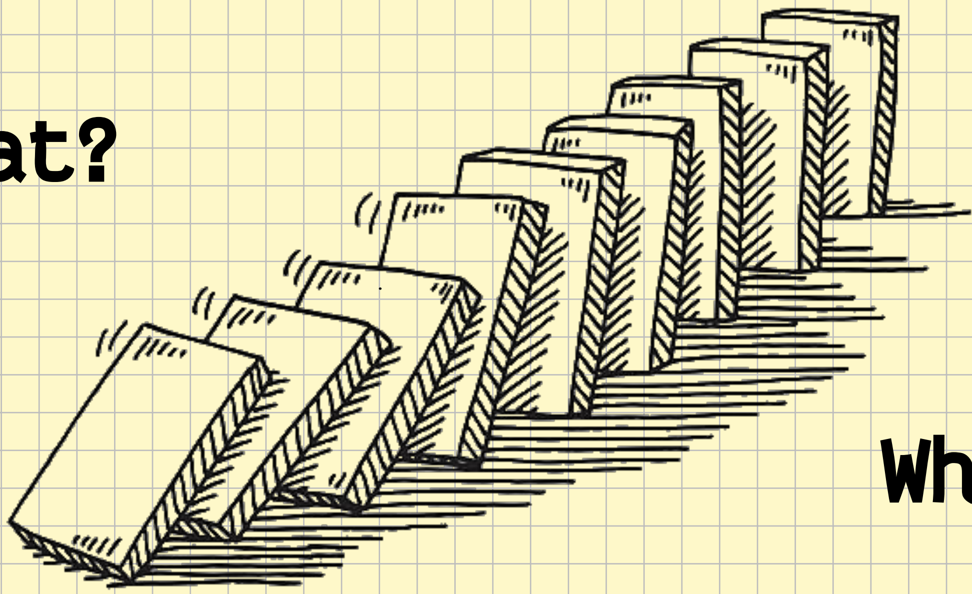
Mathematical Induction

What?



Why?

Wave of proofs

Suppose we have ∞ propositions numbered like P1,P2,P3.....
If I can prove two things
a) that P1 is true
b) the correctness of one propo-sition implies the next one one to be true.

