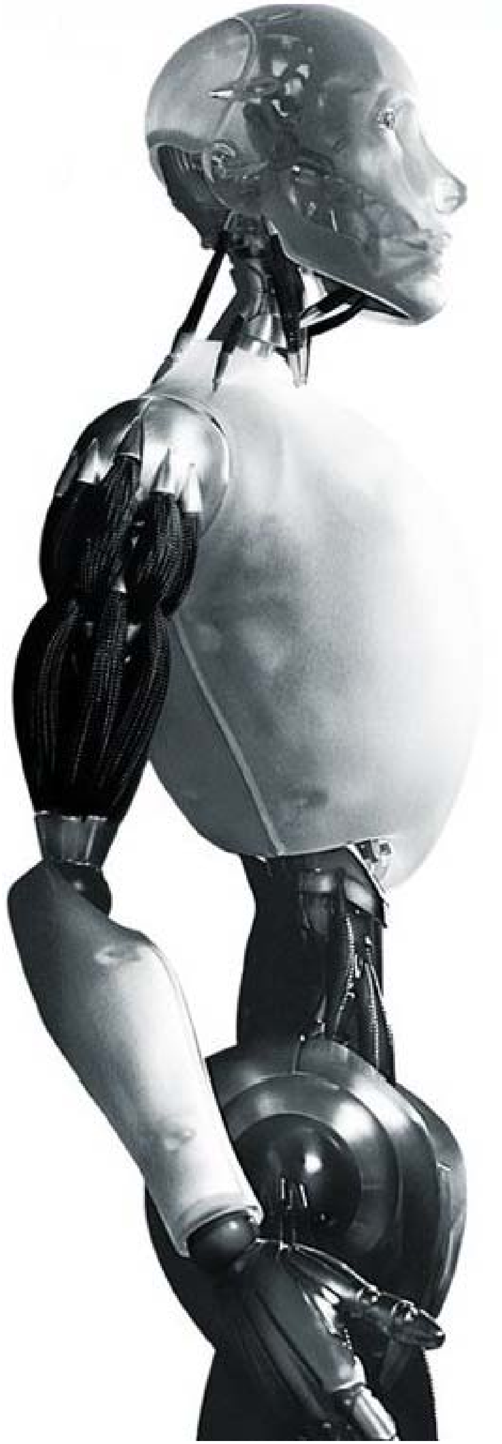


Robotics

Dynamics

TU Berlin

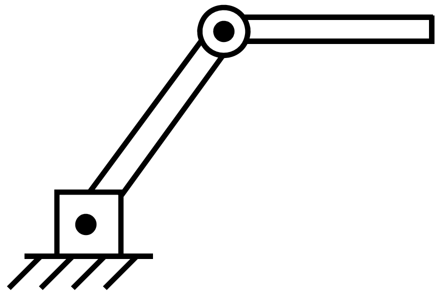
Oliver Brock

Dynamics

- *Kinematics*: branch of dynamics that deals with aspects of motion apart from considerations of mass and force
- *Dynamics*: branch of mechanics that deals with forces and their relation primarily to the motion but sometimes also to the equilibrium of bodies
- *Mechanics*: branch of physical science that deals with energy and forces and their effect on bodies
- What about the homework?

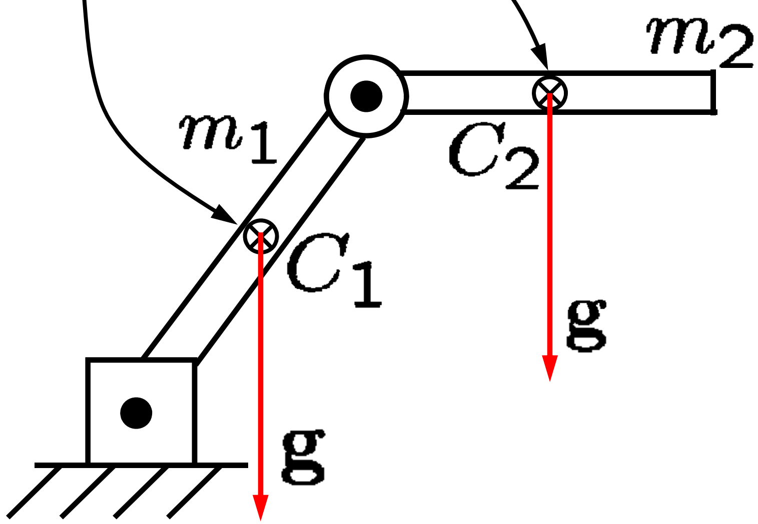
How Dynamics affect Robots

- Gravity
- Mass / Inertia
- Centrifugal Forces
- Coriolis Forces
- Dependencies:
  - Gravity: configuration
  - Mass / Inertia: acceleration
  - Centrifugal Forces: velocity
  - Coriolis Forces: velocity
- All of these depend on the inertia!



