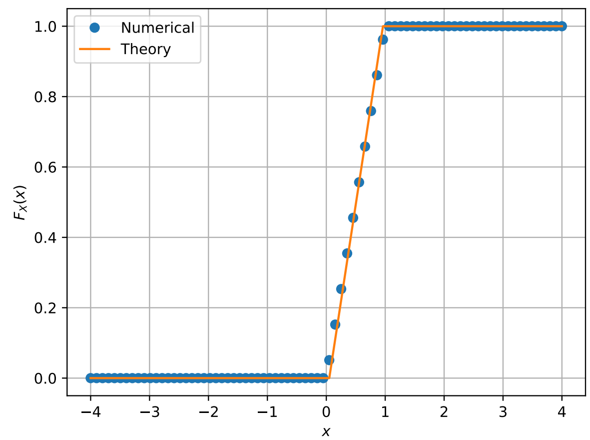

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since it is uniform distribution then pdf is given by:

$$p_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{1.2}$$

for $0 < x < 1$:

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{1.3}$$

$$= \int_{0}^{x} 1.dx \tag{1.4}$$

$$= x \tag{1.5}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 < x \end{cases} \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution: Download and excecute the code

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/1.3-1.4.c
```

Now compile and excecute

```
$ gcc -o 1.3-1.4 1.3-1.4.c
$ ./1.3-1.4
```

Experimental values:

$$\text{Mean} = 0.499772 \tag{1.9}$$
$$\text{Variance} = 0.083368 \tag{1.10}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.11}$$

Solution: Mean and variance are given by :

$$E[U] = \int_0^1 x.dx = 0.5 \tag{1.12}$$
$$Var[U] = E\left[U^2\right] - E[U]^2 = \int_0^1 x^2.dx - 0 \tag{1.13}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.14}$$
$$= \frac{1}{12} \tag{1.15}$$

By comparing these values obatined with results (1.9) and (1.10) we can conclude that simulation varies with theoritical analysis.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Soltion: Download

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.1.c
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.1.py
```

Now compile the c-code to generate the data

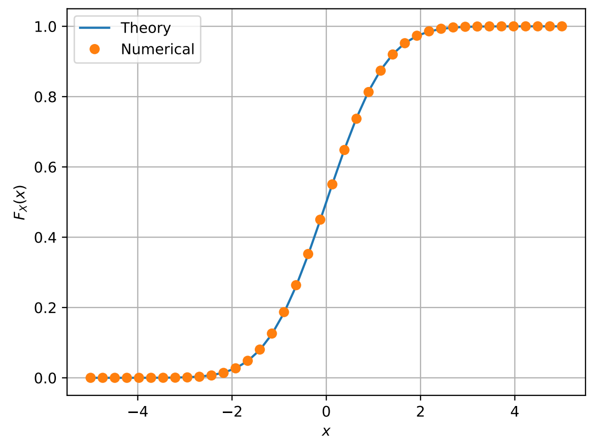

Fig. 2.2: The CDF of X

```
$ gcc -o 2.1 2.1.c
$ ./2.1
```

Now excecute the python code which take the data from output of c code:

```
$ python3 2.1.py
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2
Properties:
a) it is symmetric about $(0, 0.5)$
b) its tends to one as we approch $x = 6$

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.2.py
$ python3 2.2.py
```

$$Q(x) = \int_x^{\infty} exp\left(\frac{-x^2}{2}\right) dx \tag{2.2}$$
$$erf(x) = \int_0^{x} exp\left(\frac{-x^2}{2}\right) dx \tag{2.3}$$
$$F_X(x) = \int_{-\infty}^{x} exp\left(\frac{-x^2}{2}\right) dx \tag{2.4}$$
$$F_X(x) = 1 - Q(x) \tag{2.5}$$
$$Q(x) = \frac{1}{2}\left(1 - erf\left(\frac{x}{\sqrt{2}}\right)\right) \tag{2.6}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The

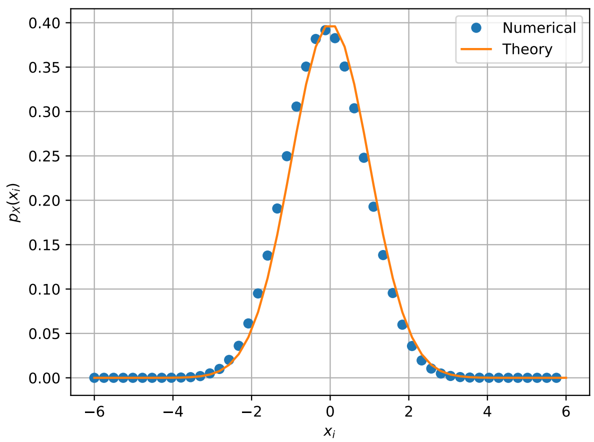

Fig. 2.3: The PDF of X

PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.7}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below
Properties:

a) it is symmetric about $x = 0$
b) its tends to zero as we approch $x = 6$

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/2.3.py
$ python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/karna-rash/A11110-
    Assignment/blob/master/codes/gau_mean.
    c
```

compile and execute:

