Term Paper 02: Kuramoto Model

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Contents

1 Introduction

The phenomenon of synchronization pervades everyday experience. The examples include the synchronization of fireflies to circadian rhythms observed in animals. The findings suggest that the fireflies can be viewed as a system of coupled oscillators. The Kuramoto model is able to describe a huge variety of examples of synchronization in the real world. We re-consider it through the framework of the network science and study the phenomenon of a particular interest, agent clustering. We propose a framework, which is able to describe clusterization in systems of intelligent agents given their outputs.

1.1 Kuramoto's Model

The Kuramoto Oscillator system is a system of N coupled periodic oscillators.[4] Each oscillator has its own natural frequency ω_i, i.e., constant angular velocity. Usually, the distribution of natural frequencies is choosen to be a gaussian-like symmetric function. A random initial (angular) position θ_i is assigned to each oscillator. The oscillator's state (position) θ_i is governed by the following differential equation:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{M_i} \sum_j A_{ij} sin\left(\theta_j - \theta_i\right) \tag{1}$$

where K is the coupling parameter and M_i is the number of oscillators interacting with oscillator i. A is the adjacency matrix enconding the interactions - typically binary and undirected (symmetric), such that if node i interacts with node j, $A_{ij} = 1$, otherwise 0. The basic idea is that, given two oscillators, the one running ahead is encouraged to slow down while the one running behind to accelerate.

Introducing a pair of new parameters to aid visualisation of the above equation and also aimed to obtain steady state analytical solution from it, we get:

$$re^{i\psi} = \frac{1}{N} \sum e^{i\theta_j}$$

$$= re^{i(\psi - \theta_i)} = \frac{1}{N} \sum e^{i(\theta_j - \theta_i)}$$

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{M_i} \sum_j A_{ij} sin\left(\theta_j - \theta_i\right)$$

In particular, the classical set up has $M = N$, since the interactions are all-to-all (i.e., a complete graph). Otherwise, M_i is the degree of node i.

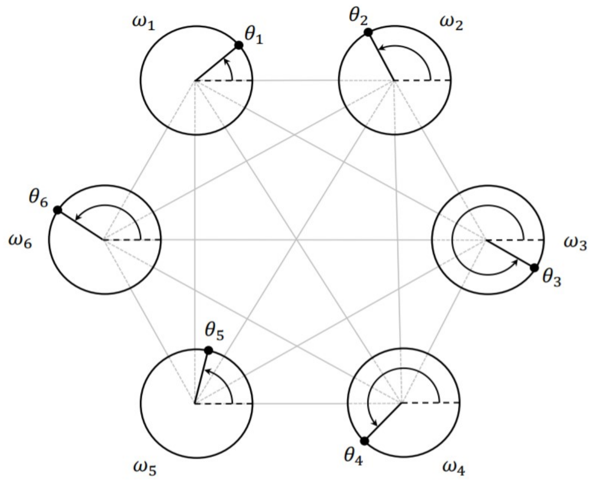

Figure 2: Interpretation of the mean-field Kuramoto model for $N = 6$. The complete graph visualizes the all-to-all coupling.

1.2 Kuramoto's Analysis

The oscillators in a partially synchronized state are split into two groups: one synchronized cluster of phase-locked oscillators rotating together with the average phase $\psi(t)$ at the mean frequency Ω, and one group of oscillators drifting relative to this cluster. Kuramoto's detailed analysis of the system yielded a self-consistency equation for the order parameter. [2] And with a symmetric $g(\omega)$ he was able to reduce the condition to

$$r = rK \int_{-\pi/2}^{\pi/2} \cos^2 \theta g(Kr \sin \theta) d\theta$$

A trivial solution of the above equation, *i.e.* $r = 0$ always exists for any value of K. This corresponds to an incoherent state for all θ and ω.

A second branch of solution for $r > 0$ corresponding to partially synchronised states is

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2 \theta g(Kr \sin \theta) d\theta \tag{2}$$

Now if we let $r \longrightarrow 0$ in the above equation and integrate we get Kuramoto's formula of the critical coupling:

$$K_c = \frac{2}{\pi g(0)} \tag{3}$$

Furthermore, with an expansion of the integrand in equation (2) around, we can deduce an approximate scaling law

$$r \sim \sqrt{\frac{16}{\pi K_c^4[-g''(0)]}}(K - K_c)^{1/2} \tag{4}$$

as $K \longrightarrow K_c$. For g unimodal and sufficiently smooth about $\omega = 0$, *i.e.* $g"(0) < 0$, the partially synchronised state bifurcates supercritically from the incoherent state for $K > K_c$, indicating a second-order phase transition.