P1 ⟶ P2 ⟶ ... ⟶ Pk ⟶ Pk+1

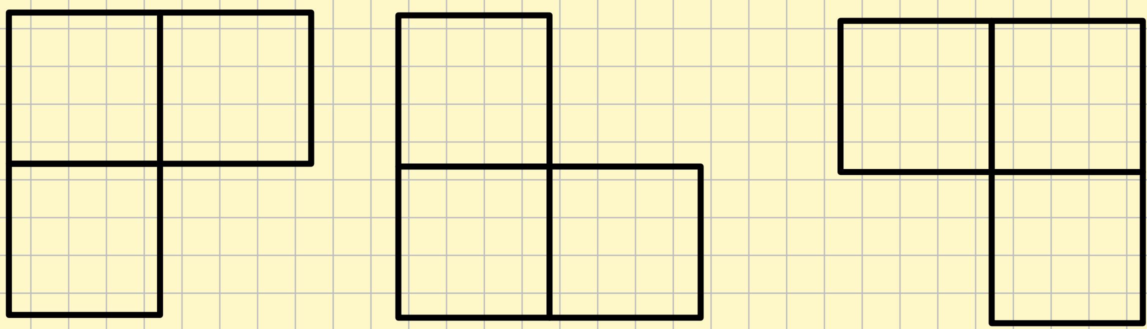
Then Pn is always true for any n

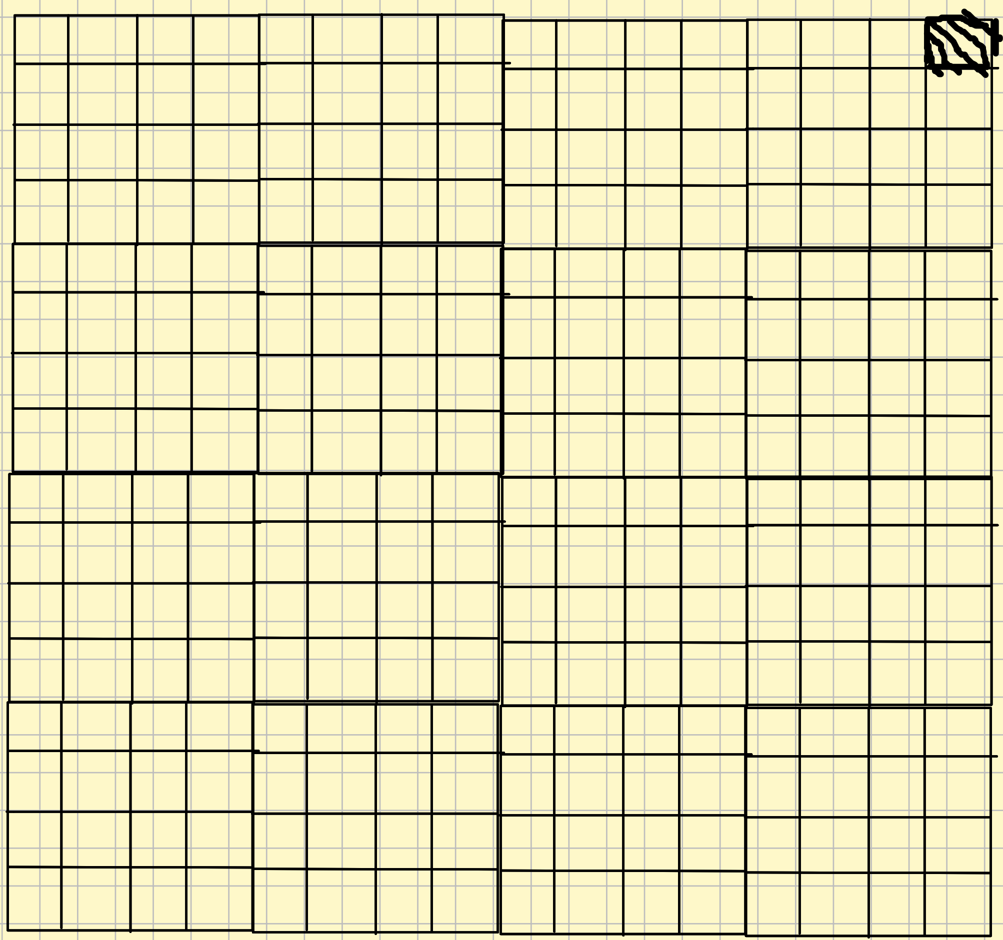
REMEMBER:

TO
TRY
THE
PROBLEM
BEFORE
SEEING
THE
SOLUTION

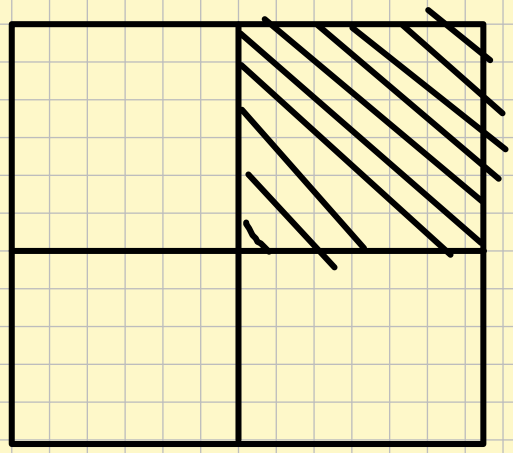
P1

If we cut 1 x 1 square from 16 x 16 square, prove that we can cut it out into trominoes?

