Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution:

```
wget https://github.com/Abhipank/Digital−
    Signal−Processing/blob/master/pingala/
    codes/jee2019.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution:

```
wget https://github.com/Abhipank/Digital−
    Signal−Processing/blob/master/pingala/
    codes/2.2.py
```

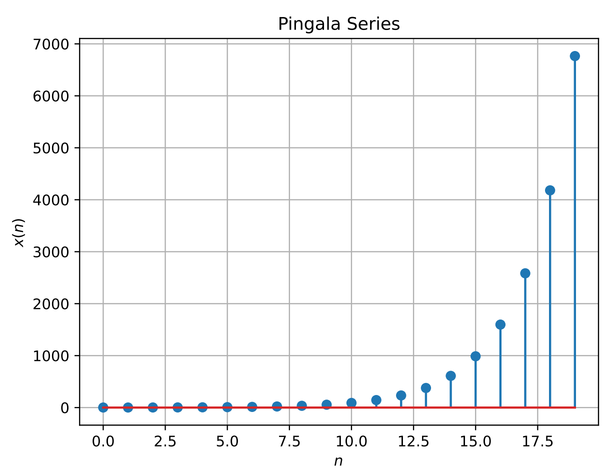

Fig. 2.2

2.3 Find $X^{+}(z)$.
Solution:

$$x(n+2) = x(n) + x(n+1) \tag{2.3}$$

$$\Rightarrow z^2(X^{+}(z) - z^{-1} - 1) = X^{+}(z) + z^1(X^{+}(z) - 1) \tag{2.4}$$

$$\Rightarrow X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.5}$$

2.4 Find $x(n)$.

Solution:

$$X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

$$\Rightarrow X^{+}(z) = \frac{z^{1}}{\alpha - \beta}\left(\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right) \tag{2.7}$$

$$\Rightarrow X^{+}(z) = \frac{\sum_{n=0}^{\infty} \alpha^{n} z^{-n+1} - \sum_{n=0}^{\infty} \beta^{n} z^{-n+1}}{\alpha - \beta} \tag{2.8}$$

$$\Rightarrow X^{+}(z) = \frac{\sum_{n=0}^{\infty} \alpha^{n+1} z^{-n} - \sum_{n=0}^{\infty} \beta^{n+1} z^{-n}}{\alpha - \beta} \tag{2.9}$$

$$\Rightarrow x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, z > \max(|\alpha|, |\beta|) \tag{2.10}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.11}$$

Solution:

```
wget https://github.com/Abhipank/Digital−Signal−Processing/blob/master/pingala/codes/2.5.py
```

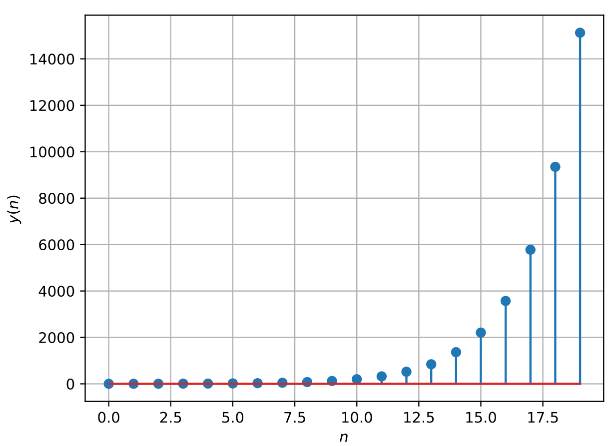

Fig. 2.5

2.6 Find $Y^{+}(z)$.
Solution:

$$y(n) = x(n+1) + x(n-1) \tag{2.12}$$

$$\Rightarrow Y^{+}(z) = z^{1}(X^{+}(z) - 1) + x(-1) + z^{-1}X^{+}(z) \tag{2.13}$$

$$\Rightarrow Y^{+}(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.14}$$