Gravity

Center of Mass



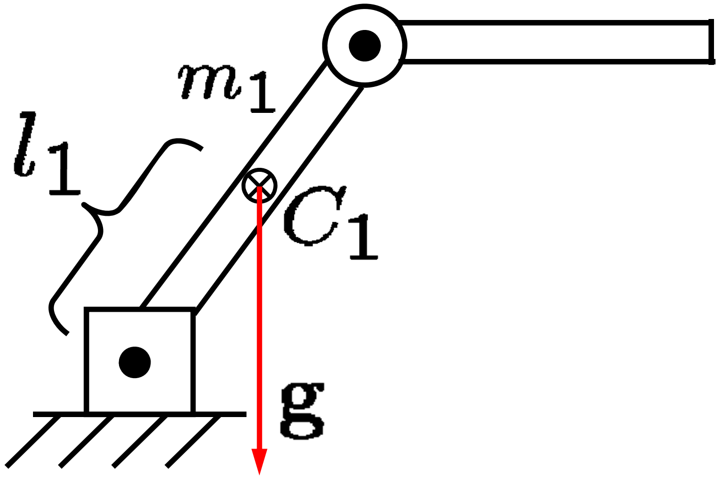
Linear part of Jacobian at Center of Mass 1

$$\tau_{m_1} = -J_{v_1}^T\,(\mathbf{g}\,m_1)$$

$$\tau_{m_2} = -J_{v_2}^T\,(\mathbf{g}\,m_2)$$

$$\mathbf{G} = -\sum_{i=0}^{n} J_{v_i}^T\,(\mathbf{g}\,m_i)$$

Gravity Example



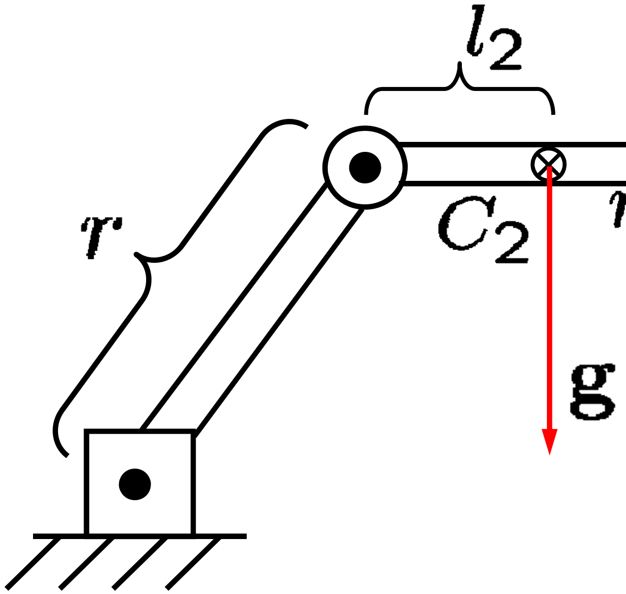
$$\tau_{m_1} = -J_{c_1}^T \, (\mathbf{g} \, m_1)$$

$${}^0\mathbf{p}_{C_1} = \begin{pmatrix} l_1 \, c_1 \\ l_1 \, s_1 \end{pmatrix}$$

$${}^0J_{v_1} = \begin{bmatrix} -l_1 \, s_1 & 0 \\ l_1 \, c_1 & 0 \end{bmatrix}$$

$$\tau_{m_1} = -\begin{bmatrix} -l \, s_1 & l \, c_1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ -\mathbf{g} \, m_1 \end{pmatrix} = \begin{pmatrix} l \, c_1 \, \mathbf{g} \, m_1 \\ 0 \end{pmatrix}$$

Gravity Example cont.