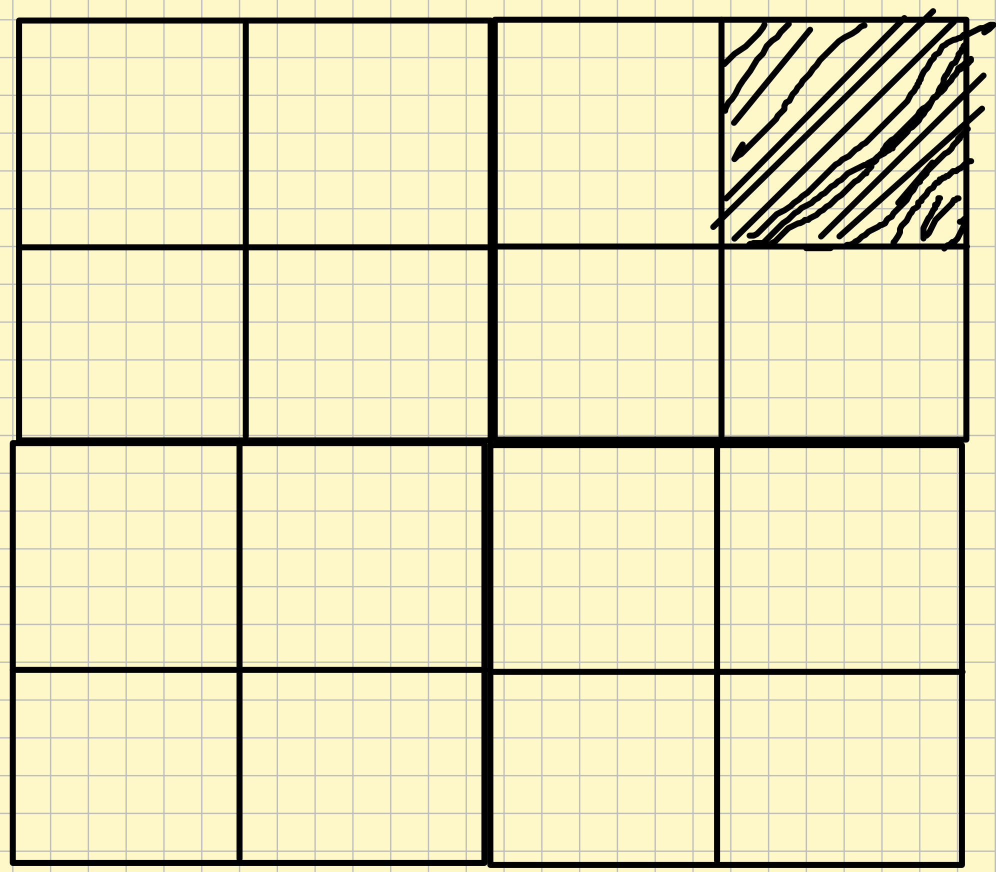




Sol.

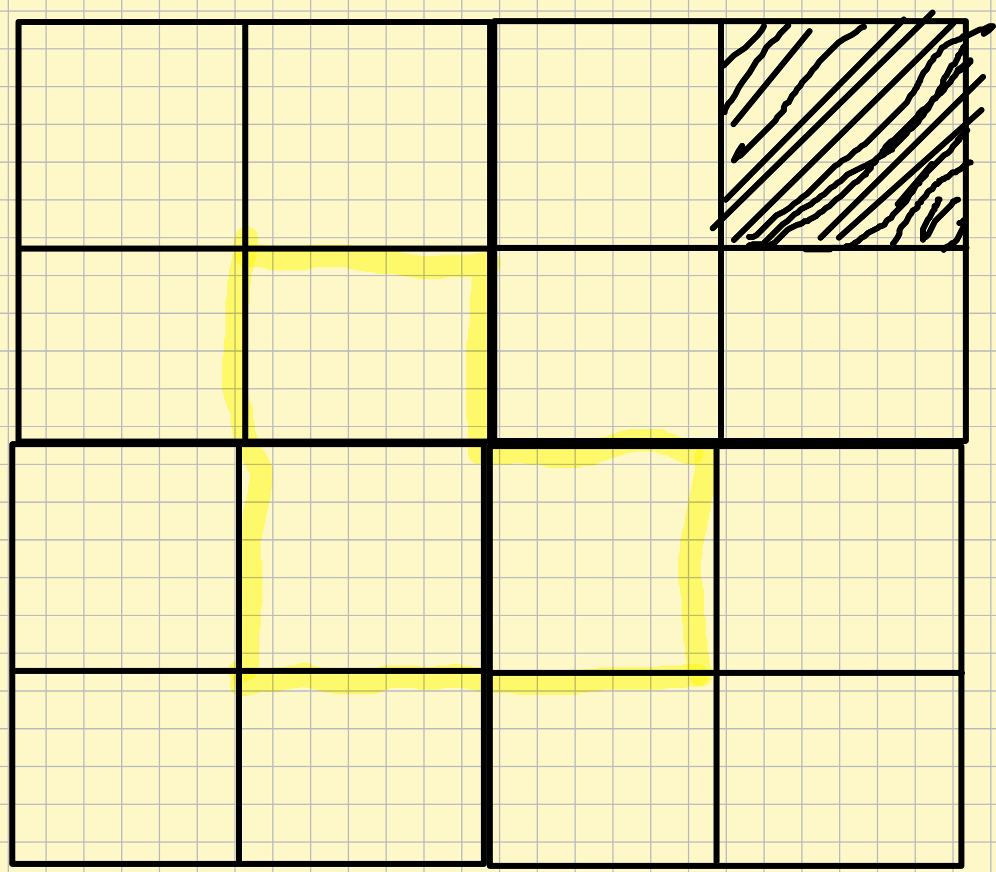


We can start from a very simple case. It seems self evident in 2 x 2. This is what we call base case. Now let's step up the game



Notice we can reduce the 4 x 4 into 4 (2 x 2) grids.

We observe we have already solved for 2 x 2, does it lead somewhere?



By cutting like this we solved the problem but does this tell us something more?
let's proceed to 8 x 8 box and we see that it can be cut into four 4 x 4 boxes.

Proceeding similarly we see that 16 x 16 too can be cut into trominoes.

$\therefore$ Proved

P2 Prove that

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$$

Sol.

Checking for n = 1.

LHS = 1 and RHS = $\frac{1(2)}{2} = 1$ This tells us that P(1) is true.

Now we can move forward by assum -ing P(k) to be true for some k belongs to N.

P(k) = $1^3 + 2^3 + 3^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$

Adding $(k+1)^3$ to both sides.

