Derivation of Jacobian for the error propagation of charged particles in magnetic fields

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Abstract

In the track reconstruction algorithms used in high-energy physics experiments, the propagation of track parameter uncertainties is an important component in fitting the tracks to the measurements. A covariance matrix is updated at every surface intersection by calculating a Jacobian that can be decoupled into sub-Jacobians for coordinate transformations and transport along the trajectory. This paper derives each sub-Jacobian in a general manner to harmonize with numerical integration methods developed for non-uniform magnetic fields. The Jacobians are validated by comparing to Jacobians from numerical differentiation.

Keywords:

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1. Introduction

In many high-energy physics experiments, charged particles traverse the detector within a magnetic field and leavefootprints from electromagnetic interactions between a particle and the detector material. The reconstruction of the trajectories, or tracks, of the particles is performed by fitting a set of measurements using the least squares method. In progressive algorithms, such as Kalman filter [1, 2], the track parameters and their uncertainties, parametrized as covariances, must be propagated along the measurements sorted in a time sequence. For every intersection with a surface, the covariance matrix defined in the local reference frame of the surface needs to be calculated before applying material interactions or the fitting process. The propagation of errors between surfaces requires a Jacobian mathematically derived under the assumption of an ideal helix under a homogeneous magnetic field [3]. However, the extension of this method to inhomogeneous magnetic fields is not straightforward. In particular, integrating the helical model with numerical integration models developed for propagation in an inhomogeneous magnetic field is challenging.

Therefore, in this paper, we mathematically derive a general solution of Jacobian which can be implemented with arbitrary numerical integration models such as the Runge-Kutta-Nyström method [4, 5, 6]. Although Jacobians might manifest differently depending on the selection of the local coordinates, we can tackle two representative cases without loss of generality: one is the local frame defined on a plane, termed the bound frame, and the other is the perigee frame [7] which is not bound to the plane[1]. The derivation of additional covariance updates from material interactions in physical planes is beyond the scope of this paper, because there are general references [8, 9] that can be easily combined with our study.

The paper is organized as follows: Section 2 mathematically derives a generalized Jacobian for bound and perigee frames and Section 3 compares the generalized Jacobians with ones evaluated by a numerical differentiation. Summary and conclusions are given in Section 5.

2. Mathematical derivation

The movements of charged particles in a magnetic field are governed by Lorentz force:

$$\frac{\partial^2 \mathbf{r}}{\partial s^2} = \psi \left(\mathbf{t} \times \mathbf{b}(\mathbf{r})\right), \tag{1}$$

where $\mathbf{r}$ is the position, s is the path length along the trajectory, $\mathbf{t}$ is the unit tangential direction, equivalent to $\frac{\partial \mathbf{r}}{\partial s}$, ψ is the charge divided by a momentum of the particle and $\mathbf{b}(\mathbf{r})$ is the magnetic field at the current position of $\mathbf{r}$. In such a non-linear system, the covariance matrix ($\mathbb{C}$) can be updated with a Jacobian ($\mathbb{J}$) assuming that the system can be linearized with the first order of Taylor expansion:

$$\mathbb{C}' = \mathbb{J}\mathbb{C}\mathbb{J}^{\mathrm{T}}. \tag{2}$$

If the covariances are updated at surface intersections, the Jacobian can be represented as the product of three sub-Jacobians, for the coordinate transformation from the local to global frame ($\mathbb{J}_{L\to G}$), transport to the next surface ($\mathbb{J}_T$), and the coordinate transformation from the global to local frame ($\mathbb{J}_{G\to L}$) as follows:

$$\mathbb{J} = \mathbb{J}_{G\to L}\mathbb{J}_T\mathbb{J}_{L\to G}. \tag{3}$$

The track parameters in the global reference frame are parametrized as $(\mathbf{r}, \mathbf{t}, \psi)$. The track parameters in the local reference frame are parametrized as $(\boldsymbol{\epsilon}, \boldsymbol{\lambda}, \psi)$ where $\boldsymbol{\epsilon}$ is the two-dimensional position in the local Cartesian coordinate system,

