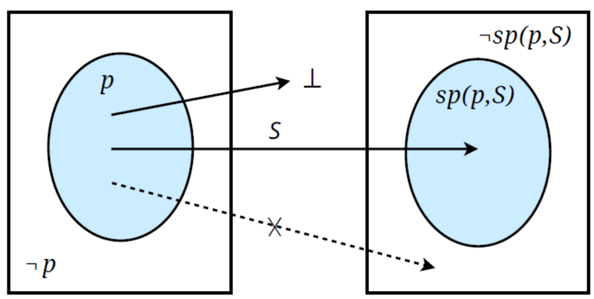
The Strongest Postcondition

- Given a precondition p and program S, **the strongest postcondition** of p and S , written as $sp(p,S)$ is (the predicate that stands for) the set of states we could terminate in if we run S starting in a state that satisfies p.
    - In symbols, using the language of states: $sp(p,S) = \{\tau \mid \tau \in M(S,\sigma) - \perp \text{ for some } \sigma \text{ where } \sigma \vDash p\}$, or equivalently $sp(p,S) = \bigcup_\sigma (M(S,\sigma) - \perp)$ where $\sigma \vDash p$.

    - From the definition of the strongest postcondition we can see that $\vDash \{p\}\, S\, \{sp(p,S)\}$; in other words, this postcondition only guarantees a valid triple under partial correctness. The reason is easy to see: the precondition p is given, and there could be states satisfying p make program S diverge or create error.

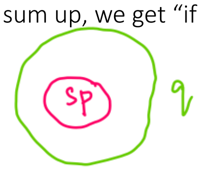


- From the above figure, we can see:
    - If $\sigma \vDash p$, then for all $\tau \in M(S,\sigma)$, either $\tau = \perp$ or $\tau \vDash sp(p,S)$.
    - If $\sigma \nvDash p$, we don't know anything interesting about $M(S,\sigma)$ and $sp(p,S)$.

1. Prove that $\vDash \{p\}\, S\, \{q\}$ **iff** $sp(p,S) \Rightarrow q$.
   It looks trivial, but our definition of $sp(p,S)$ didn't say anything about this postcondition the strongest. In this example, let us prove that $sp(p,S)$ is the strongest postcondition.

    - $\Leftarrow$: By the definition of $sp(p,S)$, we have $\vDash \{p\}\, S\, \{sp(p,S)\}$. And since $sp(p,S) \Rightarrow q$, then $\vDash \{p\}\, S\, \{q\}$ (weakening the postcondition).

    - $\Rightarrow$: Let τ be a state such that $\tau \vDash sp(p,S)$. By the definition of $sp(p,S)$, there exists some $\sigma \vDash p$ such that $\tau \in M(S,\sigma) - \perp$. $\vDash \{p\}\, S\, \{q\}$ implies that $M(S,\sigma) - \perp \vDash q$, thus $\tau \vDash q$. To sum up, we get "if $\tau \vDash sp(p,S)$, then $\tau \vDash q$", which implies "$sp(p,S) \Rightarrow q$".



Calculate *sp* for Loop-free Programs

Like wlp, we can use some algorithm/rules to calculate $sp(p,S)$ textually.

- $sp(p, \mathbf{skip}) \equiv p$.
- $sp(p, v \coloneqq e) \equiv p[v_0 / v] \wedge v = e[v_0 / v]$, where v_0 is the aged v (in other words, the old value of v before executing $v \coloneqq e$).
    - This is the forward assignment rule, so actually this rule can produce the strongest postcondition.

- $sp(p, S_1; S_2) \equiv sp(sp(p,S_1), S_2)$.

2. Calculate the following sp's.
   a. $sp\ (x > y,\ x \coloneqq x + k) \equiv (x > y)[x_0\ /\ x] \wedge x = (x + k)[x_0\ /\ x] \equiv x_0 > y \wedge x = x_0 + k$
   b. $sp\ (x_0 > y \wedge x = x_0 + k,\ y \coloneqq y + k) \equiv x_0 > y_0 \wedge x = x_0 + k \wedge y = y_0 + k$

   c. $sp\ (x > y,\ x \coloneqq x + k;\ y \coloneqq y + k) \equiv x_0 > y_0 \wedge x = x_0 + k \wedge y = y_0 + k$ #Combine a. and b.
   - By losing x_0 and y_0, we can slightly weaken the postcondition to $x > y$.

   d. $sp(x > f(x, y), x \coloneqq x + 1; x \coloneqq x + x)$
   - $sp(x > f(x, y),\ x \coloneqq x + 1) \equiv \big(x > f(x, y)\big)[x_0\ /\ x] \wedge x = (x + 1)[x_0\ /\ x]$
     $\equiv x_0 > f(x_0, y) \wedge x = x_0 + 1$