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

We notice LHS can be written as P(k+1)

$$P(k+1) = (k+1)^2 \left[\frac{k^2 + 4(k+1)}{4}\right]$$

$$P(k+1) = \frac{(k+1)^2(k+2)^2}{4}$$

This of the form which was required and so we are done here.

Elementary identity problems

H1 Prove $1+2+3.....+n = \frac{n(n+1)}{2}$ [Classic] and try to connect it to P2

H2 For every positive integer n prove that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}....+\frac{1}{n\times(n+1)}=\frac{n-1}{n}$$

H3 Prove $1+x+x^2+x^3....+x^n = \frac{x^{n+1}-1}{n+1}$

H4 $\frac{1}{a(a+b)}+\frac{1}{(a+b)(a+2b)}.....+\frac{1}{(a+(n-1)b)(a+nb))}=\frac{n}{a(a+nb)}$

These are usually the best problems for beginners in Induction but it is not what induction is. To find more identity problems here are some resources:

1) Mathematical Circles Ch 9
2) https://brilliant.org/wiki/induction/#induction-summation
   (Although we will move onto other sections too)
3) https://brilliant.org/wiki/flawed-induction-proofs/
   (Check this out too for common pitfalls.)

After some basic familiarity we move to other parts of induction.

P3

Let $\{a_n\}$ be a sequence of nat -ural numbers such that $a_1 = 5$ $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n. Prove that $a_n = 3^n + 2^n$ for all natural n.

Walkthrough:

First let us check whether our base cases are true or not. Then generalize the statement - and see whether it's true or not and then we are done.

Sol.

a1 = 3 + 2 = 5 Base cases
a2 = 9 + 4 = 13 are correct.

Assume $a_k = 3^k + 2^k$ and

$$a_{k+1} = 3^{k+1} + 2^{k+1}$$

Now put them into the relation.

$$a_{n+2} = 5a_{n+1} - 6a_n$$

$$a_{k+2} = 5(3^{k+1} + 2^{k+1}) - 6(3^k + 2^k)$$

$$a_{k+2} = 3^{k+2} + 2^{k+2}$$

And we are done here.

First let's discuss strong induction before moving onto problems and problems.

As we saw in induction that we usually chose the base point to be 1 but we can choose any other as well.

We can also any number k0 as the starting point.

Then proving from k0 to all k

Or
Assume that P(m) is true for all m < (k + 1).

These both approaches are equivalent as you can see.

STRONG INDUCTION is the popular name.

So now we can confidently move on to problems.

Actually while doing them I will try to motivate you to the solution which probably will make it clear why that approach was used.

P4

Let n be any natural number. Prove the identity.

$$\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} = \frac{1}{k^2+k+1} \sum_{k=1}^{n} k$$

Motivation: First of all notice the k term in the right term and correlate it with identity. And try to find the difference of the two consecutive sums and realize what it means?

Solution:

Replacing sum of k terms with n(n+1)/2.

We get this form of equation:

$$f(n) = \frac{n(n+1)}{2(n^2+n+1)}$$

as a function of n.

Let us assume that it is true for some k then what will be the k + 1 sum.

$$f(k+1) = \frac{(k+1)(k+2)}{2((k+1)^2+(k+1)+1)}$$

The difference should be the k + 1 term.

$$f(k+1) - f(k) = \frac{(k+1)(k+2)}{2((k+1)^2+(k+1)+1)} - \frac{k(k+1)}{2(k^2+k+1)}$$

$$= \frac{k+1}{2}\left[\frac{(k+1)+1}{(k+1)^2+(k+1)+1} - \frac{(k+1)-1}{(k+1)^2-(k+1)+1}\right]$$

$$= \frac{k+1}{2}\frac{[(k+1)^3+1]-[(k+1)^3-1]}{((k+1)^2+(k+1)+1)((k+1)^2-(k+1)+1)}$$

$$= \frac{k+1}{[(k+1)^2+1]^2-(k+1)^2}$$

$$= \frac{k+1}{(k+1)^4+(k+1)^2+1}$$

We are done now. For a twist there's also a nice solution by telescoping series method or say partial fractions. Find it.

P5

Prove the inequality for n > 1

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} > \frac{3n}{2n+1}$$

Motivation: Observe by what term does both sides increase for nth term.

Taking differences we get:

If n-1 terms were there then LHS increases by 1/n^2 and RHS is given by:

$$\frac{3n}{2n+1} - \frac{3(n-1)}{2(n-1)+1} = \frac{3}{4n^2-1}$$

Proof:

If I prove that LHS grows faster for n > 1 then I am done here.

