AI1110 Assignment 8

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Outline

Abstract

This document contains gambler's ruin problem of chapter 3 in papoullis textbook

Problem

Two players A and B plays game consecutively till one of them loses all his capital. Suppose A starts with a capital of \$a and B with a capital of \$b and the loser pays \$1 to the winner in each game. Let p represent the probability of winning each game for A and $q = 1 - p$ for player B. Find the probability of ruin for each player if no limit is set for the number of games Let N_a denote the average duration of the game for player A starting with capital a. Show that

$$N_a = \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1}\frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}} & , p \neq q \\ ab & , p = q = \frac{1}{2} \end{cases} \tag{1}$$

Solution

Let P_n denote the probability of the event X_n is "A's ultimate ruin when his wealth is \$n $(0 \leq n \leq a+b)$". A will be ruined in two ways , increase his wealth by \$1 the lose ruin later or reduce his wealth by \$1 ruin later,This can be recursive written as: Let H be event:"A succeds in the next game" Let D represent the duration of the game.

$$X_n = X_n.\left(H+\bar{H}\right) = X_nH + X_n\bar{H} \tag{2}$$

$$P_n = \Pr\left(X_n\right) = \Pr\left(X_n|H\right)\Pr\left(H\right) + \Pr\left(X_n|\bar{H}\right)\Pr\left(\bar{H}\right) \tag{3}$$

$$= pP_{n+1} + qP_{n-1} \tag{4}$$

$$P_0 = 1, P_{a+b} = 0 \tag{5}$$

Let us solve the recursion:

$$(P_{n+1} - P_n) = \frac{q}{p}(P_n - P_{n-1}) \tag{6}$$

$$= \left(\frac{q}{p}\right)^n (P_1 - 1) \tag{7}$$

try to use the initial conditions:

$$P_{a+b} - P_n = \sum_{k=n}^{a+b-1} P_{k+1} - P_k = \sum_{k=n}^{a+b-1} \left(\frac{q}{p}\right)^k (P_1 - 1) \tag{8}$$

$$= (P_1 - 1)\frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}}{1 - \frac{q}{p}} \tag{9}$$

Since $P_{a+b} = 0$ and $P_0 = 1$

$$P_n = (1 - P_1) \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}}{1 - \frac{q}{p}} \tag{10}$$

$$P_0 = 1 = (1 - P_1) \frac{\left(\frac{q}{p}\right)^0 - \left(\frac{q}{p}\right)^{a+b}}{1 - \frac{q}{p}} \tag{11}$$

Divide the last equations:

$$P_n = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}}{1 - \left(\frac{q}{p}\right)^{a+b}} \tag{12}$$

if $p = q = \frac{1}{2}$,then $P_n, P_{n-1}...$ are in A.P and equal

$$P_n = \frac{b}{a+b} \tag{13}$$

Now let us calculate the duration of the game: As we know its a recursion, with initial conditions $N_0 = N_{a+b} = 0$,

$$
\begin{aligned}
N_k &= \Pr(H)\, E(D|H) + \Pr\left(\bar{H}\right) E\left(D|\bar{H}\right) && (14)\\
&= p\,(1 + N_{k+1}) + q\,(1 + N_{k-1}) && (15)\\
&= 1 + pN_{k+1} + qN_{k-1} && (16)
\end{aligned}
$$

This is inhomogeneous linear difference equation. General solution:

$$N_k = Ck + A + B\left(\frac{q}{p}\right)^k \tag{17}$$

On using $N_0 = N_{a+b} = 0$:

$$C = \frac{1}{1-2p} \tag{18}$$

$$B = \frac{a+b}{(1-2p)\left(1-\left(\frac{q}{p}\right)^{a+b}\right)} \tag{19}$$

$$A = \frac{-B}{\left(1-\left(\frac{q}{p}\right)^{a+b}\right)} \tag{20}$$

On solving equation:

$$N_a = \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1}\frac{1-\left(\frac{p}{q}\right)^{b}}{1-\left(\frac{p}{q}\right)^{a+b}} & , p \neq q \\ ab & , p = q = \frac{1}{2} \end{cases} \tag{21}$$