Disclaimer

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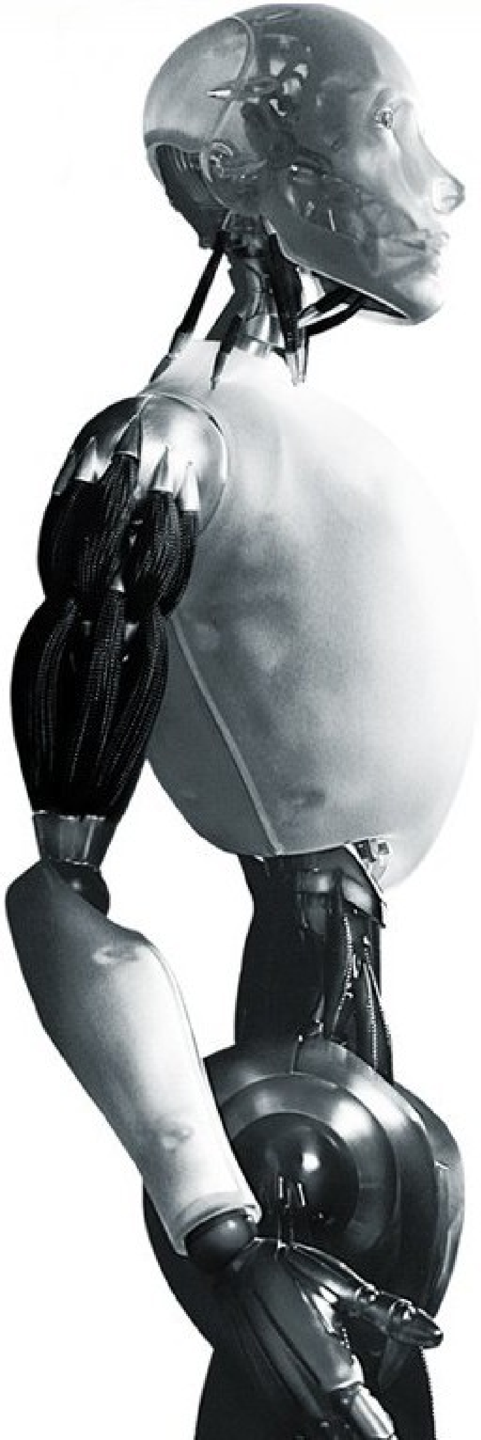

Robotics

Basics of Control – A Second Look

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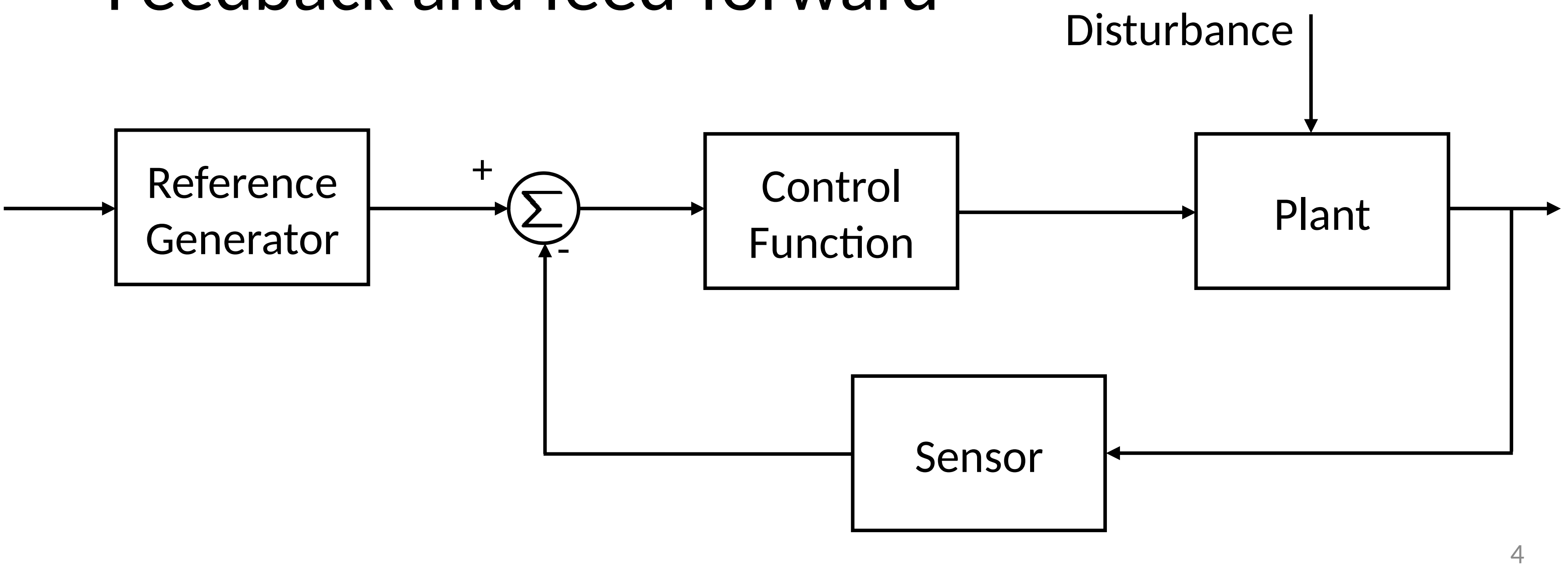
Reading for this set of slides

- Craig – Intro to Robotics (3rd Edition)
  - 1 Introduction
  - 2 Spatial descriptions and transformations (2.1 – 2.9)
  - 3 Manipulator kinematics (3.1 – 3.6)
  - 7 Trajectory generation

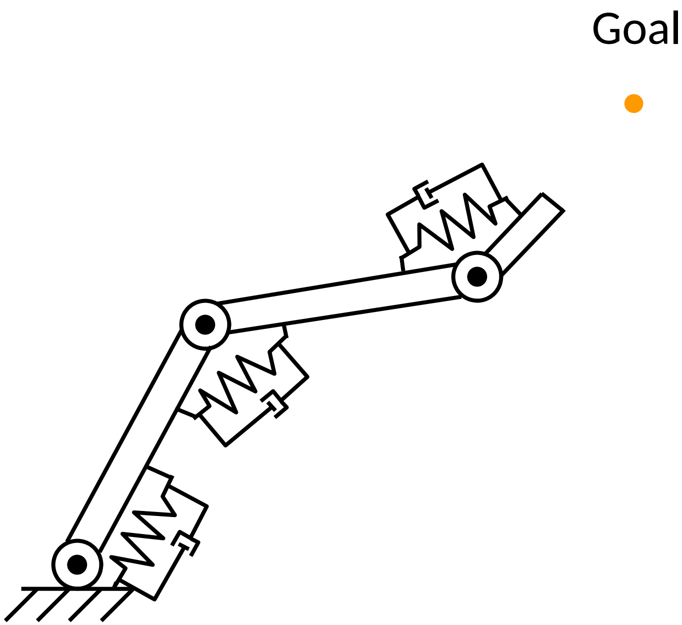
Please note that this set of slides is intended as support for the lecture, not as a stand-alone script. If you want to study for this course, please use these slides in conjunction with the indicated chapters in the text books. The textbooks are available online or in the TUB library (many copies that can be checked out for the entire semester. There are also some aspects of the lectures that will not be covered in the text books but can still be part of the homework or exam. For those It is important that you attend class or ask somebody about what was covered in class.

Control

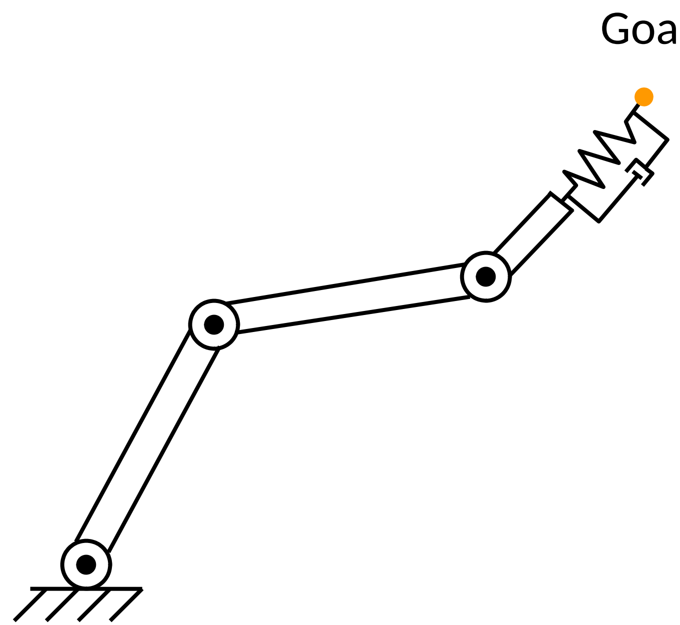
- **Control** is the process of causing a *system variable* to conform to some desired value, called a *reference value*.
- Feedback and feed-forward



