

Random Numbers

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Abstract—This manual provides a simple introduction to the generation of random numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files

```
$ wget https://github.com/karna-rash/A11110
-Assignment/blob/master/codes/1.1.c
```

Now compile and exeute

```
$ gcc -o 1.1 1.1.c
$ ./1.1
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/karna-rash/A11110
-Assignment/blob/master/codes/1.2.py
$ python3 1.2.py
```

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Since it is uniform distribution then pdf is given by:

$$p_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

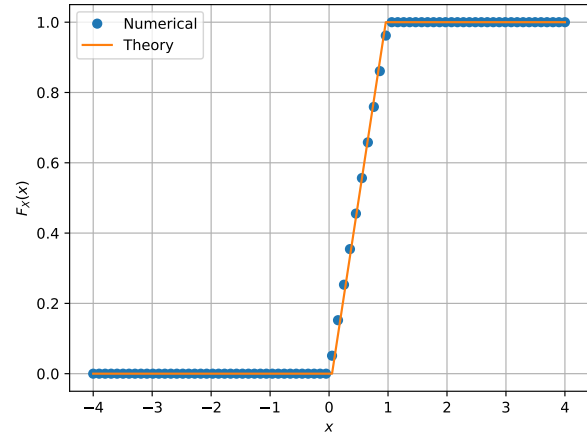


Fig. 1.2: The CDF of U

for $0 < x < 1$:

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

$$= \int_0^x 1 dx \quad (1.4)$$

$$= x \quad (1.5)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 < x \end{cases} \quad (1.6)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.7)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

Write a C program to find the mean and variance of U .

Solution:Download and execute the code

```
$ wget https://github.com/karna-rash/A11110
-Assignment/blob/master/codes/1.3-1.4.c
```

Now compile and execute

```
$ gcc -o 1.3-1.4 1.3-1.4.c
$ ./1.3-1.4
```

Experimental values:

$$\text{Mean} = 0.499772 \quad (1.9)$$

$$\text{Variance} = 0.083368 \quad (1.10)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.11)$$

Solution: Mean and variance are given by :

$$E[U] = \int_0^1 x \cdot dx = 0.5 \quad (1.12)$$

$$\text{Var}[U] = E[U^2] - E[U]^2 = \int_0^1 x^2 \cdot dx - 0 \quad (1.13)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (1.14)$$

$$= \frac{1}{12} \quad (1.15)$$

By comparing these values obtained with results (1.9) and (1.10) we can conclude that simulation varies with theoretical analysis.

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:Download

```
$ wget https://github.com/karna-rash/A11110
-Assignment/blob/master/codes/2.1.c
$ wget https://github.com/karna-rash/A11110
-Assignment/blob/master/codes/2.1.py
```

Now compile the c-code to generate the data

```
$ gcc -o 2.1 2.1.c
$ ./2.1
```

Now execute the python code which take the data from output of c code:

```
$ python3 2.1.py
```

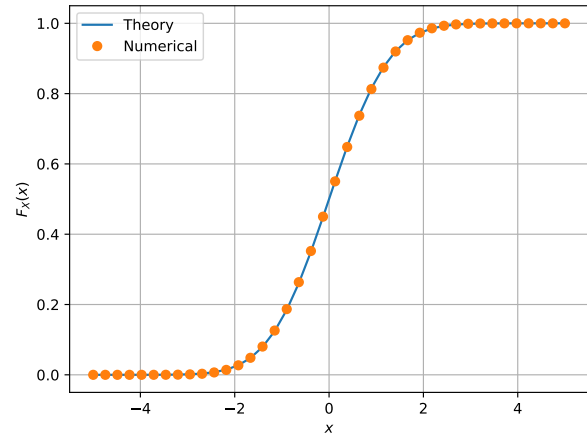


Fig. 2.2: The CDF of X

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 Properties:

- it is symmetric about (0, 0.5)
- its tends to one as we approach $x = 6$

```
$ wget https://github.com/karna-rash/A11110
-Assignment/blob/master/codes/2.2.py
$ python3 2.2.py
```

$$Q(x) = \int_x^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \quad (2.2)$$

$$\text{erf}(x) = \int_0^x \exp\left(\frac{-x^2}{2}\right) dx \quad (2.3)$$

$$F_X(x) = \int_{-\infty}^x \exp\left(\frac{-x^2}{2}\right) dx \quad (2.4)$$

$$F_X(x) = 1 - Q(x) \quad (2.5)$$

$$Q(x) = \frac{1}{2} \left(1 - \text{erf}\left(\frac{x}{\sqrt{2}}\right) \right) \quad (2.6)$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.7)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

Properties:

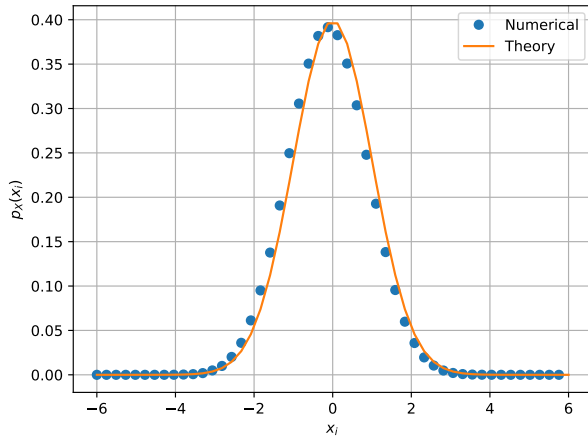


Fig. 2.3: The PDF of X

- a) it is symmetric about $x = 0$
- b) its tends to zero as we approach $x = 6$

```
$ wget https://github.com/karna-rash/A11110-
  -Assignment/blob/master/codes/2.3.py
$ python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.

Solution:

```
wget https://github.com/karna-rash/A11110-
  Assignment/blob/master/codes/gau_mean.
  c
```

compile and execute:

```
gcc -o gau_mean gau_mean.c -lm
./gau_mean
```

Experimental values

$$\text{Mean} = -0.000354 \quad (2.8)$$

$$\text{Variance} = 1.001179 \quad (2.9)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.10)$$

repeat the above exercise theoretically.

Solution:

a) Mean :

$$E(X) = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.11)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\left(\exp\left(-\frac{x^2}{2}\right)\right) \quad (2.13)$$

$$= 0 \quad (2.14)$$

b) variance:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.15)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.16)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int \exp\left(-\frac{x^2}{2}\right) dx \right) \quad (2.17)$$

$$- \frac{1}{\sqrt{2\pi}} \int \int \left(x \exp\left(-\frac{x^2}{2}\right) \right) dx dx \quad (2.18)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.19)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.20)$$

$$= 1 \quad (2.21)$$

By comparing these values obtained with results (2.8) and (2.9) we can conclude that simulation varies with theoretical analysis.

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

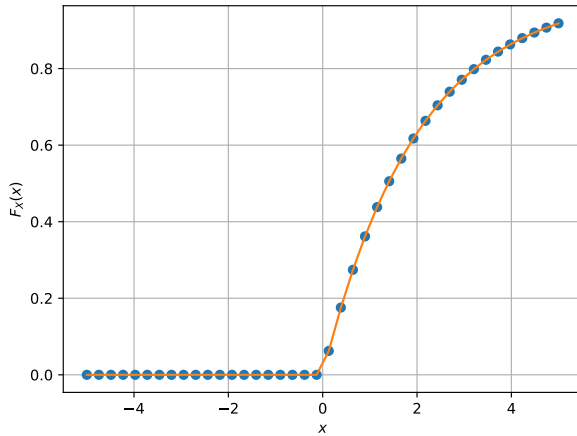
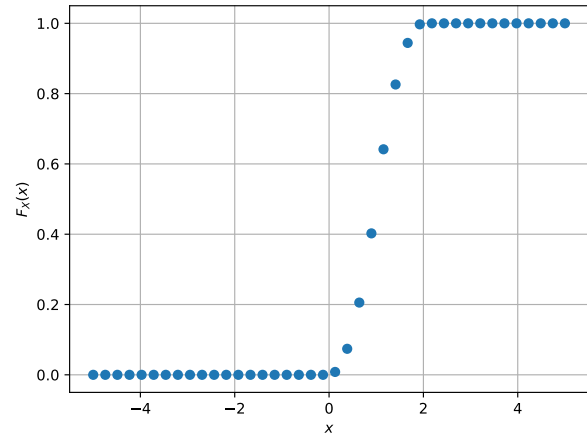
and plot its CDF.

Solution: Download the codes 3.1

```
$ wget https://github.com/karna-rash/A11110-
  -Assignment/blob/master/codes/3.1.c
$ wget https://github.com/karna-rash/A11110-
  -Assignment/blob/master/codes/3.1.py
```

Compile and execute:

```
$gcc -o 3.1 3.1.c
$./3.1
$python 3.1.py
```

Fig. 3.1: The PDF of V Fig. 4.2: The CDF of T

3.2 Find a theoretical expression for $F_V(x)$.

Solution: Let $V = G(U)$ then

$$F_V(x) = F_U(G^{-1}(x)) \quad (3.2)$$

$$= F_U(1 - e^{-\frac{x}{2}}) \quad (3.3)$$

$$(3.4)$$

$$F_V(x) = \begin{cases} G^{-1}(x) & 0 \leq 1 - e^{-\frac{x}{2}} \leq 1 \\ 0 & 1 - e^{-\frac{x}{2}} < 0 \end{cases} \quad (3.5)$$

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & 0 \leq x \leq \infty \\ 0 & x < 0 \end{cases} \quad (3.6)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

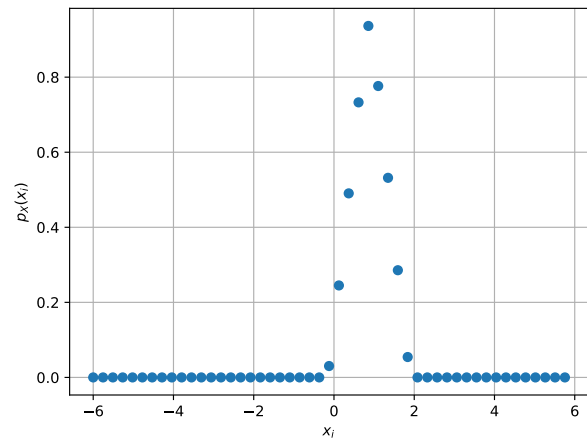
Solution: Download and execute the code

