Assignment 1

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Abhipank/Digital-
    Signal-Processing/blob/main/CODES/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: Its observable that this signal consists of high frequency noise from 5.1 KHz to 18.9 KHz because of key strokes along with background noise.. Spectrogram represents the synthesizer key tones ranging from 440 Hz to 5.1 KHz.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
wget https://github.com/Abhipank/Digital-
    Signal-Processing/blob/main/CODES/
    test1.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz that was being contributed by the noise.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/Abhipank/Digital-
    Signal-Processing/blob/main/CODES/
    xnyn.py
```

3.3 Repeat the above exercise using a C code.
Solution:

```
wget https://github.com/Abhipank/Digital-
    Signal-Processing/blob/main/CODES/
    xnync.py
```

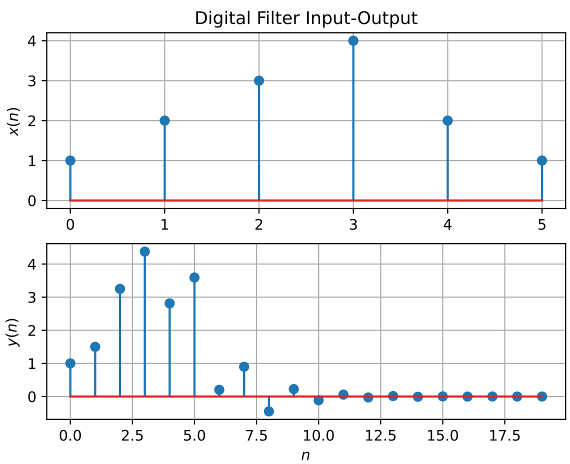

Fig. 3.2

```
wget https://github.com/Abhipank/Digital-
    Signal-Processing/blob/main/CODES/
    coeffs.h
wget https://github.com/Abhipank/Digital-
    Signal-Processing/blob/main/CODES/
    outputsig.c
```

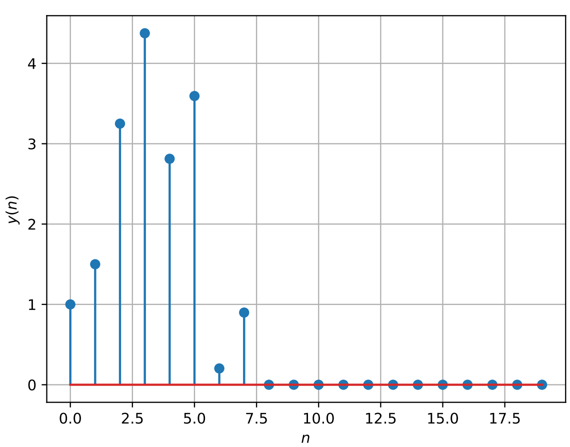

Fig. 3.3

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.6}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.8}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$\mathcal{Z}(y(n) + \frac{1}{2}y(n-1)) = \mathcal{Z}(x(n) + x(n-2)) \tag{4.11}$$

$$\Rightarrow Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.12}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.13}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.16}$$

Solution:

$$U(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.17}$$

$$= 1 \tag{4.18}$$

$$\Rightarrow \delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1, \quad 0 \le |z| < \infty \tag{4.19}$$

and from (4.15),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.20}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.21}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.22}$$

Solution:

$$U(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.23}$$

$$\Rightarrow U(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.24}$$

$$\Rightarrow a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.25}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.26}$$

Plot $|H(e^{\jmath\omega})|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6.