Controlling a Robot

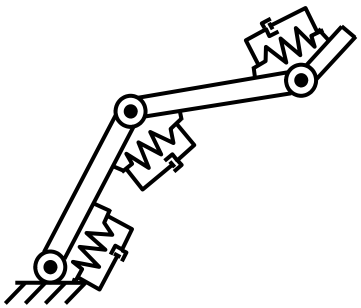


Joint Space

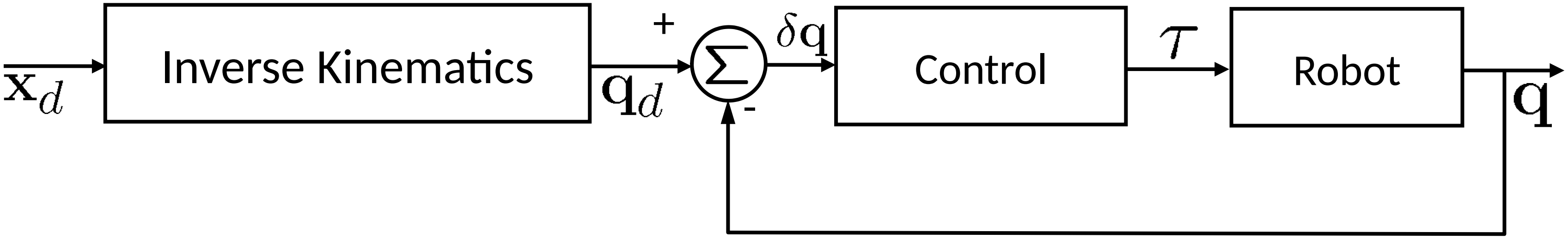


Operational Space

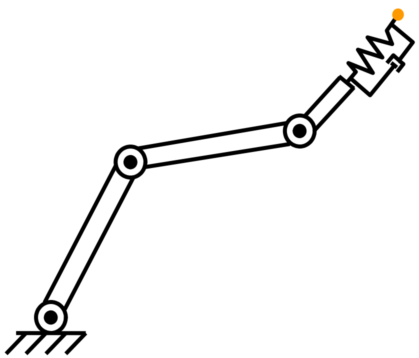
Control with Inverse Kinematics



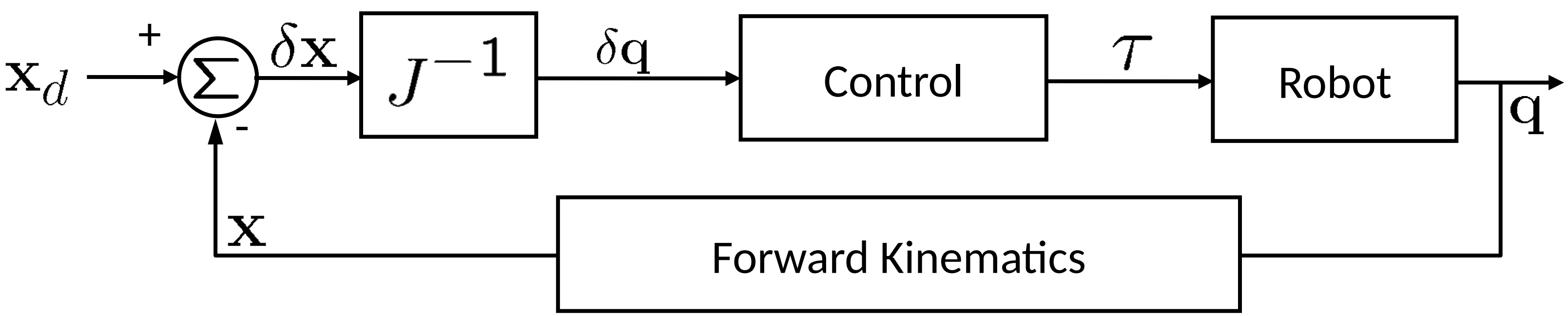
$$\mathbf{q}_{t+1} = \mathbf{q}_t + \delta\mathbf{q}$$



Operational Space Control with J^{-1}

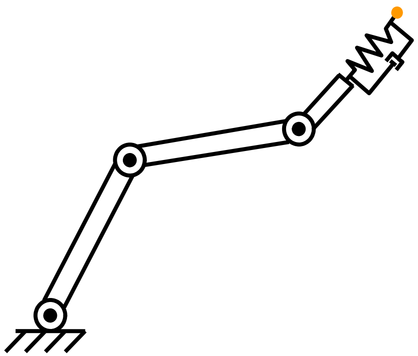


$$\delta \mathbf{q} = J^{-1}(\mathbf{q})\ \delta \mathbf{x}$$



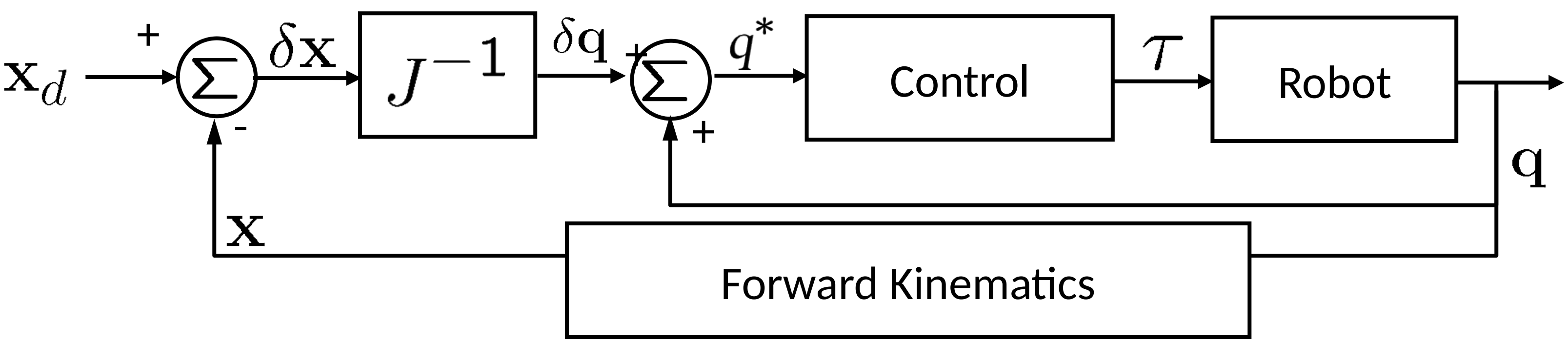
Operational Space Control with J^{-1}

Resolved-Rate Motion Control

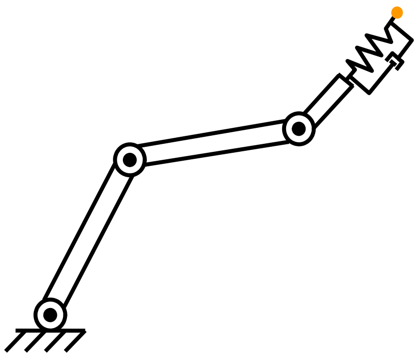


$$\dot{\mathbf{q}}^* = J(\mathbf{q})^{-1}\ \dot{\mathbf{x}}^*$$

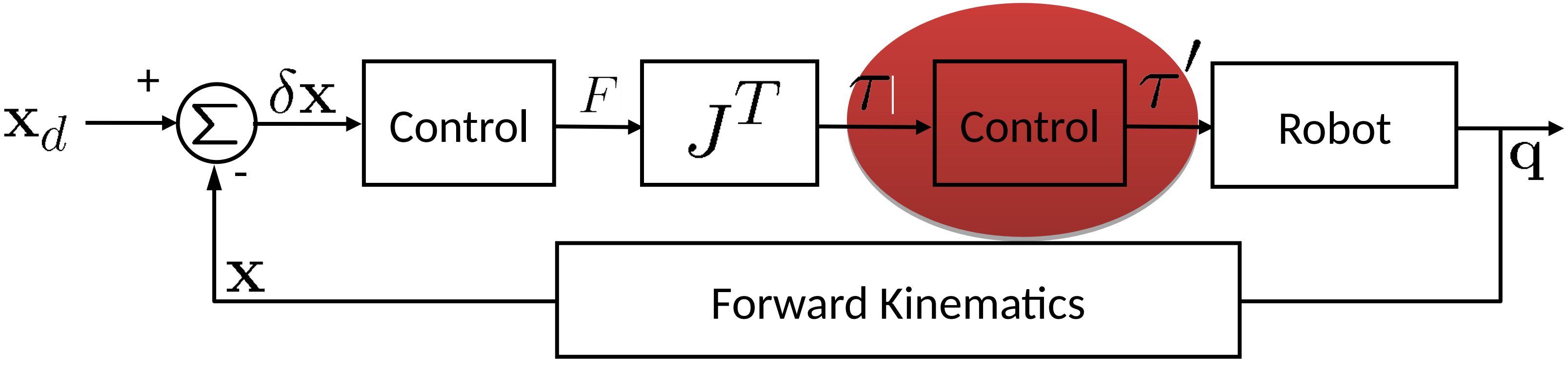
$$\mathbf{q}^*_{t+1} = \mathbf{q}_t + \delta t\ \dot{\mathbf{q}}^*_t$$