1.3 Model Behaviour

A couple of facts in order to gain intuition about the model's behaviour:

- If synchronization occurs, it happens abruptly.
- Partial synchronization is a possible outcome.
- The order parameter r_t measures global synchronization at time t. It is basically the normalized length of the sum of all vectors (oscillators in the complex plane).
- About the global order parameter r_t:
  (i) constant, in the double limit $N \to \infty$, $t \to \infty$
  (ii) independent of the initial conditions
  (iii) depends on coupling strength
  (iv) it shows a sharp phase transition (as function of coupling)
- Steady solutions can be computed assuming r_t constant. The result is basically that each oscillator responds to the mean field produced by the rest.
- In the all-to-all connected scenaria, the critical coupling K_c can be analytically computed and it depends on the spread of the natural frequencies distribution [1].
- The higher the dimension of the lattice on which the oscillators are embedded, the easier it is to synchronize. For example, there isn't any good synchronization in one dimension, even with strong coupling. In two dimensions it is not clear yet. From 3 dimensions on, the model starts behaving more like the mean field prediction.

2 Visualization

I instantanised the model with parameters with 100 oscillators and a coupling value of 3. The simulation is done and the output is the time series for all nodes.

2.1 Synchronization

The oscillators are said to synchronize if

$$\theta_i = \theta_j \to 0 \text{ as } t \to \infty \; \forall \; i, j = 1, 2, 3, ...$$

or in other words the phase differences given by $\theta_i - \theta_j \forall \; i, j = 1, 2, 3, ...$ become constant asymptotically. The order parameter $r(t)$ with $0 \leq r(t) \leq 1$ is a measure of phase coherence of the oscillator population. If the oscillators synchronize, then the parameter converges to a constant $r_\infty \leq 1$, but if the oscillators add incoherently then the order parameter r remains close to zero.

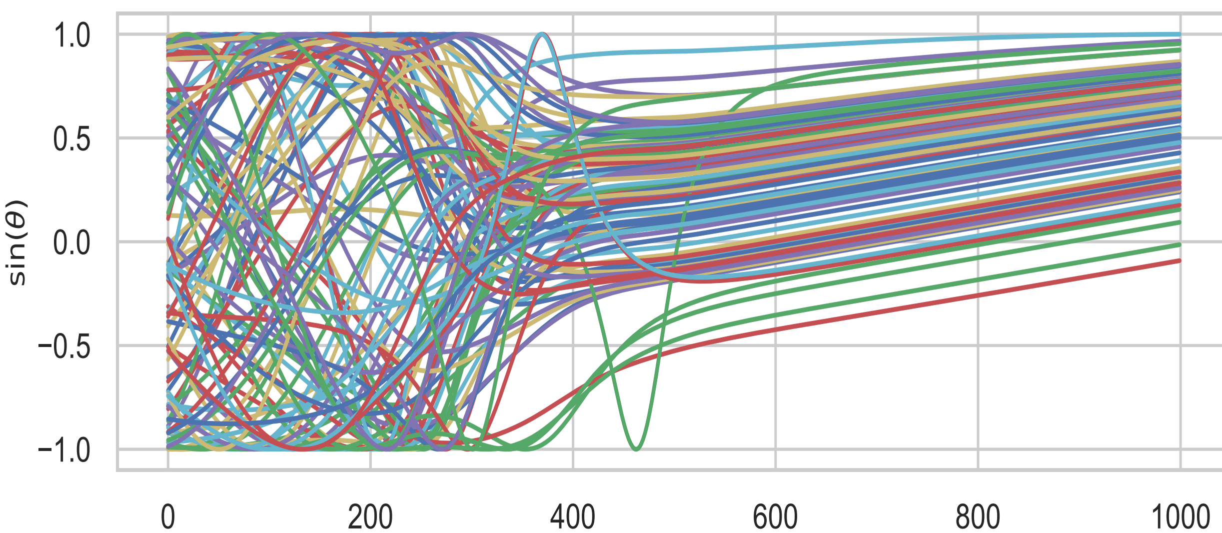

Figure 3: Plot of all the time series of the 100 oscillators with coupling parameter ($k = 3$)

To get a nice intuition about the problem, the oscillators may also be thought of as points moving on a unit circle. Imagining these oscillators on circle as points, the points then move with the same angular frequency and hence, angular distance (phase difference) between the points remain constant with time.

