Term Paper 02: Sandpile Model

submitted by

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1 Introduction

The ASM was discovered in 1987 by Bak, Tang, and Wiesenfeld as the first example of a dynamical system exhibiting self-organized criticality [1]. At a schematic level, the model consists of a d-dimensional lattice whose sites are described as containing an integer number of grains of sand. At any given site, if the number of grains is greater than some threshold value then the site “topples,” scattering its sand to its nearest neighbors

1.1 Self Organized Criticality

Self-organized criticality refers to the tendency of many dynamical systems to naturally drive themselves to a state displaying fluctuations over a wide range of scales.

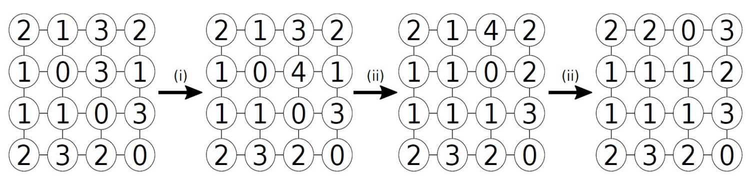

Figure 2: An illustration of the Abelian Sandpile Model (ASM) toppling rules in the simple case of a 4 × 4 square lattice with open boundary conditions, with $\triangle_{ij} = -1$ if sites i and j are nearest neighbors, $\triangle_{ij} = 4$ if i = j, and $\triangle_{ij} = 0$ otherwise.

A single simulation of the classic sandpile model on a lattice of size (100, 100) and 100000 grains of sand yields the following lattice:

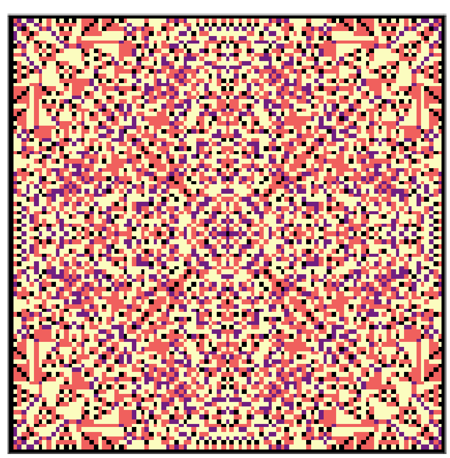

Figure 3: Lattice generated from simulation

2 Phenomenology and Plots

The time evolution of a configuration of the ASM is defined by the following rules:

- taking a configuration
- adding a particle on a randomly chosen vertex that different from q
- performing topplings until a new stable configuration is obtained.

In the Abelian Sandpile Model (ASM), there are in general three quantities which are used to study the dynamical properties. [2] First, define for a given avalanche:

(i) n, the number of topples involved in the avalanche before the system relaxes to a stable state,

(ii) t, the number of time steps required for the relaxation process (the "lifetime" of the avalanche),

(iii) l, the linear average size of an avalanche (which reaches sites at positions r_i), defined by

$$l = \frac{1}{|A|} \sum |r_i - R_{cm}| \tag{1}$$

where

$$R_{cm} = \frac{1}{|A|} \sum r_i \tag{2}$$

and the avalanche reaches A distinct points.

I have done a basic python implementation of the classic Bak–Tang–Wiesenfeld model, including the avalanche distribution plots, written for this course. To show that the ASM exhibits self-organized criticality, I performed simulations of an ASM on a 2 dimensional 100×100 square lattice with open boundary conditions, nearest-neighbour toppling rules like those in figure 2 and $\triangle_{ii} = 4$ for all i. The power-law distributions of avalanche durations and sizes result from the self-organized criticality (SOC) behavior.

2.1 Approach to criticality

Approach to criticality of the 2-dimensional ASM on a 100 $\times$ 100 square lattice, measured by the density of sand grains as a function of time t. It can be shown that the toppling rules described above give rise to relaxation rules that are abelian, in the sense that a state where multiple sites are above the threshold will always relax to the same final state, regardless of the order in which the critical sites are toppled. Below is the figure which reaches the criticality of density of sand grains ρ as a function of time t. The stationary value of $\rho = \rho_c \approx 3.12$ is typical for the ASM on a 2 dimensional lattice.