Operational Space Control with J^T

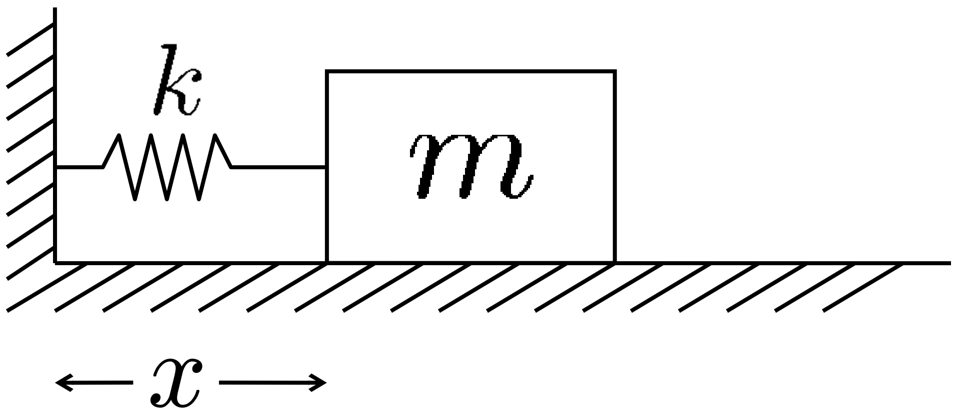


$$\tau = J^T(\mathbf{F})$$



Let's start simple…

Conservative System / Simple Harmonic Oscillator



$$K = \frac{1}{2}\, m\, \dot{x}^2$$

$$V = \frac{1}{2}\, k\, x^2$$

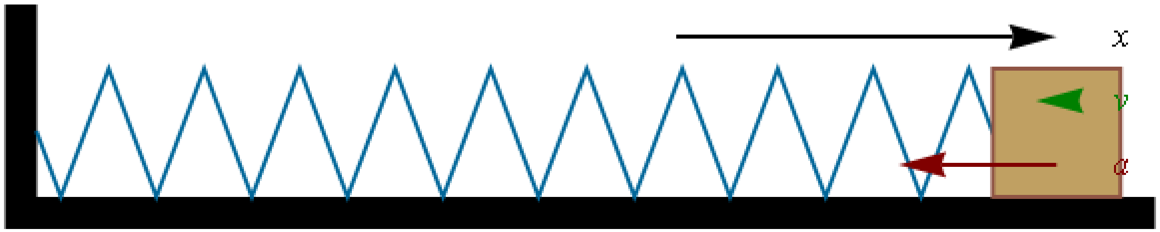
Forces are equal

$$f = m\ddot{x} = -k\,x$$

Equation of motion

$$m\ddot{x} + kx = 0$$

Harmonic Oscillator: Equation of Motion

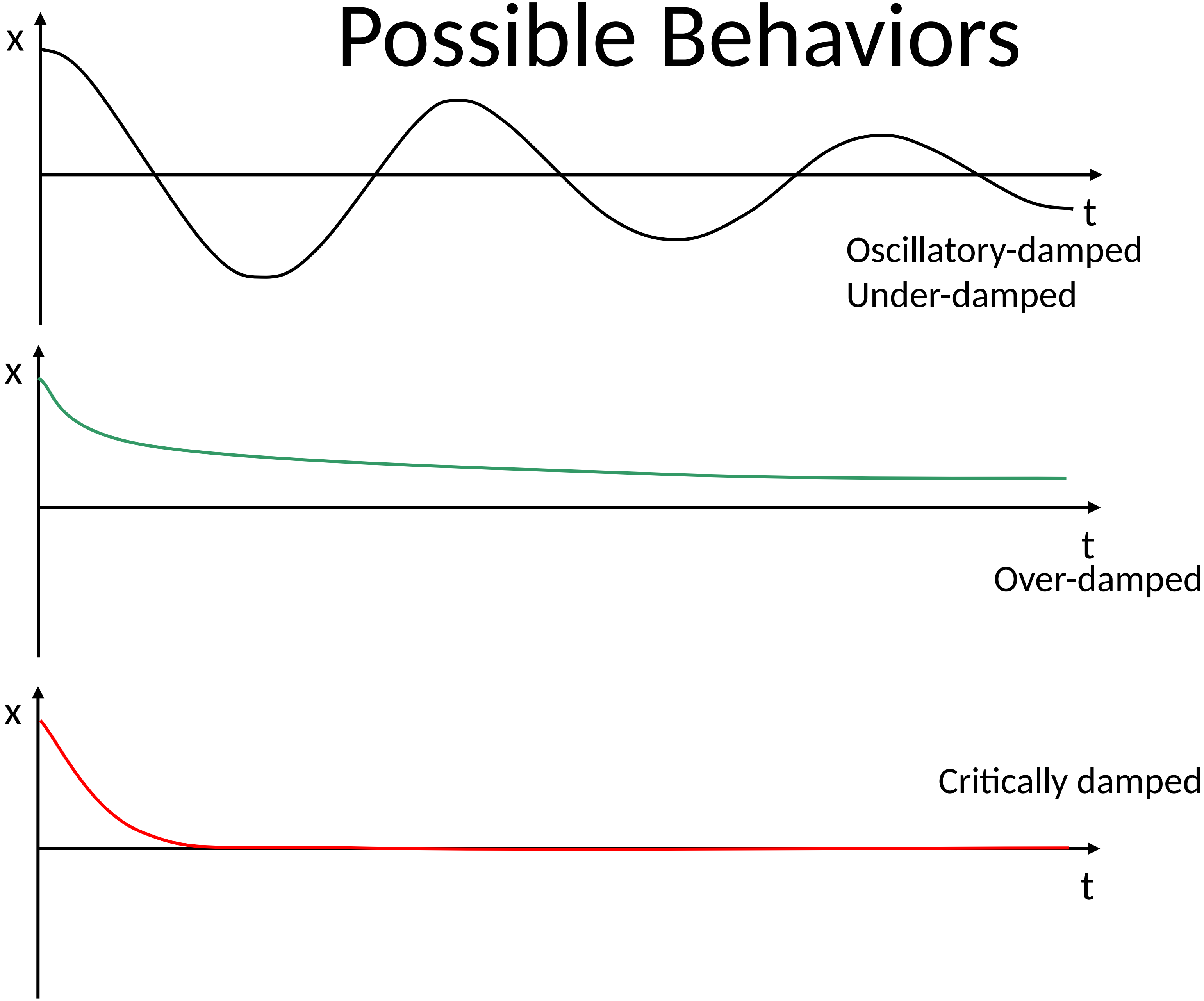


x

$\ddot{x}$

$$m\ddot{x} + kx = 0$$

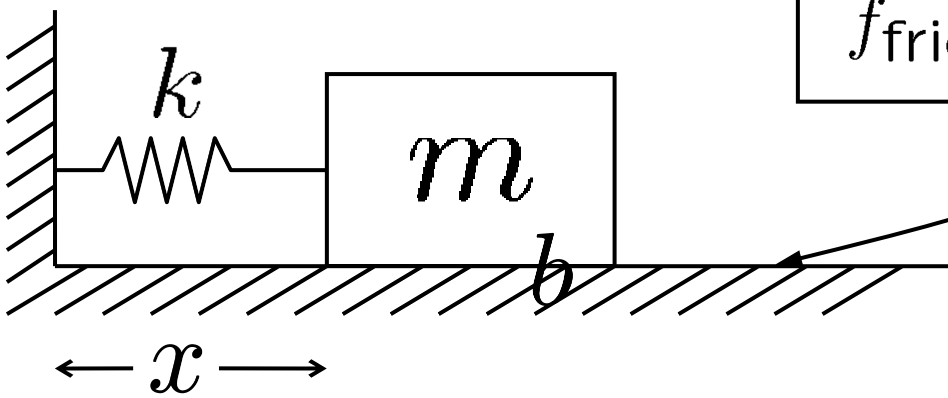
Possible Behaviors



Dissipative System: Add Friction

viscous friction

$$f_{\text{friction}} = -b\dot{x}$$



Equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solving the Equation of Motion (EOM) of a Second Order Linear System

Find trajectory $x(t)$ depending on parameters m, b, k

such that the following holds at all times: $$m\ddot{x} + b\dot{x} + kx = 0$$

We assume
$$x = e^{st}$$

Therefore
$$\dot{x} = se^{st}$$
$$\ddot{x} = s^2 e^{st}$$

$$ms^2e^{st} + bse^{st} + ke^{st} = 0$$

$$e^{st}(ms^2 + bs + k) = 0$$

$$ms^2 + bs + k = 0$$

Solve for s (easy), then $x = e^{st}$

Solving the Equation of Motion (EOM) of a Second Order Linear System

$$m\ddot{x} + b\dot{x} + kx = 0$$

Characteristic equation:

$$ms^2 + bs + k = 0$$

Roots (poles):

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m}$$

$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}$$

Solution for cases 1 & 2:

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Solutions:

1. $b^2 > 4mk$
   real and unequal roots
   overdamped
2. $b^2 < 4mk$
   complex roots
   underdamped
3. $b^2 = 4mk$
   real and equal roots
   critically damped

$$b^2 = 4mk$$

$$\frac{b^2}{m^2} = 4\frac{k}{m}$$

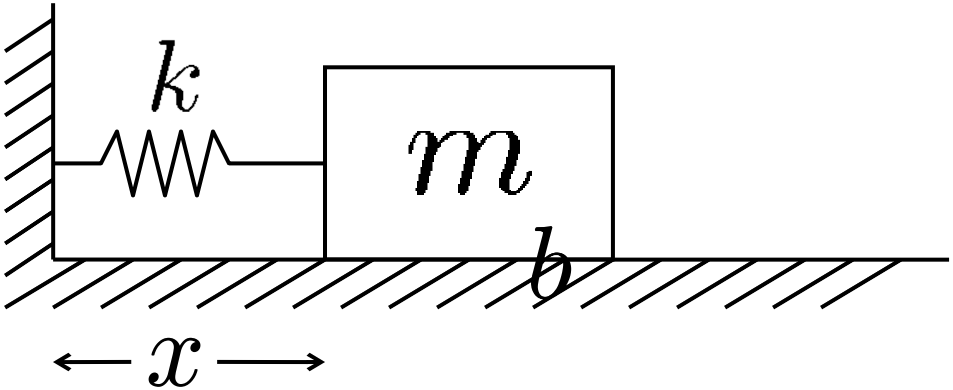
$$\frac{b}{m} = 2\sqrt{\frac{k}{m}} = 2\omega_n$$

Solution with real, unequal roots

(overdamped)

