CS536 Assignment 2

Due: Sep. 23rd, 2024

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In each question, quantified variables have domain $\mathbb{Z}$ unless otherwise specified.

1. In lecture 5, without the consideration of $\perp$, we have the following conclusions:

$$\sigma \not\vDash \exists x \in S. p \Leftrightarrow \sigma \vDash \forall x \in S. \neg p$$

$$\sigma \not\vDash \forall x \in S. p \Leftrightarrow \sigma \vDash \exists x \in S. \neg p$$

   We used "$\sigma \not\vDash p \Leftrightarrow \sigma \vDash \neg p$" during the proof to get the above conclusions, so they are not totally correct anymore with the consideration of $\perp$. Let's create some conclusions without replacing $\sigma \not\vDash p$ with $\sigma \vDash \neg p$. Remind that, the following definitions about satisfaction of quantified predicates are still correct (I rephrased them here):
   - $\sigma \vDash \exists x \in S. p$ means there exists $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha] \vDash p$.
   - $\sigma \vDash \forall x \in S. p$ means for every value $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha] \vDash p$.

   Fill in the blanks as required so that each of the following sentences is correct.
   - Fill $\underline{type\ 1}$ with one of the words: "$\boldsymbol{some}$", "$\boldsymbol{every}$" or "$\boldsymbol{this}$".
   - Fill $\underline{type\ 2}$ with either "$\vDash$" or "$\not\vDash$".

   a. $\sigma \vDash \exists x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$ and for $\underline{type\ 1}$ $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha]$ $\underline{type\ 2}$ $p$.
   b. $\sigma \vDash \forall x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$ and for $\underline{type\ 1}$ $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha]$ $\underline{type\ 2}$ $p$.
   c. $\sigma \not\vDash \exists x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$ and for $\underline{type\ 1}$ $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha]$ $\underline{type\ 2}$ $p$.
   d. $\sigma \not\vDash \forall x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$ and for $\underline{type\ 1}$ $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha]$ $\underline{type\ 2}$ $p$.
   e. $\vDash \exists x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$, it is the case that $\sigma$ $\underline{type\ 2}$ $\exists x \in S. p$.
   f. $\vDash \forall x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$, it is the case that $\sigma$ $\underline{type\ 2}$ $\forall x \in S. p$.
   g. $\not\vDash \exists x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$, it is the case that $\sigma$ $\underline{type\ 2}$ $\exists x \in S. p$.
   h. $\not\vDash \forall x \in S. p$ means for $\underline{type\ 1}$ state $\sigma$, it is the case that $\sigma$ $\underline{type\ 2}$ $\forall x \in S. p$.

2. True or false? Justify your answers briefly.
   a. $\{x = 2, y = 3\} \vDash x < 2 \rightarrow y < 3$
   b. $\{b = (2,5,4,8)\} \vDash \exists m. 0 \le m < 4 \wedge b[m] < 2$
   c. $\{x = 2,\ b = (2,3)\} \vDash \exists y. \forall 0 \le x \le 1. b[x] = y$
   d. $\{x = 1, b = (5,3,6)\} \vDash \forall x. \forall 0 \le k < 3. x < b[k]$

3. Translate the following Java programs into our programming language.
   a. $\textbf{for}\ (\textbf{int}\ i = 0; i < b.length; i++)\ \{b[i] = i;\}$
   b. $\textbf{while}\ (x\ != 1)\ \{\textbf{if}\ (x \,\%\, 2 == 0)\ \{x = x/2;\}\ \textbf{else}\ \{x++;\}\}$
   c. $\textbf{int}\ m = 8, p = 1, y = 1;\ \textbf{while}\ (++m < 20)\ \{p = p * (y++);\}$

4. Evaluate each of the following configurations to completion. Do not use $\rightarrow^*$ or $\rightarrow^n$ in your solutions for this question.
   a. $\langle \textbf{if}\ x < 2\ \textbf{then}\ x := y + 1; w := x + 2\ \textbf{fi}, \{x = 3, y = 3, w = 4\}\rangle$
   b. $\langle \textbf{while}\ x < 2\ \textbf{do}\ x := y + 1; w := x + 2\ \textbf{od},\ \{x = 1, y = 3, w = 4\}\rangle$

c. $\langle x \coloneqq y+1; y \coloneqq x+1,\ \sigma \rangle$. Here, $x$ and $y$ are both defined in state $\sigma$.