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[1]The perigee frame is included in this study due to its importance in the vertex reconstruction and track parametrization with wire measurements in drift chambers and straw tubes.

and λ are the azimuthal (ϕ) and polar (θ) angles in the global spherical coordinate system. As $\mathbf{t}$ is defined under the implicit condition of a unit vector, the number of degrees of freedom of global track parameters remains six as expected. Meanwhile, the local coordinate frame has five degrees of freedom because one spatial dimension is constrained by a surface. The general forms of coordinate transformation Jacobians at the surface intersections are the following:

$$\mathbb{J}_{L\to G} = \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial \boldsymbol{\epsilon}} & \frac{\partial \mathbf{r}}{\partial \lambda} & \frac{\partial \mathbf{r}}{\partial \psi} \\ \frac{\partial \mathbf{t}}{\partial \boldsymbol{\epsilon}} & \frac{\partial \mathbf{t}}{\partial \lambda} & \frac{\partial \mathbf{t}}{\partial \psi} \\ \frac{\partial \psi}{\partial \boldsymbol{\epsilon}} & \frac{\partial \psi}{\partial \lambda} & \frac{\partial \psi}{\partial \psi} \end{pmatrix}, \quad \mathbb{J}_{G\to L} = \begin{pmatrix} \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{r}} & \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{t}} & \frac{\partial \boldsymbol{\epsilon}}{\partial \psi} \\ \frac{\partial \lambda}{\partial \mathbf{r}} & \frac{\partial \lambda}{\partial \mathbf{t}} & \frac{\partial \lambda}{\partial \psi} \\ \frac{\partial \psi}{\partial \mathbf{r}} & \frac{\partial \psi}{\partial \mathbf{t}} & \frac{\partial \psi}{\partial \psi} \end{pmatrix}. \tag{4}$$

As the momentum at the intersections of the surfaces is independent of the position or momentum we can consider the particle derivatives between them to be zero, leading to simplified representations of $\mathbb{J}_{L\to G}$ and $\mathbb{J}_{G\to L}$:

$$\mathbb{J}_{L\to G} = \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial \boldsymbol{\epsilon}} & \frac{\partial \mathbf{r}}{\partial \lambda} & \mathbb{O} \\ \frac{\partial \mathbf{t}}{\partial \boldsymbol{\epsilon}} & \frac{\partial \mathbf{t}}{\partial \lambda} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & 1 \end{pmatrix}, \quad \mathbb{J}_{G\to L} = \begin{pmatrix} \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{r}} & \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{t}} & \mathbb{O} \\ \frac{\partial \lambda}{\partial \mathbf{r}} & \frac{\partial \lambda}{\partial \mathbf{t}} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & 1 \end{pmatrix}, \tag{5}$$

where $\mathbb{O}$ is a zero submatrix with the appropriate dimension.

For the transport Jacobian, the variations of state $(\mathbf{r}_{\mathtt{f}}, \mathbf{t}_{\mathtt{f}}, \psi_{\mathtt{f}})$ at the destination surface (f) with respect to the variations of state $(\mathbf{r}_{\mathtt{i}}, \mathbf{t}_{\mathtt{i}}, \psi_{\mathtt{i}})$ at the departure surface (i) is expressed as follows [3]:

$$d\mathbf{r}_{\mathtt{f}} = \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} \cdot d\mathbf{r}_{\mathtt{i}} + \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} \cdot d\mathbf{t}_{\mathtt{i}} + \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \cdot d\psi_{\mathtt{i}} + \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial s} \cdot ds, \tag{6}$$

$$d\mathbf{t}_{\mathtt{f}} = \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} \cdot d\mathbf{r}_{\mathtt{i}} + \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} \cdot d\mathbf{t}_{\mathtt{i}} + \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \cdot d\psi_{\mathtt{i}} + \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial s} \cdot ds, \tag{7}$$

$$d\psi_{\mathtt{f}} = \frac{\partial \psi_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} \cdot d\mathbf{r}_{\mathtt{i}} + \frac{\partial \psi_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} \cdot d\mathbf{t}_{\mathtt{i}} + \frac{\partial \psi_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \cdot d\psi_{\mathtt{i}} + \frac{\partial \psi_{\mathtt{f}}}{\partial s} \cdot ds. \tag{8}$$

