Random Numbers

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AI21BTECH11003

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
solution: Download the following files and execute the C program.

```
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    coeffs.h
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    exrand.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1}$$

solution: The following code plots Fig. 1.2

```
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    cdf_plot1.1.py
```

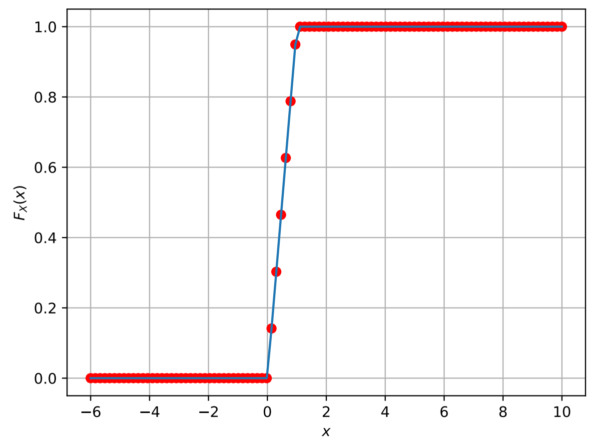

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
solution:

$$p_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x < 0, x > 1 \end{cases} \tag{2}$$

$$F_U(x) = Pr(U \le x) = \begin{cases} \int_0^1 1dx = x & 0 \le x \le 1 \\ 0 & x < 0, x > 1 \end{cases} \tag{3}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{4}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{5}$$

Write a C program to find the mean and variance of U
solution: The following code finds mean and variance of U

```
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    que1.4.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{6}$$

solution:

$$E[U] = \int_{-\infty}^{\infty} x^1 dF_U(x) = E[U] = \int_0^1 x^1 p_X(x) dx$$
$$\implies E[U] = 0.5$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x)$$
$$\implies E[U^2] = \int_0^1 x^2 p_X(x) dx$$
$$\implies E[U^2] = \int_0^1 x^2 1 dx$$
$$\implies E[U^2] = 1/3$$

$$var[U] = E[U - E[U]]^2 \tag{7}$$
$$\implies var[U] = E[U^2 + E[U]^2 - 2UE[U]]$$
$$\implies var[U] = E[U^2] - E[U]^2$$
$$\implies var[U] = 1/3 - 1/4$$
$$\implies var[U] = 1/12$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{8}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
solution: Download the following files and execute the C program.

```
wget https://github.com/Abhipank/probability
    -and-random-variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    coeffs.h
wget https://github.com/Abhipank/probability
    -and-random-variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    exrand.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
solution The CDF of X is plotted in Fig. 2.2

```
wget https://github.com/Abhipank/probability
    -and-random-variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    que2.2.py
```

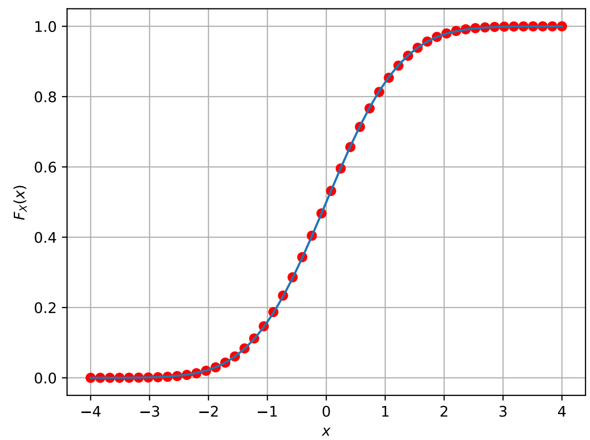

Fig. 2.2: The CDF of X

CDF is denoted by $\phi(x)$.

$$\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{(2\pi)} \times \sigma} e^{-(x-\mu)^2/2} dx \tag{9}$$
$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{(2\pi)} \times \sigma} e^{-(x-\mu)^2/2} dx \tag{10}$$
$$\implies \phi(x) = 1 - Q(x) = Q(-x) \tag{11}$$

Here are properties for CDF of normal distribution:

a) $\lim_{x\to+\infty} \phi(x) = 1$
b) $\lim_{x\to-\infty} \phi(x) = 0$
c) $\phi(x) = 1 - \phi(-x)$
d) $\phi(x)$ is non decreasing.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{12}$$

What properties does the PDF have?
solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Abhipank/probability
    -and-random-variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    pdf_plot.py
```