```
gcc -o gau_mean gau_mean.c -lm
./gau_mean
```

Experimental values

$$\text{Mean} = -0.000354 \tag{2.8}$$
$$\text{Variance} = 1.001179 \tag{2.9}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.10}$$

repeat the above exercise theoretically.
Solution:

a) Mean :

$$E(X) = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.11}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} xexp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\left(exp\left(-\frac{x^2}{2}\right)\right) \tag{2.13}$$
$$= 0 \tag{2.14}$$

b) variance:

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.15}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 exp\left(-\frac{x^2}{2}\right) dx \tag{2.16}$$
$$= \frac{1}{\sqrt{2\pi}}\left(x \int xexp\left(-\frac{x^2}{2}\right) dx\right) \tag{2.17}$$
$$- \frac{1}{\sqrt{2\pi}} \int \int \left(xexp\left(-\frac{x^2}{2}\right)\right) dx.dx \tag{2.18}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2}\right) dx \tag{2.19}$$
$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \tag{2.20}$$
$$= 1 \tag{2.21}$$

By comparing these values obatined with results (2.8) and (2.9) we can conclude that simulation varies with theoritical analysis.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the codes 3.1

```
$ wget https://github.com/karna-rash/A11110
    -Assignment/blob/master/codes/3.1.c
```

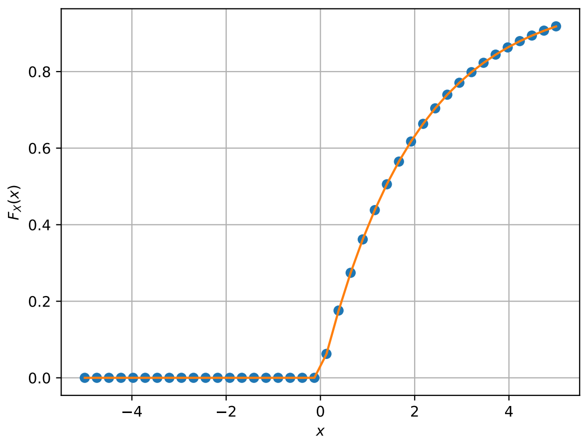

Fig. 3.1: The PDF of V

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/3.1.py
```

Compile and execute:

```
$gcc −o 3.1 3.1.c
$./3.1
$python 3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$. **Soltion:** Let $V = G(U)$ then

$$F_V(x) = F_U(G^{-1}(x)) \tag{3.2}$$
$$= F_U(1 - e^{-\frac{x}{2}}) \tag{3.3}$$
$$\tag{3.4}$$

$$F_V(x) = \begin{cases} G^{-1}(x) & 0 \leq 1 - e^{-\frac{x}{2}} \leq 1 \\ 0 & 1 - e^{-\frac{x}{2}} < 0 \end{cases} \tag{3.5}$$

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & 0 \leq x \leq \infty \\ 0 & x < 0 \end{cases} \tag{3.6}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download and excecute the code

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/4.1.c
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/functions.
    h
$ gcc 4.1.c −lm
$ ./a.out
```

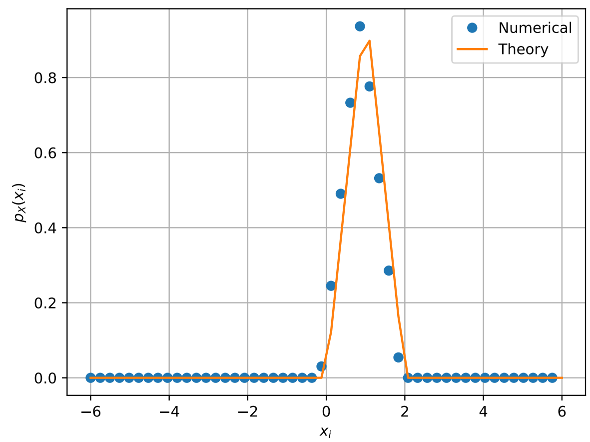

Fig. 4.2: The PDF verify of T

4.2 Find the CDF of T.
Solution: Download and excecute the code .it gives the following plot Fig.4.3

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/4.2.py
$ python 4.2.py
```

$$F_T(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ \frac{(2-t)^2}{2} & 1 \leq t \leq 2 \\ 1 & 2 \leq t \end{cases} \tag{4.2}$$

4.3 Find the PDF of T.
Solution: Download and excecute the code. It gives the following plot Fig.4.2

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/4.3.py
$ python 4.3.py
```

$$p_T(t) = \begin{cases} t & 0 \leq t < 1 \\ 2 - t & 1 \leq t \leq 2 \\ 0 & otherwise \end{cases} \tag{4.3}$$