The last terms with the path length variation compensate for overstepping or understepping to the destination surface, which can be caused by track displacements at the departure surface.

2.1. Bound frame

In the bound frame, the local position of the surface of intersection is a vector span of $\mathbf{u}$ and $\mathbf{v}$ which are the orthonormal basis of the plane. The relationship between the global and local position at the surface intersections can be represented as follows:

$$\mathbf{r} = l_0\mathbf{u} + l_1\mathbf{v} + \mathbf{c}, \tag{9}$$

where l_0 and l_1 are the two components of $\boldsymbol{\epsilon}$, and $\mathbf{c}$ is a vector from the origin to the surface. The track parametrization of the bound frame is illustrated in Fig. 1.

For the transform Jacobians of the coordinates, the remaining off-diagonal submatrices are zero matrices because the surface

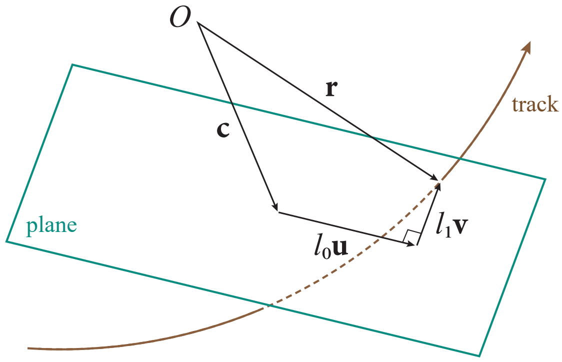

Figure 1: Track parametrization for the bound frame where O is the global origin.

intersection point does not depend on the direction and vice versa. Thus, $\mathbb{J}_{L\to G}$ and $\mathbb{J}_{G\to L}$ are further simplified into block diagonal matrices:

$$\mathbb{J}_{L\to G} = \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial \boldsymbol{\epsilon}} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \frac{\partial \mathbf{t}}{\partial \lambda} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & 1 \end{pmatrix}, \quad \mathbb{J}_{G\to L} = \begin{pmatrix} \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{r}} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \frac{\partial \lambda}{\partial \mathbf{t}} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & 1 \end{pmatrix}. \tag{10}$$

$\frac{\partial \mathbf{r}}{\partial \boldsymbol{\epsilon}}$ and $\frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{r}}$ can be obtained from Eq. (9):

$$\frac{\partial \mathbf{r}}{\partial \boldsymbol{\epsilon}} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}, \quad \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{r}} = \begin{pmatrix} \mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}} \end{pmatrix}. \tag{11}$$

The exact forms of $\frac{\partial \mathbf{t}}{\partial \lambda}$ and $\frac{\partial \lambda}{\partial \mathbf{t}}$ are skipped in this paper because they are the well-known transformations between the Cartesian and spherical coordinates.

