Elliptic Curve Cryptography in Bitcoin

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Group Theory Recap

Groups

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Example

- Modulo n addition on $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$

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  - $\mathbb{Z}_n^*$ is cyclic if n is a prime

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- **Lagrange's Theorem:** If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.
- Example: Check the cardinalities of the subgroups of $\mathbb{Z}_6$.
- **Corollary:** If a group has prime order, then every non-identity element is a generator.

Elliptic Curves Over Real Numbers

Elliptic Curves over Reals

The set E of real solutions (x, y) of

$$y^2 = x^3 + ax + b$$

along with a "point of infinity" $\mathcal{O}$. Here $4a^3 + 27b^2 \neq 0$.

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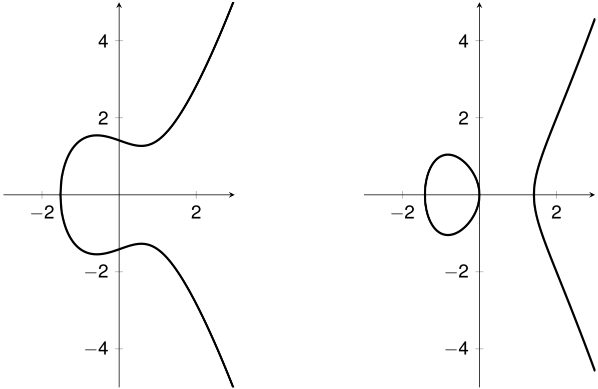
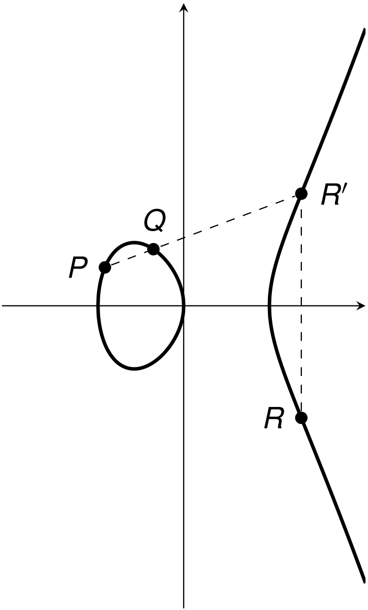


$y^2 = x^3 - x + 2$ $\qquad$ $y^2 = x^3 - 2x$

Point Addition (1/3)



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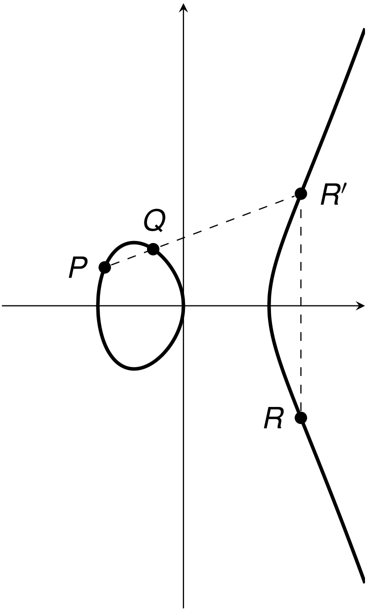

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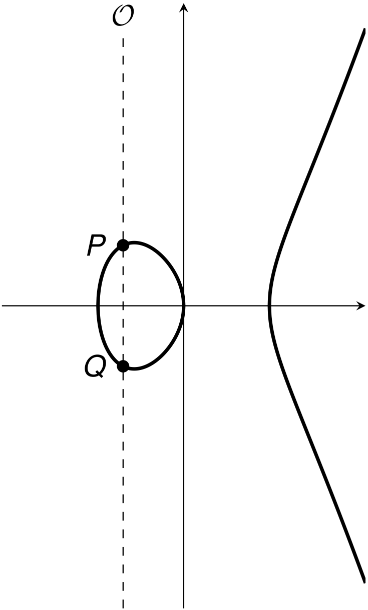
$$R = (x_3, y_3)$$
$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2$$
$$y_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3) - y_1$$

Point Addition (2/3)



$$P = (x_1, y_1), Q = (x_2, y_2)$$
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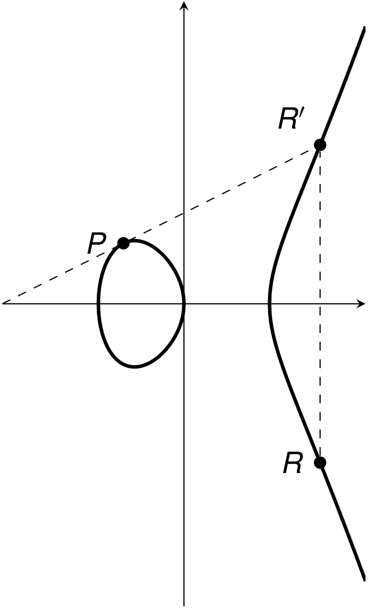
$$P + Q = \mathcal{O}$$