Here's where induction comes in: Assume for some k > 1 this is true:

$$\frac{1}{k^2} > \frac{3}{4k^2-1}$$

We can even verify this by putting in a base case and then proceeding forward. Cross mult -plying both sides with no fear of signs.

$$4k^2 - 1 > 3k^2$$

$$k^2 > 1$$

Which is no doubt a true statement.

Since (k+1)^2 too would have to be strictly greater than 1 we can reconstruct the inequality:

$$(k+1)^2 > 1$$

$$4(k+1)^2 - 3(k+1)^2 - 1 > 0$$

$$4(k+1)^2 - 1 > 3(k+1)^2$$

$$\frac{1}{(k+1)^2} > \frac{3}{4(k+1)^2 - 1}$$

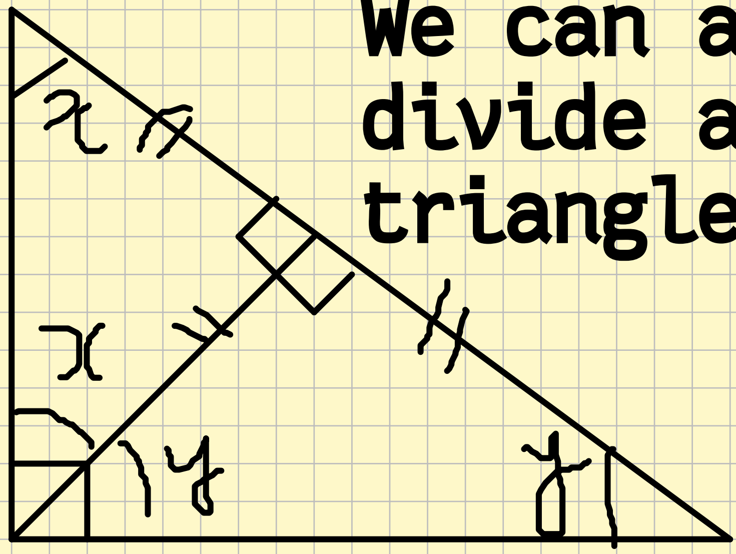
$$\therefore Proved$$

This completes the induction and hence the proof that LHS grows faster.

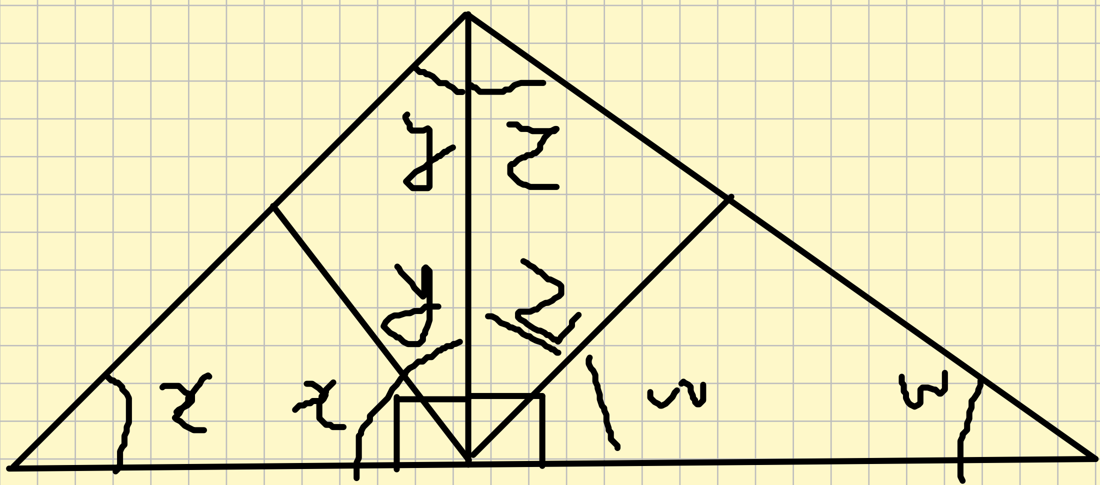
P6

Prove that for all n ⩾ 4, it is possible to dissect any non-equilateral triangle into n isosceles triangles.

Motivation:



We can always divide a right triangle like this.



