

## Circular Separator

We need to check existence of a circle that separates two sets of points  $\{(x_{r1}, y_{r1}), \dots, (x_{rn}, y_{rn})\}$  (red) and  $\{(x_{g1}, y_{g1}), \dots, (x_{gm}, y_{gm})\}$  (green). Treat these together as a 'signed' dataset of the form  $\{(x_1, y_1, s_1), \dots, (x_{n+m}, y_{n+m}, s_{n+m})\}$  where  $s_t = +1$  for the red points and  $s_t = -1$  for the green ones. Transform this dataset into the 'quadratic'-space —  $\{(\mathbf{p}_1, s_1), \dots, (\mathbf{p}_{n+m}, s_{n+m})\}$  where  $\mathbf{p}_t = (x_t^2, y_t^2, x_t \cdot y_t, x_t, y_t, 1)$ . A circular separator for the original dataset will be of the form  $(x - c_x)^2 + (y - c_y)^2 = c \equiv x^2 + y^2 - 2c_x \cdot x - 2c_y \cdot y + (c_x^2 + c_y^2 - c) = 0$ . This is equivalent to finding a separator of the form  $(1, 1, 0, w_1, w_2, w_3) \circ \mathbf{v} = 0$  where  $\circ$  denotes the inner product between the two vectors and  $\mathbf{v} \in \mathcal{R}^6$  with  $v_6 = 1$ . For this we require that for  $1 \leq t \leq (n + m)$

$$s_t \cdot ((1, 1, 0, w_1, w_2, w_3) \circ \mathbf{p}_t) > 0 \equiv s_t \cdot (x_t^2 + y_t^2 + w_1 \cdot x_t + w_2 \cdot y_t + w_3) = \epsilon_t, \epsilon_t > 0 \quad (1)$$

Now notice that any matrix of the form  $\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{bmatrix}$  must be non-singular as long as the three points  $(u_1, v_1), (u_2, v_2)$  and  $(u_3, v_3)$  are not collinear. Find any three points  $(x_i, y_i), (x_j, y_j), (x_k, y_k)$  that are not collinear. If the entire dataset (red and green points together) is collinear, then it is easy to check if the two sets are separable by a circle — they will be iff at least one of groups appears contiguously on the line. Having picked the three non collinear points we get

$$\begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \epsilon_i - x_i^2 - y_i^2 \\ \epsilon_j - x_j^2 - y_j^2 \\ \epsilon_k - x_k^2 - y_k^2 \end{bmatrix} \Rightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \epsilon_i - x_i^2 - y_i^2 \\ \epsilon_j - x_j^2 - y_j^2 \\ \epsilon_k - x_k^2 - y_k^2 \end{bmatrix}$$

This means we can write each  $w_l, 1 \leq l \leq 3$  as  $w_l = a_{li} \cdot \epsilon_i + a_{lj} \cdot \epsilon_j + a_{lk} \cdot \epsilon_k + b_l$ . Equation 1 after substitution for the  $w_l$  in terms of  $\epsilon_i, \epsilon_j, \epsilon_k$  would become:

$$\sum_{l \in \{i, j, k\}} s_t \cdot (x_t \cdot a_{tl} + y_t \cdot a_{2l} + a_{3l}) \epsilon_l = \epsilon_t - s_t \cdot (x_t^2 + y_t^2 + b_1 \cdot x_t + b_2 \cdot y_t + b_3), \quad \forall 1 \leq t \notin \{i, j, k\} \leq (n + m)$$

These are  $(m+n-3)$  equations in  $(m+n)$  variables  $\epsilon_1, \dots, \epsilon_{n+m}$ . Note that plugging in the expressions for  $w_l$  (in terms of  $\epsilon_l$ ) will result in identities for the equations for  $t = i, j, k$ . We can rewrite the above equations more legibly as

$$\alpha_{ti} \cdot \epsilon_i + \alpha_{tj} \cdot \epsilon_j + \alpha_{tk} \cdot \epsilon_k + \beta_t = \epsilon_t > 0, \quad \forall 1 \leq t \notin \{i, j, k\} \leq (n + m)$$

for constants  $\alpha_{ti}, \alpha_{tj}, \alpha_{tk}, \beta_t$ . So we need to check if there exist values  $\epsilon_i, \epsilon_j, \epsilon_k > 0$  such that

$$\alpha_{ti} \cdot \epsilon_i + \alpha_{tj} \cdot \epsilon_j + \alpha_{tk} \cdot \epsilon_k + \beta_t > 0, \quad \forall 1 \leq t \notin \{i, j, k\} \leq (n + m) \quad (2)$$

This is easy to do with a variant of Gaussian elimination.