$$\tau_{m_2} = -J_{c_2}^T \left(\mathbf{g}\, m_2\right)$$

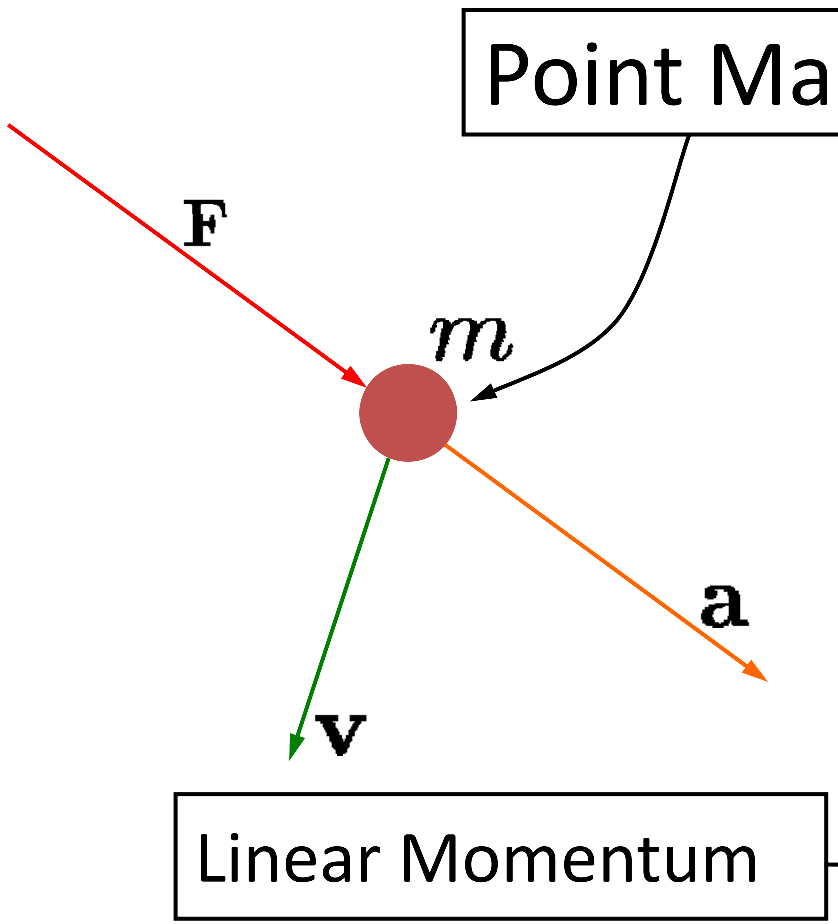
$$^0\mathbf{p}_{C_2} = \begin{pmatrix} r\, c_1 + l_2\, c_{12} \\ r\, s_1 + l_2\, s_{12} \end{pmatrix}$$

$$^0J_{v_2} = \begin{bmatrix} -r\, s_1 - l_2\, s_{12} & -l_2\, s_{12} \\ r\, c_1 + l_2\, c_{12} & l_2\, c_{12} \end{bmatrix}$$

$$\tau_{m_2} = -\begin{bmatrix} -r\, s_1 - l_2\, s_{12} & r\, c_1 + l_2\, c_{12} \\ -l_2\, s_{12} & l_2\, c_{12} \end{bmatrix} \begin{pmatrix} 0 \\ -\mathbf{g}\, m_2 \end{pmatrix} =$$

$$\begin{pmatrix} (r\, c_1 + l_2\, c_{12})\, \mathbf{g}\, m_2 \\ l_2\, c_{12}\, \mathbf{g}\, m_2 \end{pmatrix}$$