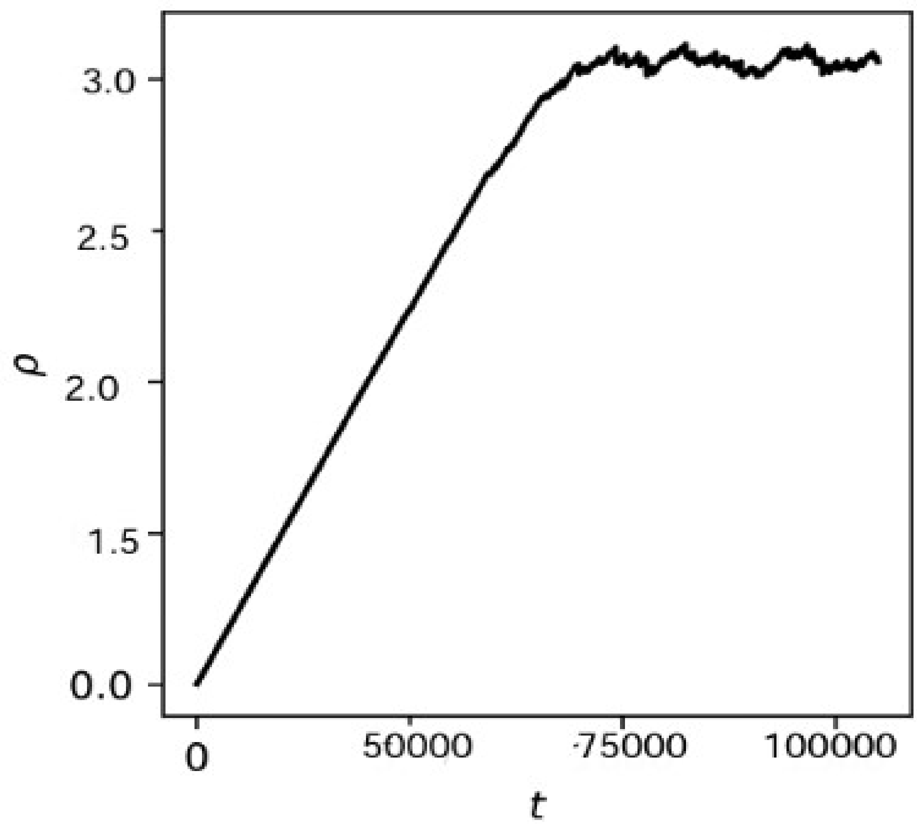

Figure 4: Approach to the criticality Abelian Sandpile Model measured by the density of sand grains ρ as a function of time t.

2.2 Avalanche Duration

Histogram of avalanche lifetimes after 100000 drive periods on a 2-dimensional 100×100 square ASM with open boundary conditions, nearest-neighbor toppling rules like figure 2 and $\triangle_{ii} = 4$ for all i. The histograms show power law behaviour. Deviations from power law behavior are likely due to finite-size effects.

We want to see if the lifetime follows a power law, i.e, if $P(T) = T^{-\alpha}$. To observe the power law better, we plot the natural logarithm of both the Avalanche Times (x-axis) and the frequency (y axis). The graph is fitted and we and the slope gives us the value of the exponential The value of α comes out to be 1.065 which is approximately 1 as expected.