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Example (real, unequal roots)



$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$m = 1,\ b = 5,\ k = 6,\ x(0) = -1,\ \dot{x}(0) = 0$

$$\ddot{x} + 5\dot{x} + 6x = 0$$

$$s^2 + 5s + 6 = 0$$

$$s_1 = -2, s_2 = -3$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$x(0) = -1$$

$$c_1 + c_2 = -1$$

$$\dot{x}(0) = 0$$

$$-2c_1 - 3c_2 = 0$$

$$c_1 = -3 \quad c_2 = 2$$

$$x(t) = -3e^{-2t} + 2e^{-3t}$$

Solution with complex roots

(underdamped)

$$s_1 = \lambda + \mu i$$
$$s_2 = \lambda - \mu i$$

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Euler's formula: $e^{ix} = \cos x + i \sin x$

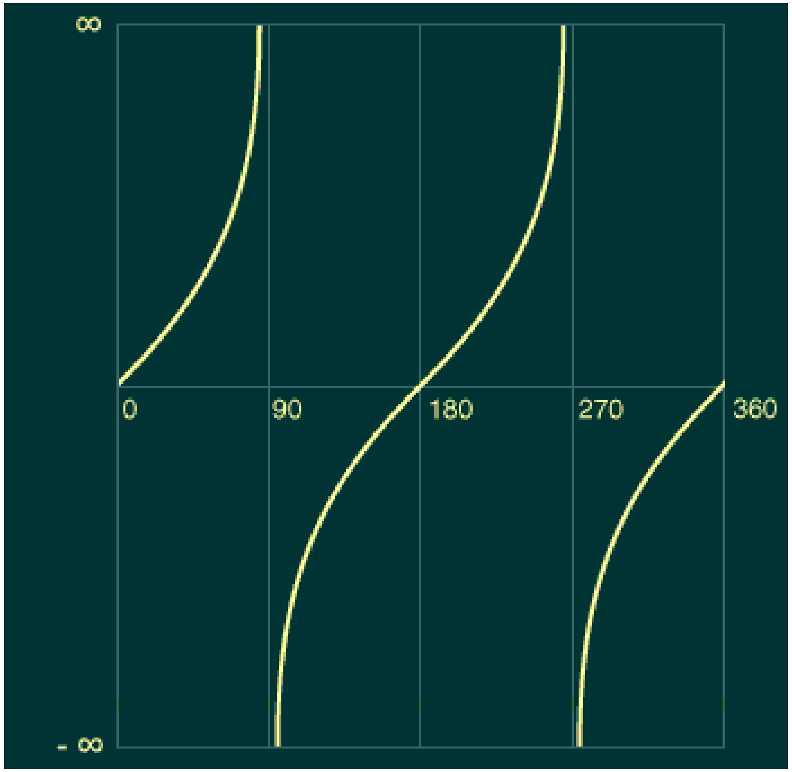
$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

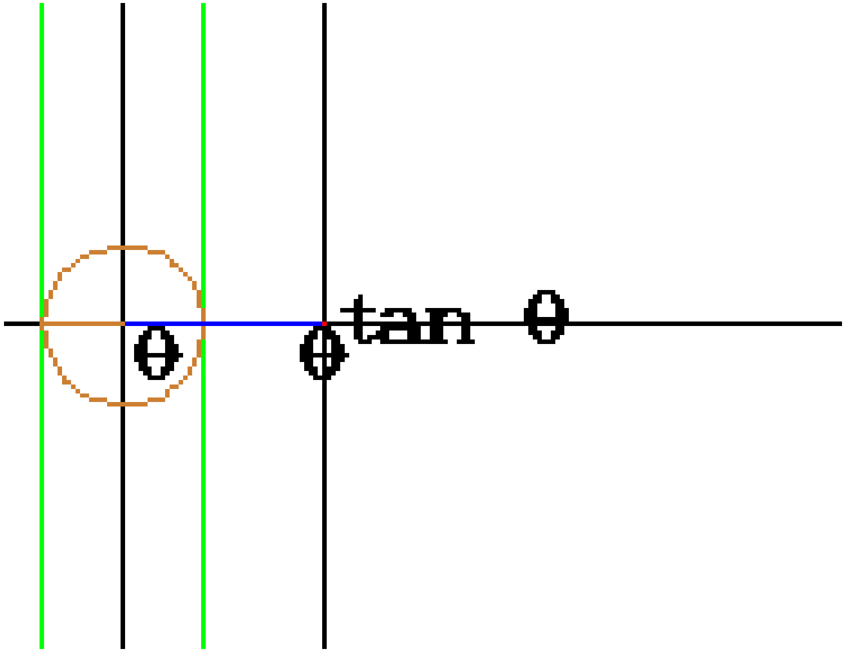
with
$$c_1 = r \cos \delta$$
$$c_2 = r \sin \delta$$

becomes $x(t) = r e^{\lambda t} \cos(\mu t - \delta)$

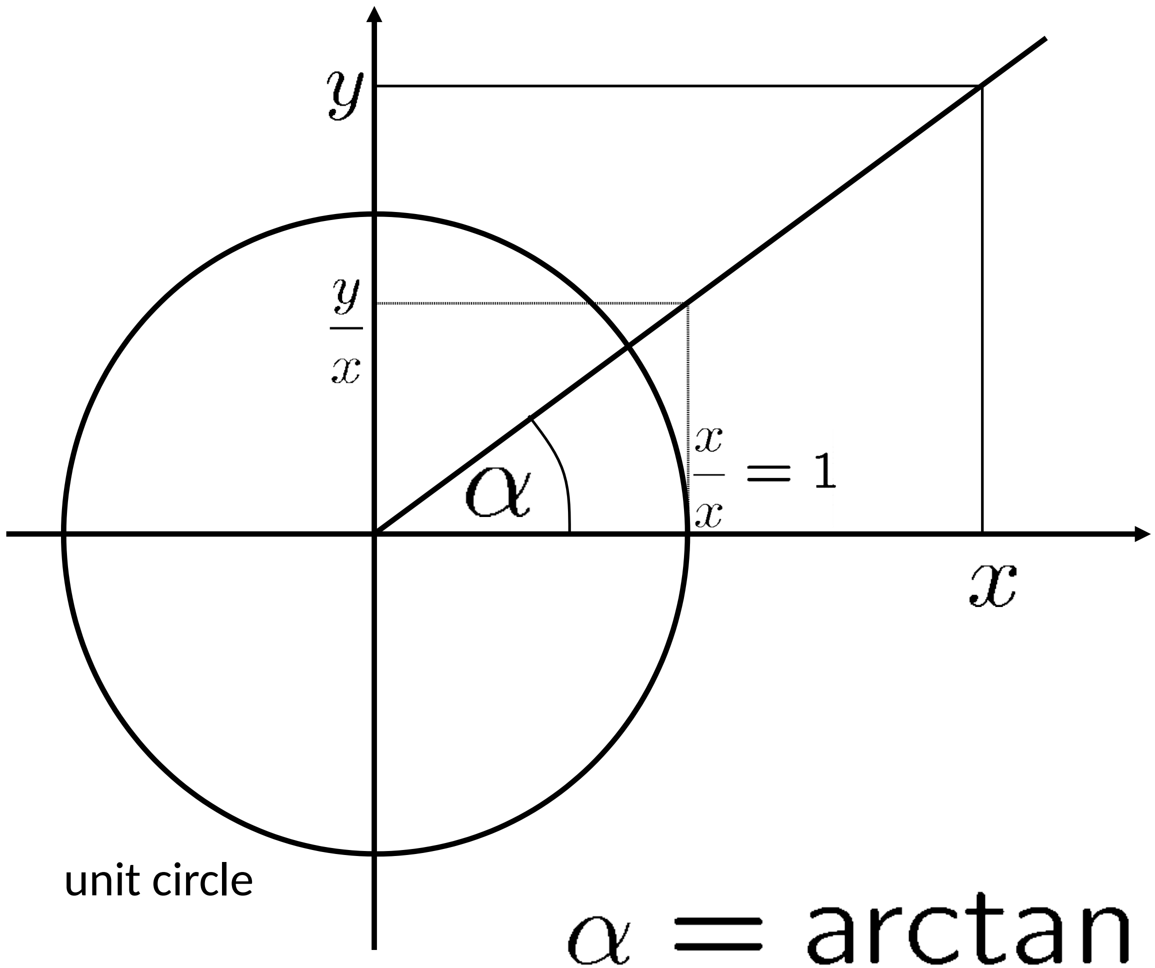
where
$$r = \sqrt{c_1^2 + c_2^2}$$
$$\delta = \operatorname{atan2}(c_2, c_1)$$

Sidebar: tangent



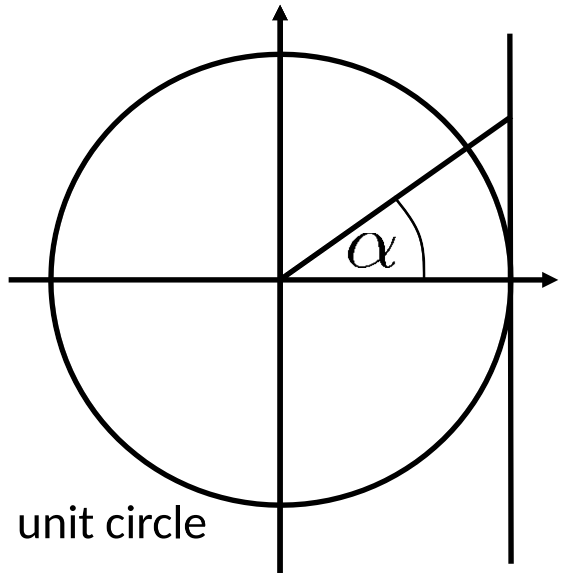


arctangent



$$\alpha = \arctan\left(\frac{y}{x}\right)$$

atan2(y,x)



$$\mathrm{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \mathrm{sign}(y)\left(\pi - \arctan(|\frac{y}{x}|)\right) & \text{if } x < 0 \\ \mathrm{sign}(y)\frac{\pi}{2} & \text{if } x = 0 \text{ and } y \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Solution with real, repeated roots

