Robust Trajectory Tracking Control of a Multi-Rotor UAV Carrying a Cable Suspended Load - Appendix

N S Abhinay[1], Kaushik Das[2], and Debasish Ghose[3]

APPENDIX I
DYNAMICS DERIVATION

The dynamics of the aerial slung load system are derived using the Lagrangian formulation. The first step in deriving the dynamics is to form the Lagrangian,

$$\mathcal{L} = \mathcal{KE} - \mathcal{PE} \tag{1}$$

wherein $\mathcal{KE}$ and $\mathcal{PE}$ denote the total kinetic and potential energy of the system. Only the translational dynamics of the UAV are considered here and thereby only its translational kinetic energy is included in the Lagrangian.

$$\mathcal{KE} = (1/2)m_q\mathbf{v_q}^T\mathbf{v_q} + (1/2)m_p\mathbf{v_p}^T\mathbf{v_p} \tag{2}$$

In (2), $\mathbf{v_q}$ and $\mathbf{v_p}$ represent the velocities of the UAV and the payload, in the inertial frame. The velocity of the payload is obtained by differentiating (**??**). The expressions for load position and linear velocity are given below in (3).

$$\mathbf{x_p} = \begin{bmatrix} x - l\sin(\beta) \\ y + l\sin(\alpha)\cos(\beta) \\ z - l\cos(\alpha)\cos(\beta) \end{bmatrix} \tag{3a}$$

$$\mathbf{v_p} = \begin{bmatrix} \dot{x} - l\cos(\beta)\dot{\beta} \\ \dot{y} + l\cos(\alpha)\cos(\beta)\dot{\beta} - l\sin(\alpha)\sin(\beta)\dot{\beta} \\ \dot{z} + l\sin(\alpha)\cos(\beta)\dot{\alpha} + l\cos(\alpha)\sin(\beta)\dot{\beta} \end{bmatrix}^T \tag{3b}$$

The potential energy of the system can be written as,

$$\mathcal{PE} = m_q g z + m_p g(z - l\cos(\alpha)\cos(\beta)) \tag{4}$$

The dynamics are then derived using the Euler-Lagrange equation which is given as,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\right) - \frac{\partial\mathcal{L}}{\partial\mathbf{q}} = \mathcal{Q} \tag{5}$$

in which $\mathbf{q} = [x, y, z, \alpha, \beta]$ represents the generalized coordinates and $\mathcal{Q}$ represents the generalized forces acting on the system, consisting of the control forces $\boldsymbol{\tau}$ and the external disturbances. The dynamics of the system can be written as

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q}) = \boldsymbol{\tau} + \boldsymbol{\xi}(t) \tag{6}$$

where $\boldsymbol{\tau} = [f_x, f_y, f_z, 0, 0]^T \in \mathbb{R}^5$, f_i, $i \in \{x, y, z\}$, is the force applied by the UAV in the inertial frame, and $\boldsymbol{\xi}(t)$ is the external disturbance like wind. The inertia matrix $M(\mathbf{q})$, coriolis and centrifugal forces $C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$, and forces due to gravity $G(\mathbf{q})$ are,

$$M(\mathbf{q}) = \begin{bmatrix} m_{qp} & 0 & 0 & 0 & -m_p l c\beta \\ \star & m_{qp} & 0 & m_p l c\alpha c\beta & -m_p l s\alpha s\beta \\ \star & \star & m_{qp} & m_p l s\alpha c\beta & m_p l c\alpha s\beta \\ \star & \star & \star & m_p l^2 c\beta^2 & 0 \\ \star & \star & \star & \star & m_p l^2 \end{bmatrix} \tag{7a}$$

$$C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} = \begin{bmatrix} m_p l s\beta\dot{\beta}^2 \\ -m_p l s\alpha c\beta\dot{\alpha}^2 - m_p l s\alpha c\beta\dot{\beta}^2 - 2m_p l c\alpha s\beta\dot{\alpha}\dot{\beta} \\ m_p l c\alpha c\beta\dot{\alpha}^2 + m_p l c\alpha c\beta\dot{\beta}^2 - 2m_p l s\alpha s\beta\dot{\alpha}\dot{\beta} \\ -2m_p l^2 s\beta c\beta\dot{\alpha}\dot{\beta} \\ -m_p l^2 s\beta c\beta\dot{\alpha}^2 \end{bmatrix} \tag{7b}$$

$$G(\mathbf{q}) = \begin{bmatrix} 0 & 0 & m_{qp}g & m_p l g s\alpha c\beta & m_p l g c\alpha s\beta \end{bmatrix}^T \tag{7c}$$

In (7), $ca = \cos(a)$, $sa = \sin(a)$, and m_{ij} is the entry of the inertia matrix in i^{th} row and j^{th} column. $m_{ij} = \star \Rightarrow m_{ij} = m_{ji}$, since the inertia matrix is symmetric and positive definite and $m_{qp} \triangleq m_q + m_p$.

