

Pingala Series

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CONTENTS

Abstract—This manual provides a simple introduction to Transforms

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

Solution:

```
wget https://github.com/Abhipank/Digital-
Signal-Processing/blob/master/pingala/
codes/jee2019.py
```

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \quad (2.2)$$

Generate a stem plot for $x(n)$.

Solution:

```
wget https://github.com/Abhipank/Digital-
Signal-Processing/blob/master/pingala/
codes/2.2.py
```

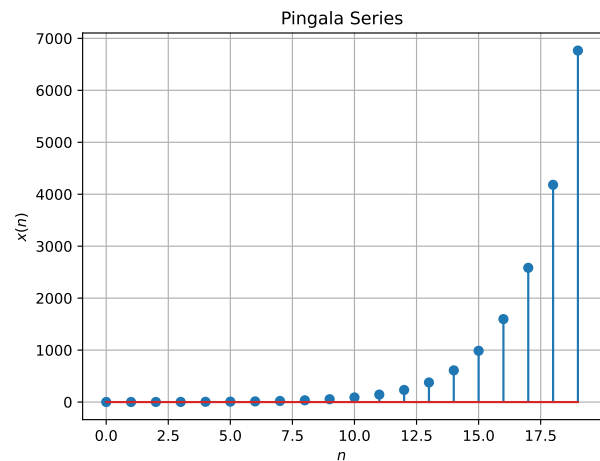


Fig. 2.2

2.3 Find $X^+(z)$.

Solution:

$$x(n+2) = x(n) + x(n+1) \quad (2.3)$$

$$\Rightarrow z^2(X^+(z) - z^{-1} - 1) = X^+(z) + z^1(X^+(z) - 1) \quad (2.4)$$

$$\Rightarrow X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.5)$$

2.4 Find $x(n)$.

Solution:

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.6)$$

$$\Rightarrow X^+(z) = \frac{z^1}{\alpha - \beta} \left(\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right) \quad (2.7)$$

$$\Rightarrow X^+(z) = \frac{\sum_{n=0}^{\infty} \alpha^n z^{-n+1} - \sum_{n=0}^{\infty} \beta^n z^{-n+1}}{\alpha - \beta} \quad (2.8)$$

$$\Rightarrow X^+(z) = \frac{\sum_{n=0}^{\infty} \alpha^{n+1} z^{-n} - \sum_{n=0}^{\infty} \beta^{n+1} z^{-n}}{\alpha - \beta} \quad (2.9)$$

$$\Rightarrow x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, z > \max(|\alpha|, |\beta|) \quad (2.10)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.11)$$

Solution:

wget <https://github.com/Abhipank/Digital-Signal-Processing/blob/master/pingala/codes/2.5.py>

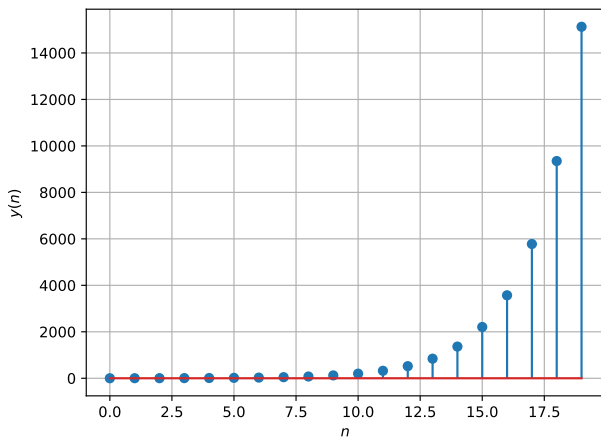


Fig. 2.5

2.6 Find $Y^+(z)$.

Solution:

$$y(n) = x(n+1) + x(n-1) \quad (2.12)$$

$$\Rightarrow Y^+(z) = z^1(X^+(z) - 1) + x(-1) + z^{-1}X^+(z) \quad (2.13)$$

$$\Rightarrow Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (2.14)$$

2.7 Find $y(n)$.

Solution:

$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (2.15)$$

$$\Rightarrow Y^+(z) = \frac{\frac{\alpha+2}{1-\alpha z^{-1}} - \frac{\beta+2}{1-\beta z^{-1}}}{\alpha - \beta} \quad (2.16)$$

$$\Rightarrow y(n) = \frac{\alpha^{n+1} - \beta^{n+1} + 2\alpha^n - 2\beta^n}{\alpha - \beta} \quad (2.17)$$

$$\Rightarrow y(n) = \alpha^{n+1} + \beta^{n+1}, z > \max(|\alpha|, |\beta|) \quad (2.18)$$