Linear Momentum



$$E_{kin} = \frac{1}{2} m \mathbf{v}^2$$

$$\mathbf{a} = \frac{\mathbf{F}}{m}$$

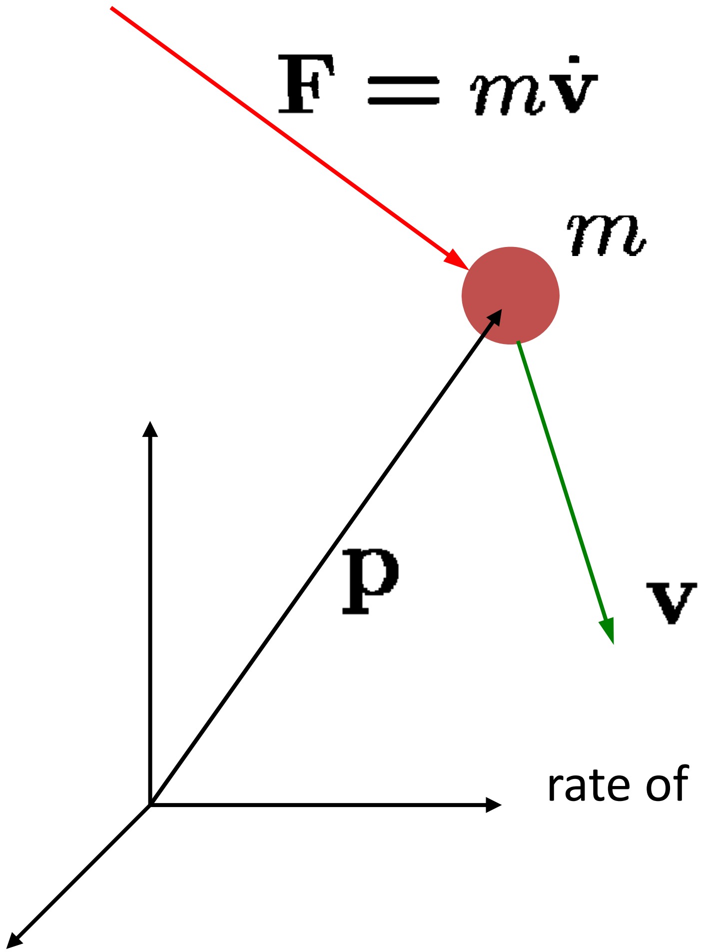
$$m\,\mathbf{a} = \mathbf{F}$$

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$

$$m\dot{\mathbf{v}} = \mathbf{F}$$

rate of change of linear momentum = applied force

Angular Momentum



$$N = \mathbf{p} \times m\dot{\mathbf{v}} = \mathbf{p} \times \mathbf{F}$$

$$\begin{aligned} \frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) &= \mathbf{p} \times m\dot{\mathbf{v}} + \mathbf{v} \times m\mathbf{v} \\ &= \mathbf{p} \times m\dot{\mathbf{v}} \end{aligned}$$

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{N}$$

rate of change of angular momentum = applied moment

Linear and Angular Momentum

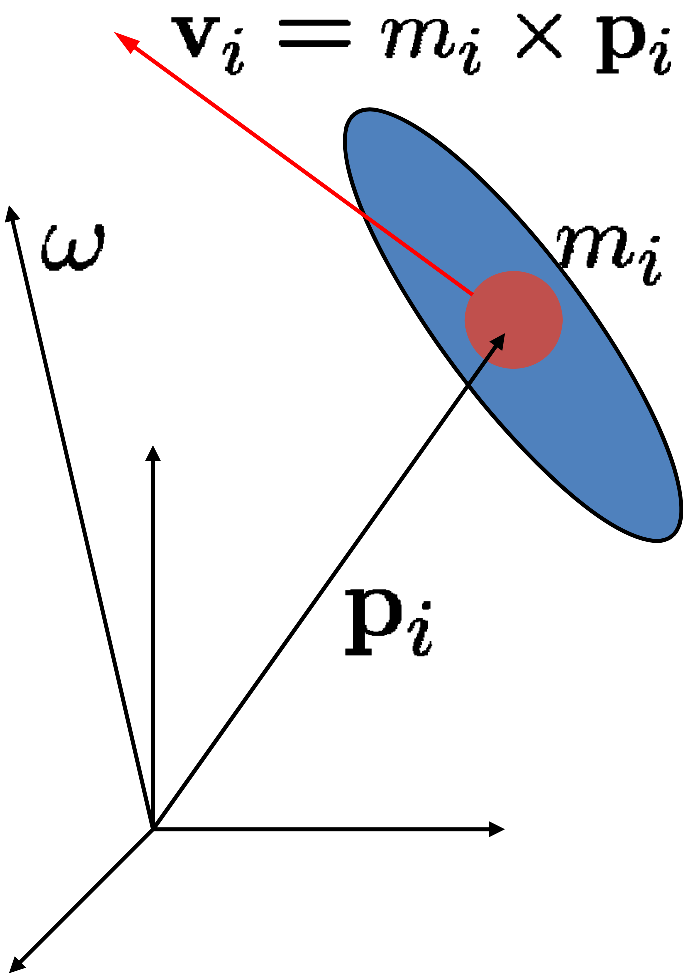
$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$

rate of change of linear momentum = applied force

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{N}$$

rate of change of angular momentum = applied moment

Angular Momentum Φ of Rigid Body



$$\mathbf{\Phi} = \sum_i m_i \mathbf{p}_i \times (\omega \times \mathbf{p}_i)$$

$$\mathbf{\Phi} = \int_V \mathbf{p} \times (\omega \times \mathbf{p})\ \rho\, dv$$

$$\mathbf{\Phi} = \omega \underbrace{\int_V -\hat{\mathbf{p}}\ \hat{\mathbf{p}}\ \rho\, dv}_{\text{Inertia Tensor } I}$$

$$\mathbf{\Phi} = I\ \omega$$

$$\dot{\mathbf{\Phi}} = I\ \dot{\omega} + \omega \times I\ \omega = \mathbf{N}$$

Euler's Equation

Newton-Euler Formulation

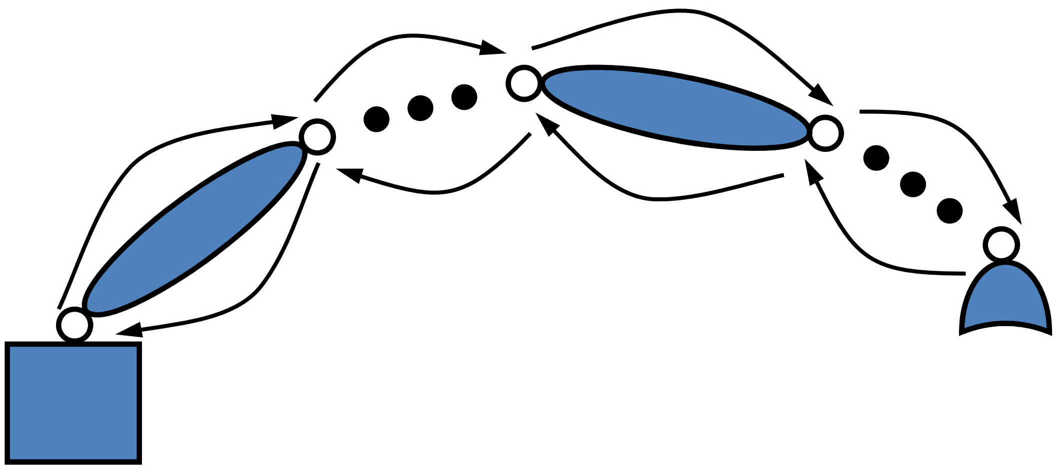
$$m\,\dot{\mathbf{v}} = m\,\mathbf{a} = \mathbf{F}$$