APPENDIX II
PROOF OF THEOREM 1

Proof: We design s_1 and s_2 to be linear sliding surfaces as in [1].

$$s_1 = n_{1x}\dot{e}_x + n_{2x}e_x;\;\; s_2 = n_{1\beta}l\dot{e}_\beta + n_{2\beta}le_\beta \tag{8a}$$

where n_{ix} are the sliding surface parameters, $sgn(*)$ defines the signum function and l is the length of the cable used to attach the load to the aerial vehicle. e_x, $\dot{e}_x$ denote the position and velocity errors of the aerial vehicle along the x-direction and e_β, $\dot{e}_\beta$ denote the angular position and velocity errors of the swing angle β which are defined as

$$e_x = x - x_{des},\; \dot{e}_x = \dot{x} - \dot{x}_{des},\; e_\beta = \beta,\; \dot{e}_\beta = \dot{\beta} \tag{9}$$

Without loss of generality (by absorbing the constants k_1, k_2 into the constants n_{ix}), the overall sliding surface is

$$S = s_1 + s_2 = n_{1x}\dot{e}_x + n_{2x}e_x + n_{1\beta}l\dot{e}_\beta + n_{2\beta}le_\beta \tag{10}$$

This way of combining the actuated and unactuated state variables in the sliding surface design is called hierarchical sliding mode technique in the literature [2]. The control law is synthesized, using the Lyapunov method, in such a way that the stability of the system is guaranteed.

[1]N S Abhinay is with the Robotics and Autonomous Systems group of TCS Research, TCS India, and is also a PhD student at Department of Aerospace Engineering, Indian Institute of Science, India ns.abhinay@tcs.com

[2]Kaushik Das is with the Robotics and Autonomous Systems group of TCS Research, TCS India. kaushik.da@tcs.com

[3]Debasish Ghose is with the RBCCPS & Department of Aerospace Engineering, Indian Institute of Science, India dghose@iisc.ac.in

Let the Lyapunov candidate function and its derivative be

$$V = (1/2)S^2; \quad \dot{V} = S\dot{S} \tag{11}$$

$$\dot{S} = \dot{s}_1 + \dot{s}_2 = n_{1x}\ddot{e}_x + n_{2x}\dot{e}_x + n_{1\beta}l\ddot{e}_\beta + n_{2\beta}l\dot{e}_\beta \tag{12}$$

The definitions of e_x, $\dot{e}_x$, e_β, $\dot{e}_\beta$ are given in (9); using (**??**) $\ddot{e}_x$ and $\ddot{e}_\beta$ can be written as

$$\ddot{e}_x = \ddot{x} - \ddot{x}_{des},\ \ddot{e}_\beta = \ddot{\beta} \tag{13a}$$

$$\ddot{x} = f_1 + g_1 f_x + d_1,\ \ddot{\beta} = f_2 + g_2 f_x + d_2 \tag{13b}$$

$$f_1 \triangleq -(m_p/m_q)g\beta,\ g_1 \triangleq (1/m_q), \tag{13c}$$

$$f_2 \triangleq -[(m_p + m_q)/(m_q l)]g\beta,\ g_2 \triangleq 1/(m_q l) \tag{13d}$$

In (13), d_1 and d_2 represent the disturbances. As stated in **??**, the effect of wind is modeled through drag acting on the bodies. Hence d_1 and d_2 are components of the vector $-M^{-1}\mathbf{d}$. Using (13) in (12), and defining

$$\dot{s}_r \triangleq n_{2x}\dot{e}_x + n_{2\beta}l\dot{e}_\beta \tag{14}$$

the following expression for $\dot{S}$ can be obtained,

$$\begin{aligned}\dot{S} = (n_{1x}g_1 + n_{1\beta}lg_2)f_x + (n_{1x}f_1 + n_{1\beta}lf_2) + \\ \dot{s}_r - n_{1x}\ddot{x}_{des} + (n_{1x}d_1 + n_{1\beta}ld_2)\end{aligned} \tag{15}$$

