Graph Theory

* **Graph**: A graph G is defined as pair of sets (V, E) where V = set of all vertices (nodes) in G and E = set of all edges in G.
  $|V(G)|$ = order of G = Number of vertices in G
  $|E(G)|$ = size of G = Number of edges in G
  K = Connected Component (K) _ no of maximal connected subgraphs

* **Directed Graph (diagraph)**: The elements of E are ordered pairs of vertices. In this case an edge (u, v) is said to be from u to v.

* **Non-directed Graph**: In a undirected graph an edge $\{u, v\}$ is said to join u and v (or) to be between u and v.

* **Null Graph**: A null graph of order n is a graph with n vertices and no edges.

* **Loop**: An edge drawn from a vertex to itself is called a loop.

* **Parallel edges**: In a graph, if a pair of vertices are joined by more than one edge then those edges are called parallel edges.

Graphs → Directed, Undirected → Simple, Multi, Pseudo

	loop	Parallel edges
Simple	✗	✗
Multi	✗	✓
Pseudo	✓	✓

Theo: Max-edge (SG) $\leq \frac{n(n-1)}{2}$

Theo: Max-deg (SG) $\leq (n-1)$

Obser: Total no of graphs possible with 'n' vertices & 'e' edges $\longrightarrow {}^{\frac{n(n-1)}{2}}C_e$

* **Degree of a vertex (deg(v))**: no of edges incident on the vertex, self loops counted 2

Theo: # odd deg vertex = EVEN

Theo: In simple Graph, atleast 2 vertex will have same degree. ($n \geq 2$)

Theo: $\sum_{\forall v_i \in G} d(v_i) = 2e$ [Handshaking Theorem]

Theo: $\sum \text{indeg}(v_i) = \sum \text{outdeg}(v) = E$ [Handshaking for Digraphs]

* **Lone vertex**: vertex with deg 0.
* **Pendant vertex**: vertex with deg 1 (Leaf nodes in trees)

Theo: A K-regular graph with n vertices has $\frac{nK}{2}$ edges. $\Rightarrow K \cdot |V| = 2 \cdot |E|$

Theo: No graph of odd order can be K-regular when n is odd.

* **Degree Sequence**: the degrees of all vertices in some sequence (asc or desc) many graphs correspond to 1 degree sequence.

* **Havell-Hakimi Theo**: A deg seq $|d_1 d_2 d_3 \ldots d_n|$ in decreasing order is a graphical seq. iff $|d_2-1, d_3-1, d_4-1, \ldots, d_{i+1}-1, d_{i+2}, d_{i+3}, \ldots, d_n|$ is a graphical (Start from max deg)
  * Any graphical seq. must have atleast one repetition.

* **Min deg, Max deg Theo**: $\delta \leq \frac{2e}{n} \leq \Delta$

$\frac{2e}{n}$ ⇝ average degree = $\frac{\text{Total degree}}{\text{no of vertices}}$

max deg : $\Delta(G)$
min deg : $\delta(G)$

max value of $\delta = \left\lfloor \frac{2e}{n} \right\rfloor$

min value of $\Delta = \left\lceil \frac{2e}{n} \right\rceil$

Inequality Theorem:

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$$\boxed{\underbrace{\text{Vertex connectivity}}_{K(G)} \leq \underbrace{\text{edge connectivity}}_{\lambda(G)} \leq \delta(G) \leq \frac{2e}{n} \leq \Delta(G) \leq n-1}$$

Special Graphs	Notations	$V(G)$ Vertices	$E(G)$ Edges	Definition	$\chi(G)$ Chromatic no.	Diameter
Null Graph	ϕ_n	n	0	$\lvert e\rvert = 0$	1	∞
Complete Graph	K_n	n	$n(n-1)/2$	$\lvert e\rvert = {}^{n}C_2$	n	1
Regular Graph	K-reg	n	$nk/2$	$\delta = \Delta = K$	–	–
Cycle Graph	C_n	n	n	Polygon	$\begin{cases}2, & n \text{ even}\\ 3, & n \text{ odd}\end{cases}$	$\lfloor \frac{n}{2} \rfloor$
Wheel Graph	W_n	n	$2(n-1)$		$\begin{cases}3, & n \text{ is odd}\\ 4, & n \text{ is even}\end{cases}$	$\begin{cases}1, & n=4\\ 2, & n>4\end{cases}$
n-Cubes	Q_n	2^n	$n2^{n-1}$	Boolean Algebra	2	n
Complete Bipartite Graph.	$K_{m,n}$	$m+n$	mn	$e \in G \Leftrightarrow V_1 - V_2$	2 (1, iff $\lvert e\rvert = 0$)	2

Not for complete Bipartite graph

↳ In general a complete bipartite graph is not a complete graph.