**Step 0:** If  $\alpha_{ti} < 0$  for all  $t, l$ , then declare NO if any  $\beta_t < 0$  else declare YES. The first part is easy to see — positive values for  $\epsilon_i, \epsilon_j, \epsilon_k$  will need  $\beta_t > 0$  for the inequality to hold. If all  $\beta_t > 0$  then the following assignment works

$$0 < \epsilon_i = \epsilon_j = \epsilon_k < \min \left( \frac{\beta_t}{|\alpha_{ti} + \alpha_{tj} + \alpha_{tk}|} \right)$$

**Step 1:** Check if for some  $l \in \{i, j, k\}$ ,  $\alpha_{tl} > 0$  for all  $t$ . If some such  $l$  exists (WLG say  $l = i$ ), declare YES else go to Step 2. Reason: to assign positive values to  $\epsilon_i, \epsilon_j, \epsilon_k$  such that the Inequalities 2 are true, assign some arbitrary positive values (say 1) to  $\epsilon_j$  and  $\epsilon_k$ . Assign

$$\epsilon_i = \max \left( 1, \max_t \left( \frac{-(\alpha_{tj} \cdot \epsilon_j + \alpha_{tk} \cdot \epsilon_k + \beta_t)}{\alpha_{ti}} \right) + 1 \right) \quad (3)$$

**Step 2:** Let  $I_N$  denote the set of indices  $\{t : \alpha_{ti} < 0\}$ . Also WLG assume  $i \neq 1$ . Form the set of inequalities:

$$\left( \alpha_{tj} + \frac{|\alpha_{ti}| \cdot \alpha_{1j}}{\alpha_{1i}} \right) \cdot \epsilon_j + \left( \alpha_{tk} + \frac{|\alpha_{ti}| \cdot \alpha_{1k}}{\alpha_{1i}} \right) \cdot \epsilon_k + \left( \beta_t + \frac{|\alpha_{ti}| \cdot \beta_1}{\alpha_{1i}} \right) > 0, \quad \forall 1 \leq t \in I_N \quad (4)$$

Notice that these are just the inequalities obtained by adding a  $\frac{|\alpha_{ti}|}{\alpha_{1i}}$  multiple of the equation for  $t = 1$  to every other  $t \in I_N$ . This eliminates  $\epsilon_i$  and also keeps the sign intact. Repeat Steps 0, 1 and 2 above to Inequalities 4. If Inequalities 4 admit a positive solution for  $\epsilon_j, \epsilon_k$  we can always compute a matching positive value for  $\epsilon_i$  using Equation 3. Similar to Step 1, if not all the coefficients of  $\epsilon_j$  across the inequalities are positive then we can get a smaller set of inequalities with only  $\epsilon_k$  of the form

$$\eta_t \cdot \epsilon_k + \theta_t > 0, \quad \text{for } t \in J_N \subset I_N \quad (5)$$

**Step 3:** This is just for completeness. If for some  $t \in J_N$  both  $\eta_t, \theta_t < 0$ , then declare NO. Otherwise let  $N = \{t \in J_N | \eta_t < 0\}$  and the  $P = \{t \in J_N | \eta_t > 0\}$  be the sets of indices for which the corresponding  $\eta_t$  is negative and positive respectively. For the  $N$  set, we would require that  $\epsilon_k < \min_t \left( \frac{-\theta_t}{|\eta_t|} \right)$  and for the  $P$  set it must be that  $\epsilon_k > \max_t \left( \frac{-\theta_t}{\eta_t} \right)$ . Therefore we declare YES if

$$\max_t \left( \frac{-\theta_t}{\eta_t} \right) < \min_t \left( \frac{-\theta_t}{|\eta_t|} \right) \Rightarrow \max_t \left( \frac{-\theta_t}{\eta_t} \right) < \epsilon_k < \min_t \left( \frac{-\theta_t}{|\eta_t|} \right)$$

and declare NO otherwise.

**Complexity:** The algorithm works in linear time —  $O(n + m)$ . Transformation in Equation 1 takes linear time (one scan through the datapoints). Finding three non-collinear points with which to eliminate  $w_1, w_2, w_3$  and form inequalities 2 takes linear time again (again one scan with constant time to invert a  $(3 \times 3)$  matrix). Finally each of the Steps 0,1,2,3 take one scan through the dataset each. Incidentally this algorithm can even be extended to higher dimensions with the running time blowing up again linearly with  $d$ .