(critically damped)

Solution: $$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

with: $$s_1 = s_2 = -\frac{b}{2m} t$$

$$x(t) = (c_1 + c_2 t)\ e^{-\frac{b}{2m} t}$$

L'Hôpital's rule: $$\text{if } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty \text{ and } \lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists}$$

$$\text{then } \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$$\lim_{t \to \infty} (c_1 + c_2 t) e^{-at} = 0 \quad \text{for any } c_1, c_2, a$$

Damping Ratio & Natural Frequency

Original characteristic equation: $$ms^2 + bs + k = 0$$

Alternative characteristic equation: $$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

with:

$$\zeta = \frac{b}{2\sqrt{km}}$$ damping ratio

$$\omega_n = \sqrt{\frac{k}{m}}$$ natural frequency

Relationship to λ and μ from first characteristic eqn.:

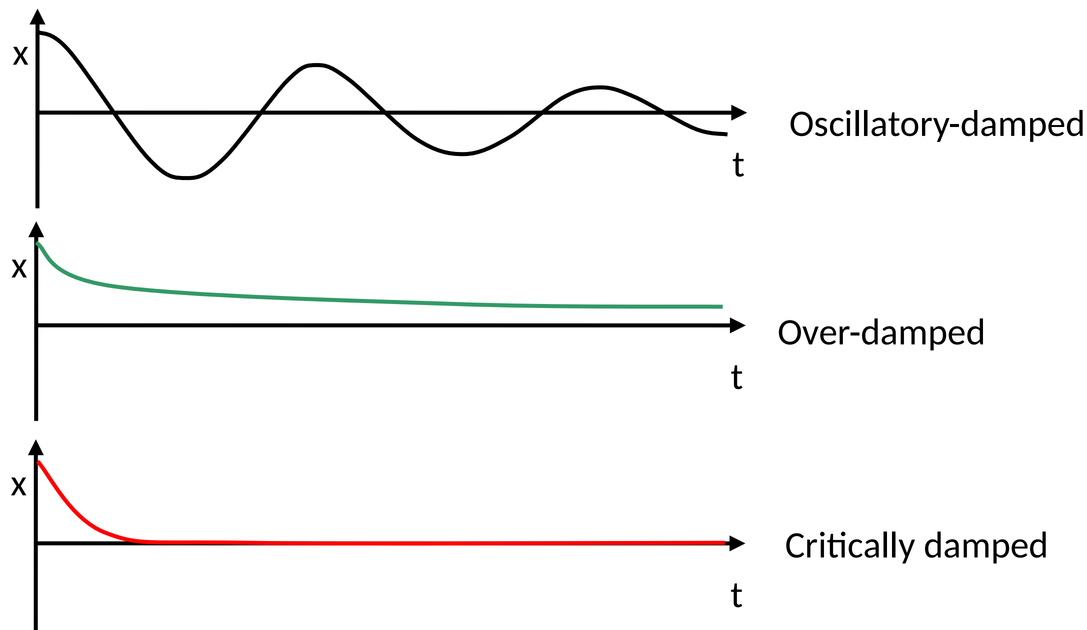
$$\lambda = -\zeta\omega_n$$

$$\mu = \omega_n\sqrt{1-\zeta^2}$$ damped natural frequency

Values of ζ

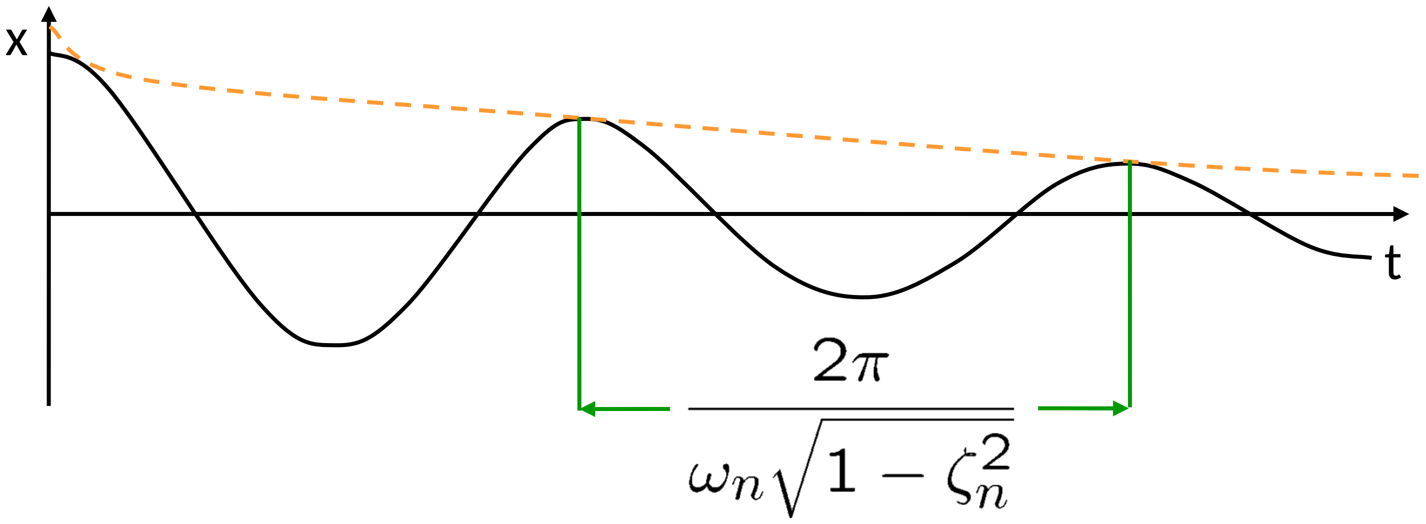
- ζ> 1 : over-damped
- ζ = 1 : critically damped
- ζ < 1 : under-damped

$$\zeta_n = \frac{b}{2\,\omega_n m} = \frac{b}{2\sqrt{km}}$$



Damped Natural Frequency ω

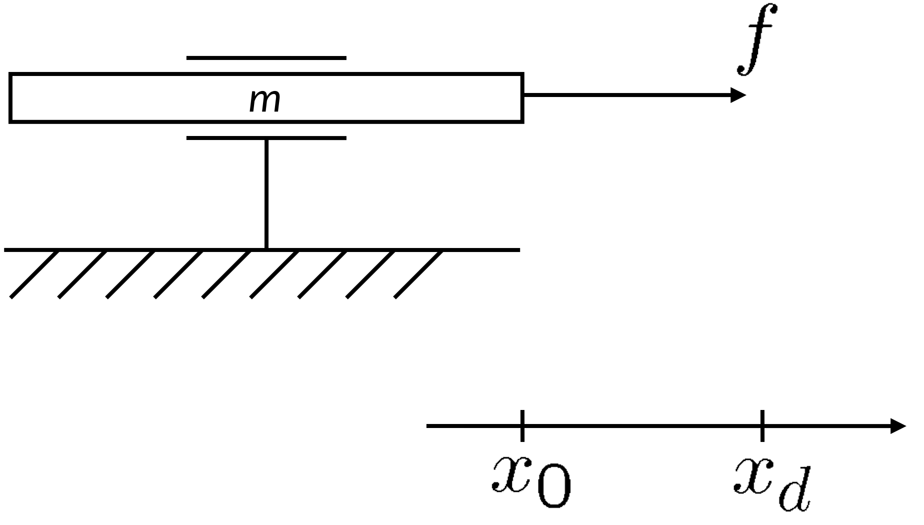
$$x(t) = ce^{-\zeta_n t}\ \cos\left(t\,\omega_n \sqrt{1-\zeta_n^2} + \phi\right)$$