rate of change of linear momentum = applied force

$$\dot{\Phi} = I\,\dot{\omega} + \omega \times I\,\omega = \mathbf{N}$$

rate of change of angular momentum = applied moment

Iterative Newton-Euler Dynamic Formulation



Iteration outward: compute linear and angular link accelerations

At the same time: use Newton-Euler to compute *inertial force* and *inertial torque* acting at the center of mass of each link

Propagate inwards: compute forces and torques at each joint

Inertia Tensor

$$\Phi = \omega \underbrace{\int_V -\hat{\mathbf{p}}\,\hat{\mathbf{p}}\,\rho\,dv}_{\text{Inertia Tensor } I} \qquad \mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$-(\hat{\mathbf{p}}\,\hat{\mathbf{p}}) = (\mathbf{p}^T\mathbf{p})\,I_3 - \mathbf{p}\mathbf{p}^T = \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Inertia Tensor cont.

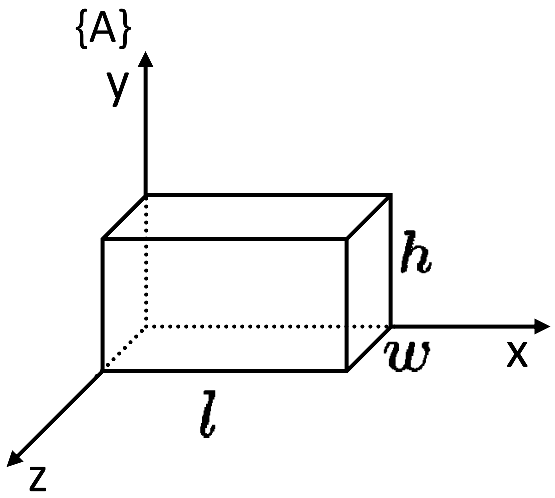
$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

frame-dependent!

$$\begin{aligned}
I_{xx} &= \int\int\int (y^2 + z^2)\, \rho\, dx\, dy\, dz \\
I_{yy} &= \int\int\int (x^2 + z^2)\, \rho\, dx\, dy\, dz \\
I_{zz} &= \int\int\int (x^2 + y^2)\, \rho\, dx\, dy\, dz \\
I_{xy} &= \int\int\int xy\, \rho\, dx\, dy\, dz \\
I_{xz} &= \int\int\int xz\, \rho\, dx\, dy\, dz \\
I_{yz} &= \int\int\int yz\, \rho\, dx\, dy\, dz
\end{aligned}$$

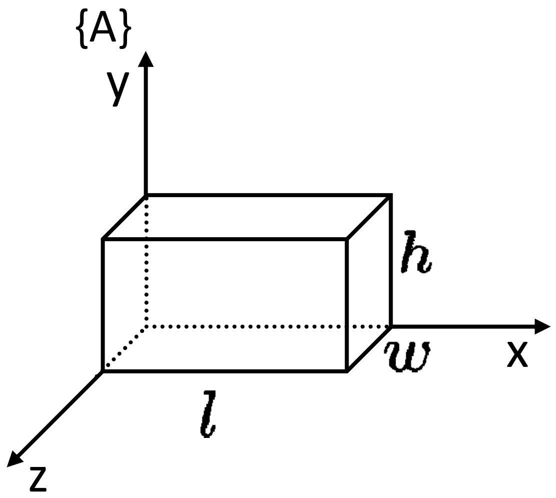
diagonal elements: mass moments of inertia
off-diagonal elements: mass products of inertia