2.7 Find $y(n)$.

Solution:

$$Y^{+}(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.15}$$

$$\Rightarrow Y^{+}(z) = \frac{\frac{\alpha+2}{1-\alpha z^{-1}} - \frac{\beta+2}{1-\beta z^{-1}}}{\alpha - \beta} \tag{2.16}$$

$$\Rightarrow y(n) = \frac{\alpha^{n+1} - \beta^{n+1} + 2\alpha^{n} - 2\beta^{n}}{\alpha - \beta} \tag{2.17}$$

$$\Rightarrow y(n) = \alpha^{n+1} + \beta^{n+1}, z > \max(|\alpha|, |\beta|) \tag{2.18}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$x_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, n \geq 0 \tag{3.2}$$

$$\Rightarrow x_{n-1} = \frac{\alpha^{n} - \beta^{n}}{\alpha - \beta}, n \geq 1 \tag{3.3}$$

$$\Rightarrow a_k = x(k-1), k \geq 1 \tag{3.4}$$

$$\Rightarrow \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} x(k-1) \tag{3.5}$$

$$\Rightarrow \sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.6}$$

$$\Rightarrow \sum_{k=0}^{n-1} x(k) = \sum_{k=0}^{n-1} x(k)u(n-1-k) \tag{3.7}$$

$$\Rightarrow \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.8}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) = x(n) * u(n-1) \tag{3.9}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.10}$$

can be expressed as

$$[x(n+1)-1]\,u(n) \tag{3.11}$$

Solution:

$$a_k = x(k-1), k \geq 1 \tag{3.12}$$

$$\Rightarrow a_{n+2} - 1 = x(n+1) - 1, n \geq 1 \tag{3.13}$$

$$\Rightarrow a_{n+2} - 1 = [x(n+1) - 1]u(n) \tag{3.14}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.15}$$

Solution:

$$a_k = x(k-1), k \geq 1 \tag{3.16}$$

$$\Rightarrow \frac{a_k}{10^k} = \frac{x(k-1)}{10^k} \tag{3.17}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=1}^{\infty} \frac{x(k-1)}{10^k} \tag{3.18}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.19}$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} x(k) 10^{-k} \tag{3.20}$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.21}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.22}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.23}$$

and find $W(z)$.

Solution:

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n), n \geq 1 \tag{3.24}$$

$$\Rightarrow w(n-1) = (\alpha^n + \beta^n)\, u(n-1), n \geq 0 \tag{3.25}$$

$$\Rightarrow w(n-1) = (\alpha^n + \beta^n) \tag{3.26}$$

$$w(n) = y(n) \tag{3.27}$$

$$\Rightarrow W(z) = Y^+(z) \tag{3.28}$$

$$\Rightarrow W(z) = \frac{1 + 2z^{-2}}{1 - z^{-1} - z^{-2}} \tag{3.29}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.30}$$

Solution:

$$y(n) = x(n-1) + x(n+1), n \geq 0 \tag{3.31}$$

$$\Rightarrow y(n-1) = x(n-2) + x(n), n \geq 1 \tag{3.32}$$

$$\Rightarrow y(n-1) = x(n-2) + x(n), n \geq 2 \tag{3.33}$$

$$a_{k-1} = x(k-1), k \geq 2 \tag{3.34}$$

$$a_{k+1} = x(k+1), k \geq 0 \tag{3.35}$$

$$\Rightarrow y(n-1) = a_{n-1} + a_{n+1}, n \geq 2 \tag{3.36}$$

$$\Rightarrow y(n-1) = b(n), n \geq 2 \tag{3.37}$$

$$y(0) = x(-1) + x(0) = 1 = b_1 \tag{3.38}$$

$$\Rightarrow y(n-1) = b_n, n \geq 1 \tag{3.39}$$

$$\Rightarrow \frac{b_k}{10^k} = \frac{x(k-1)}{10^k} \tag{3.40}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=1}^{\infty} \frac{y(k-1)}{10^k} \tag{3.41}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.42}$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} y(k) 10^{-k} \tag{3.43}$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.44}$$

3.6 Solve the JEE 2019 problem.

Solution:

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1), n \geq 1 \tag{3.45}$$

$$a_{n+2} - 1 = [x(n+1) - 1]u(n), n \geq 1 \tag{3.46}$$

$$\Rightarrow a_{n+2} - 1 = [x(n+1)u(n) - u(n+1)], n \geq 1 \tag{3.47}$$

$$\mathcal{Z}\{x(n) * u(n-1)\} = \mathcal{Z}\{[x(n+1)u(n) - u(n+1)]\} \tag{3.48}$$

$$\Rightarrow a_{n+2} - 1 = \sum_{k=1}^{n} a_k \tag{3.49}$$

Hence 1.1 is correct.Using (3.21) 1.2 is correct.Using (3.44) 1.4 is incorrect.Using (3.39) and (2.18) 1.3 is correct.