Damped Natural Frequency:

$$\omega = \omega_n\sqrt{1-\zeta_n^2}$$

Application to Robot Control



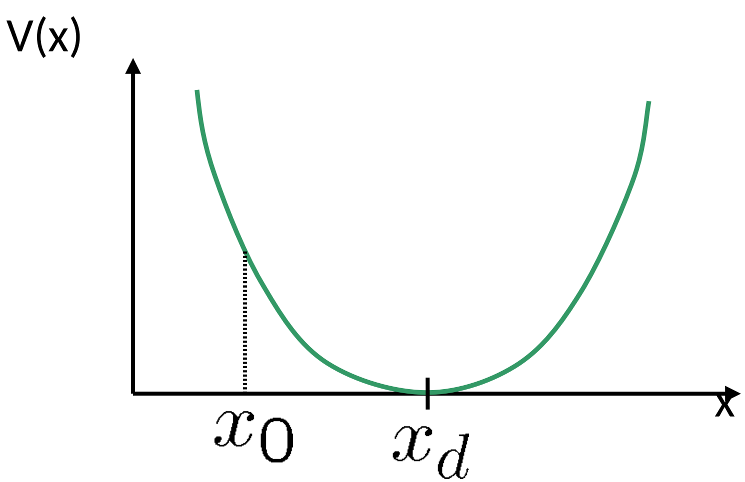
Proportional Control

Idea: apply force proportional to error

$$f = -\boxed{k_p}(x - x_d)$$

position gain

$$V(x) = \frac{1}{2}k_p(x - x_d)^2$$

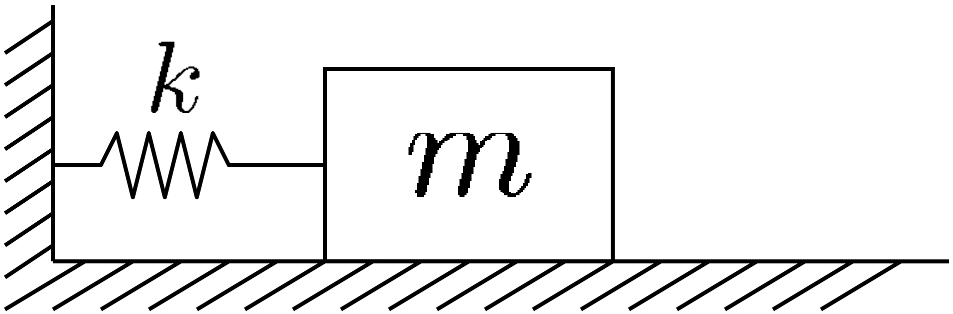


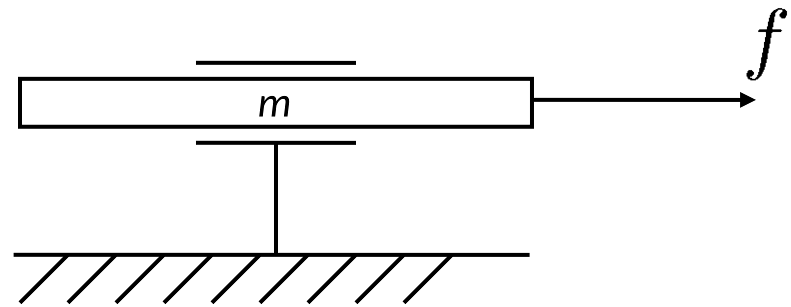
$$\mathbf{F} = -\nabla V(x) = -\frac{\partial V}{\partial x}$$

$$m\ddot{x} = \mathbf{F} = -\frac{\partial}{\partial x}\left[\frac{1}{2}k_p(x - x_p)^2\right]$$

$$m\ddot{x} + k_p(x - x_d) = 0$$

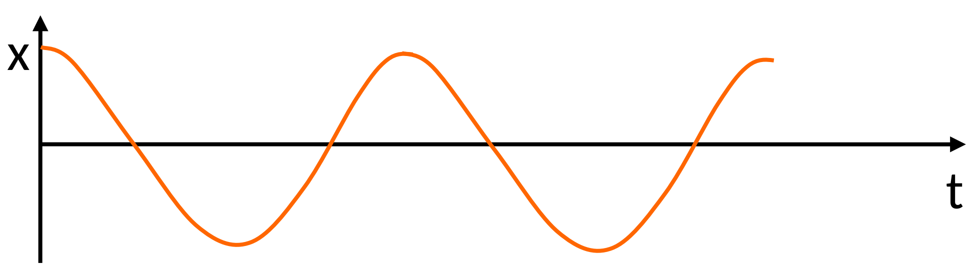
Comparison





$$m\ddot{x} + kx = 0$$

$$m\ddot{x} + k_p(x - x_d) = 0$$



closed loop frequency

$$\omega = \sqrt{\frac{k_p}{m}}$$

Introduction of Dissipation

Idea: apply force opposing velocity

$$f = -k_p(x - x_d) - \boxed{k_v}\,\dot{x}$$

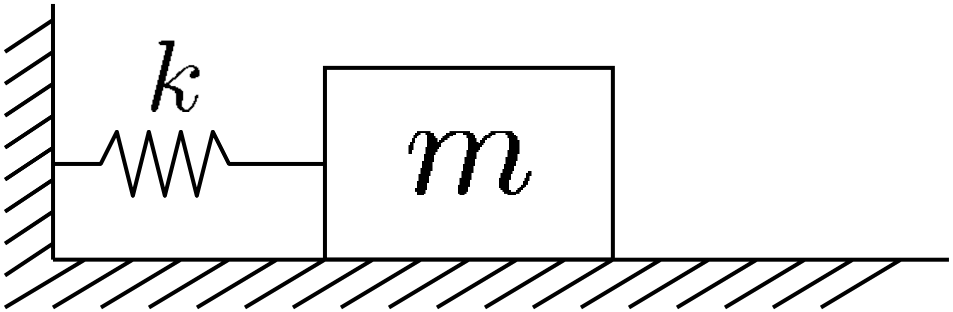
velocity gain

Asymptotic stability condition:

$$\dot{x}^T\ \tau_{\text{dissipation}} < 0, \qquad \text{for } \dot{x} \neq 0$$

$$\dot{x}^T(-k_v\dot{x}) = -k_v\dot{x}^2 < 0, \qquad \text{for } k_v > 0, \dot{x} \neq 0$$

Designing a Linear Controller



$$f = m\ddot{x} + b\dot{x} + kx$$

$$f = -k_p x - k_v\,\dot{x}$$

$$m\ddot{x} + b\dot{x} + kx = -k_p x - k_v \dot{x}$$

$$m\ddot{x} + (b + k_v)\dot{x} + (k + k_p)x = 0$$

physical system

control parameters

Linear Controller cont.

$$m\ddot{x} + (b + k_v)\dot{x} + (k + k_p)x = 0$$

$$m\ddot{x} + b'\dot{x} + k'x = 0$$

determines damping

closed-loop stiffness

for critical damping: $$b' = 2\sqrt{mk'}$$

Example: $$m = b = k = 1 \quad k' = 16$$

$$b' = 2\sqrt{mk'} = 2\sqrt{1 \cdot 16} = 8 \quad \text{for critical damping}$$

$$\Rightarrow \quad k_p = 15 \quad k_v = 7$$

Proportional Derivative (PD) Control

$$f = -k_p(x - x_d) - k_v\,\dot{x}$$

Propotional to reduce error
Derivative (velocity) to introduce dissipation

$$m\ddot{x} + k_v\dot{x} + k_p x = k_p x_d$$

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = \omega^2 x_d$$

closed-loop frequency

closed-loop damping ratio

$$\omega^2 = \frac{k_p}{m} \qquad \zeta = \frac{k_v}{2\sqrt{k_p m}}$$

$$k_p = m\,\omega^2 \qquad k_v = m\,(2\zeta\omega)$$

The Real EOM in Joint Space

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau$$

Linear versus Nonlinear Control

- Linear control $m\ddot{x} + b\dot{x} + kx = 0$
  - systems described by linear differential eqns.
  - commonly used in industrial robots
  - dynamics are non-linear
  - often this approximation is useful
- Nonlinear control
  - various forms of linearization
  - address nonlinearities

Control Law Partitioning

- Idea: decomposition of control into **model** and **servo** portions
- Extracts physical parameters from control problem $\Rightarrow$unit mass system without friction etc.
- Will be used to deal with nonlinear systems

Control Law Partitioning

model portion

$$f = \alpha f' + \beta$$

servo portion

$$m\ddot{x} + b\dot{x} + kx = \alpha f' + \beta$$

$$\alpha = m \qquad \beta = b\dot{x} + kx$$

Equations of Motion of unit mass:

$$\ddot{x} = f'$$

Unit Mass Controller

$$f' = -k_v \dot{x} - k_p x$$

with $\ddot{x} = f'$ yields

$$\ddot{x} + k_v \dot{x} + k_p x = 0$$

for critical damping: $k_v = 2\sqrt{k_p}$

independent of physical system!

Trajectory or Motion Control

A trajectory specifies as a function of time:

$$x_d(t), \dot{x}_d(t), \ddot{x}_d(t)$$

Error is defined as

$$e \stackrel{\text{def.}}{=} e(t) = x_d(t) - x(t)$$

$$f' = \ddot{x} = \ddot{x}_d + k_v \dot{e} + k_p e$$

$$\ddot{e} + k_v \dot{e} + k_p e = 0$$

Disturbance Rejection

$$\ddot{e} + k_v \dot{e} + k_p e = f_{\text{disturbance}}$$

for bounded disturbances we can guarantee stability

Steady state error:

$$k_p e = f_{\text{disturbance}} \Rightarrow e = \frac{f_{\text{disturbance}}}{k_p}$$

Error will never be zero in the presence of a disturbance since k_p cannot be ∞

Integral Term gives PID Control

$$f' = \ddot{x}_d + k_v\,\dot{e} + k_p\,e + k_i\int e\,dt$$

$$\ddot{e} + k_v\,\dot{e} + k_p\,e + k_i\int e\,dt = f_{\text{disturbance}}$$

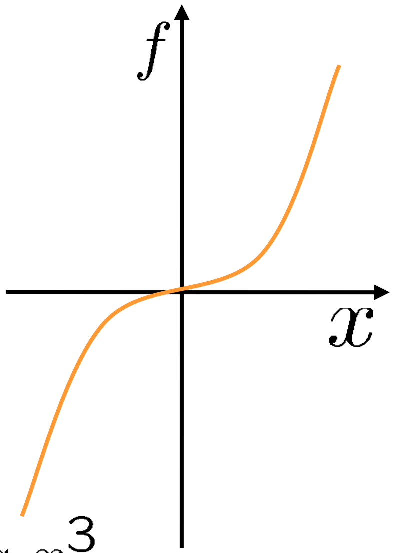
P = proportional
I = integral
D = derivative
control (feedback)

For simplicity, we will not consider the integral term.