If the control input f_x is chosen as

$$f_x = u_{eq} + u_{smc} \tag{16a}$$

$$u_{eq} = -(n_{1x}g_1 + n_{1\beta}lg_2)^{-1}(n_{1x}f_1 + n_{1\beta}lf_2 + \dot{s}_r - n_{1x}\ddot{x}_{des}) \tag{16b}$$

then $\dot{S}$ becomes

$$\dot{S} = u_{smc} + (n_{1x}d_1 + n_{1\beta}ld_2) \tag{17}$$

Many choices are available for u_{smc}, but here super-twisting sliding mode control is chosen because it is a second order sliding mode control algorithm that helps in the reduction of chattering since the discontinuous $sgn(S)$ appears in the control law through an integral term[3] as shown in the expression, for the super-twisting control law, given below

$$u_{sta} = -\gamma_1|S|^{0.5}sgn(S) + w; \quad \dot{w} = -\gamma_2 sgn(S) \tag{18}$$

If in (18) γ_1 and γ_2 are chosen depending on the bounds of the disturbances d_1 and d_2 in (17), then it can be shown that $S, \dot{S} \to 0$ in finite time [4].

Appendix III
Proof of Theorem 2

Proof: While in sliding mode, the UAV position error dynamics along x-axis can be written as shown in (19).

$$S = 0 \Rightarrow \dot{e}_x = -\frac{n_{2x}}{n_{1x}}e_x - \frac{n_{1\beta}l}{n_{1x}}\dot{e}_\beta - \frac{n_{2\beta}l}{n_{1x}}e_\beta \tag{19}$$

Substituting f_x from (**??**) in (20),

$$\ddot{e}_\beta = \ddot{\beta} = f_2 + g_2 f_x + d_2 \tag{20}$$

β error dynamics can be written as

$$\ddot{e}_\beta = \frac{1}{l\,(n_{1x} + n_{1\beta})}(-n_{1x}g\beta + \dot{s}_r - n_{1x}\ddot{x}_{des} + z_1 + d_2) \tag{21}$$

Now consider the error dynamics, excluding the effect of the disturbance and the desired acceleration of the aerial vehicle, that is, the β error dynamics given in (22) and the position error dynamics given in (19).

$$\ddot{e}_\beta = [1/(l\,(n_{1x} + n_{1\beta}))](-n_{1x}g\beta + \dot{s}_r) \tag{22}$$

Let $\Gamma \triangleq [e_x, e_\beta, \dot{e}_\beta]^T$. It is possible to write the error dynamics in (19) and (22) in the form

$$\dot{\Gamma} = A\Gamma \tag{23a}$$

$$A = \begin{bmatrix} k1 & k2 & k3 \\ 0 & 0 & 1 \\ k4 & k5 & k6 \end{bmatrix} \tag{23b}$$

$$k1 = -\frac{n_{2x}}{n_{1x}},\ k2 = -\frac{n_{1\beta}l}{n_{2x}},\ k3 = -\frac{n_{2\beta}}{n_{1x}} \tag{23c}$$

$$k4 = \frac{n_{2x}^2}{ln_{1x}(n_{1x} + n_{1\beta})},\ k5 = \frac{ln_{2x}n_{2\beta} - n_{1x}^2 g}{ln_{1x}(n_{1x} + n_{1\beta})}, \tag{23d}$$

$$k6 = (n_{2x}n_{1\beta} - n_{2\beta}n_{1x})/(n_{1x}(n_{1x} + n_{1\beta})) \tag{23e}$$

and if the parameters n_{ix} and $n_{i\beta}$ are chosen such that A is Hurwitz, then the error dynamics are guaranteed to converge to zero. In order for A to be Hurwitz, all of its eigenvalues must have negative real parts or equivalently the characteristic equation of $A - \lambda I$ should satisfy the Routh-Hurwitz criterion. Applying the RH criterion we obtained the conditions for stability as

$$\begin{aligned}&\frac{n_{2x} + n_{2\beta}}{n_{1x} + n_{1\beta}} > 0,\ \frac{n_{2x}}{n_{1x} + n_{1\beta}} > 0,\ \frac{n_{1x}}{n_{1x} + n_{1\beta}} > 0, \\ &\frac{n_{1x}}{n_{1x} + n_{1\beta}} > \frac{n_{2x}}{n_{1x} + n_{1\beta}}\end{aligned} \tag{24}$$