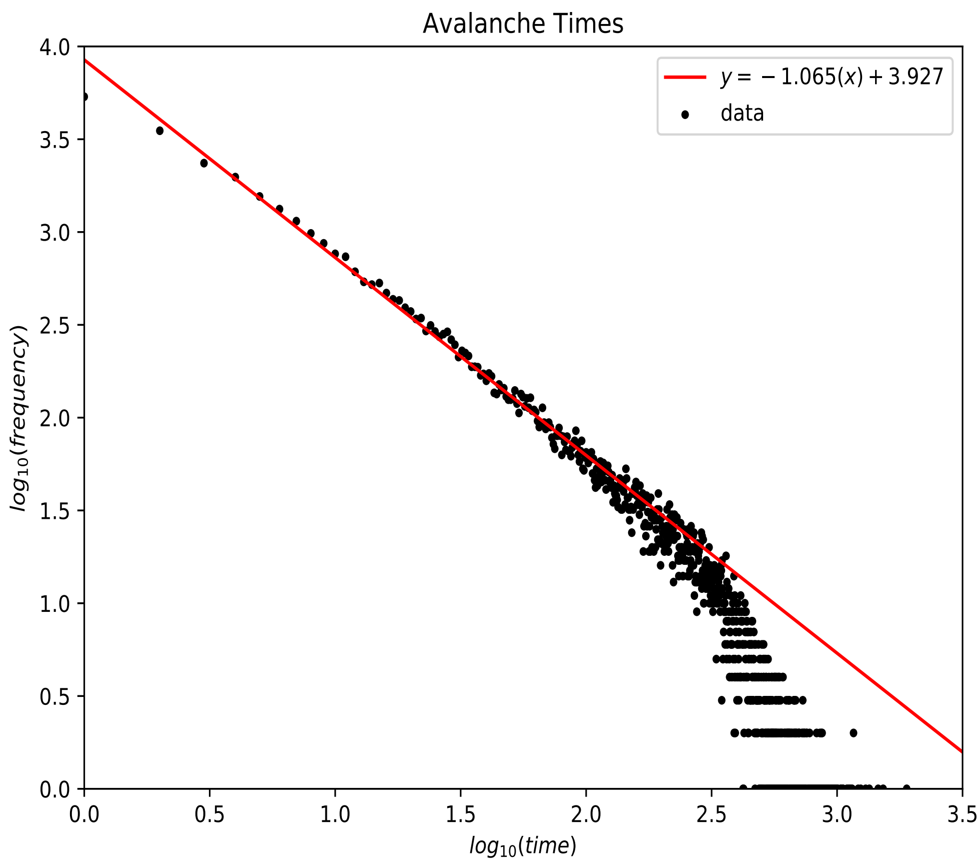

Figure 5: Histogram of avalanche lifetimes

2.3 Avalanche Sizes

Histogram of avalanche sizes after 100000 drive periods on a 2-dimensional 100×100 square ASM with open boundary conditions, nearest-neighbor toppling rules like figure 2 and $\triangle_{ii} = 4$ for all i. The histograms show power law behaviour. Deviations from power law behavior are likely due to finite-size effects.

We want to see if the Avalanche Sizes follows a power law, i.e, if $P(T) = T^{-\alpha}$. To observe the power law better, we plot the natural logarithm of both the Avalanche Sizes (x-axis) and the frequency (y axis). The graph is fitted and we and the slope gives us the value of the exponential The value of α comes out to be 1.046 which is approximately 1 as expected.

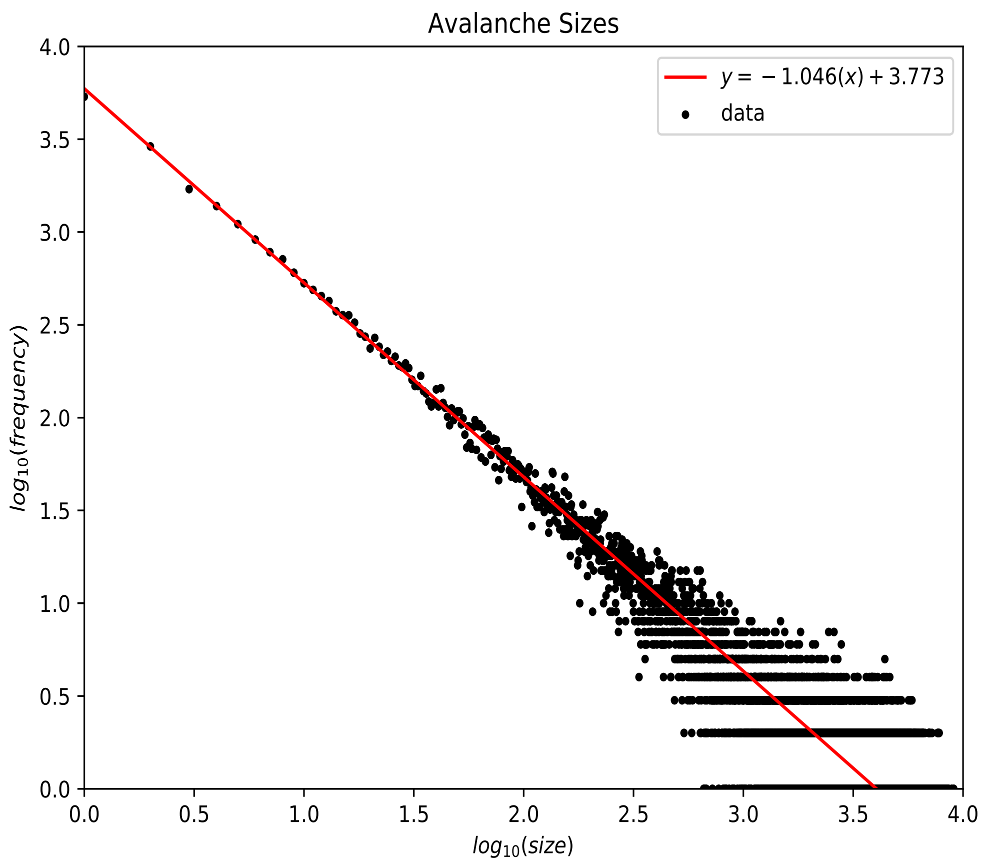

Figure 6: Histogram of avalanche sizes (in number of topples)

3 Conclusions

In this term paper, I have described the simplest model exhibiting Self Organized Criticality, the Abelian Sandpile Model, and discussed the model phenomenologically in terms of critical exponents and distribution functions. The Avalanche Lifetimes and the Avalanche Sizes follow the Power Law Distribution (as expected).

References

[1] Chao; Wiesenfeld Kurt Bak, Per; Tang. Self-organized criticality: An explanation of the 1/f noise. *Physical Review Letters*, 59, 1987.

[2] Deepak Dhar. The abelian sandpile and related models. *Physica A: Statistical Mechanics and its Applications*, 263, 1999.