Mathematical logic

Connections

- $\wedge$ conjunction
- $\vee$ disjunction
- $\rightarrow$ ....if....
- $\leftrightarrow$ ....iff....
- $\neg$ $\sim$ negation

$p \rightarrow q$

- Converse : $q \rightarrow p$
- Inverse : $\sim p \rightarrow \sim q$
- Contrapositive : $\sim q \rightarrow \sim p$

- $p \rightarrow q \equiv \sim q \rightarrow \sim p \equiv \sim p \vee q$
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv \neg p \leftrightarrow \neg q$
- $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

Type-2
- $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

Logical Equivalences

$p \wedge T \equiv p$	$p \vee F \equiv p$	Identity
$p \vee T \equiv T$	$p \wedge F \equiv F$	Domination
$p \vee p \equiv p$	$p \wedge p \equiv p$	Idempotent
$\neg(\neg p) \equiv p$		Double negation
$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$	Commutative
$(p \vee q) \vee r \equiv p \vee (q \vee r)$		Associative
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$		Associative
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$		Distributive
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$		
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's
$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$	Absorption
$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$	Negation

- $p \vee q \equiv \neg p \rightarrow q$
- $p \wedge q \equiv \neg(p \rightarrow \neg q)$
- $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Type-2
- $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
- $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

- $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Precedence of operators :- $() > ' > \wedge > \vee > \rightarrow > \leftrightarrow$

Inference rule — Consider premises as TRUE. Check for Conclusion is coming as TRUE or FALSE

①	②	③	④	⑤	⑥	⑦
p	$\neg q$	$p \rightarrow q$	$p \vee q$	p	p	$p \vee q$
$p \rightarrow q$	$p \rightarrow q$	$q \rightarrow r$	$\neg p$		q	$\neg p \vee r$
q	$\neg p$	$p \rightarrow r$	q	$p \vee q$	$p \wedge q$	$q \vee r$

① $(p \wedge (p \rightarrow q)) \rightarrow q$

② $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

③ $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

④ $((p \vee q) \wedge \neg p) \rightarrow q$

⑤ $p \rightarrow (p \vee q)$

⑥ $((p) \wedge (q)) \rightarrow (p \wedge q)$

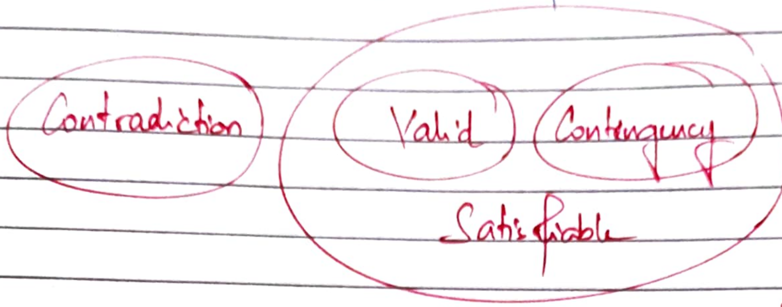
✱ $(p \wedge q) \rightarrow p$

⑦ $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

* **Number of non equivalent propositional function possible with "n" propositional variable** $\longrightarrow$ 2^{2^n}

Page No.

Satisfiable	Tautology	Contradiction	Contengency
Atleast 1 true	All true	All false	Neither tautology nor contradiction



Predicate logic & Quantifiers

Quantifiers → Universal ($\forall$)
Quantifiers → Existential ($\exists$)

Imp
$\forall x\, P(x) = P(x_1) \wedge P(x_2) \wedge \ldots P(x_n)$ ..... Universal
$\exists x\, P(x) = P(x_1) \vee P(x_2) \vee \ldots P(x_n)$ ... Existential

$\sim \forall x P(x) = \exists x \sim P(x)$
$\sim \exists x P(x) = \forall x \sim P(x)$
} Negation

$\forall x P(x) \longrightarrow \exists x P(x)$

English to Predicate
ALL $\rightarrow$
SOME $\wedge$

Rules

$\exists x[P(x) \vee Q(x)] \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

$\forall x[P(x) \wedge Q(x)] \equiv \forall x P(x) \vee \exists x\, Q(x)$

$\exists x\, [P(x) \wedge Q(x)] \longrightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$