```
$ wget https://github.com/karna-rash/A11110
  -Assignment/blob/master/codes/4.1.c
$ wget https://github.com/karna-rash/A11110
  -Assignment/blob/master/codes/functions.
  h
$ gcc 4.1.c -lm
$ ./a.out
```

4.2 Find the CDF of T .

Solution: Download and execute the code .it gives the following plot Fig.4.2

```
$ wget https://github.com/karna-rash/A11110
  -Assignment/blob/master/codes/4.2.py
$ python 4.2.py
```

Fig. 4.3: The PDF of T

4.3 Find the PDF of T .

Solution: Download and execute the code. It gives the following plot Fig.4.3

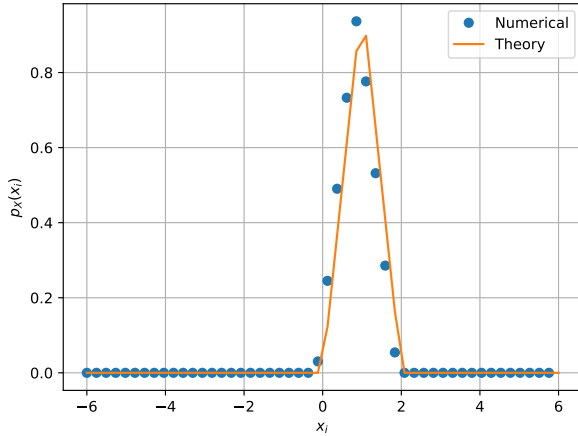
```
$ wget https://github.com/karna-rash/A11110
  -Assignment/blob/master/codes/4.3.py
$ python 4.3.py
```

Solution: Download and execute the code. It gives the following plot Fig.4.2

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: Let us use convolution of $p_{U_1}(x)$ and $p_{U_2}(x)$

$$p_T(t) = \int_{-\infty}^{\infty} p_{U_1}(T - \alpha) p_{U_2}(\alpha) d\alpha \quad (4.2)$$

Fig. 4.5: The PDF verify of T

$$p_T(t) = \begin{cases} \int_0^t 1.d\alpha & 0 \leq t < 1 \\ \int_{t-1}^1 1.d\alpha & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

$$= \begin{cases} t & 0 \leq t < 1 \\ 2 - t & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad (4.4)$$

Now we can get CDF from PDF

$$F_T(t) = \Pr(T \leq t) \quad (4.5)$$

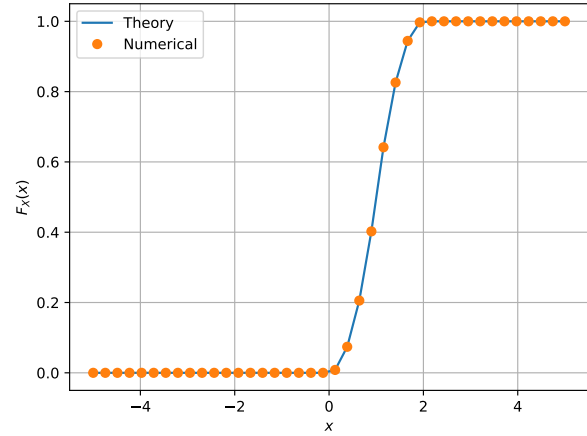
$$= \int_{-\infty}^t p_T(x) dx \quad (4.6)$$

$$F_T(t) = \begin{cases} 0 & t \leq 0 \\ \int_0^t x.d\alpha & 0 \leq t < 1 \\ \int_1^t 2 - x.d\alpha + \frac{1}{2} & 1 \leq t \leq 2 \\ 1 & 2 \leq t \end{cases} \quad (4.7)$$

$$= \begin{cases} 0 & t \leq 0 \\ \frac{x^2}{2} & 0 \leq t < 1 \\ \frac{(2-x)^2}{2} & 1 \leq t \leq 2 \\ 1 & 2 \leq t \end{cases} \quad (4.8)$$

4.5 Verify your results through a plot.

Solution: After Compare the empirical and theoretical plots in 4.5 and 4.5

Fig. 4.5: The CDF Verify of T

where $A = 5$ dB, $X \in (1, -1)$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

Solution: Download and execute the code

```
$ wget https://github.com/karna-rash/A11110
  -Assignment/blob/master/codes/5.1.c
$ wget https://github.com/karna-rash/A11110
  -Assignment/blob/master/codes/functions.
  h
$ gcc 5.1.c -lm
$ ./a.out
```

5.2 Plot Y .

5.3 Guess how to estimate X from Y .

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

5.5 Find P_e .

5.6 Verify by plotting the theoretical P_e .

5 MAXIMUL LIKELIHOOD

5.1 Generate

$$Y = AX + N, \quad (5.1)$$