Inertia Tensor Example



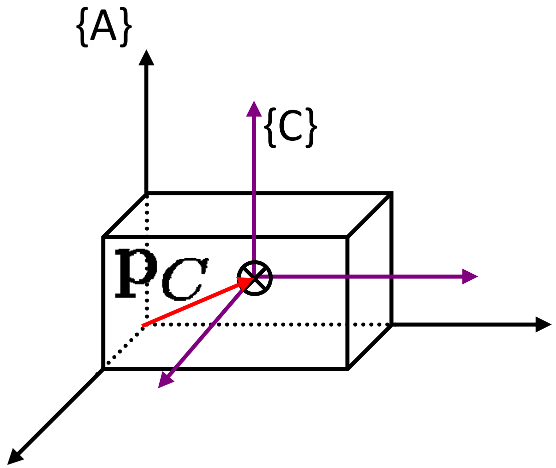
$$
\begin{aligned}
{}^{A}I_{xx} &= \int_0^h \int_0^l \int_0^w (y^2 + z^2)\ \rho\ dx\ dy\ dz \\
&= \int_0^h \int_0^l (y^2 + x^2)\ w\ \rho\ dy\ dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l\right) w\ \rho\ dz \\
&= \left(\frac{hl^3w}{3} + \frac{h^3lw}{3}\right)\ \rho \\
&= \frac{m}{3}(l^2 + h^2)
\end{aligned}
$$

Inertia Tensor Example cont.



$$
{}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & -\frac{m}{4}wl & -\frac{m}{4}hw \\ -\frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}hw & -\frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}
$$

Parallel Axis Theorem



$$ {}^{A}I = {}^{C}I + m\left[\left(\mathbf{p}_C^T\mathbf{p}_C\right) I_3 - \mathbf{p}_C\mathbf{p}_C^T\right] $$

Translation of inertia tensor expressed
at center of mass

Rotation Transformation

$$^{B}\Phi = {}^{A}I\ {}^{A}\omega$$

$$\begin{aligned}
^{B}\Phi &= {}^{B}_{A}R\ {}^{A}\Phi \\
&= {}^{B}_{A}R\ {}^{A}I\ {}^{A}\omega \\
&= {}^{B}_{A}R\ {}^{A}I\ ({}^{A}_{B}R\ {}^{B}\omega) \\
&= {}^{B}_{A}R\ {}^{A}I\ ({}^{B}_{A}R^{T}\ {}^{B}\omega) \\
&= \left[{}^{B}_{A}R\ {}^{A}I\ {}^{B}_{A}R^{T}\right]\omega
\end{aligned}$$

$$^{B}I = \left[{}^{B}_{A}R\ {}^{A}I\ {}^{B}_{A}R^{T}\right]$$

Facts about Inertia Tensors

- Moments of inertia are positive.
- Sum of moments does not depend on orientation of frame
- Products of inertia can be made zero by varying reference frame; this gives rise to
  - principal moments
  - principal axis
- Eigenvalues: principal moments
- Eigenvectors: principal axis

Lagrange Formulation

$$L = K - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \tau$$

force balance versus energy balance

Derivation

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} &= \tau \\
\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial K}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} &= \tau \\
K &= \frac{1}{2}\,\dot{\mathbf{q}}^T\, M\, \dot{\mathbf{q}} \\
\frac{\partial K}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}}\,\left(\frac{1}{2}\,\dot{\mathbf{q}}^t\, M\, \dot{\mathbf{q}}\right) &= M\,\dot{\mathbf{q}} \\
\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) &= M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}}
\end{aligned}
$$

Derivation cont.

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) = M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial K}{\partial \mathbf{q}} = M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial K}{\partial \mathbf{q}} = M\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}})$$

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \tau$$

Equations of Motion

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \tau$$

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}]$$

$$[\dot{\mathbf{q}}^2]_{(n\times 1)} = [\dot{q}_1^2 \;\; \dot{q}_2^2 \;\; \cdots \;\; \dot{q}_n^2]^T$$

$$[\dot{\mathbf{q}}\dot{\mathbf{q}}]_{(n(n-1)/2\times 1)} = [\dot{q}_1\dot{q}_2 \;\; \dot{q}_1\dot{q}_3 \;\; \cdots \;\; \dot{q}_1\dot{q}_n \;\; \dot{q}_2\dot{q}_3 \;\; \cdots \;\; \dot{q}_2\dot{q}_n \;\; \cdots \;\; \dot{q}_{n-1}\dot{q}_n]^T$$

$$\boxed{M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau}$$

Mass Matrix

$$K = \sum_{i=1}^{n} K_i \qquad\qquad K_i = \frac{1}{2}(m_i \mathbf{v}_i^T \mathbf{v}_i + \omega_i^T \, {}^C I_i \, \omega_i)$$

$$\mathbf{v}_i = J_{v_i}\dot{\mathbf{q}} \qquad\qquad \omega_i = J_{\omega_i}\dot{\mathbf{q}}$$

$$\begin{aligned} K &= \frac{1}{2} \sum_{i=1}^{n} (m_i \dot{\mathbf{q}}^T J_{v_i}^T J_{v_i} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T J_{\omega_i}^T \, {}^C I_i \, J_{\omega_i} \dot{\mathbf{q}}) \\ &= \frac{1}{2} \dot{\mathbf{q}}^T \left[\sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T \, {}^C I_i \, J_{\omega_i}) \right] \dot{\mathbf{q}} \end{aligned}$$

$$M = \sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T \, {}^C I_i \, J_{\omega_i})$$