Solution: Download and excecute the code. It gives the following plot Fig.4.3

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Let us use convulsion of

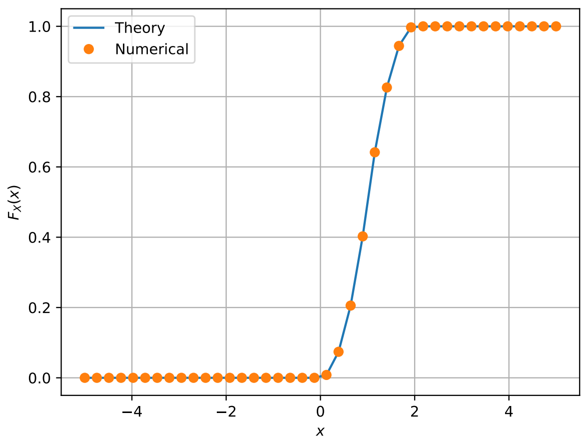

Fig. 4.3: The CDF Verify of T

$p_{U_1}(x)$ and $p_{U_2}(x)$

$$p_T(t) = \int_{-\infty}^{\infty} p_{U_1}(T-\alpha)\, p_{U_2}(\alpha)\, d\alpha \tag{4.4}$$

$$p_T(t) = \begin{cases} \int_0^t 1.d\alpha & 0 \le t < 1 \\ \int_{t-1}^1 1.d\alpha & 1 \le t \le 2 \\ 0 & otherwise \end{cases} \tag{4.5}$$

$$= \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t \le 2 \\ 0 & otherwise \end{cases} \tag{4.6}$$

Now we can get CDF from PDF

$$F_T(t) = \Pr(T \le t) \tag{4.7}$$

$$= \int_{-\infty}^{t} p_T(x)\, dx \tag{4.8}$$

$$F_T(t) = \begin{cases} 0 & t \le 0 \\ \int_0^t x.dx & 0 \le t < 1 \\ \int_1^t 2-x.dx + \frac{1}{2} & 1 \le t \le 2 \\ 1 & 2 \le t \end{cases} \tag{4.9}$$

$$= \begin{cases} 0 & t \le 0 \\ \frac{t^2}{2} & 0 \le t < 1 \\ \frac{(2-t)^2}{2} & 1 \le t \le 2 \\ 1 & 2 \le t \end{cases} \tag{4.10}$$

4.5 Verify your results through a plot.
Solution: After Compare the empirical and theorical plots in 4.3 and 4.2

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$. **Solution:** Download and excecute the code

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/5.1.c
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/functions.
    h
$ gcc 5.1.c −lm
$ ./a.out
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0,1)$. **Solution:** Download and excecute the code

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/5.2.c
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/functions.
    h
$ gcc 5.2.c −lm
$ ./a.out
```

5.3 Plot Y using a scatter plot. Download and execute the code

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/5.3.py
$ python 5.3.py
```

5.4 Guess how to estimate X from Y.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.6 Find P_e assuming that X has equiprobable symbols.

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.4}$$

5.10 Repeat the above exercise using the MAP criterion.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Download and excecute the code

```
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/5.1.c
$ wget https://github.com/karna−rash/A11110
    −Assignment/blob/master/codes/functions.
    h
$ gcc 6.1.c −lm
$ ./a.out
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α. **Solution:** Since $X1$ and $X2$ are independent normal distribution:
For $x \geq 0$ Let $X1 = R\cos(\theta)$ and $X2 = R\sin(\theta)$ and And the jacobian matrix J:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \theta} \end{pmatrix} \tag{6.3}$$

$$= \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix} \tag{6.4}$$

Now let us find the

$$p_{R,\theta}(r,\theta) = p_{X1,X2}(x1,x2)\,|J| \tag{6.5}$$

$$= p_{X1}(x1)\,p_{X2}(x2) \times r \tag{6.6}$$

$$= \frac{r}{2\pi} e^{-\frac{x_1^2}{2}} e^{-\frac{x_2^2}{2}} \tag{6.7}$$

$$= \frac{r}{2\pi} e^{-\frac{r^2}{2}} \tag{6.8}$$

Remove θ and get in R:

$$P_R(r) = \int_0^{2\pi} p_{R,\theta}(r,\theta)\,.d\theta \tag{6.9}$$

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d.\theta \tag{6.10}$$

$$= 2\pi.\frac{r}{2\pi} e^{-\frac{r^2}{2}} \tag{6.11}$$

$$= re^{-\frac{r^2}{2}} \tag{6.12}$$

Now transform into Y:

$$p_Y(y) = p_R(r)\,.\frac{dr}{dy} \tag{6.13}$$

$$= \sqrt{y}e^{-\frac{y}{2}}.\frac{1}{2\sqrt{y}} \tag{6.14}$$

$$= \frac{e^{-\frac{y}{2}}}{2} \tag{6.15}$$

from pdf we can get cdf:
For $0 \leq x \leq 1$

$$F_V(x) = \int_0^x \frac{e^{-\frac{u}{2}}}{2} du \tag{6.16}$$

$$= 1 - e^{-\frac{x}{2}} \tag{6.17}$$

$$\tag{6.18}$$

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases} \tag{6.19}$$

So Thus confirmed that $\alpha = \frac{1}{2}$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.20}$$