Assignment 3

Due: Oct 6th, 2024

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1. Calculate denotational semantics for the following nondeterministic programs.
   a. Let $S \equiv \mathbf{if}\ x > y \rightarrow x := x - 1\ \square\ x > y \rightarrow y := y + 1\ \square\ x + y = 4 \rightarrow x := y/x\ \square\ x + y = 4 \rightarrow x := x/y\ \mathbf{fi}$, and let $\sigma = \{x = 3,\ y = 1\}$. Calculate $M(S, \sigma)$ and show your work.
   b. Let $W \equiv \mathbf{do}\ x > y \rightarrow x := x - 1\ \square\ x > y \rightarrow y := y + 1\ \square\ x + y = 4 \rightarrow x := y/x\ \square\ x + y = 4 \rightarrow x := x/y\ \mathbf{od}$, and let $\sigma = \{x = 3,\ y = 1\}$. Calculate $M(W, \sigma)$ and show your work.

2. Let b be an array of size $n \geq 1$, and $\forall 0 \leq i < n. b[i] = 0 \vee b[i] = 1$. Decide which number ($0\ or\ 1$) is the majority in b <u>**without counting their quantities**</u>.

   Write a program named $MAJORITY$ in our language that can solve the above problem and bind the majority among 0 and 1 to variable $major$. You can assume that b is written in the memory state; and to simplify the question, we artificially define $b[n] = 100$, so you don't need to worry about a possible runtime error when the array index reaches n. Your program doesn't have to be deterministic. Be careful of the grammar in our programming language.
   Here are some hints:
   1) We can use the following linear-search-like algorithm: scan the array b to pair up each 0 with a 1. Once we have some $1's$ left over, then 1 is the majority; once we have some $0's$ left over, then 0 is the majority; if we can pair up all numbers, then either of them can be the majority.
   2) A student Jason wrote a partial solution that can be useful to solve the above problem:
   $$J \equiv\ \mathbf{do}\ b[k_0] = 1 \rightarrow k_0 := k_0 + 1\ \square\ b[k_1] = 0 \rightarrow k_1 := k_1 + 1\ \mathbf{od}$$
   Consider k_0 and k_1 as pointers for number 0 and 1 respectively. What program J does is to find the next k_0 and k_1 such that $b[k_0] = 0$ and $b[k_1] = 1$. You can use J inside of your program.

3. True or false. Justify your answer briefly.
   a. If $M(S, \sigma)$ contains exactly one state, then S must be a deterministic statement.
   b. If $\sigma \vDash \{p\}\ S\ \{q\}$, then $\sigma \vDash p$.
   c. If $\sigma \nvDash_{tot} \{p\}\ S\ \{q\}$, then $\sigma \nvDash p$.
   d. If $\sigma \vDash \{p\}\ S\ \{q\}$, then $M(S, \sigma) \vDash q$.
   e. If $\sigma \nvDash \{p\}\ S\ \{q\}$, then $\sigma \nvDash_{tot} \{p\}\ S\ \{q\}$.

4. Let predicate function $P(k, s) \equiv s^2 \leq k \leq (s + 1)^2$. For each of the following triples, decide whether it is valid under total correctness, justify your answer briefly.
   a. $\{P(k, s + 1)\}\ s := s + 1\ \{P(k, s)\}$
   b. $\{P(k, s)\}\ s := s + 1\ \{P(k, s + 1)\}$
   c. $\{P(k, s) \wedge s < 0\}\ s := s + 1; k := k + 1\ \{P(k, s)\}$
   d. $\{P(k, s) \wedge s = s_0\}\ s := s + 1\ \{P(k, s_0)\}$
   e. $\{P(k + 1, s + 1)\}\ s := s + 1; k := k + 1\ \{P(k, s)\}$

5. Answer the following questions and justify your answer briefly.
   a. Let $\sigma \vDash \{x \neq 0\}\ \mathbf{while}\ x \neq 0\ \mathbf{do}\ x := x - 2\ \mathbf{od}\ \{x < 0\}$, what are the possible values of $\sigma(x)$?
   b. Let $\sigma \vDash_{tot} \{x \neq 0\}\ \mathbf{while}\ x \neq 0\ \mathbf{do}\ x := x - 2\ \mathbf{od}\ \{x < 0\}$, what are the possible values of $\sigma(x)$?

6. Let $\sigma \vDash_{tot} \{p_1\}\, S\, \{q_1\}$ and $\sigma \vDash_{tot} \{p_2\}\, S\, \{q_2\}$. Decide whether σ necessarily satisfies the following triples under total correctness, justify your answer briefly.
   a. $\{p_1 \wedge p_2\}\, S\, \{q_1 \vee q_2\}$
   b. $\{p_1 \vee p_2\}\, S\, \{q_1 \wedge q_2\}$
   c. $\{p_1 \vee p_2\}\, S\, \{q_1 \vee q_2\}$

7. Let $\vDash \{p_1\}\, S\, \{q_1\}$ and $\vDash \{p_2\}\, S\, \{q_2\}$. Decide whether the following triples are valid under partial correctness, justify your answer briefly.
   a. $\{p_1 \wedge p_2\}\, S\, \{q_1 \wedge q_2\}$
   b. $\{p_2\}\, S\, \{q_1 \to q_2\}$
   c. $\{\neg p_1 \to p_2\}\, S\, \{\neg q_1 \to q_2\}$

8. Let $w \Leftrightarrow wp(S, q)$, and let S be a deterministic program. Decide whether each of the following statements is true or false, justify your answer briefly. We assume $\tau(q) \neq \perp$ for any well-formed state τ.
   a. $\vDash_{tot} \{w\}\, S\, \{q\}$
   b. $\vDash \{w \wedge q\}\, S\, \{q\}$
   c. There exists some state σ such that $\sigma \nvDash w$ but $M(S, \sigma) \vDash q$.
   d. If $\sigma \vDash w$, then $M(S, \sigma) \neq \{\perp\}$.
   e. If $\sigma \nvDash w$, then $\sigma \vDash \{\neg w\}\, S\, \{\neg q\}$.

You don't have to logically simplify your answers to questions 9 and 10.

9. Let $S \equiv y := y \,\%\, x$ and $q \equiv sqrt(y) > x$.
   a. Calculate $wlp(S, q)$.
   b. Calculate $wp(S, q)$.

10. Let $S \equiv \mathbf{if}\ y \geq 0 \to x := y/x\ \square\ x \geq 0 \to x := x/y\ \mathbf{fi}$ and $q \equiv x < y < z$.
    a. Calculate $wlp(S, q)$.
    b. Calculate $wp(S, q)$.