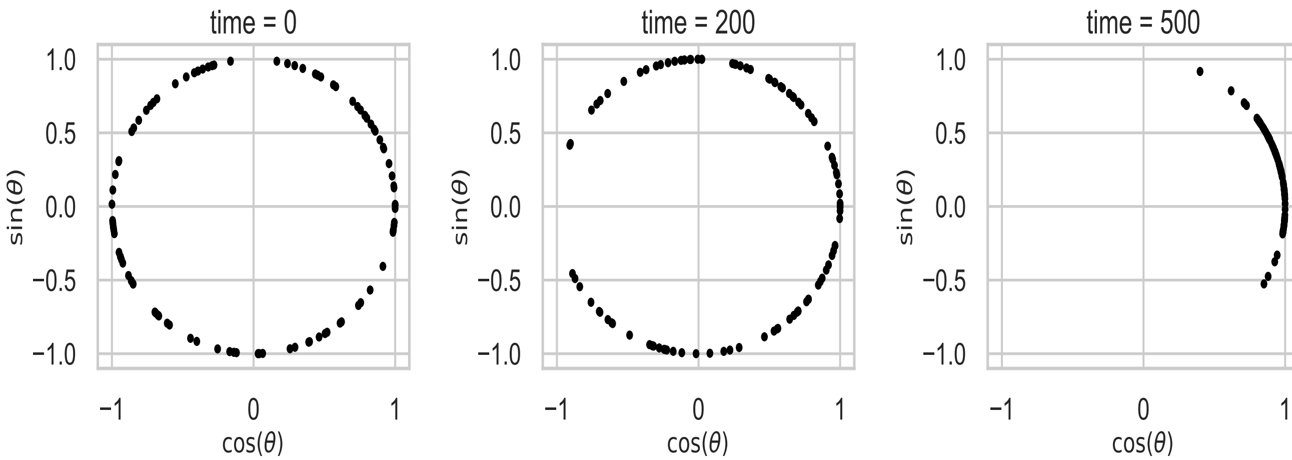

Figure 4: A visualisation of the Kuramoto model that shows phase synchronisation over time. The phases of the oscillators are shown as points on the unit circle.

2.2 Checking Natural Frequencies

The model was run with different coupling (K) parameters and the natural frequencies taken from the random Gaussian distribution was checked.