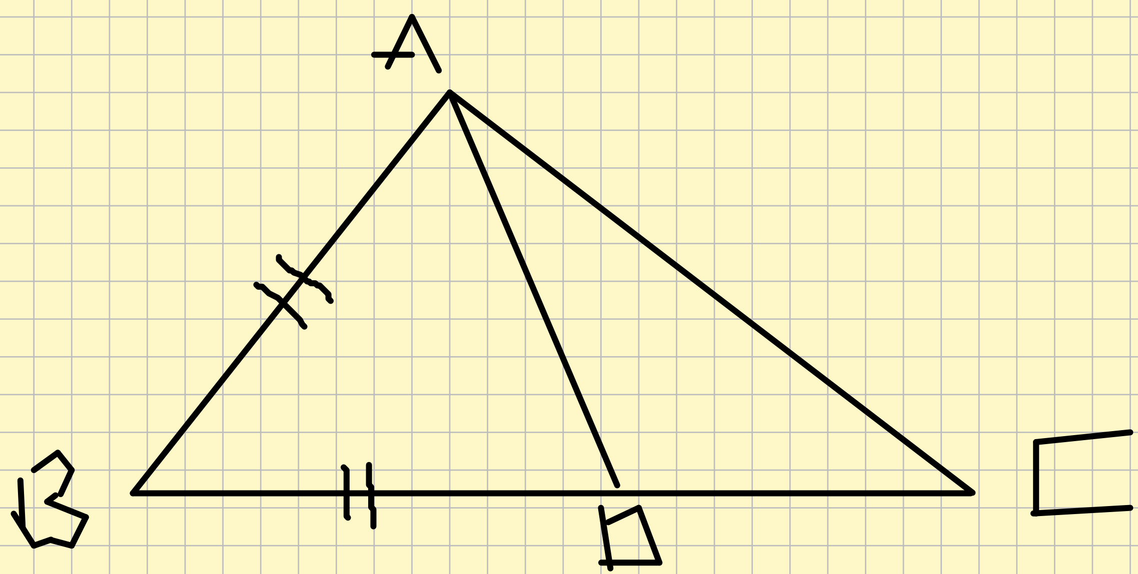
We can divide any non-eq triangle like this as above.

Proof:

We saw a base case above in the motivation so we can move now.

Suppose all non-eq triangles can be divided into m isosceles triangles for $m \geqslant 4$.

Suppose we have a non-eq triangle ABC with shortest side AB and we take a point D on BC such that BD = AB.



Then we see that ADC is not equilateral why? See angle C is the largest and cannot be 60 since ABC is not equi. Dividing ADC into m isotriangles we have successfully divided ABC into m + 1 triangles and QED.

P7

Let a be a real number such that sin a + cos a is a rational number, prove that

$$sin^n a + cos^n a$$

is a rational number for all n which are natural numbers.

Motivation: Get back to trig basics and remember:

$$sin^2 a + cos^2 a = 1$$

We now have two base cases which we can use to proceed further.

Proof:

Using the fact that

$$(\sin a + \cos a)^2 = 1 + 2 \sin a \cos a$$

This shows that sum and product both are rational.

Assuming that $sin^n a + cos^n a$ is rational we go to prove this:

$$\sin^{n+1} a + \cos^{n+1} a$$
$$= (\sin a + \cos a)(\sin^n a + \cos^n a) - \sin a \cos a(\sin^{n-1} a + \cos^{n-1} a)$$

We know that product of rational numbers is rational and since we proved that sin a x cos a is rational too and we can express the terms in bracket too to be rational and by PMI we are done here.

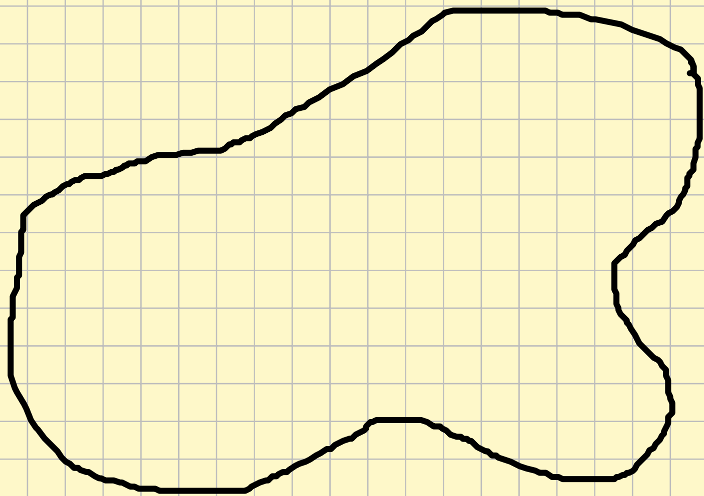
P8

I'm playing the color-country game against Bob. We take turns, on my turn, I draw in a country. On Bob's turn, he chooses any color for the country but he must make sure that he never colors adjacent countries the same color. Is it possible for me to force Bob to use more than 4 colors? Or more generally more than N colors?

Motivation: This does not seem like a problem to apply induction on but suppose we try to generalize it then what happens?