Linearization

- Control a nonlinear system
- Use control law partitioning to extract nonlinear portion
- Counteract or cancel nonlinear portion
- Achieve overall linear behavior of system
- Allows to treat the system as a unit mass

Linearization using Partitioning



$f = q\,x^3$

$$m\,\ddot{x} + b\,\dot{x} + q\,x^3 = f$$

$$\alpha = m$$
$$\beta = b\,\dot{x} + q\,x^3$$

$$f' = \ddot{x}_d + k_v\,\dot{e} + k_p\,e$$

Computed Torque Method

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau$$

$$\tau = \alpha\,\tau' + \beta$$

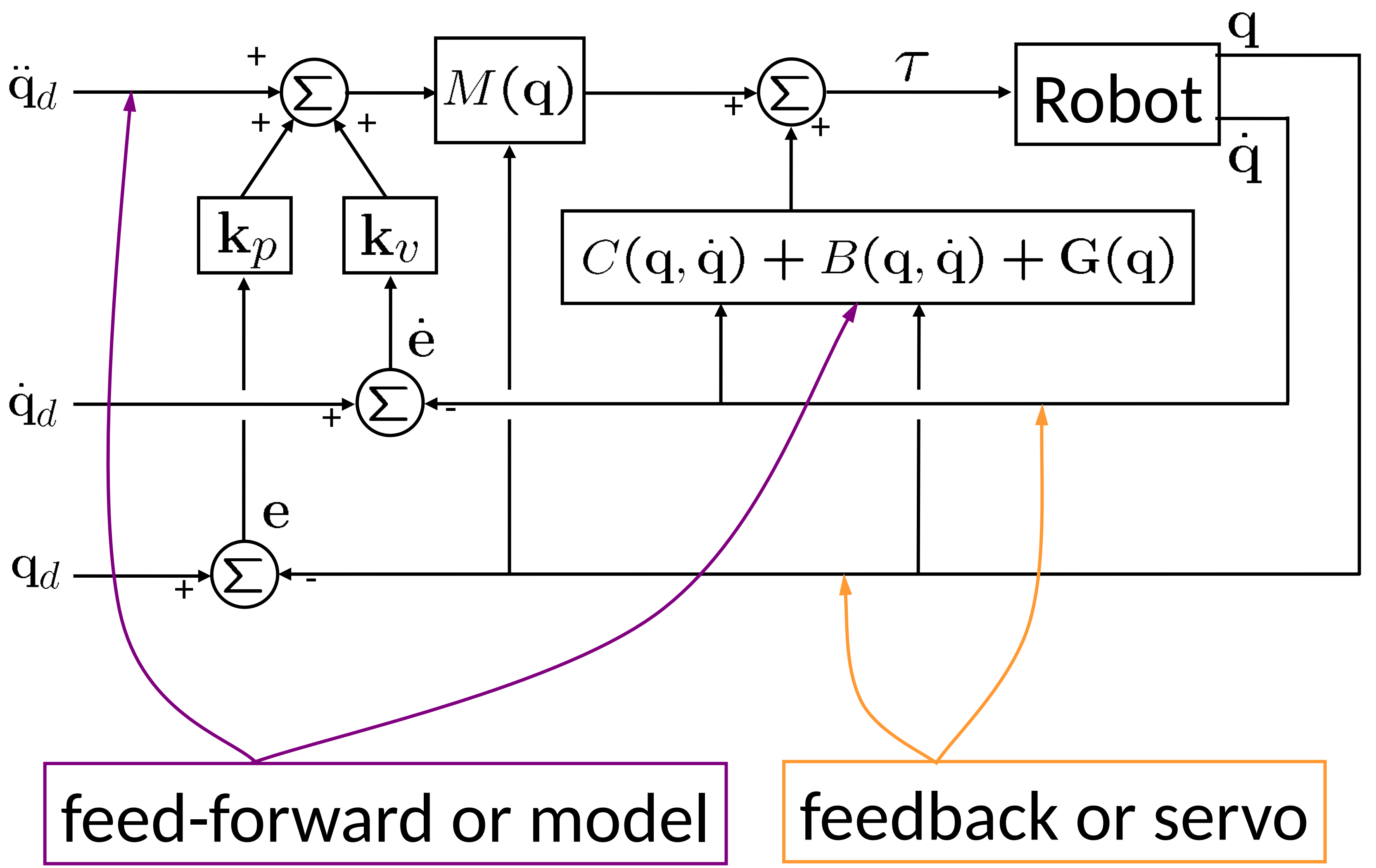
$$\alpha = M(\mathbf{q})$$

$$\beta = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q})$$

$$\tau' = \ddot{\mathbf{q}}_d + \mathbf{k}_v\,\dot{\mathbf{e}} + \mathbf{k}_p\,\mathbf{e}$$

Note that we have gone from a single mass to a system of masses!

The Controller



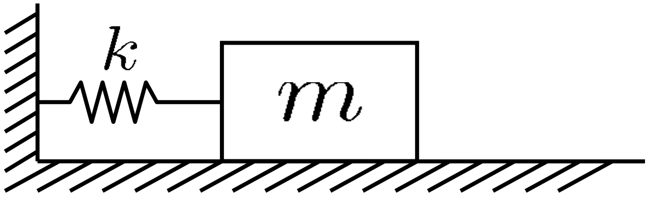
Recap

- Single mass with spring (and damper)
- Characteristics of motion
- Design controller to achieve desired behavior for linear system
- Partitioning for linearization of nonlinear system
- "Vectorization" for unified approach to *controlling a manipulator* with many d.o.f.

Stability Analysis

- In a linear system stability requires $k_v > 0$
- Assuming bounded disturbance we can make certain guarantees
- Analysis more complex in nonlinear systems
- Linearization is not always possible
  - inaccurate models
  - unknown models

Energy-Based Stability Analysis



$$m\ddot{x} + b\dot{x} + kx = 0$$

$$E = \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}k\,x^2$$

$$\begin{aligned}\dot{E} &= m\ddot{x}\dot{x} + kx\dot{x} \\ &= (-b\dot{x} - kx)\dot{x} + kx\dot{x} \\ &= -b\dot{x}^2 \\ &< 0\end{aligned}$$

Energy of system is reduced until it comes to rest at x = 0

Lyapunov Stability Theory

- Energy based example is an instance of Lyapunov method
- Applies to linear and nonlinear systems
- Stability analysis, but no performance analysis
- Aleksandr Mikhailovich Lyapunov, (1857-1918), friend of Markov and Chebychev

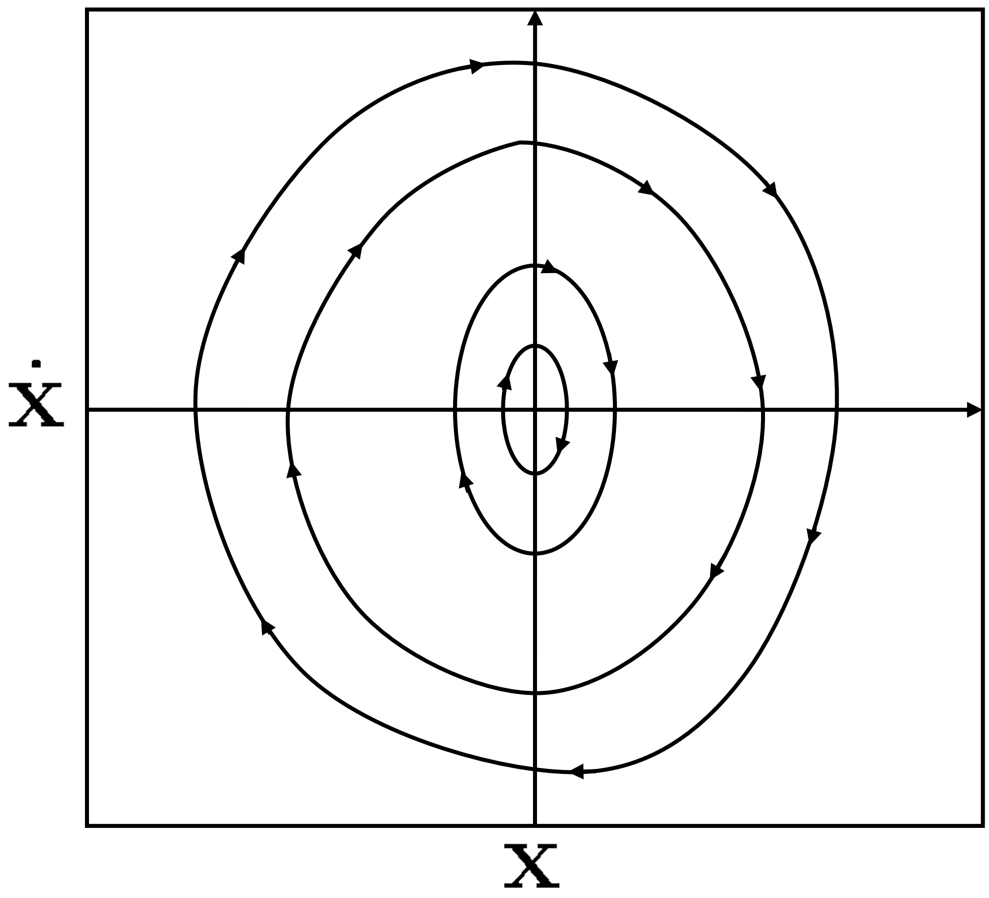
Lyapunov's Second Method

- Also called "direct" method
- Determines stability of differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