PDF is denoted by $f(x)$.Here are properties for PDF of normal distribution:

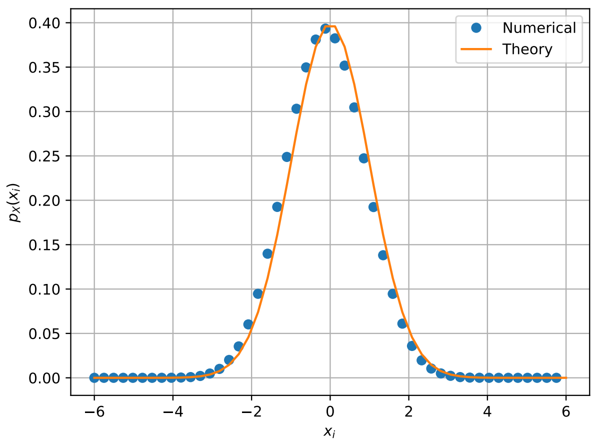

Fig. 2.3: The PDF of X

a) All three mean,mode,median are the same and are under the peak of pdf curve.
b) $f(x) = f(-x)$.The function is symmetric about the y-axis.
c) Within one standard deviation, 68% of all observations lie.
d) $\int_{-\infty}^{\infty} f(x)dx = 1$
e) $\forall x \in R$ f(x) ≥ 0

2.4 Find the mean and variance of X by writing a C program.
solution: The following code finds mean and variance of X

```
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    que2.4.c
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (13)$$

repeat the above exercise theoretically.

solution:

$$E[X] = \int_{-\infty}^{\infty} x^1 dF_X(x) = E[X] = \int_{-\infty}^{\infty} x^1 p_X(x)dx$$

$$\Rightarrow E[X] = \int_{-\infty}^{\infty} x^1 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$\Rightarrow$ E[X] =0 because integrand is odd function.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 dF_X(x)$$

$$\Rightarrow E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x)dx$$

$$\Rightarrow E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$\Rightarrow E[X^2] = -x[\frac{1}{\sqrt{(2\pi)}} e^{-x^2/2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-x^2/2} dx$$

$\Rightarrow$ $E[X^2] = 1$ by integration by parts and $\int_{-\infty}^{\infty} p_X(x)dx = 1$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \quad (14)$$

and plot its CDF.
solution: Download the following files and execute the C program.

```
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    coeffs.h
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    que3.1.c
```

The CDF of V is plotted in Fig. 3.1

```
wget https://github.com/Abhipank/probability
    −and−random−variables/blob/main/
    SIMULATION%20ASSIGNMENT/codes/
    que3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$.
solution:

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (15)$$

$$F_V(x) = Pr(V \leq x) = \begin{cases} \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (16)$$

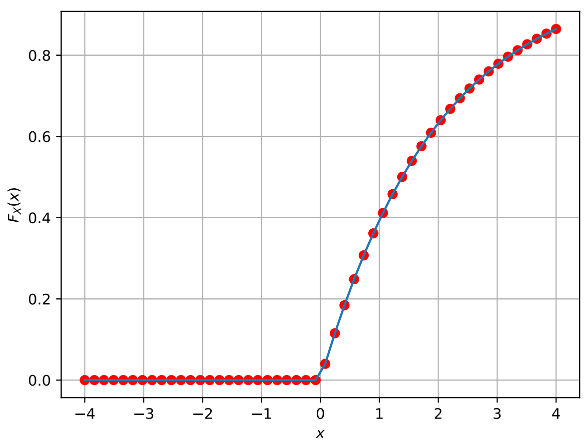

Fig. 3.1: The CDF of V