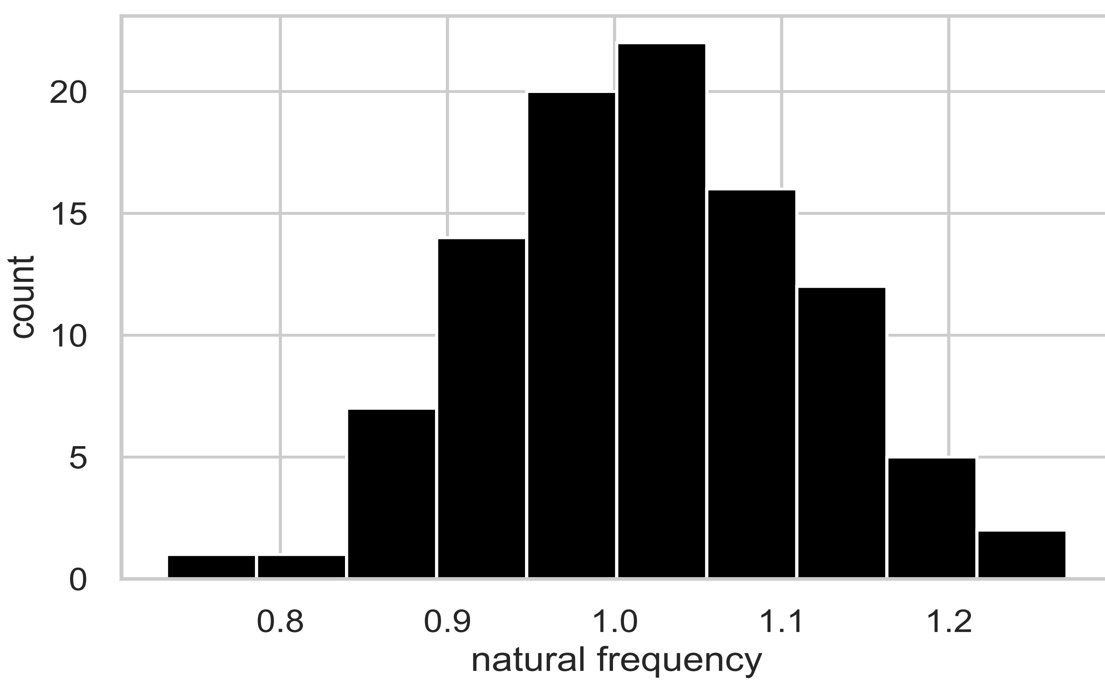

Figure 5: Gaussian Natural Frequencies for 100 oscillators

3 Phenomenology and Plots

The Kuramoto model displays the phenomenology of synchronization. This is first investigated for the mean-field Kuramoto model where several choices of the natural frequency distribution are considered. In particular, we follow Kuramoto's original analysis in identifying the critical coupling for the case that the natural frequency distribution is symmetric and unimodal.

3.1 Time Evolution of r

Let us now look in more detail at how this synchronization process is revealed by the value of the phase coherence r. For this, we follow Kuramoto's assumptions, namely, we suppose that g is symmetric and unimodal around the zero frequency. The latter property means that $\omega \to \mathrm{g}(\omega)$ is strictly increasing on $(-\infty, 0]$ and strictly decreasing on $[0, \infty)$. In this case, simulations carried out for eqn 1 show that $r(t)$ has a typical evolution, exemplified by the simulations in the figure generated below using Python.

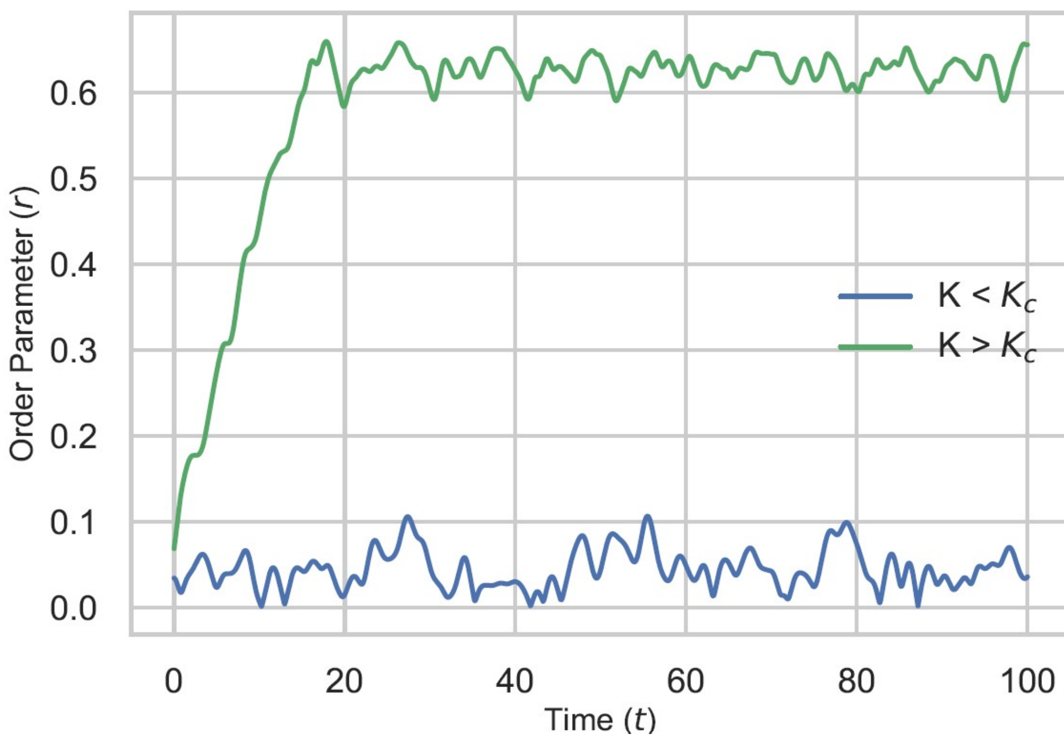

Figure 6: Typical evolution of $r(t)$ for the model eq 1 with $K < K_C$ and $K > K_C$. These simulation results are obtained with $N = 100$ oscillators and g being the standard Gaussian density (in this case $K_C \approx 1.6$) and blue and green correspond to K=1 and K=2 respectively.

