Math Course Notes

Your Name

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Real Analysis

Limits and Continuity

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous* at a point $c \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$, it follows that $|f(x) - f(c)| < \epsilon$.

Differentiation

Theorem

If f is differentiable at $c \in \mathbb{R}$, then f is continuous at c.

Proof

Let $\epsilon > 0$ be given. Since f is differentiable at c, there exists $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, we have

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \frac{\epsilon}{|x - c|}$$

Thus,

$$|f(x) - f(c)| < \epsilon$$

which shows that f is continuous at c.

Examples

Example 1

Consider the function $f(x) = x^2$. This function is continuous at every point $c \in \mathbb{R}$. To see this, note that

$$|f(x) - f(c)| = |x^2 - c^2| = |x - c||x + c|$$

Since $|x+c|$ is bounded near c, $|f(x)-f(c)|$ can be made arbitrarily small by choosing x sufficiently close to c.