* **Cyclic Graph**:- A graph with atleast one cycle is called Cyclic graph.

* **Acyclic Graph**:- having no cycle.

* **Connected Graph**:- A undirected graph G is called connected if there is a path b/w every pair of distinct vertices in G.

NOTE → A graph which is not connected has atleast 2 connected components.

→ A connected graph with no cycles is called a Tree.

→ A tree with n vertices has $(n-1)$ edges.

→ A tree with $n (n>1)$ vertices has atleast 2 vertices of degree 1.

→ An acyclic graph which is not connected is called a forest.

* **Bipartite Graph**:- $G(V,E)$ s.t. V can be divided into 2 sets V_1 & V_2 such that edges will be from one set to another but not in same set.

Theo:- Bipartite Graph cannot contain odd-len cycle. Then either it consist of EVEN length cycle(s) or no cycle.

Theo:- Every even-len cyc. can be converted to Bi-partite.

* **Complete Bipartite Graph**:- $(K_{m,n})$

$\lVert v_1 \rVert = m \quad \lVert v_2 \rVert = n$

→ $\delta(K_{m,n}) = \min(m,n)$

→ $\Delta(K_{m,n}) = \max(m,n)$

→ $V = m+n$

$E = m * n$

→ $\max(e) = \left\lfloor \frac{n^2}{4} \right\rfloor$ ← when split b/w m & n is as close as possible.

* **Star Graph**:- A complete bipartite graph of the form $K_{1,n-1}$ is called a star graph.

↳ A star graph is a tree.

* **Complement of a graph**:-

→ $K_n - G = \overline{G}$

→ $\boxed{\text{one of } G \text{ or } \overline{G} \text{ will always be connected.}}$

→ if G = disconnected ; $\overline{G}$ = connected

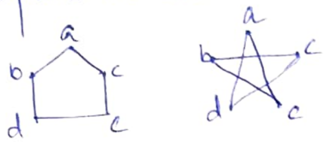
if G = connected ; $\overline{G}$ = disconnected.

i.e. if G is graph with 'x' edges then $\overline{G}$ will have ${}^{n}C_2 - x$.

$G_1 \cup G_2 = K_n$ and $G_1 \cap G_2 = \phi_n$ $\Longleftrightarrow$ G_1 & G_2 are complement of each other
where n is no of vertices in G_1 and G_2

* Complement of a Null graph is complete graph.
* If deg seq. of a graph G with order n is $(a\ b\ c\ d.\ e \ldots)$ then deg. seq. of $\bar{G}$ is $(n-a-1 \quad n-b-1 \quad n-c-1 \ldots\ldots)$

* Self Complemented Graph :- 2 graphs are self complemented graph if they are isomorphic to each other. { Q_n is not self complemented graph }

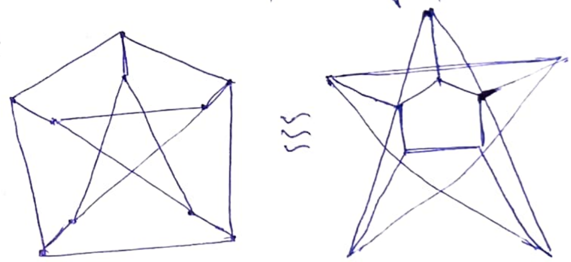
eg  self complementary graphs

* If graph with vertices in a self complementary graph then no of edges $= \dfrac{n(n-1)}{4}$
ie. no of edges either in the form of $4n+1$ or $4n$ but the converse ie not true. ie $C_5 \cong \overline{C_5}$

* Isomorphic :- G_1, G_2 two graph are said to isomorphic iff $f: G_1 \rightarrow G_2$ is a Bijective functions
  (i) $V(G_1) = V(G_2)$
  (ii) $e(G_1) = e(G_2)$
  (iii) degree sequence same.
  (iv) No of cycles same.
  (v) Adjacency preserved.

eg. $K_{3,3}$ is isomorphic to K_5
Note: $K_{3,3}$ and K_5 are non-planner graph.

peterson graph:



* Topological Sorting :- The topological sort algo takes a directed acyclic graph and returns an array of the nodes where each node appears before all the nodes it points to.