Centrifugal and Coriolis Forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial K}{\partial \mathbf{q}} = M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix}$$

$$= M\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}})$$

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix}$$

Centrifugal and Coriolis cont.

$$\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix}$$

Define:

$$m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$

Element of *M*

Element of ***q***

$$\frac{d\, m_{ij}}{dt} = \sum_{k=1}^{n} m_{ijk}\dot{q}_k$$

Example

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$

$$\mathbf{v} \;=\; \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\,\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\,\dot{\mathbf{q}} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{21} & \dot{m}_{22} \end{bmatrix}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{\mathbf{q}} \end{bmatrix}$$

Example cont.

$$\mathbf{v} = \begin{bmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{21} & \dot{m}_{22} \end{bmatrix} \dot{\mathbf{q}} - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix} \dot{\mathbf{q}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix} \begin{pmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{pmatrix} +$$

$$\begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix} [\dot{q}_1 \ \dot{q}_2]$$

Example cont. cont.

$$\mathbf{v} = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix} \begin{pmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{pmatrix} + \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix} [\dot{q}_1 \ \dot{q}_2]$$

Christoffel Symbols

Define: $$b_{ijk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

$$\mathbf{v} = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{pmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{pmatrix} + \begin{bmatrix} 2\, b_{112} \\ 2\, b_{212} \end{bmatrix} [\dot{q}_1 \ \dot{q}_2]$$

Centrifugal and Coriolis Computation

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}]$$

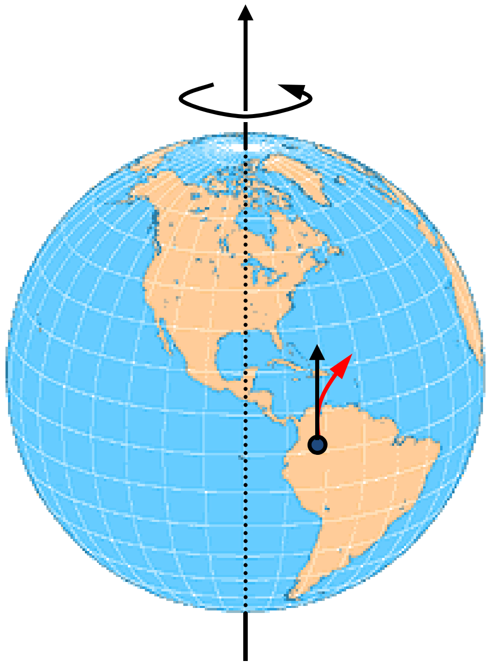
$$[\dot{\mathbf{q}}^2]_{(n\times 1)} = [\dot{q}_1^2 \;\; \dot{q}_2^2 \;\; \cdots \;\; \dot{q}_n^2]^T$$

$$[\dot{\mathbf{q}}\dot{\mathbf{q}}]_{(n(n-1)/2\times 1)} = [\dot{q}_1\dot{q}_2 \;\; \dot{q}_1\dot{q}_3 \;\; \cdots \;\; \dot{q}_1\dot{q}_n \;\; \dot{q}_2\dot{q}_3 \;\; \cdots \;\; \dot{q}_2\dot{q}_n \;\; \cdots \;\; \dot{q}_{n-1}\dot{q}_n]^T$$

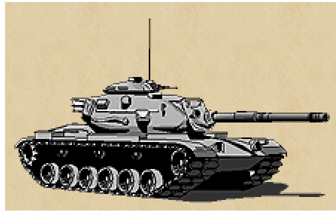
$$C(\mathbf{q})_{(n\times n)} = \begin{bmatrix} b_{111} & b_{122} & \cdots & b_{1nn} \\ b_{211} & b_{222} & \cdots & b_{2nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n11} & b_{n22} & \cdots & b_{nnn} \end{bmatrix}$$

$$B(\mathbf{q})_{(n\times n(n-1)/2)} = \left[\begin{array}{cccc|ccc|c|c} 2b_{112} & 2b_{113} & \cdots & 2b_{11n} & 2b_{123} & \cdots & 2b_{12n} & \cdots & 2b_{1(n-1)n} \\ 2b_{212} & 2b_{213} & \cdots & 2b_{21n} & 2b_{223} & \cdots & 2b_{22n} & \cdots & 2b_{2(n-1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2b_{n12} & 2b_{n13} & \cdots & 2b_{n1n} & 2b_{n23} & \cdots & 2b_{n2n} & \cdots & 2b_{n(n-1)n} \end{array}\right]$$