For the derivation of the transport Jacobian, we start from the constraint that $d\mathbf{r}$ is always orthogonal to the surface normal vector ($\mathbf{w}$). From $\mathbf{w} \cdot d\mathbf{r} = 0$, ds can be written as a function of track parameters by taking an inner product between $\mathbf{w}$ and Eq. (6):

$$ds = -\frac{1}{\mathbf{w}_{\mathtt{f}} \cdot \mathbf{t}_{\mathtt{f}}} \mathbf{w}_{\mathtt{f}} \cdot \left(\frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} \cdot d\mathbf{r}_{\mathtt{i}} + \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} \cdot d\mathbf{t}_{\mathtt{i}} + \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \cdot d\psi_{\mathtt{i}} \right), \tag{12}$$

where $\mathbf{w}_{\mathtt{f}}$ is the normal vector of the destination surface. By feeding the above equation back to Eq. (6), (7) and (8), we can obtain $\mathbb{J}_T$ as a product of two matrices:

$$\mathbb{J}_T = \begin{pmatrix} \mathbf{I} + \mathbf{P}_{rr} & \mathbb{O} & \mathbb{O} \\ \mathbf{P}_{tr} & \mathbf{I} & \mathbb{O} \\ \mathbf{P}_{\psi r} & \mathbb{O} & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} & \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} & \frac{\partial \mathbf{r}_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \\ \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} & \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} & \frac{\partial \mathbf{t}_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \\ \frac{\partial \psi_{\mathtt{f}}}{\partial \mathbf{r}_{\mathtt{i}}} & \frac{\partial \psi_{\mathtt{f}}}{\partial \mathbf{t}_{\mathtt{i}}} & \frac{\partial \psi_{\mathtt{f}}}{\partial \psi_{\mathtt{i}}} \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} \mathbf{P}_{rr} &= -\frac{1}{\mathbf{w}_{\mathtt{f}} \cdot \mathbf{t}_{\mathtt{f}}} \mathbf{t}_{\mathtt{f}} \mathbf{w}_{\mathtt{f}}^{\mathrm{T}}, \\ \mathbf{P}_{tr} &= -\frac{1}{\mathbf{w}_{\mathtt{f}} \cdot \mathbf{t}_{\mathtt{f}}} \psi_{\mathtt{f}} (\mathbf{t}_{\mathtt{f}} \times \mathbf{b}(\mathbf{r}_{\mathtt{f}})) \mathbf{w}_{\mathtt{f}}^{\mathrm{T}}, \\ \mathbf{P}_{\psi r} &= -\frac{1}{\mathbf{w}_{\mathtt{f}} \cdot \mathbf{t}_{\mathtt{f}}} \frac{\psi_{\mathtt{f}} E_{\mathtt{f}}}{p_{\mathtt{f}}^2} \left(-\frac{\partial E_{\mathtt{f}}}{\partial s} \right) \mathbf{w}_{\mathtt{f}}^{\mathrm{T}}, \end{aligned} \tag{14}$$

where $p_{\mathtt{f}}$, $E_{\mathtt{f}}$ are the momentum and energy of the particle at the destination surface. $-\frac{\partial E_{\mathtt{f}}}{\partial s}$ can be regarded as the stopping power of the material [10] in which the particle is propagating.

The first matrix of Eq. (13) is from the surface constraint and the second matrix is the free space transport Jacobian which can be either analytically calculated [3] or numerically integrated [6, 11]. The transport Jacobian diverges when the track intersects surface almost parallel to its plane. It makes sense because even a very small variation of track parameter at the departure surface can make a large offset to the intersection points at the destination surface.

2.2. Perigee frame

A perigee frame describes a local coordinate system for the point of the closest approach of a track to a line. The condition of the closest approach is met when the shortest segment between the track and the line is perpendicular to track direction and the line direction. Under this condition, the perigee frame can be handled as a bound frame while satisfying Eq. (9): $\mathbf{c}$ is a center of the line, $\mathbf{v}$ is a unit vector parallel to the line, $\mathbf{u}$ is $\frac{\mathbf{v}\times\mathbf{t}}{|\mathbf{v}\times\mathbf{t}|}$ and the sign of l_0 follows the sign of $\mathbf{u}\cdot(\mathbf{r}-\mathbf{c})$. The track parametrization of the perigee frame is illustrated in Fig. 2. There are a few differences in the perigee frame definition between this paper and the original reference [7], such as the sign convention and a constant multiplier, but the basic principle remains the same.

