Finite Groups

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Groups

Definition

A set G with a binary operation $\star$ defined on it is called a group if

- the operation $\star$ is associative,
- there exists an identity element $e \in G$ such that for any $a \in G$

$$a \star e = e \star a = a,$$

- for every $a \in G$, there exists an element $b \in G$ such that

$$a \star b = b \star a = e.$$

Example

- Modulo n addition on $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$

Cyclic Groups

Definition
A finite group is a group with a finite number of elements. The order of a finite group *G* is its cardinality.

Definition
A cyclic group is a finite group *G* such that each element in *G* appears in the sequence

$$\{g, g \star g, g \star g \star g, \dots\}$$

for some particular element $g \in G$, which is called a generator of *G*.

Example
$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is a cyclic group with a generator 1

Group Isomorphism

Example

- $\mathbb{Z}_2 = \{0, 1\}$ is a group under modulo 2 addition
- $R = \{1, -1\}$ is a group under multiplication

$\mathbb{Z}_2$	R
$0 \oplus 0 = 0$	$1 \times 1 = 1$
$1 \oplus 0 = 1$	$-1 \times 1 = -1$
$0 \oplus 1 = 1$	$1 \times -1 = -1$
$1 \oplus 1 = 0$	$-1 \times -1 = 1$

Definition

Groups *G* and *H* are isomorphic if there exists a bijection
$\phi : G \to H$ such that

$$\phi(\alpha \star \beta) = \phi(\alpha) \otimes \phi(\beta)$$

for all $\alpha, \beta \in G$.

Cyclic Groups and $\mathbb{Z}_n$

Theorem
Every cyclic group G of order n is isomorphic to $\mathbb{Z}_n$

Proof.
Let *g* be a generator of *G*. Define $g^i = \underbrace{g \star g \star \cdots \star g}_{i \text{ times}}$.

The function $\phi : G \to \mathbb{Z}_n$ defined by $\phi(g^i) = i \bmod n$ is a
bijection. $\qquad \square$

Corollary
Every finite cyclic group is abelian.

Subgroups

Definition
A nonempty subset *S* of a group *G* is called a subgroup of *G* if

- $\alpha + \beta \in S$ for all $\alpha, \beta \in S$
- $-\alpha \in S$ for all $\alpha \in S$

Example
$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ has subgroups

- $\{0\}$
- $\{0, 3\}$
- $\{0, 2, 4\}$
- $\{0, 1, 2, 3, 4, 5\}$

Lagrange's Theorem

Theorem
If S is a subgroup of a finite group G, then $|S|$ divides $|G|$.

Definition
Let S be a subgroup of a group G. For any $g \in G$, the set
$S \oplus g = \{s \oplus g | s \in S\}$ is called a coset of S.

Example
$S = \{0, 3\}$ is a subgroup of $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. It has cosets
$$S \oplus 0 = \{0, 3\}, \quad S \oplus 1 = \{1, 4\}, \quad S \oplus 2 = \{2, 5\},$$
$$S \oplus 3 = \{0, 3\}, \quad S \oplus 4 = \{1, 4\}, \quad S \oplus 5 = \{2, 5\}.$$

Lemma
Two cosets of a subgroup are either equal or disjoint.

Lemma
If S is finite, then all its cosets have the same cardinality.

Application of Lagrange's Theorem

Prove that $2^{p-1} = 1 \bmod p$ for any prime $p > 2$.

- Consider the group $\mathbb{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$ under the operation

  $$a \odot b = ab \bmod p$$

- Consider the subgroup S generated by 2

  $$\left\{ 2, 2^2, 2^3, \ldots, 2^{n-1}, 2^n = 1 \right\}$$

- What can we say about the order of S?

Subgroups of Cyclic Groups

Theorem
Every subgroup of a cyclic group is cyclic.

Theorem
If G is a finite cyclic group with $|G| = n$, then G has a unique subgroup of order d for every divisor d of n.

Proof.

- If $G = \langle g \rangle$ and d divides n, then $\langle g^{n/d} \rangle$ has order d
- Every subgroup of G is of the form $\langle g^k \rangle$ where k divides n
- If k divides n, $\langle g^k \rangle$ has order $\frac{n}{k}$
- If a subgroup has order d, it is equal to $\langle g^{n/d} \rangle$

$\square$

Number of Generators of a Cyclic Group

Examples

- $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ has four generators $1, 2, 3, 4$
- $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ has two generators $1, 5$
- $\mathbb{Z}_{10} = \{0, 1, 2, \ldots, 9\}$ has four generators $1, 3, 7, 9$

Theorem
A cyclic group of order n has $\phi(n)$ generators where

$\phi(n) = $ *No. of integers in* $\{0, 1, \ldots, n-1\}$ *relatively prime to n*

Corollary

$$n = \sum_{d\,:\,d|n} \phi(d)$$

Questions? Takeaways?