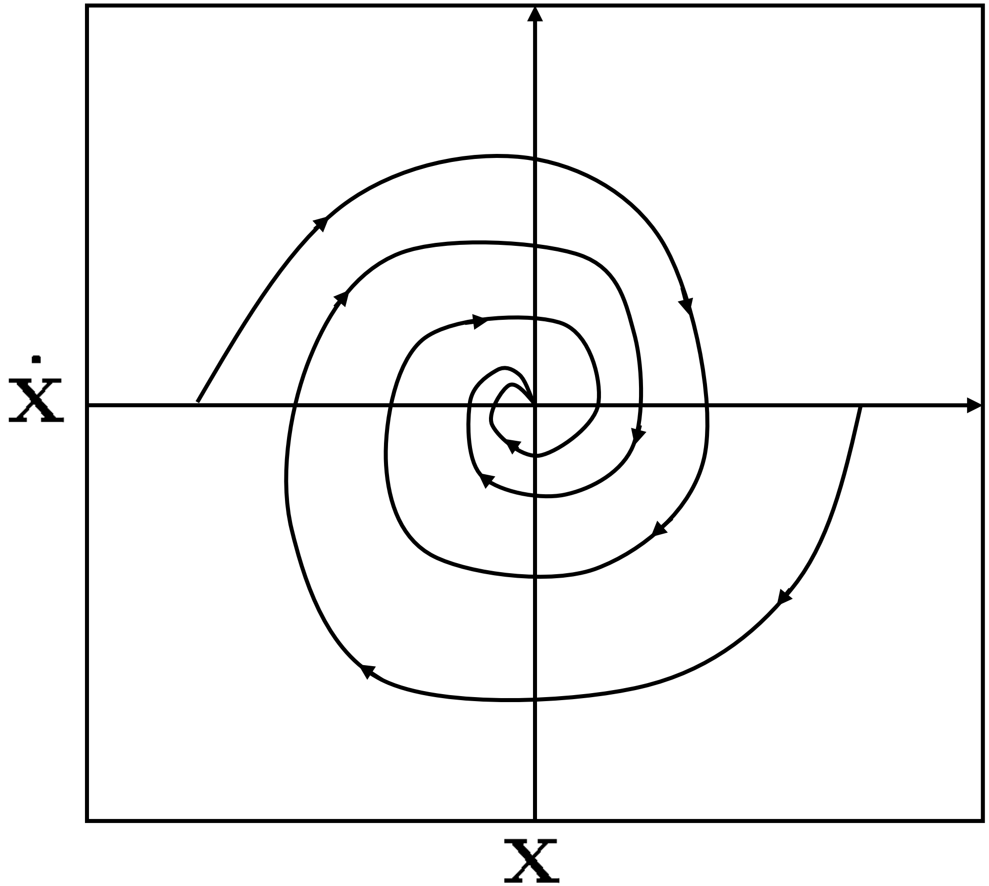
- Requires energy function $E(\mathbf{x})$
  - with continuous first partial derivatives
  - $\forall \mathbf{x} : E(\mathbf{x}) > 0$ except for $E(0) = 0$
  - and such that $\dot{E}(\mathbf{x}) \leq 0$
- Energy-like function that always decreases

Phase Plot: Lyapunov Stable

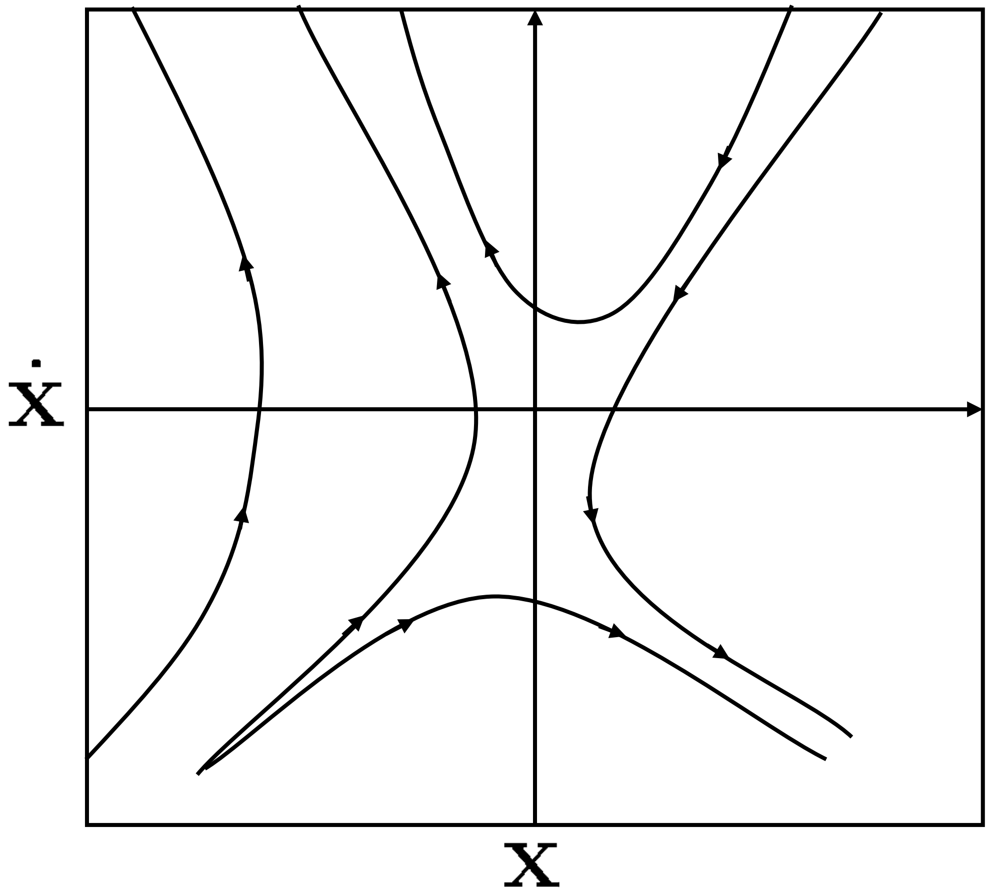


phase plot for $\dot{E}(\mathbf{x}) = 0$

Phase Plot: Asymptotically Stable



Phase Plot: Unstable



Stability of Computed Torque

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \tau$$

$$\tau = K_p\mathbf{e} - K_v\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q})$$

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) + K_v\dot{\mathbf{q}} + K_p\mathbf{q} = K_p\,\mathbf{q}_d$$

Energy function:

$$E = \frac{1}{2}\dot{\mathbf{q}}^T M(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}\mathbf{e}^T K_p\mathbf{e}$$

always positive because M, K_p positive definite

$$\begin{aligned}\dot{E} &= \frac{1}{2}\dot{\mathbf{q}}^T\dot{M}(\mathbf{q})\dot{\mathbf{q}} + \dot{\mathbf{q}}^T M(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{e}^T K_p\dot{\mathbf{q}} \\ &= \frac{1}{2}\dot{\mathbf{q}}^T\dot{M}(\mathbf{q})\dot{\mathbf{q}} - \dot{\mathbf{q}}^T K_v\dot{\mathbf{q}} - \dot{\mathbf{q}}^T\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) \\ &= -\dot{\mathbf{q}}^T K_v\dot{\mathbf{q}}\end{aligned}$$

$$\frac{1}{2}\dot{\mathbf{q}}^T\dot{M}(\mathbf{q})\dot{\mathbf{q}} = \dot{\mathbf{q}}^T\mathbf{v}(\mathbf{q},\dot{\mathbf{q}})$$

always non-positive for K_v positive definite

Asymptotic Stability?

$$\dot{E} = -\dot{\mathbf{q}}^T K_v \dot{\mathbf{q}} = 0 \quad \Rightarrow \quad \ddot{\mathbf{q}} = \dot{\mathbf{q}} = 0$$

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) + K_v \dot{\mathbf{q}} + K_p \mathbf{q} = K_p \, \mathbf{q}_d$$

$$K_p \mathbf{e} = 0 \quad \Rightarrow \quad \mathbf{e} = 0$$

YES!

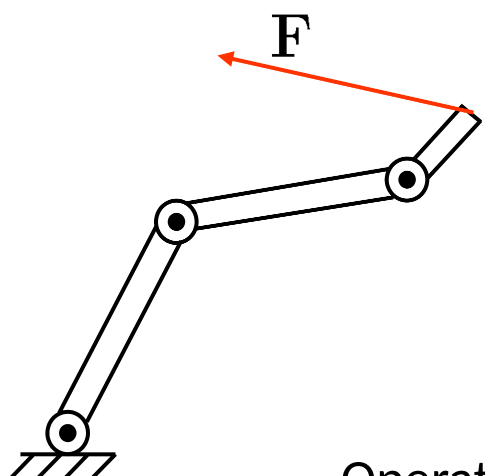
Lyapunov’s “First” Method

- Called indirect method of Lyapunov
- Uses linearization for nonlinear systems
- Stability of local linearization determines stability of original nonlinear equations
- We won’t look at it here…

Effector Inertia Matrix Λ

Joint space: $M(\mathrm{q})\ddot{\mathrm{q}} + C(\mathrm{q})[\dot{\mathrm{q}}^2] + B(\mathrm{q})[\dot{\mathrm{q}}\dot{\mathrm{q}}] + \mathbf{G}(\mathrm{q}) = \tau$

Inertia perceived at the joints



$m = ?$

Inertia perceived at effector?

Operational space inertia matrix $\Lambda(\mathbf{x})$

Equations of Motion

Joint Space

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \tau$$

Operational Space

$$\Lambda(\mathbf{x})\ddot{\mathbf{x}} + \mu(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{p}(\mathbf{x}) = \mathbf{F}$$