3.2 Time Evolution with varying Coupling Values

We plot the time evolution of the order parameter r (dynamics of $r(t)$) and colour coded with the range of coupling values and the results are plotted below. The darker shades represent the coupling close to 1.0 and vice versa for the lighter shades.

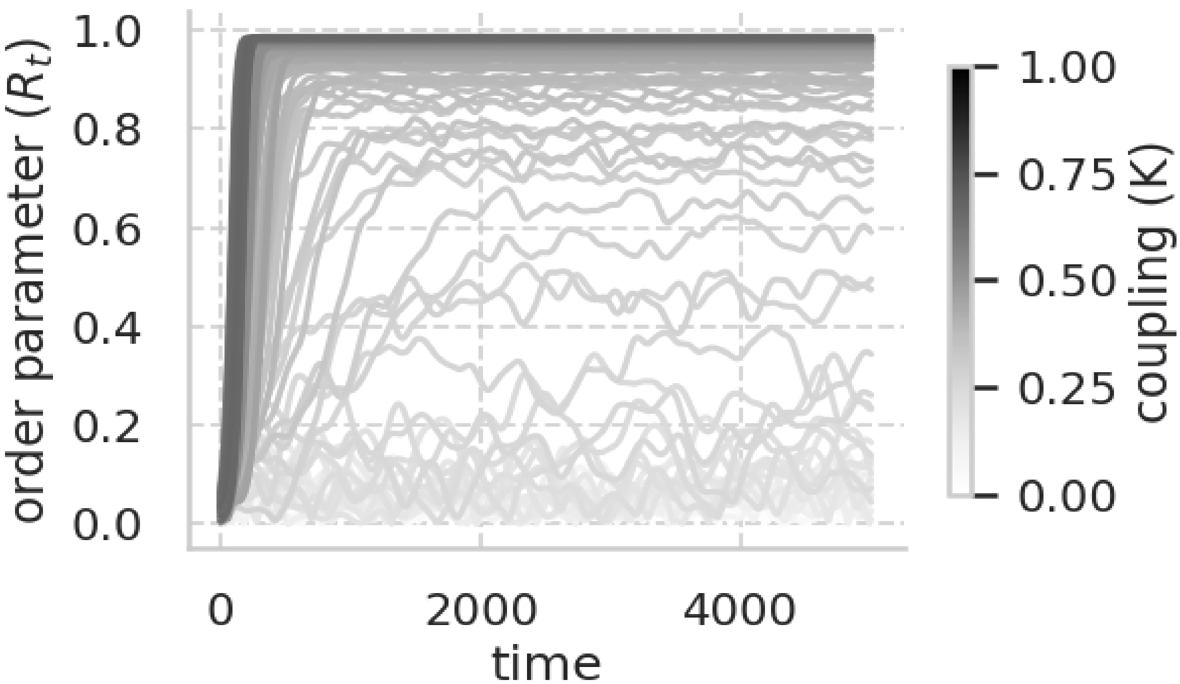

Figure 7: Plot of all time series for all coupling values (the color coded shows the strength of coupling value)

3.3 Continuum Limit

In the continuum limit case where $N \to \infty$, Kuramoto showed that there exists a value of the coupling gain K such that for all $K < K_C$, the oscillators are incoherent (or remain unsynchronized), but for $K > K_C$ the incoherent state becomes unstable, the oscillators start synchronizing and eventually $r(t)$ settles at some $r_\infty(K) < 1$. Kuramoto calculated closed form solutions for the gain K_C (the critical gain for the onset of synchronization), and $r_\infty(K)$. Furthermore it has been shown via simulations that for $K > K_C$, the population of oscillators divides into two groups. The oscillators whose natural frequencies is close to the mean frequency, lock on to form a synchronized cluster and start rotating with the mean frequency Ω, while those whose natural frequencies are far way from the mean of the group, drift relative to the synchronized cluster oscillators. [3]

In the light of the numerical results discussed in Subsection above, we are interested in the stationary solutions of the system. Note that it has the trivial stationary solution

$$\rho(\theta, \omega) = \frac{1}{2\pi}, \quad r = 0$$

independently of the choice I make for K and g. Obviously, this solution corresponds to the incoherent state as for instance shown in Figure 6 To find the stationary solutions that correspond to the partially synchronized states, it is illuminating to follow Kuramoto's original analysis in identifying the critical coupling K_C.