```
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/dtft.py
```

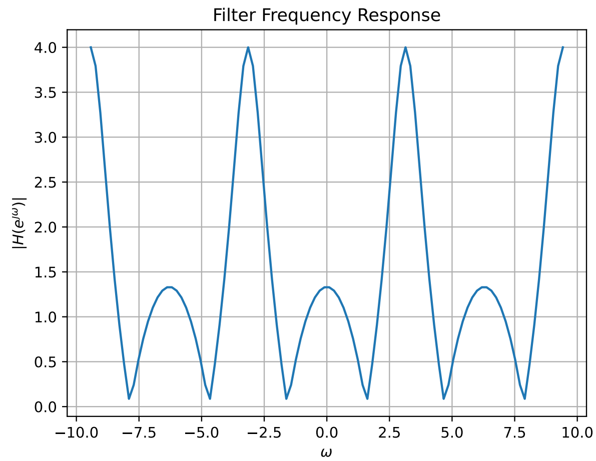

Fig. 4.6: $|H(e^{\jmath\omega})|$

$$|H(e^{-jw})| = \frac{|1 + e^{-2jw}|}{|1 + \frac{1}{2}e^{-jw}|} \tag{4.27}$$

$$= \sqrt{\frac{(1 + \cos 2w)^2 + (\sin 2w)^2}{(1 + \frac{1}{2}\cos w)^2 + (\frac{1}{2}\sin w)^2}} \tag{4.28}$$

$$= \frac{2|\cos w|}{\sqrt{\frac{5}{4} + \cos w}} \tag{4.29}$$

$$\tag{4.30}$$

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.
Solution: Since $H(e^{-jw})$ is DTFT of $h(n)$ we can do Inverse Fourier Transform.

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} dw \tag{4.31}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.13)
Solution:

$$H(z) = 1 - \frac{1}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} + ... \tag{5.2}$$

$$h(n) = \{1, -\frac{1}{2}, \frac{5}{4}, -\frac{5}{8}, \frac{5}{16}\} \tag{5.3}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.4}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.13), Using uniqueness of DTFT

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.5}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.6}$$

using (4.22) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/hn.py
```

```
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/coeffs.h
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/h.c
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/hnC.py
```

From its plot,boundedness can be seen. To justify h(n) as bounded,convergence of h(n) using ratio test is enough.

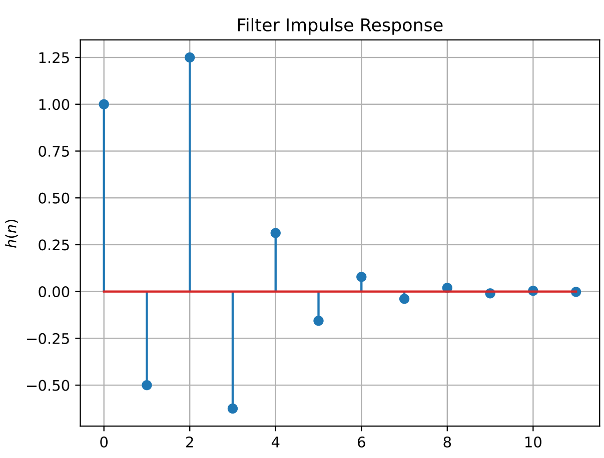

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 Convergent? Justify using the ratio test.
Solution:

$$\lim_{n\to-\infty} \frac{|h(n+1)|}{|h(n)|} = \frac{1}{2} \tag{5.7}$$
$$< 1 \tag{5.8}$$

Thus,Convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.9}$$

Is the system defined by (3.2) stable for the impulse response in 5.4.
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} h(n) \tag{5.10}$$
$$= \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n + \sum_{n=2}^{\infty}\left(\frac{-1}{2}\right)^{n-2} \tag{5.11}$$
$$= \frac{4}{3} \tag{5.12}$$
$$< \infty \tag{5.13}$$

5.6 Verify the above result with the help of python.
Solution:

```
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.14}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/hndef.py
```

```
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/coeffs.h
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/5.7.c
wget https://github.com/Abhipank/Digital-Signal-Processing/blob/main/CODES/hndefc.py
```

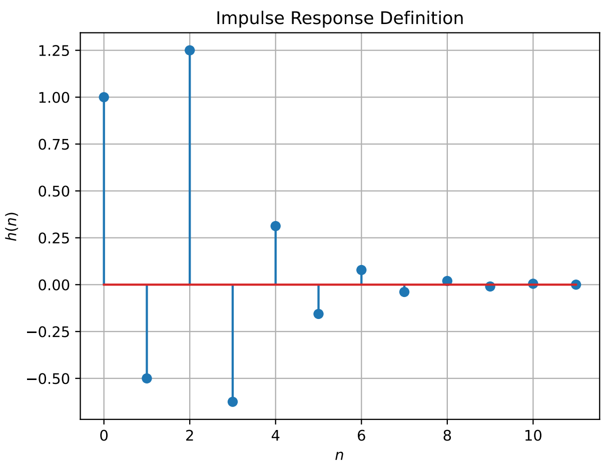

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.15)$$

Comment. The operation in (5.15) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.py
```

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using Toeplitz matrices.

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.16)$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

6.5 Wherever possible, express all the above equations as matrix equations.

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (7.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.3

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.