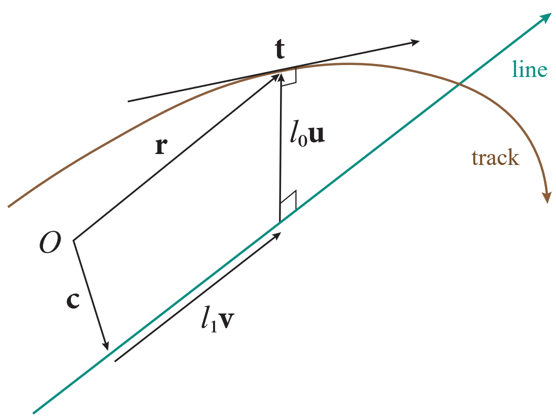

Figure 2: Track parametrization for the perigee frame where O is the global origin.

As for the bound frame, we can start with finding additional zero submatrices of coordinate transform Jacobians. $\frac{\partial\mathbf{t}}{\partial\boldsymbol{\epsilon}}$ and $\frac{\partial\lambda}{\partial\mathbf{r}}$ are zero matrices because the direction is fixed in these partial derivatives. The same idea can be applied to $\frac{\partial\boldsymbol{\epsilon}}{\partial\mathbf{t}}$ where the variation of $\mathbf{t}$ is a rotation around $\mathbf{u}$, and the global point of the intersection is preserved. However, $\frac{\partial\mathbf{r}}{\partial\lambda}$ is a nonzero matrix because the global intersection point can rotate around $\mathbf{v}$ while conserving the elements of $\boldsymbol{\epsilon}$. In summary, the coordinate transform Jacobians can be written as the following:

$$\mathbb{J}_{L\to G} = \begin{pmatrix} \frac{\partial\mathbf{r}}{\partial\boldsymbol{\epsilon}} & \frac{\partial\mathbf{r}}{\partial\lambda} & \mathbb{O} \\ \mathbb{O} & \frac{\partial\mathbf{t}}{\partial\lambda} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & 1 \end{pmatrix}, \quad \mathbb{J}_{G\to L} = \begin{pmatrix} \frac{\partial\boldsymbol{\epsilon}}{\partial\mathbf{r}} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \frac{\partial\lambda}{\partial\mathbf{t}} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & 1 \end{pmatrix}. \tag{15}$$

The same discussion of Section 2.1 can be applied to the diagonal submatrices. For the nonzero off-diagonal components, $\frac{\partial\mathbf{r}}{\partial\lambda}$ is equal to $l_0\frac{\partial\mathbf{u}}{\partial\lambda}$ because the last two terms of Eq. (9) will vanish in the partial derivative. Here, we only present the final form of $\frac{\partial\mathbf{r}}{\partial\phi}$, which also holds for $\frac{\partial\mathbf{r}}{\partial\theta}$:

$$\frac{\partial\mathbf{r}}{\partial\phi} = \frac{l_0}{|\mathbf{v}\times\mathbf{t}|}\left[-\mathbf{u}\left(\mathbf{u}\cdot\mathbf{v}\times\frac{\partial\mathbf{t}}{\partial\phi}\right) + \mathbf{v}\times\frac{\partial\mathbf{t}}{\partial\phi}\right]. \tag{16}$$

It is easy to confirm that $\mathbb{J}_{G\to L}\mathbb{J}_{L\to G}$ is the identity matrix now that Eq. (16) is orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

The derivation of transport Jacobian for the perigee frame can also follow the same procedure of the bound frame by finding the constraint from surface intersections. The condition for the closest approach indicates an implicit function between the global position and direction:

$$F(\mathbf{r},\mathbf{t}) = \mathbf{t}\cdot[(\mathbf{r}-\mathbf{c}) - ((\mathbf{r}-\mathbf{c})\cdot\mathbf{v})\mathbf{v}] = 0. \tag{17}$$