$\forall x\, P(x) \vee \forall x\, Q(x) \longrightarrow \forall x\, [P(x) \vee Q(x)]$

$\forall x\, [P(x) \rightarrow Q(x)] \longrightarrow \forall x\, P(x) \longrightarrow \forall x\, Q(x)$

$\forall x\, [P(x) \leftrightarrow Q(x)] \longrightarrow \forall x P(x) \longleftrightarrow \forall x\, Q(x)$

$(\forall x)\, P(x) \wedge (\forall x)\, Q(x) \longleftrightarrow \forall x\, [P(x) \wedge Q(x)]$

$(\forall x)\, P(x) \vee (\forall x)\, Q(x) \longleftrightarrow \forall x\, \forall y\, [P(x) \vee Q(y)]$

$(\exists x)\, P(x) \wedge (\exists x)\, Q(x) \longleftrightarrow \exists x\, \exists y\, [P(x) \wedge Q(y)]$

$(\exists x) P(x) \vee (\exists x)\, Q(x) \longleftrightarrow \exists x\, [P(x) \vee Q(x)]$

$(\forall x)\, P(x) \vee (\exists x)\, Q(x) \longleftrightarrow (\forall x)(\exists y)\, [P(x) \wedge Q(y)]$

$(\forall x) P(x) \vee (\exists x)\, Q(x) \longleftrightarrow \forall x\, \exists y\, [P(x) \vee Q(y)]$

$A \vee \forall x\, P(x) \longleftrightarrow \forall x\, [A \vee P(x)]$

$A \vee \exists x\, P(x) \longleftrightarrow \exists x\, [A \vee P(x)]$

$A \wedge (\forall x)\, P(x) \longleftrightarrow \forall x\, [A \wedge P(x)]$

$A \wedge (\exists x)\, P(x) \longleftrightarrow \exists x\, [A \wedge P(x)]$

Implication

[T → F is only F]

- p implies q
- q, if p
- if p then q
- q follows from p
- q is necessary for p
- p is sufficient for q
- for p, q is necessary
- p only if q

$\equiv p \to q$

Biconditional

- if and only if (iff)
- p is necessary & sufficient for q
- p is same as q
- p is equivalent to q

$\equiv p \leftrightarrow q$

1. when ≡ whenever ≡ if
2. But ≡ and ≡ nevertheless ≡ as well ≡ also.
3. Unless ≡ OR.
4. required ≡ necessary ≡ must.
5. enough ≡ sufficient.

for any argument :-

if [one premise is false] OR [conclusion is TRUE]

⇒ always a valid argument.

*** while checking for the fun^n, we should check for Dependencies and Domain

Domain : Real No.

$P(x, y) = x^2 + y = 10$

$\forall y \exists x . \ x = \sqrt{10 - y}$

For $y = 11$, $x = \sqrt{-1}$ = complex ≠ Real hence false.

*** Existential quantifier variable must be independent of Data on Right of it and may depend on Data on the left

$\forall y \exists x \forall z$ ≡ x can depend on y but x cannot depend on z.

* $\forall x \left[P(x) \to Q(x) \right] \Rightarrow \left[\forall x P(x) \to \forall x Q(x) \right]$

* $\left[(\exists x P(x)) \to (\exists x Q(x)) \right] \Rightarrow \exists x (P(x) \to Q(x))$

Very Imp.

* Demorgan's law :-

$\overline{(p \wedge q)} = p' \vee q'$

$\overline{(p \vee q)} = p' \wedge q'$

* Duality :-

$p \vee q = T \xleftrightarrow{\text{Dual}} p \wedge q = F$

* change $\wedge$ with $\vee$ and 0 with 1 and vice-versa.

* Both Commutative & Associative :-

AND, OR, XOR, $\leftrightarrow$

* Commutative but not Associative :-

NAND $\uparrow$ and NOR $\downarrow$

* Neither commutative nor Associative :-

$\rightarrow$

* Functionally complete set :-

minimal $\{\wedge, \sim\}$, $\{\vee, \sim\}$, $\{\rightarrow, \sim\}$

Smallest minimal $\{\uparrow\}$, $\{\downarrow\}$

* Minimal FC set :-

→ Should be FC

→ No subset is FC (proper)

* Principal Disjunction Normal form :-

Canonical sum of products

* Principal Conjunction Normal form :-

Canonical product of sum.