   - $sp(x > f(x, y), x \coloneqq x + 1; x \coloneqq x + x)$
     $\equiv sp(x_0 > f(x_0, y) \wedge x = x_0 + 1,\ x \coloneqq x + x)$
     $\equiv (x_0 > f(x_0, y) \wedge x = x_0 + 1)[x_1\ /\ x] \wedge x = (x + x)[x_1\ /\ x]$
     $\equiv x_0 > f(x_0, y) \wedge x_1 = x_0 + 1 \wedge x = x_1 + x_1$

- Let us think about sp in a conditional statement with an example:
  $sp(T,\ \textbf{if}\ x \geq y + z\ \textbf{then}\ x \coloneqq x - 1\ \textbf{else}\ y \coloneqq y + 2\ \textbf{fi}) \equiv ?$

  Following intuition, it is quite straightforward to come up with the following solution:
  - When the if condition is true, we should have $sp(T \wedge x \geq y + z,\ x \coloneqq x - 1) \equiv T \wedge x_0 \geq y + z \wedge x = x_0 - 1$.
  - When the if condition is false, we should have $sp(T \wedge x < y + z,\ y \coloneqq y + 2) \equiv T \wedge x < y_0 + z \wedge y = y_0 + 2$.
  - The sp for the whole statement should one of the above, thus:
    $"sp"(T,\ \textbf{if}\ x \geq y + z\ \textbf{then}\ x \coloneqq x - 1\ \textbf{else}\ y \coloneqq y + 2\ \textbf{fi})$
    $\equiv (T \wedge x_0 \geq y + z \wedge x = x_0 - 1) \vee (T \wedge x < y_0 + z \wedge y = y_0 + 2)$

  - Is this postcondition the strongest? No, it can be stronger since we didn't include that y is not updated in the true branch and x is not updated in the false branch. We can add this information by aging more variables.

- To calculate the sp for a conditional statement, we need to calculate some variable sets first:
  - $lhs(S) =$ the set of variables that appear as the lhs of assignments in statement S.
  - $rhs(S) =$ the set of variables that appear as the rhs of assignments in statement S.
  - $free(p) =$ the set of variables that are free in precondition p.
  - $aged(p, S) = lhs(S) \cap \big(rhs(S) \cup\ free(p)\big)$ is the set of variables whose assignments cause aging.

- Let $IF \equiv\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}$, and let $aged(p, IF) = \{x, y, \dots\}$. Then $sp(p, IF) \equiv sp(p_0 \wedge B,\ S_1) \vee sp(p_0 \wedge \neg B,\ S_2)$, where $p_0 = p \wedge x = x_0 \wedge y = y_0 \dots$

- Let $NF \equiv\ \textbf{if}\ B_1 \rightarrow S_1\ \ \square\ B_2 \rightarrow S_2\ \textbf{fi}$, and let $aged(p, NF) = \{x, y, \dots\}$. Then $sp(p, NF) \equiv sp(p_0 \wedge B_1,\ S_1) \vee sp(p_0 \wedge B_2,\ S_2)$, where $p_0 = p \wedge x = x_0 \wedge y = y_0 \dots$

3. Calculate $sp(T,\ \textbf{if}\ x \geq y + z\ \textbf{then}\ x \coloneqq x - 1\ \textbf{else}\ y \coloneqq y + 2\ \textbf{fi})$
   - Let $p \equiv T, S \equiv\ \textbf{if}\ x \geq y + z\ \textbf{then}\ x \coloneqq x - 1\ \textbf{else}\ y \coloneqq y + 2\ \textbf{fi}$
     - $lhs(S) = \{x, y\}$
     - $rhs(S) = \{x, y\}$

- $free(p) = \emptyset$
- $aged(p, S) = \{x, y\}$

- $sp(T \wedge x = x_0 \wedge y = y_0 \wedge x \geq y + z,\ x \coloneqq x - 1)$
  $\equiv T \wedge x_0 = x_0 \wedge y = y_0 \wedge x_0 \geq y + z \wedge x = x_0 - 1$

- $sp(T \wedge x = x_0 \wedge y = y_0 \wedge x < y + z,\ y \coloneqq y + 2)$
  $\equiv T \wedge x = x_0 \wedge y_0 = y_0 \wedge x < y_0 + z \wedge y = y_0 + 2$