The implicit function theorem leads to the first order ordinary differential equation of ($\frac{\partial F}{\partial\mathbf{r}}\cdot d\mathbf{r} + \frac{\partial F}{\partial\mathbf{t}}\cdot d\mathbf{t} = 0$), which yields the following constraint of the global parameter variations:

$$(\mathbf{t}-(\mathbf{t}\cdot\mathbf{v})\mathbf{v})\cdot d\mathbf{r} + l_0\mathbf{u}\cdot d\mathbf{t} = 0. \tag{18}$$

The above equation helps to obtain ds by linearly superposing the Eq. (6) and (7).

$$\begin{aligned} ds = &-\frac{1}{1-(\mathbf{t}_{\mathtt{f}}\cdot\mathbf{v}_{\mathtt{f}})^2 + l_{0\mathtt{f}}\mathbf{u}_{\mathtt{f}}\cdot\psi_{\mathtt{f}}(\mathbf{t}_{\mathtt{f}}\times\mathbf{b}(\mathbf{r}_{\mathtt{f}}))} \\ &\times\left[(\mathbf{t}_{\mathtt{f}}-(\mathbf{t}_{\mathtt{f}}\cdot\mathbf{v}_{\mathtt{f}})\mathbf{v}_{\mathtt{f}})\cdot\left(\frac{\partial\mathbf{r}_{\mathtt{f}}}{\partial\mathbf{r}_{\mathtt{i}}}\cdot d\mathbf{r}_{\mathtt{i}} + \frac{\partial\mathbf{r}_{\mathtt{f}}}{\partial\mathbf{t}_{\mathtt{i}}}\cdot d\mathbf{t}_{\mathtt{i}} + \frac{\partial\mathbf{r}_{\mathtt{f}}}{\partial\psi_{\mathtt{i}}}\cdot d\psi_{\mathtt{i}}\right)\right. \\ &\left. + l_{0\mathtt{f}}\mathbf{u}_{\mathtt{f}}\cdot\left(\frac{\partial\mathbf{t}_{\mathtt{f}}}{\partial\mathbf{r}_{\mathtt{i}}}\cdot d\mathbf{r}_{\mathtt{i}} + \frac{\partial\mathbf{t}_{\mathtt{f}}}{\partial\mathbf{t}_{\mathtt{i}}}\cdot d\mathbf{t}_{\mathtt{i}} + \frac{\partial\mathbf{t}_{\mathtt{f}}}{\partial\psi_{\mathtt{i}}}\cdot d\psi_{\mathtt{i}}\right)\right], \end{aligned} \tag{19}$$

where $\mathbf{u}_{\mathtt{f}}$ and $\mathbf{v}_{\mathtt{f}}$ are the orthonormal basis of the destination surface, and $l_{0\mathtt{f}}$ is $\mathbf{u}_{\mathtt{f}}$ component of the local position. Afterwards, ds can be directly put into Eq. (6) and (7) to decouple the transport Jacobian into two matrices.

$$\mathbb{J}_T = \begin{pmatrix} \mathbf{I}+\mathbf{Q}_{rr} & \mathbf{Q}_{rt} & \mathbb{O} \\ \mathbf{Q}_{tr} & \mathbf{I}+\mathbf{Q}_{tt} & \mathbb{O} \\ \mathbf{Q}_{\psi r} & \mathbf{Q}_{\psi t} & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\mathbf{r}_{\mathtt{f}}}{\partial\mathbf{r}_{\mathtt{i}}} & \frac{\partial\mathbf{r}_{\mathtt{f}}}{\partial\mathbf{t}_{\mathtt{i}}} & \frac{\partial\mathbf{r}_{\mathtt{f}}}{\partial\psi_{\mathtt{i}}} \\ \frac{\partial\mathbf{t}_{\mathtt{f}}}{\partial\mathbf{r}_{\mathtt{i}}} & \frac{\partial\mathbf{t}_{\mathtt{f}}}{\partial\mathbf{t}_{\mathtt{i}}} & \frac{\partial\mathbf{t}_{\mathtt{f}}}{\partial\psi_{\mathtt{i}}} \\ \frac{\partial\psi_{\mathtt{f}}}{\partial\mathbf{r}_{\mathtt{i}}} & \frac{\partial\psi_{\mathtt{f}}}{\partial\mathbf{t}_{\mathtt{i}}} & \frac{\partial\psi_{\mathtt{f}}}{\partial\psi_{\mathtt{i}}} \end{pmatrix}, \tag{20}$$