* Connectivity of a graph :-
  $\forall x, y \in G$ $\exists$ a path from x to y
  Smallest connected graph $\longrightarrow$ • a
  Disconnected graph. $\exists\, x, y \in G$ s.t. no path b/w x and y
* If in any graph there is exactly 2 vertices of odd degree, then there is a ~~graph~~ path. b/w them.

* Connected & disconnected :-

→ Path exists b/w all pairs of vertices. ⟶ Conn.

→ $(n-1) \le e \le \frac{n(n-1)}{2}$ ⟶ for conn graph; $(K=1)$

→ $\underbrace{(n-K)}_{\text{upto pseudo graph.}} \le e \le \underbrace{\frac{(n-K)(n-K+1)}{2}}_{\text{for simple graph.}}$ ⟶ for disconn graph; $(K \ge 2)$

* max no of edges in a disconnected (no of components) graph with n vertices $= \frac{(n-1)(n-2)}{2}$

→ if config of disconn graph given: then calculate max edges manually.

* Cut-vertex :- a vertex is cut-vertex if on removal of it graph is disconnected (articulation point or cut node) no of components

* Cut-edge :- an edge $\in G$ if its removal disconnects the graph.
  * also known as Bridge
  * a pendent edge is always cut-edge.

* Theo: if cut edge exists then cut vertex also exists $(n \ge 3)$

* $K_{1,n}$, no of cut edges $= n$ and no of cut vertex $= 1$.

* Theo: Star graph generates most no of components on removal of one vertex.

* A Tree with n vertices $(n \ge 2)$ will have $(n-1)$ cut edges.

* A Tree with n vertices $(n \ge 2)$ will have no of cut vertex $= i$ where $i =$ no of internal nodes $[i = n - L]$

* No cut vertex exists in cyclic graph, if a. is a cut edge it must be included in every spanning tree.

* A vertex v is a cut vertex iff $\exists\, x, y \in G$, such that every path b/w x and y includes v.

* In a rooted Full m-ary Tree $\boxed{n = mi + 1}$ every internal node will have m children and a root which is not child of anyone.

* Cut-set :- A minimal set of edges whose removal disconnects the graph a n-edge cutset can never a K-1 edge cutset.

* Vertex-connectivity & edge-connectivity :-

↳ min no of vertex to be removed from the graph to disconnect it. (attack starting from minimum degree)

edge connectivity - min no of edges to be removed to make the graph disconnected. (Less than min degree always)

$\therefore\ VC \le EC \le \delta$

Theo: A graph is separable iff VC = 1

* A graph is n-connected → VC = n and K line connected → EC = K
for a complete graph, EC = n-1
VC does not exist but we assume it to be n-1.

* Strongly connected ∀ x, y path from x → y and y → x should exist.
* Unilateral connected ∀ x, y path from x → y or y → x exist.
* weak connected graph must have single component.

• Walk / Path / Trail :-

	Repeat E	Repeat V
Walk →	✓	✓
Trail →	✗	✓
path →	✗	✗

* Walk – Seq of vertices such that b/w 2 successive vertices there should be an edge and edge should not be repeated.

* path :- No vertices should repeat but starting and vertices should be same and no edge should repeat.

* Cycle :- Every Closed path is a cycle

* Directed Euler Graph :-
a connected graph which contains a directed Euler ~~graph~~ cycle. OR it is unilaterally connected and $\forall_{v \in G}$, indeg(v) = outdeg(v)

* Girth :- Length of smallest cycle in the graph.
girth of an acyclic graph is 'O', girth of cycle graph is n.