* SOP and POS are not unique but canonical POS/SOP are unique

* No of distinct. PCNF / PDNF / Boolean functions / Truth tables for Boolean f^n of n variables = 2^{2^n}

⇒ for given assignment of variables, no of minterms = 0 ≡ no of maxterms = 1 ≡ Only one.

*** $p \Rightarrow (q \Rightarrow r) \equiv p' + q' + r = q' + p' + r = q \Rightarrow (p \Rightarrow r) = (p \wedge q) \Rightarrow r$

* $\forall x (P(x) \wedge Q(x)) \longrightarrow \forall x P(x) \wedge \forall x P(x)$ — TRUE

* $\forall x \{ P(x) \vee \forall x Q(x) \longrightarrow \forall x (P(x) \vee Q(x))$ — TRUE

Only one way

Logical Equivalences Diagram

$\exists y \forall x \rightarrow \forall x \exists y$

$\forall x \exists y \not\rightarrow \exists y \forall x$

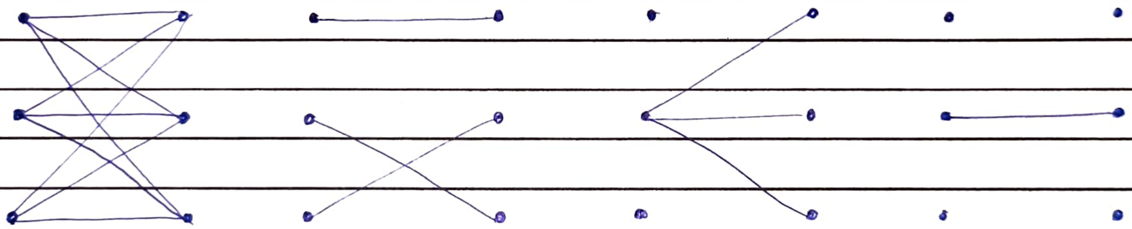
1 girl is gf of all the boys

$$\begin{tikzcd}
\forall x \forall y \arrow[r, leftrightarrow] \arrow[d] & \forall y \forall x \arrow[d] \\
\exists y \forall x \arrow[d] & \exists x \forall y \arrow[d] \\
\forall x \exists y \arrow[d] & \forall y \exists x \arrow[d] \\
\exists y \exists x \arrow[r, leftrightarrow] & \exists x \exists y
\end{tikzcd}$$

$P(x,y)$

Nested Quantifier

$\forall x \forall y$ $\quad$ $\forall x \exists y$ $\quad$ $\exists y \forall x$ $\quad$ $\exists x \exists y$



Translation: $\exists, \forall$ has highest Precedence.

$\exists \rightarrow$ There exists, some, atleast one $\rightarrow$ implication never used inside.
$\forall \rightarrow$ For all, every, any $\rightarrow$ may cause trouble if used with '$\wedge$'

No Hardworking person is poor $\equiv$ Not (a hardworking person is poor)

Special Case: Diamonds & Pearls are precious $\equiv \forall x\ (D(x) \vee P(x) \longrightarrow P_r(x))$
and actually means 'OR'

* Exactly One:

• $\exists!\ x\, A(x)$ (shortcut)

$\exists x\ (A(x) \wedge \forall y\ (A(y) \rightarrow y = x))$ | $[\exists x\ [A(x) \wedge \forall y\ ((x \neq y) \rightarrow \sim A(y))]]$

* Atleast 2: $[\exists x\, \exists y\ (A(x) \wedge A(y) \wedge x \neq y)]$

* Exactly 2: $\exists x\ (A(x) \wedge \exists y\ (A(y) \wedge x \neq y) \wedge \forall z\ (A(z) \Rightarrow x = z \vee y = z)$

OR

$\exists x\ ((A(x) \wedge A(y) \wedge x \neq y) \wedge \forall z\ (x \neq z \text{ or } x \neq y \rightarrow \sim A(z))$

* Atmost 2: $\forall x\, \forall y\, \forall z\ (A(x) \wedge A(y) \wedge A(z)) \Rightarrow x \neq y \text{ or } y \neq z \text{ or } x \neq z)$

OR

$\exists x\, \exists y\ (x \neq y \wedge \forall z\ (A(z) \Longleftrightarrow z = x \text{ or } z = y)$

atleast 2 object

significant