5. Let $S \equiv \textbf{if } x > 0 \textbf{ then } x \coloneqq x+1 \textbf{ else } y \coloneqq -2 * x \textbf{ fi}$ and let $W \equiv \textbf{while } x > y \textbf{ do } S \textbf{ od.}$ Evaluate each of the following configurations to completion. You may use $\rightarrow^*$ and/or $\rightarrow^n$ in your solutions for this question.
   a. $\langle W, \sigma_1 \rangle$ where $\sigma_1 \vDash y < x \le 0$
   b. $\langle W, \sigma_2 \rangle$ where $\sigma_2 \vDash x > 0 \wedge y \le 0$

6. Let $W \equiv \textbf{while } x < 3 \textbf{ do } S \textbf{ od}$, where $S \equiv x \coloneqq x+1; y \coloneqq y * x$.
   a. Calculate $M(S, \tau)$. Here $\tau$ is a state with $x$ and $y$ defined.
   b. Calculate $M(W, \sigma)$, where $\sigma(x) = 4$ and $\sigma(y) = 1$.
   c. Calculate $M(W, \sigma)$, where $\sigma \vDash x = 1 \wedge y = 1$.

7. Let $W \equiv \textbf{while } x > 0 \textbf{ do } S \textbf{ od}$, where $S \equiv \textbf{if } x < y \textbf{ then } x \coloneqq y/x \textbf{ else } x \coloneqq x-1; y \coloneqq b[y] \textbf{ fi}$. Answer the following questions.
   a. Calculate $M(S, \sigma)$ where $\sigma(x) = -2$ and $\sigma(y) = -1$.
   b. Calculate $M(W, \sigma)$ where $\sigma = \{x = 2,\ y = 2,\ b = (0,1,2)\}$.
   c. Calculate $M(W, \sigma)$ where $\sigma = \{x = 8,\ y = 2,\ b = (4,2,0)\}$.
   d. Is there any state $\sigma$ such that $M(W, \sigma) = \{\perp_e\}$ because of the "division by zero" error?

8. Let $S \equiv x \coloneqq sqrt(x)\ /\ b[y]$ and let $\sigma = \{b = (3,0,-2,4), x = \alpha, y = \beta\}$. Here, $\alpha$ and $\beta$ are two named integer constants. Find all possible states $\sigma$ such that $M(S, \sigma) = \{\perp_e\}$.

   Hint: You can describe such states by describing $\alpha$ and/or $\beta$, or describing $\sigma(x)$ and/or $\sigma(y)$. For example, you can say: $\sigma$ with $\sigma(x) < 0$ and $\sigma(y) =$ any arbitrary integer can satisfy $M(S, \sigma) = \{\perp_e\}$, because we have a "square root of negative number" error while evaluating $sqrt(x)$.

9. In the following statements, $\sigma$ represents some well-formed state and $p$ represents some predicate. Decide true or false for each of them, justify your answers briefly.
   a. $\perp \vDash T$
   b. $\perp \nvDash F$
   c. If $\sigma(p) \ne \perp$, then $\sigma \vDash p$
   d. If $\sigma(p) = \perp$, then $\nvDash \neg p$
   e. If $\vDash p$, then $\neg \exists \sigma. \sigma(p) = \perp$

10. Let $\Sigma$ be the collection of all well-formed states. Decide true or false for each of the following statements, justify your answers briefly.
   a. Let $\Sigma_0 \subset \Sigma$ and $\Sigma_0 \vDash p$, also let $\tau \vDash p$; then $\Sigma_0 \cup \{\tau\} \vDash p$
   b. $\emptyset \vDash p$ and $\emptyset \vDash \neg p$ ($\emptyset$ represents an empty collection of states)
   c. Let $\tau \in \Sigma$, then $\tau \vDash p$ or $\tau \vDash \neg p$
   d. Let $\Sigma_0 \subset \Sigma$, then $\Sigma_0 \vDash p$ or $\Sigma_0 \vDash \neg p$
   e. Let $\sigma_1 \vDash p_1$ and $\sigma_2 \vDash p_2$, then $\{\sigma_1,\ \sigma_2\} \vDash p_1 \vee p_2$