* Euler's Graph :- [Closed trail cover all edges exactly one time].
→ G is Euler iff it is connected & $\boxed{\forall_{v \in G} \deg(v) = \text{EVEN}}$ → Theo
→ G is Euler iff it contains euler ~~graph~~ cycle (atleast one)
→ A graph contains euler's path iff it is connected and contains exactly 2 vertices of odd degree [P_n is not Euler]. [Only for conn. graph] Theo

→ K-regular graph is euler iff K is even and graph is connected.
→ a wheel graph is never euler
→ Q_n is euler graph when n is even.
→ a Bipartite complete graph is euler if both m and n are even.

* No of odd degree vertices = 0 → Euler, 2 → Unicursal

* Euler Line / Path :- [Open ~~trai~~ trail cover all edges exactly once].

* Unicursal path :- a connected graph which contains euler path.

if a graph contains Euler's cycle, then will be no euler path.

* Traversable graph :- a connected graph which either is Euler or unicursal.

* **Hamiltonian Graph :-** (H.G) [Closed path that covers all vertices]

Plan

G is Hamiltonian iff it contains a cycle in which every vertex should be traversed atleast once.

* Theo: K_n $\forall (n \geq 3)$ in HG
$\forall (n \geq 2)$ in HP.

→ A Null P_n graph is not Hamiltonian.

→ K-regular graph may or may not be Hamn.

→ A complete graph with $n \geq 3$ is always Hamiltonian.
no of closed paths = n!

→ A cycle graph is always Hamn

Q_2 and Q_3 are Hamiltonian graphs.

→ A wheel graph is always Hamn

→ Q_n - may or may not be Hamn

Properties - A Hamiltonian graph never has any pendent edge.

* **Hamiltonian path :-** Open path that covers all vertices (H.P.)

→ Every HG contains HP, but reverse need not be true.

* **Dirac's Theo :-** if a graph is connected & $\forall_{v \in G}$ $deg(v) \geq n/2$ then graph is Hamiltonian..

* **ORE's Theo :-** if a graph is connected and $\forall_{u,v}$ where u and v are non adjacent, if $deg(u) + deg(v) \geq n \rightarrow G$ is Hamiltonian.

K_n → no of Ham cycles = n!

→ no of Ham cycles starting from a particular vertex = (n-1)!

→ no of unordered Ham cycles = (n-1)! * $\frac{n!}{n}$ ← symmetry

→ no of edge disjoint Hamn cycles = $\left\lfloor \frac{n-1}{2} \right\rfloor$

* for all $n \geq 3$, the number of distinct Hamn cycles in a complete graph (K_n) is → $\frac{(n-1)!}{2}$

Adjacency Matrix - can represent SG, MG and PG

→ for SG and MG $\sum_{\forall i \in n} \sum_{\forall j \in n} a_{ij} = 2 \times |E|$ not for PG

→ Row sum ~~and~~ or col^n sum represents the degree of corresponding vertex.

* For directed Graph -
row sum → out degree
column sum → in degree

$\sum_{\forall i \in N} \sum_{\forall j \in N} a_{ij} = E$ for directed graph works for PG as well

* **Incidence matrix :**

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} c_1 \; c_2 \; c_3 \ldots c_5 \\ \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

→ each col^n will contain exactly 2 1's except for self loops

→ row sum corresponds to degree of vertex.

Planar Graphs :- A graph G is planar iff ∃ a planar representation (embedding) such that no 2 edges are crossing each other.

→ K_4 is planar, K_5 is not planar, $K_{3,3}$ is not planar

Euler's formula for planar graphs

for disconn graph → $\boxed{r = e - n + k + 1}$

e = no of edges
n = no of vertices
r = no of regions
k = no of components

* for a connected planar graph

→ $\boxed{r = e - n + 2}$

⇒ In all the regions, one will be open and all the others will be closed.

* Kuratowski's Theorem :- A graph G is planar iff it does not contains any subgraph homoeomorphic to K_5 or $K_{3,3}$.

NOTE: K_5 is smallest non-planar graph in terms of vertices and $K_{3,3}$ is smallest non-planar graph in terms of edges.

⇒ An isomorphic graph is always Homoeomorphic.

⇒ A homoeomorphic graph may or may not be isomorphic.