3 POWER OF THE Z TRANSFORM

3.1 Show that

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \quad (3.1)$$

Solution:

$$x_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, n \geq 0 \quad (3.2)$$

$$\Rightarrow x_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1 \quad (3.3)$$

$$\Rightarrow a_k = x(k-1), k \geq 1 \quad (3.4)$$

$$\Rightarrow \sum_{k=1}^n a_k = \sum_{k=1}^n x(k-1) \quad (3.5)$$

$$\Rightarrow \sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(k) \quad (3.6)$$

$$\Rightarrow \sum_{k=0}^{n-1} x(k) = \sum_{k=0}^{n-1} x(k)u(n-1-k) \quad (3.7)$$

$$\Rightarrow \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \quad (3.8)$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) = x(n) * u(n-1) \quad (3.9)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.10)$$

can be expressed as

$$[x(n+1) - 1]u(n) \quad (3.11)$$

Solution:

$$a_k = x(k-1), k \geq 1 \quad (3.12)$$

$$\Rightarrow a_{n+2} - 1 = x(n+1) - 1, n \geq 1 \quad (3.13)$$

$$\Rightarrow a_{n+2} - 1 = [x(n+1) - 1]u(n) \quad (3.14)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.15)$$

Solution:

$$a_k = x(k-1), k \geq 1 \quad (3.16)$$

$$\Rightarrow \frac{a_k}{10^k} = \frac{x(k-1)}{10^k} \quad (3.17)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=1}^{\infty} \frac{x(k-1)}{10^k} \quad (3.18)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.19)$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} x(k) 10^{-k} \quad (3.20)$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.21)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.22)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.23)$$

and find $W(z)$.

Solution:

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n), n \geq 1 \quad (3.24)$$

$$\Rightarrow w(n-1) = (\alpha^n + \beta^n)u(n-1), n \geq 0 \quad (3.25)$$

$$\Rightarrow w(n-1) = (\alpha^n + \beta^n) \quad (3.26)$$

$$w(n) = y(n) \quad (3.27)$$

$$\Rightarrow W(z) = Y^+(z) \quad (3.28)$$

$$\Rightarrow W(z) = \frac{1 + 2z^{-2}}{1 - z^{-1} - z^{-2}} \quad (3.29)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.30)$$

Solution:

$$y(n) = x(n-1) + x(n+1), n \geq 0 \quad (3.31)$$

$$\Rightarrow y(n-1) = x(n-2) + x(n), n \geq 1 \quad (3.32)$$

$$\Rightarrow y(n-1) = x(n-2) + x(n), n \geq 2 \quad (3.33)$$

$$a_{k-1} = x(k-1), k \geq 2 \quad (3.34)$$

$$a_{k+1} = x(k+1), k \geq 0 \quad (3.35)$$

$$\Rightarrow y(n-1) = a_{n-1} + a_{n+1}, n \geq 2 \quad (3.36)$$

$$\Rightarrow y(n-1) = b(n), n \geq 2 \quad (3.37)$$

$$y(0) = x(-1) + x(0) = 1 = b_1 \quad (3.38)$$

$$\Rightarrow y(n-1) = b_n, n \geq 1 \quad (3.39)$$

$$\Rightarrow \frac{b_k}{10^k} = \frac{x(k-1)}{10^k} \quad (3.40)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=1}^{\infty} \frac{y(k-1)}{10^k} \quad (3.41)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.42)$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} y(k) 10^{-k} \quad (3.43)$$

$$\Rightarrow \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.44)$$

3.6 Solve the JEE 2019 problem.

Solution:

$$\sum_{k=1}^n a_k = x(n) * u(n-1), n \geq 1 \quad (3.45)$$

$$a_{n+2} - 1 = [x(n+1) - 1]u(n), n \geq 1 \quad (3.46)$$

$$\Rightarrow a_{n+2} - 1 = [x(n+1)u(n) - u(n+1)], n \geq 1 \quad (3.47)$$

$$\mathcal{Z}\{x(n) * u(n-1)\} = \mathcal{Z}\{[x(n+1)u(n) - u(n+1)]\} \quad (3.48)$$

$$\Rightarrow a_{n+2} - 1 = \sum_{k=1}^n a_k \quad (3.49)$$

Hence 1.1 is correct. Using (3.21) 1.2 is correct. Using (3.44) 1.4 is incorrect. Using (3.39) and (2.18) 1.3 is correct.