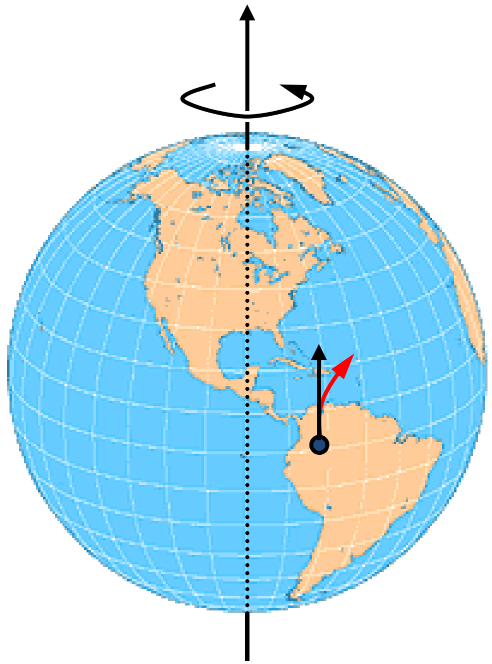
What are Coriolis Forces anyway?



$$\mathbf{F}_{\text{Coriolis}} = -2\, m\, \omega_{\{A\}} \times {}^{A}\mathbf{v}$$



Our Planet – The Earth

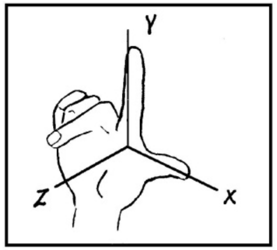


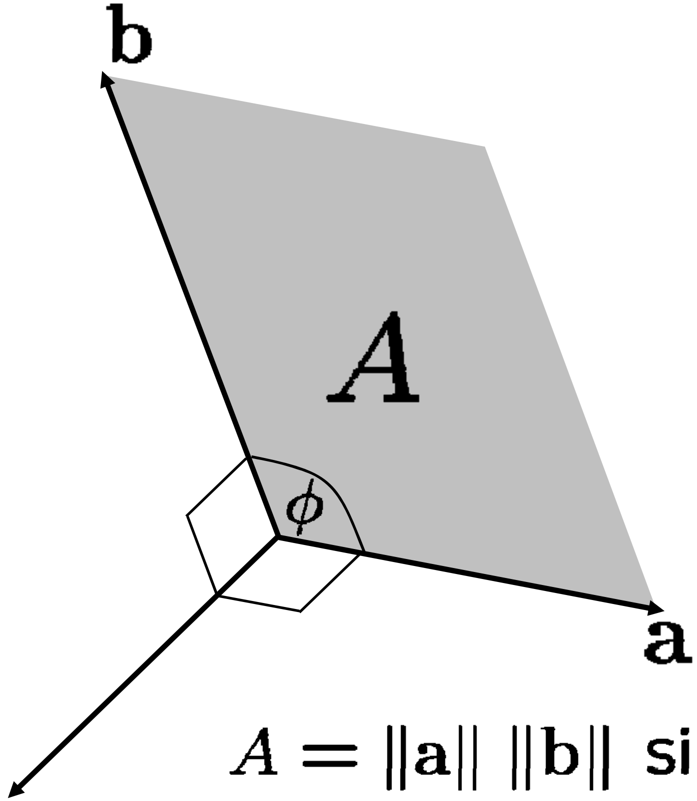
Velocity at equator:
1040 mph
1673 km/h
464 m/s
Radius of earth at equator:
6,378.14 km
Angular velocity of earth:
$|\Omega| = |v| \, / \, |r| = 464 \, / \, 6{,}378{,}140 =$
$7.27 \cdot 10^{-5}$ rad/s

Of course: 2π in 24 hours!

Reminder: Cross Product

$$\mathbf{F}_{\text{Coriolis}} = -2\, m\, \omega_{\{A\}} \times {}^{A}\mathbf{v}$$





$$A = \|\mathbf{a}\|\ \|\mathbf{b}\|\ \sin\phi = \|\mathbf{a} \times \mathbf{b}\|$$

Estimate of Coriolis Forces

$$\|\mathbf{F}_{\text{Coriolis}}\| = 2\,m\,\|\omega_{\{A\}}\|\;\|{}^{A}\mathbf{v}\|\;\sin\phi$$

$|F| \approx 2 \cdot 1 \cdot 7.27 \cdot 10^{-5} \cdot 0.1 \cdot 1 = 1.45 \cdot 10^{-5}$ N

$|F| \approx 2 \cdot 50 \cdot 7.27 \cdot 10^{-5} \cdot 277 \cdot 0.087 = 0.17$ N

sin(5°)

$|F| \approx 2 \cdot 40 \cdot 2 \cdot 1 \cdot 1 = 160$ N

Summary

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau$$

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