3.4 Bifurcation Diagram

Combining the different analytic approaches, I, here present a full bifurcation diagram of the Kuramoto model with a bimodal frequency distribution that is compounded by two compact distributions. No level of sync can be established over a range of coupling parameters extending up to the 'critical' coupling. At that point something new happens; the system is suddenly able to form a broad cluster that 'travels' together on the unit circle. Synchronization is now possible, and the degree of synchronization depends sensitively on the coupling strength.

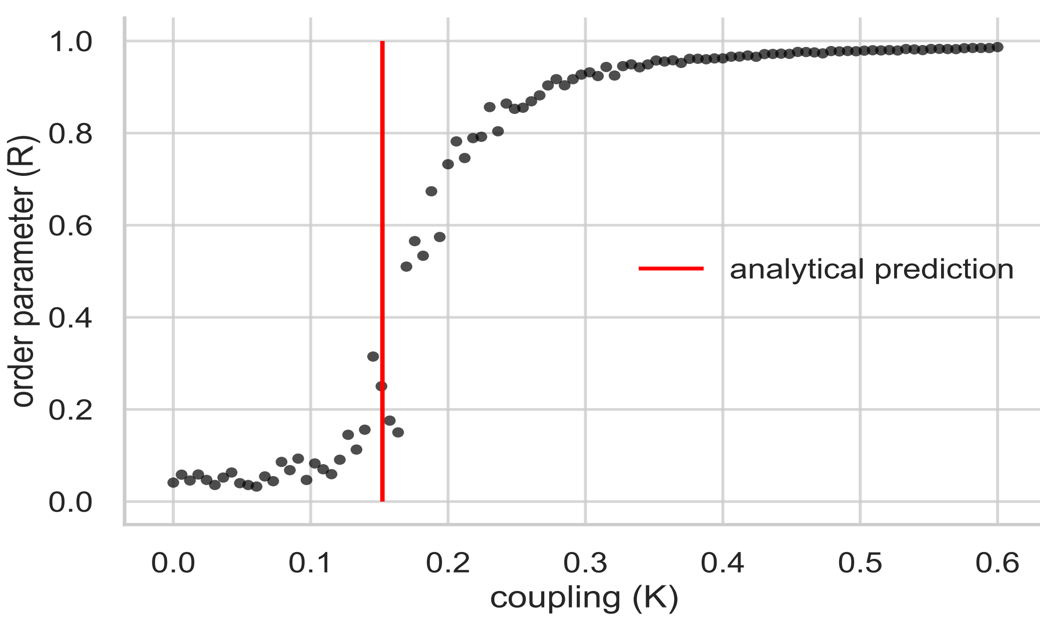

Figure 8: Phase diagram for 100 oscillators computed for K in range 1-5 with r averaged over last 100 timesteps. A supercritical bifurcation is seen around $K = K_C$

3.5 Phase Diagram with varying Standard Deviation

I plotted a phase diagrams with varying standard deviations with 1000 oscillators. Considering a unimodal frequency distribution in form of a Gaussian distribution function with $\sigma = 0.8, 1.0, 1.2$ and 1.4. By using the formula for the frequency distribution:

$$g(\omega) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\omega^2/2\sigma^2}$$

$g(0)$ for this distribution is:

$$g(0) = \frac{1}{\sigma\sqrt{2\pi}} \tag{5}$$

Using equation (5) critical coupling can be evaluated easily.