- $sp(p, S)$
  $\equiv (T \wedge x_0 = x_0 \wedge y = y_0 \wedge x_0 \geq y + z \wedge x = x_0 - 1) \vee (T \wedge x = x_0 \wedge y_0 = y_0 \wedge x < y_0 + z \wedge y = y_0 + 2)$
  $\Leftrightarrow (y = y_0 \wedge x_0 \geq y + z \wedge x = x_0 - 1) \vee (x = x_0 \wedge x < y_0 + z \wedge y = y_0 + 2)$

4. Calculate $sp(p, S)$ where $p \equiv (x = y)$ and $S \equiv \textbf{if } y \geq 1 \rightarrow x \coloneqq 1 \ \Box\ y \leq 1 \rightarrow z \coloneqq 0 \textbf{ fi}$.
   - $lhs(S) \equiv \{x, z\}$
   - $rhs(S) \cup free(p) \equiv \{x, y\}$
   - $aged(p, S) \equiv \{x\}$

   - $sp(x = y \wedge x = x_0 \wedge y \geq 1,\ x \coloneqq 1) \equiv x_0 = y \wedge x_0 = x_0 \wedge y \geq 1 \wedge x = 1$
   - $sp(x = y \wedge x = x_0 \wedge y \leq 1,\ z \coloneqq 0) \equiv x = y \wedge x = x_0 \wedge y \leq 1 \wedge z = 0$
   - $sp(p, S) \equiv (x_0 = y \wedge x_0 = x_0 \wedge y \geq 1 \wedge x = 1) \vee (x = y \wedge x = x_0 \wedge y \leq 1 \wedge z = 0)$

Forward Assignment vs. Backward Assignment

- With backward assignment rule, we can get partially valid triple $\{q[e\ /\ v]\}\ v \coloneqq e\ \{q\}$; and with forward assignment rule we get partially valid triple $\{p\}\ v \coloneqq e\ \{p[v_0\ /\ v] \wedge v = e[v_0\ /\ v]\}$. What if we apply the "opposite" assignment rules on each of these two triples?

- First, let us calculate the $sp(q[e\ /\ v], v \coloneqq e)$, where this precondition is calculated from the backward assignment rule.
  - $sp(q[e\ /\ v], v \coloneqq e)$
    $\equiv q[e\ /\ v][v_0\ /\ v] \wedge v = e[v_0\ /\ v]$
    $\Leftrightarrow q[e[v_0\ /\ v]\ /\ v] \wedge v = e[v_0\ /\ v]$
    $\Rightarrow q[v\ /\ v]$
    $\Leftrightarrow q$
    - $sp(q[e\ /\ v], v \coloneqq e) \Rightarrow q$. This implies that $\{q[e\ /\ v]\}\ v \coloneqq e\ \{q\}$ is a valid triple under partial correctness. Note that, we don't have $sp(q[e\ /\ v], v \coloneqq e) \Leftrightarrow q$.

- Then, let us calculate the $wlp(v \coloneqq e,\ p[v_0\ /\ v] \wedge v = e[v_0\ /\ v])$, where this postcondition is calculated from the forward assignment rule.
  - $wlp(v \coloneqq e,\ p[v_0\ /\ v] \wedge v = e[v_0\ /\ v])$
    $\equiv (p[v_0\ /\ v] \wedge v = e[v_0\ /\ v])[e\ /\ v]$
    $\equiv p[v_0\ /\ v] \wedge e = e[v_0\ /\ v]$
    - $p \wedge v = v_0$
      $\Leftrightarrow p \wedge T \wedge v = v_0$
      $\Leftrightarrow p \wedge e = e \wedge v = v_0$
      $\Leftrightarrow p[v\ /\ v] \wedge e = e[v\ /\ v] \wedge v = v_0$
      $\Rightarrow p[v_0\ /\ v] \wedge e = e[v_0\ /\ v]$
      $\equiv wlp(v \coloneqq e,\ p[v_0\ /\ v] \wedge v = e[v_0\ /\ v])$

- $(p \wedge v = v_0) \Rightarrow wlp(v \coloneqq e,\ p[v_0 / v] \wedge v = e[v_0 / v])$. This implies that $\{p\}\ v \coloneqq e\ \{p[v_0 / v] \wedge v = e[v_0 / v]\}$. is a valid triple under partial correctness. Note that, we don't have $(p \wedge v = v_0) \Leftrightarrow wlp(v \coloneqq e,\ p[v_0 / v] \wedge v = e[v_0 / v])$.

- Here we showed that these two assignment rules can derive from each other: these two rules are equally strong, if one can create a partial valid triple then the other can also create a partially valid triple. We didn't find anything interesting between sp and wlp in general.