My turn:

N = 1

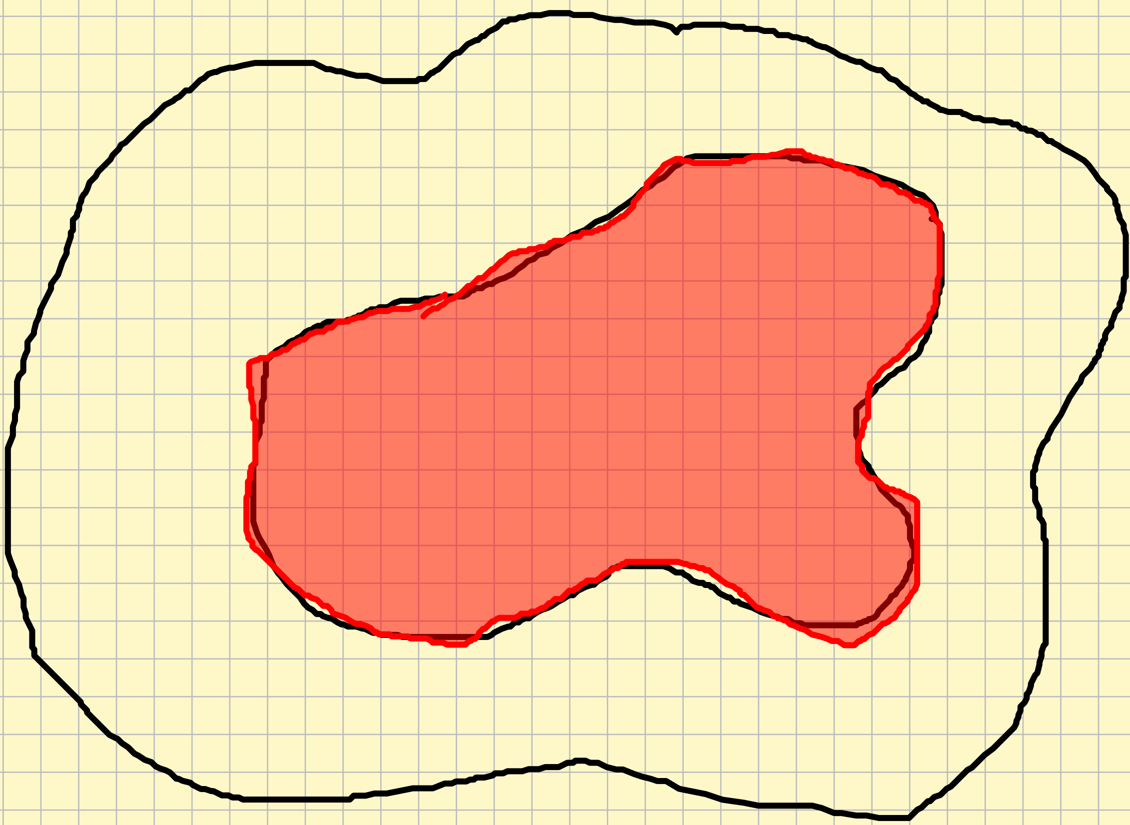


Bob's Turn:

This seems too simple. But lets try to force him to use second color.

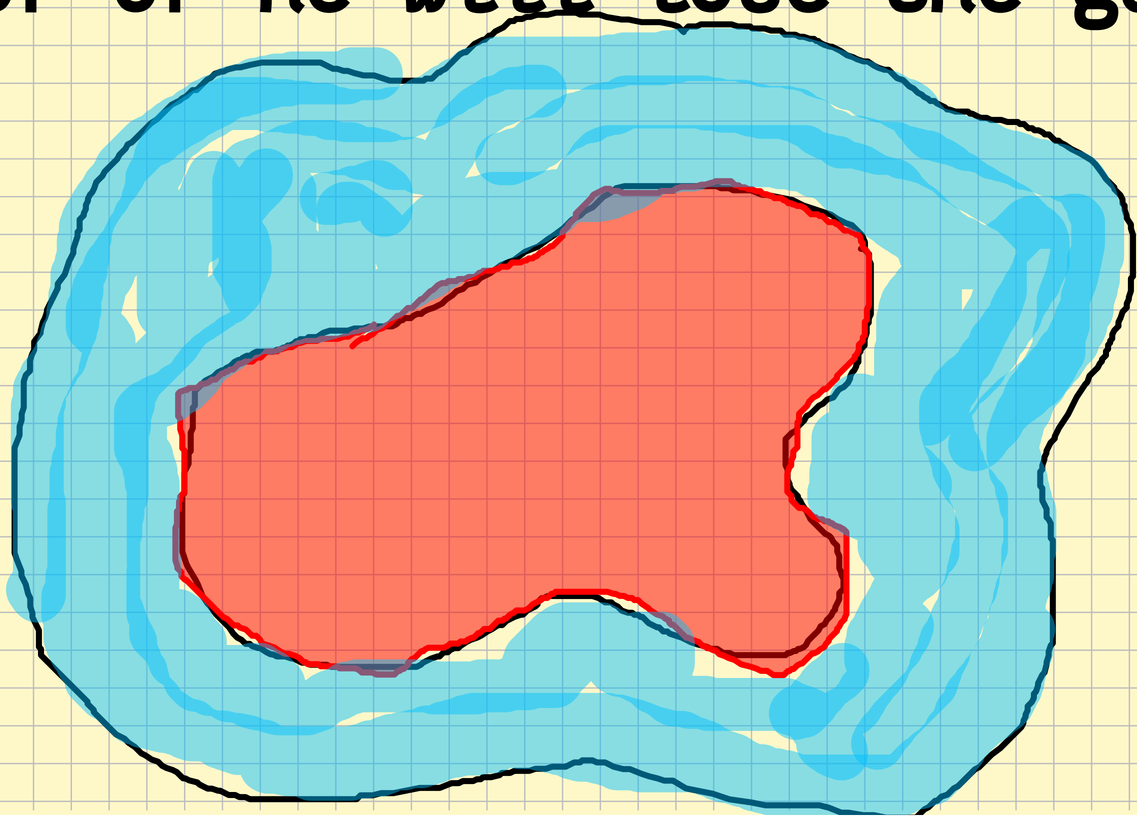
Again my turn:

Time to become smart :)



Bob's Turn:

Now he is forced to use another color or he will lose the game.



As you might have guessed by now what is my strategy? To make him use N colors on countries which are disconnected and then make a large country engulfing them all.

Now you will ask what if Bob tries to choose same color when we have disconnected countries?

The solution is that we will make Bob use 1 of the N colors in one blob of country and another of the N colors in another blobs in exterior and once N colors are used we will engulf them whole for the N+1 th colors.

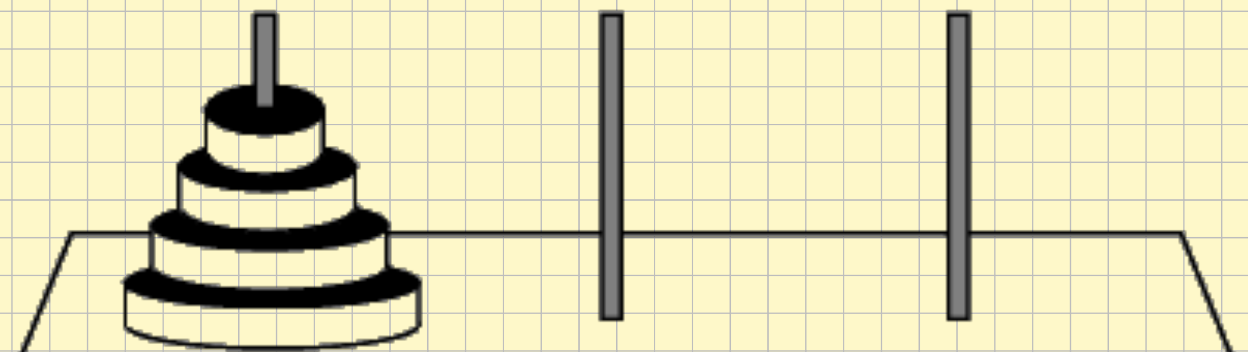
Proof:
We have already proved for the base case in the motivation.

Let us assume that we can force Bob to use N colors.
Then will proceed as above to make him use N colors in exterior and thus we will make him use the N+1 th color and QED.

Now specializing this we get it proved for 4 colors too.

PROBLEMS

H5



Tower of Hanoi

https://www.transum.org/Maths/Investigation/Tower_Of_Hanoi/Default.asp?Level=1

Try this out.

Rules are simple. Move n disks to second pole in the same order such that during the process no bigger disk ever lies on a smaller one.

Here we have three subtasks

a) Prove that it is possible to win this game

b) Prove that it is possible to do so in $2^n - 1$ moves.

c) And not possible in lesser moves.

H6

A set of n points is taken on a circle and each pair is connected by a segment. It happens that no three of these segments meet at the samepoint. Into how many parts do they divide the interior of the circle?

H7

Prove that if a,b,c,d ⩾ 0 then

$$\frac{3}{a+1}+\frac{3}{b+1}+\frac{2}{c+1}+\frac{1}{d+1} \leq 6+\frac{1}{a+b+1}+\frac{1}{a+b+c+1}+\frac{1}{a+b+c+d+1}$$

H8

Prove that for all integer n ≥ 3, the equation
1/x1 + 1/x2 + ... + 1/xn = 1
is solvable in distinct positive integers.

H9

Prove that for every positive integer n
there exists an n-digit number divisible by 5^n
all of whose digits are odd.

H10

Let $a_1, a_2, \cdots$ be a sequence of real numbers satisfying

$$a_{i+j} \leq a_i + a_j$$

for all $i, j = 1, 2, \ldots$ Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.