* Homoemorphism :-

H is homoemorphism to G iff

→ $\alpha : V(G) \longrightarrow V(H)$

→ need not bijective

→ $xy \in E(G) \longrightarrow \alpha(x)\alpha(y) \in E(H)$

* Isomorphism :-

H is isomorphism to G iff:

→ $\alpha : V(G) \longrightarrow H(G)$

→ Bijective

→ $xy \in E(G) \longleftrightarrow \alpha(x)\alpha(y) \in E(H)$

→ Adjancency & non-adjacency is preserved.

→ if a graph is planar : $\boxed{3f \le 2e}$

$\therefore$ $\boxed{\begin{array}{l} e \le (3v - 6) \\ \delta \le 5 \end{array}}$

→ if a graph is planar and no triangle

$\therefore$ $\boxed{\begin{array}{l} e \le 2n - 4 \\ \delta \le 3 \end{array}}$

→ in any connected planar + bipartite : $\boxed{4f \le 2e}$

$\therefore$ $\boxed{e \le (2v - 4)}$

→ dirac : $\forall v_i \in G ;\ d(v_i) \ge n/2$

→ Ores : $d(u) + d(v) \ge n \ \forall (u, v) \in E$

* **Tree** - A graph is a Tree iff it is connected and acyclic (minimally conn graph)
Satisfying only 2 of connected, acyclic and (n-1) edges will imply 3rd and it will be tree.

$\forall x, y \; \exists$ exactly one path b/w x and y.

every tree with 2 or more values is Dichromatic/Bipartite.

Fundamental cycle of a tree - in a tree if an edge is added b/w any 2 vertices, then ~~actually~~ exactly one cycle will be generated and that cycle is known as fundamental cycle. [True for trees only.]

* no of fundamental cycles in a Tree with n vertices $= {}^nC_2$
* Depth (vertex): Distance b/w root and vertex
* Height (tree): maximum depth of any vertex in the tree
* Levels (tree) = height +1
* m-ary tree - $0 \le$ children of node $s \le m$
* full m-ary tree - 0 or m children of any node.
* Complete m-ary - all levels except the last level should be completely filled.

→ A complete m-ary tree may not be full m-ary tree and vice versa

* Rank and Nullity of a graph: for a graph with n vertices, e edges and K components

$Rank(G) = n-K$ $\qquad$ Rank + Nullity = no of edges = e

$Nullity(G) = e-n+K$

Rank (G) = no of edges in the spanning tree of the graph.
or
no of edges in the spanning forest of the disconnected graph.

Nullity = no of edges to be removed from a graph to convert it into spanning forest/tree.

* Branch Set - set of edges taken into the tree, $|BS| = Rank$
* Chord Set - set of edges which are not taken, $|CS| = Nullity$.
* Radius and diameter of a tree:

→ eccentricity (v): Distance of the farthest vertex from v.

→ Centre (Tree): vertex with minimum eccentricity

→ Radius: min^n eccentricity; diameter: max^n eccentricity

→ A tree can be either unicentric or Bicentric only.

for a graph :- eccentricity : shortest distance to the farthest node.

with n vertices, no of labelled graphs = $2^{\frac{n(n-1)}{2}}$

no of simple labelled graphs with edges = ${}^{\frac{n(n-1)}{2}}C_e$

no of labelled trees possible = n^{n-2}

no. of spanning trees of $K_n = n^{n-2}$

no of rooted Labelled tree = n^{n-1}

no of labelled subgraph for a K_n graph . $\sum_{r=1}^{n} {}^nC_r \, 2^{\frac{r(r-1)}{2}}$

* Graph No.'s :-

1. Chromatic No: $\chi(G)$ - minimum no of colours required to ~~do~~ colour a graph such that no adjacent vertices have same colour.

$\chi(\text{Tree} \mid \text{Bipartite}) = 2$ $\quad \chi(C_n) = \begin{cases} 2 & \text{even} \\ 3 & \text{odd} \end{cases}$ $\quad \chi(W_n) = \begin{cases} 4 & \text{even} \\ 3 & \text{odd} \end{cases}$

$\chi(\text{isolated}) = 1$.

2. Independence Set :- Set of the vertices which are not adjacent to each other. "NonAdjacent Vertices"

*also known as Clique.

* a single vertex is always independent set.
* an independent set is maximal if even one of the remaining vertex added should violate the condition for independent set.
* Largest maximal independent set is not unique but independence no. is unique
- To draw independence set → Start with min^n deg vertex.
  → start adding vertices by checking adjacency

* Independence No:- size of largest maximal independence set.

