AI1110 Assignment 9

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Abstract

This document contains problem of chapter 6 in papoullis textbook

Problem

Chapter 6-6.56

x and yare zero mean independent random variables with variances σ_1^2 and σ_2^2 respectively,that is , x $\sim N\left(0, \sigma_1^2\right), y \sim N\left(0, \sigma_2^2\right)$ Let

$$z = ax + by + c \qquad c \neq 0 \tag{1}$$

1. Find the characterstic equation $\phi_z\left(u\right)$ of z.
2. Using $\phi_z\left(u\right)$ conclude that z is also a normal random variable.
3. Find the mean and variance of z.

Theory

Gaussian Normal Distribution

A normal distribution (also known as Gaussian, Gauss, or Laplace–Gauss distribution) is a type of continuous probability distribution for a real-valued random variable. The general form of its probability density function is:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \tag{2}$$

Charteristic Function

For a scalar random variable X the characteristic function is defined as the expected value of e^{itX},where i is the imaginary unit, and t $\in$ R is the argument of the characteristic function

Theory

$$\begin{aligned}
\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{j\omega x} \mathrm{d}x \\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{j\omega x} \mathrm{d}x + \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{j\omega x} \mathrm{d}x \\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{j\omega x} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{-j\omega x} \mathrm{d}x \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \cos(\omega x) \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}'(\omega) &= -\frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{e}^{-\frac{x^2}{2}} x \sin(\omega x) \mathrm{d}x = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sin(\omega x) \mathrm{d}\left(\mathrm{e}^{-\frac{x^2}{2}}\right) \\
&= \frac{2}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \sin(\omega x)\Big|_{0}^{\infty} - \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{e}^{-\frac{x^2}{2}} \omega \cos(\omega x) \mathrm{d}x \\
&= -\omega \mathcal{F}(\omega)
\end{aligned}$$

Final Result

The solution to so obtained ODE, $\mathcal{F}'(\omega) = -\mathcal{F}(\omega)$ is $\mathcal{F}(\omega) = ce^{-\frac{\omega^2}{2}}$ for normal distribution $\mathcal{N}\left(\mu, \sigma^2\right)$

$$\phi_x(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2} \tag{3}$$

Solution

1. The characterstic equation is given by:

$$
\begin{aligned}
\phi_Z(u) &= E\left(e^{juZ}\right) && (4)\\
&= E\left(e^{ju(aX+bY+c)}\right) && (5)\\
&= E\left(e^{ju(aX)}\right) E\left(e^{ju(AY)}\right) e^{juc} && (6)\\
&= \phi_X(au)\,\phi_Y(bu)\,e^{juc} && (7)\\
&= e^{juc-\frac{\left((a^2\sigma_1^2+b^2\sigma_2^2)u^2\right)}{2}} && (8)
\end{aligned}
$$

2. On comparing with general characterstic equation for normal density We can say:

$$Z \sim \mathcal{N}\left(c, a^2\sigma_1^2 + b^2\sigma_2^2\right) \tag{9}$$

3. for normal distribution $\mathcal{N}\left(\mu, \sigma^2\right)$

$$E\left(X\right) = \mu, Variance = \sigma^2 \tag{10}$$

Here for distribution of Z:

$$E\left(X\right) = c, Variance = a^2\sigma_1^2 + b^2\sigma_2^2 \tag{11}$$