What is the minimum value of the following function, where $t \in \{1, 2, 3, \ldots\}$:

$$\frac{1}{2}\left(1 - 2^{-t}\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{t}\right)^{t}\right)$$

(a) $\frac{1}{3}$

(b) $\frac{1}{2}$

(c) $\frac{2}{3}$

(d) $\frac{3}{4}$

d.

What best describes the functions $g_1(x)$ and $g_2(x)$ in the following plot (Figure 1).

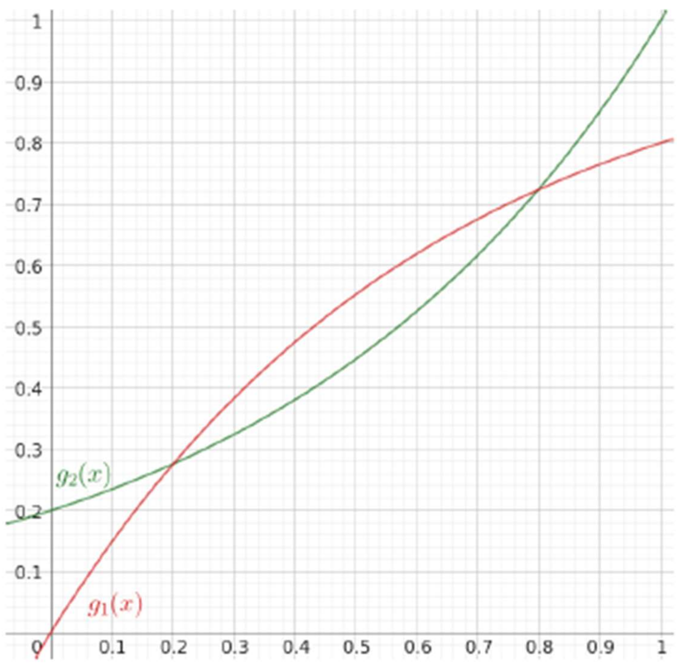

Figure 1: Problem 2

(a) $g_1(x) = c^{x-1}$, $g_2(x) = 1 - c^{-x}$ for some $c > 4$

(b) $g_1(x) = 1 - c^{-x}$, $g_2(x) = c^{x-1}$ for some $c > 4$

(c) $g_1(x) = c^{x-1}$, $g_2(x) = 1 - c^{-x}$ for some $c < 4$

(d) $g_1(x) = 1 - c^{-x}$, $g_2(x) = c^{x-1}$ for some $c < 4$

b.

Which of the following is **INCORRECT** for a concave continuous function $f(x)$ defined over all real numbers?

(a) $f''(x) \leqslant 0$
(b) $\forall x \in (0,1)$, $f(x) \geqslant f(0) + x \cdot (f(1) - f(0))$
(c) $\forall x, y \in \mathbb{R}$, $f\left(\frac{x+y}{2}\right) \geqslant \frac{f(x)+f(y)}{2}$
(d) $\forall x \in \mathbb{R}, f(x) \geqslant f''(x)$

d.

What best describes the term 'Integrality Gap' for an optimization problem with some ILP formulation and LP relaxation? Assume standard notations.

(a) The extreme value of $\frac{\mathsf{ILP\text{-}OPT}(\mathcal{I})}{\mathsf{LP\text{-}OPT}(\mathcal{I})}$ over all instances $\mathcal{I}$
(b) The average of $\frac{\mathsf{ILP\text{-}OPT}(\mathcal{I})}{\mathsf{LP\text{-}OPT}(\mathcal{I})}$ over all instances $\mathcal{I}$
(c) The extreme value of $\{\mathsf{OPT}(\mathcal{I}) - \lfloor \mathsf{OPT}(\mathcal{I}) \rfloor\}$ over all instances $\mathcal{I}$
(d) The average of $\{\mathsf{OPT}(\mathcal{I}) - \lfloor \mathsf{OPT}(\mathcal{I}) \rfloor\}$ over all instances $\mathcal{I}$

a.