where

$$
\begin{aligned}
\mathbf{Q}_{rr} &= -\frac{1}{1-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})^2+\kappa}\mathbf{t}_\mathrm{f}(\mathbf{t}_\mathrm{f}-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})\mathbf{v}_\mathrm{f})^\mathrm{T},\\
\mathbf{Q}_{rt} &= -\frac{1}{1-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})^2+\kappa}\mathbf{t}_\mathrm{f}(l_{0\mathrm{f}}\mathbf{u}_\mathrm{f})^\mathrm{T},\\
\mathbf{Q}_{tr} &= -\frac{1}{1-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})^2+\kappa}\psi_\mathrm{f}(\mathbf{t}_\mathrm{f}\times\mathbf{b}(\mathbf{r}_\mathrm{f}))(\mathbf{t}_\mathrm{f}-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})\mathbf{v}_\mathrm{f})^\mathrm{T},\\
\mathbf{Q}_{tt} &= -\frac{1}{1-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})^2+\kappa}\psi_\mathrm{f}(\mathbf{t}_\mathrm{f}\times\mathbf{b}(\mathbf{r}_\mathrm{f}))(l_{0\mathrm{f}}\mathbf{u}_\mathrm{f})^\mathrm{T},\\
\mathbf{Q}_{\psi r} &= -\frac{1}{1-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})^2+\kappa}\frac{\psi_\mathrm{f}E_\mathrm{f}}{p_\mathrm{f}^2}\left(-\frac{\partial E_\mathrm{f}}{\partial s}\right)(\mathbf{t}_\mathrm{f}-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})\mathbf{v}_\mathrm{f})^\mathrm{T},\\
\mathbf{Q}_{\psi t} &= -\frac{1}{1-(\mathbf{t}_\mathrm{f}\cdot\mathbf{v}_\mathrm{f})^2+\kappa}\frac{\psi_\mathrm{f}E_\mathrm{f}}{p_\mathrm{f}^2}\left(-\frac{\partial E_\mathrm{f}}{\partial s}\right)(l_{0\mathrm{f}}\mathbf{u}_\mathrm{f})^\mathrm{T}, \qquad (21)
\end{aligned}
$$

where κ is defined as $l_{0\mathrm{f}}\mathbf{u}_\mathrm{f}\cdot\psi_\mathrm{f}(\mathbf{t}_\mathrm{f}\times\mathbf{b}(\mathbf{r}_\mathrm{f}))$.

The exact divergence condition is complicated, but it can be simplified for tracks with low curvatures by approximating κ to zero. It is a fair assumption because, for example, a particle with the transverse momentum of 1 GeV/c under the longitudinal magnetic field of 1 T has the radius of 3.33 m in xy-plane, and detecting ranges of wire measurements are usually around 1 cm, where the term gets negligible. Under this assumption, the divergence condition is met when the track is almost parallel to the line direction, for the same reason explained in Section 2.1.

3. Comparisons with a numerical differentiation

4. Discussions

Mention the other coordinate of local frame and how to implement correction term with vector representation.

Also add me to the section 1.

5. Summary and conclusions

Acknowledgements

This work was supported by the National Science Foundation under Cooperative Agreements OAC-1836650 and PHY-2323298.

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