Appendix IV
Altitude Control Law

From (**??**), motion of the UAV along z-axis is independent of the motion of the payload. The altitude control input f_z is computed using a conventional first-order sliding surface and super-twisting control law. Hence, for altitude control, we use the conventional first-order sliding surface.

$$s_z = n_{1z}\dot{e}_z + n_{2z}e_z \tag{25}$$

The force along the z-axis of the inertial frame can be computed as [5]

$$f_z = -(m_q + m_p)(-g - n_{1z}\ddot{z}_{des} + n_{2z}\dot{e}_z + f_{z_{smc}}) \tag{26}$$

$$f_{z_{smc}} = \gamma_1|s_z|^{0.5}sign(s_z) - w_z;\ \dot{w}_z = \gamma_2 sign(s_z) \tag{27}$$

There is no disturbance estimator term in the altitude control law because we consider the disturbance to affect only the $x - y$ motion of the UAV.

Appendix V
Sliding Mode Control Law Using Non-Linear Dynamics

The sliding surface vector for the nonlinear system is formulated as the combination of the actuated and the unactuated position and velocity errors

$$\mathbf{s} = N_{1a}\dot{\tilde{\mathbf{q}}}_{\mathbf{a}} + N_{2a}\tilde{\mathbf{q}}_{\mathbf{a}} + N_{1u}l\dot{\tilde{\mathbf{q}}}_{\mathbf{u}} + N_{2u}l\tilde{\mathbf{q}}_{\mathbf{u}} \tag{28a}$$

$$N_{1a} = \begin{bmatrix} n_{1x} & 0 \\ 0 & n_{1y} \end{bmatrix}, \; N_{2a} = \begin{bmatrix} n_{2x} & 0 \\ 0 & n_{2y} \end{bmatrix} \tag{28b}$$

$$N_{1u} = \begin{bmatrix} n_{1\beta} & 0 \\ 0 & n_{1\alpha} \end{bmatrix}, \; N_{2u} = \begin{bmatrix} n_{2\beta} & 0 \\ 0 & n_{2\alpha} \end{bmatrix} \tag{28c}$$

Following a similar mathematical discourse as done for linear dynamics case, the control law can be derived to be

$$\mathbf{u} = -M_s^{-1}[\mathbf{f_s} + \dot{\mathbf{s}}_{\mathbf{r}} + \mathbf{u_{smc}}] \tag{29}$$

$$M_s = N_{1a}{M'}_{aa}^{-1} - N_{1u}l{M'}_{uu}^{-1}M_{au}^T M_{aa}^{-1} \tag{30}$$

$$\mathbf{f_s} = N_{1a}{M'}_{aa}^{-1}\mathbf{f}'_{\mathbf{a}} + N_{1u}l{M'}_{uu}^{-1}\mathbf{f}'_{\mathbf{u}} \tag{31}$$

Closed loop error dynamics in sliding mode are given as

$$\mathbf{s}{=}0 \rightarrow \dot{\tilde{\mathbf{q}}}_{\mathbf{a}}{=} - N_{1a}^{-1}N_{2a}\tilde{\mathbf{q}}_{\mathbf{a}} - N_{1a}^{-1}N_{1u}l\dot{\tilde{\mathbf{q}}}_{\mathbf{u}} - N_{1a}^{-1}N_{2u}l\tilde{\mathbf{q}}_{\mathbf{u}} \tag{32}$$

$$\ddot{\mathbf{q}}_{\mathbf{u}} = {M'}_{uu}^{-1}(\mathbf{f}'_{\mathbf{u}} + M_{au}^T M_{aa}^{-1} M_s^{-1}(\mathbf{f_s} + \dot{\mathbf{s}}_{\mathbf{r}})) \tag{33}$$

Linearized nominal closed-loop error dynamics in sliding mode can be written in matrix form as

$$\dot{\mathbf{\Gamma}} = A\mathbf{\Gamma}, \; \mathbf{\Gamma} = \begin{bmatrix} y_1, & y_2, & y_3, & y_4, & y_5, & y_6 \end{bmatrix}^T, \tag{34}$$

$$y_1{=}e_x, \; y_2{=}e_y, \; y_3{=}e_\beta, \; y_4{=}e_\alpha, \; y_5{=}\dot{e}_\beta, \; y_6{=}\dot{e}_\alpha \tag{35}$$

Assuming A to be Hurwitz for stability, conditions on sliding surface parameters are the same as for linear dynamics.

References

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