$$K_c = \sqrt{\frac{8}{\pi}}\sigma$$

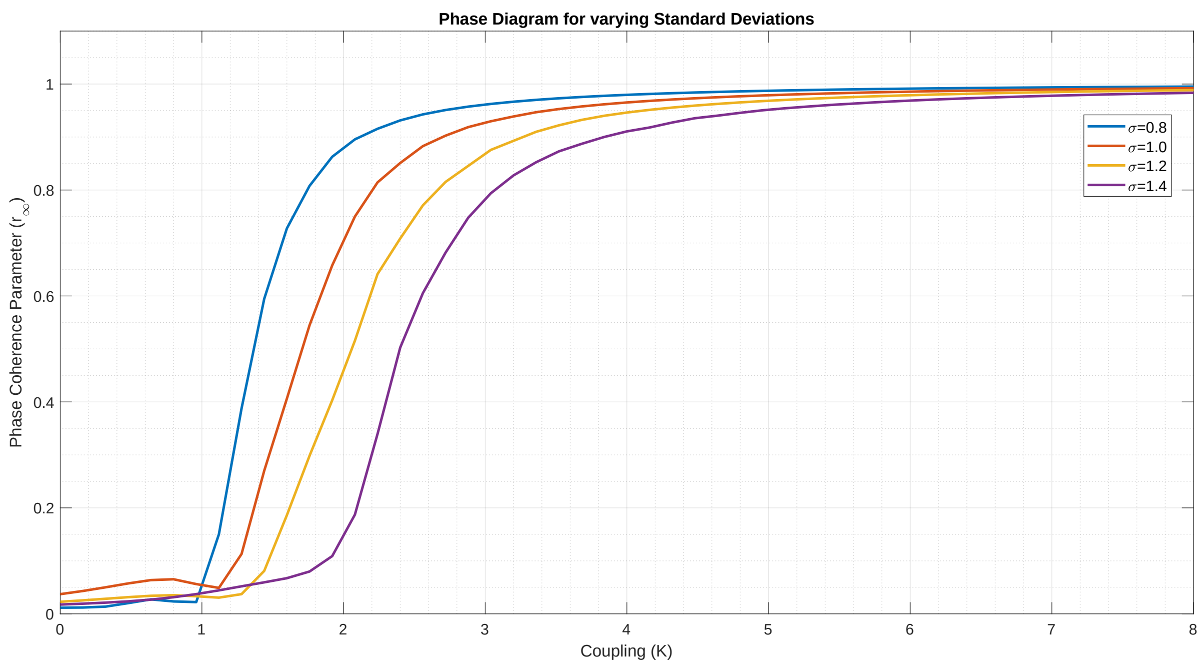

Figure 9: Phase diagram for 1000 oscillators and varying standard deviations
0.8, 1.0, 1.2 & 1.4

4 Analysis & Inferences

Section 3 and its different subsections deals with all the analysis and computational results. Here are some of the analytical results:

- We saw that for K smaller than a certain critical value K_C, the oscillators do not appear to feel the mutual interactions and just rotate around the unit circle near their natural frequencies. For each initial distribution of the phases, the oscillators therefore uniformly spread over the circle resulting in $r(t)$ decreasing to zero.
- Gaussian distribution with different variances (mean = 0) are tried for plotting the time evolution $r(t)$. . It is clear from the graphs that for $K < K_C$, $r(t)$ settles faster to zero as the variance increases. On the other hand, for $K > K_C$, $r(t)$ settles to a smaller value as variance increases. Also when the variance is large, the time taken for $r(t)$ to settle is also long for $K > K_C$.
- Phase lock or phase synchronization: This is the case where all angles $\theta_i(t)$ converge exponentially to a common angle θ_∞ as $t \to \infty$. This can only occur when all the natural frequencies are identical.
- Phase cohesiveness: This is the case where each pairwise distance $|\theta_i(t)| - |\theta_j(t)|$ converges to a constant value, which may be not be zero.
- Frequency synchronization: This is the case when all the frequencies $\frac{d\theta_i}{dt}$ converge exponentially fast to a common frequency $\frac{d\theta_i}{dt}$ as $t \to \infty$.
- Exponential synchronization: This is a combination of phase lock and frequency synchronization or phase cohesiveness and frequency synchronization.

5 Conclusions

In this paper we studied the phenomenon of synchronization in the Kuramoto model with an arbitrary but finite number of oscillators. A necessary condition in the form of a lower bound on the coupling gain $K = K_C$ was established for the onset of synchronization in the Kuramoto model. From the results of simulations, the dynamics of r were plotted for various coupling strengths corresponding to Gaussian distribution. A phase analysis is also performed by averaging r for each investigated K.

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