3. Dominating set :- Set of vertex from where whole graph is covered in one move. "He or my friend"

Maximal independent set ⟶ Dominating set (not neccessarily minimal)

converse is not true.

- To draw minimal dominating set → take vertices with maximum degree
  → add vertices which ~~are~~ are uncovered.

* Domination no. :- size of smallest minimal dominating set.

* Domination No. ≤ Independence No. * $[\alpha(G) \leq \beta(G)]$
* Independence No ⇒ $\beta_g \geq \frac{n}{K}$ [pigeonhole principle]
* $\frac{n}{\beta(G)} \leq \chi(G) \leq n+1-\beta(G)$

* Thm : Chromatic no of a graph is atleast equal to largest clique.

There is no meaning of chromatic no. with pseudo graph.

* Chromatic no of multigraph is equal to chromatic no of its equivalent simple graph.
* Chromatic no of a planar graph is atmost 4 $\quad \boxed{K_G \leq 1 + \Delta \leq n}$

for any graph.

→ "Non Adjacent edges"

* **Matching**: Set of edges none of which are adjacent to other.

* NOTE: $m(K_n) = \lfloor \frac{n}{2} \rfloor$

* **Matching no.**: size of largest maximal matching
(always unique)
↳ obtain → Start with the edges with less no of adjacency and continue.

→ "Atleast 1 Marriage proposal"

* **Covering**: Set of edges which covers all vertices are included (adj) to atleast 1 Edge of that set.

* **Covering no.**: size of smallest minimal cover for edges.

* The covering of graph is atleast equal to $n/2$ i.e. $|cover| \geq \lceil \frac{n}{2} \rceil$
* A pendant edge will always be a part of cover
* A matching need not be covering and a covering need not be matching.

* **Perfect Matching (PM)**: Matching + covering
→ Maximal matching ~~st~~ st. induced deg of all vertices to 1.
* perfect Matching is possible only for a graph with even no of vertices] Theo

$$\text{No of perfect Matching} = \frac{(2n)!}{n! \times 2^n}$$

Theo: A cover is minimal iff every component is a star graph.
* Cubic graph → 3-regular graph.
* A simple graph G (with n vertices and K edges) is a forest iff G has $(n-K)$ edges.

* The Hamming distance Relation on bit strings has a n-cube (Q_n) structure.
* the components of all cyclic graphs C_n, $n \geq 5$ are connected.

Theo: Every PM is minimal cover but the converse is not true.

Theo: A non-Null graph is Bipartite iff it is Bichromatic

Observation: A tree with 2 or more vertices is always Bipartite.
↳ While ~~ch~~ checking for Bipartite, look for odd length cycles.

1. $m(C_n) = m(W_n) = m(K_n) = \lfloor \frac{n}{2} \rfloor$

2 colourable ⟷ Bipartite graph.

2. $m(K_{m,n}) = \min(m, n)$

* $\chi(G) \leq \beta(G)$; neither complete graph nor ODD length Cycle.
* Chromatic No. $\chi(G) \leq \Delta(G) + 1$
* If $\deg(x) + \deg(y) \geq n$, then, Hamiltonian cycle x, y non adjacent.

Some Theorems & Observations :-

1) If a graph will have "n" no of edges then $\begin{cases} \frac{(n-1)(n-2)}{2}+1 & \text{Guarantee for Connectivity} \\ \frac{(n-1)(n-2)}{2} & \text{No Guarantee for Conn.} \end{cases}$

2) If cut edge exist it should not belongs to cycle.

3) If cut edge exist cut vertex will also be exist for $n \geq 3$.

4) If a SUBSET of any SET is "cut edge" or "cut set" then, it is not necessary that whole set is "cutset".

5) If $\deg(v) \geq n/2$ then Hamiltonian cycle.

6) If $G_{n \geq 3}$ if $|E| = \frac{(n-1)(n-2)}{2} + 2$ then $G \rightarrow$ Ham^n Cycle.

7) K_n will $\left(\frac{n-1}{2}\right)$ Edge disjoint cycle.

8) K_n will $\frac{(n-1)!}{2}$ Cycle.

9) $K_{n,n}$ will $\frac{n!(n-1)!}{2}$ cycle.