Let X be a uniform random variable with the range $[0.2, 1]$. What is the expected value of $\frac{1}{X}$?

(a) $\frac{1}{0.6} \approx 1.6667$
(b) ∞
(c) $\frac{5}{4} \ln 5 \approx 2.1180$
(d) $\ln 5 \approx 1.6094$

c.

Recall the randomized rounding algorithm for the prize collecting Steiner tree problem on the graph $G(V, E)$. Assuming standard notation, let (x^*, y^*) be an optimal solution to the LP relaxation. Let $U = \{v \in V \mid y_v^* \geqslant \alpha\}$ for some $\alpha \in (0,1)$. Let $T(V_T, E_T)$ be a Steiner tree with the terminal set U. Then, which of the following is correct?

(a) $V_T \subseteq U$ and V_T may or may not be equal to U.
(b) $U \subseteq V_T$ and V_T may or may not be equal to U.
(c) $V_T = U$
(d) $V_T = V \setminus U$

b.

What was the method used to derandomize the 2.54-factor approximation algorithm for the prize collecting Steiner tree problem on $G(V, E)$, keeping the approximation guarantee unchanged? Assume that the framework is to find an optimal solution (x^*, y^*), choose all vertices $v \in V$ with $y_v^* \geqslant \alpha$, find a Steiner tree on these vertices and return it.

(a) Set $\alpha = 2/3$.

(b) Set $\alpha = \dfrac{1}{1 - e^{-1/2}}$.

(c) Run the algorithm $|V|$ times, for all choices of $\alpha \in \{y_v^* \mid v \in V\}$.

(d) Since integrality gap is 3, we cannot get a 2.54-factor deterministic algorithm.

c.

Consider the ILP formulation and its LP relaxation of the prize collecting Steiner tree problem where ILP-OPT and LP-OPT are their respective optimal values. On a cycle of n vertices, where each vertex has cost ∞, and each edge has cost 1, which of the following is true regarding the values of ILP-OPT and LP-OPT?

(a) $\text{ILP-OPT} = n - 1$ and $\text{LP-OPT} \leqslant \dfrac{n}{2}$

(b) $\text{ILP-OPT} \leqslant \dfrac{n}{2}$ and $\text{LP-OPT} = n - 1$

(c) $\text{ILP-OPT} = n - 1$ and $\text{LP-OPT} = n - 1$

(d) $\text{ILP-OPT} \leqslant \dfrac{n}{2}$ and $\text{LP-OPT} \leqslant \dfrac{n}{2}$

a.

Consider the 3-factor approximation algorithm for the uncapacitated facility location problem using randomized rounding of LP. Let $c_{i,j}$ be the cost of assigning client j to facility i, and let f_i be the cost of opening the i-th facility. Following the notation discussed in class, let (x^*, y^*) and (v^*, w^*) be optimal primal and dual solutions of the corresponding LP's respectively. At the k-th iteration of the algorithm, how is the client j_k chosen among the unassigned clients? Note that we denote the set of facilities by F.

(a) j_k is chosen with probability $v_{j_k}^*$

(b) j_k is chosen such that it produces the minimum value of v_j^* among all unassigned j

(c) j_k is chosen with probability $v_{j_k}^* + \sum_{i \in F} c_{i,j_k} x_{i,j_k}^*$

(d) j_k is chosen such that it produces the minimum value of $v_j^* + \sum_{i \in F} c_{i,j} x_{i,j}^*$ among all unassigned j

d.

Identify the **INCORRECT** statement regarding the 3-factor approximation algorithm for the uncapacited location problem discussed in class.

(a) The set of facilities opened in each iteration is random and may be different for different runs of the algorithm.

(b) The set of clients, whose neighbourhoods are considered in each iteration, is random and can be different for different runs of the algorithm.

(c) Both the relaxed primal LP as well as its dual LP needs to be solved using an LP solver as a subroutine of this algorithm.

(d) The algorithm ends by opening some subset of facilities, and assigns every client to exactly one open facility.

b.