Point Addition (3/3)



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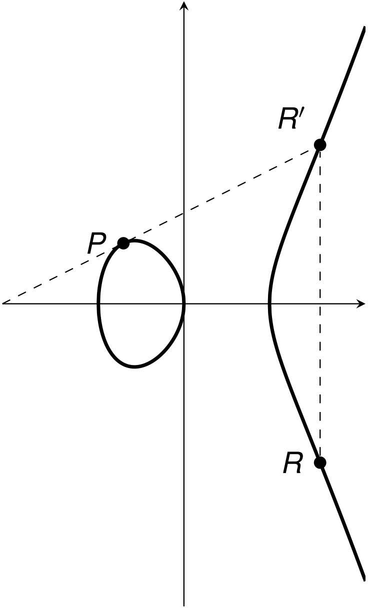

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$$R = (x_3, y_3)$$
$$x_3 = \left(\frac{3x_1^2 + a}{2y_1}\right)^2 - 2x_1$$
$$y_3 = \left(\frac{3x_1^2 + a}{2y_1}\right)(x_1 - x_3) - y_1$$

Elliptic Curves Over Finite Fields

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A finite field is a field with a finite cardinality.

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- $\mathbb{F}_5$

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$*$	0	1	2	3	4
0	0	0	0	0	0
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*	0	1	2	3	4
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2	0	2	4	1	3
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- In fields, division is multiplication by multiplicative inverse

$$\frac{x}{y} = x * y^{-1}$$

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Theorem

The characteristic of a finite field is prime

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For char$(F) \neq 2, 3$, the set E of solutions (x, y) in $\mathbb{F}^2$ of

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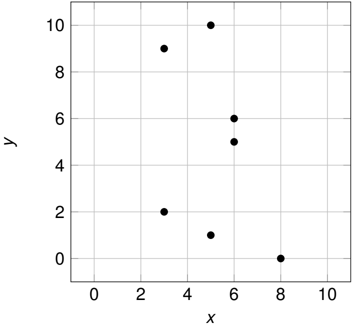
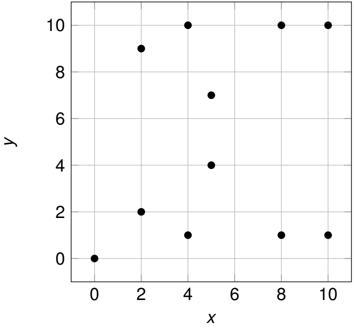


$y^2 = x^3 + 10x + 2$ over $\mathbb{F}_{11}$



$y^2 = x^3 + 9x$ over $\mathbb{F}_{11}$

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$(3,2)$	$(3,2)$	$(6,6)$	$\mathcal{O}$	$(6,5)$	$(8,0)$	$(3,9)$	$(5,10)$	$(5,1)$
$(3,9)$	$(3,9)$	$\mathcal{O}$	$(6,5)$	$(8,0)$	$(6,6)$	$(5,1)$	$(3,2)$	$(5,10)$
$(5,1)$	$(5,1)$	$(6,5)$	$(8,0)$	$(6,6)$	$\mathcal{O}$	$(5,10)$	$(3,9)$	$(3,2)$
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$(6,6)$	$(6,6)$	$(5,10)$	$(3,2)$	$(3,9)$	$(5,1)$	$\mathcal{O}$	$(8,0)$	$(6,5)$
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- In fact, its a group

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$$= 2^{256} - 2^{32} - 2^9 - 2^8 - 2^7 - 2^6 - 2^4 - 1$$

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n = \ &\texttt{FFFFFFFF}\ \texttt{FFFFFFFF}\ \texttt{FFFFFFFF}\ \texttt{FFFFFFFE} \\
&\texttt{BAAEDCE6}\ \texttt{AF48A03B}\ \texttt{BFD25E8C}\ \texttt{D0364141}
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- Public key is kP where $P = (x, y)$

$$x = \texttt{79BE667E F9DCBBAC 55A06295 CE870B07}$$
$$\texttt{029BFCDB 2DCE28D9 59F2815B 16F81798},$$
$$y = \texttt{483ADA77 26A3C465 5DA4FBFC 0E1108A8}$$
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  - Set $N = P$ and $Q = \mathcal{O}$
  - for $i = 0, 1, \ldots, m$
    - if $k_i = 1$, set $Q \leftarrow Q + N$
    - Set $N \leftarrow 2N$
  - Return Q

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  - 173 bits instead of 1024 bits, 373 bits instead of 4096 bits

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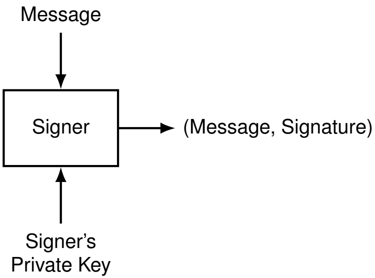

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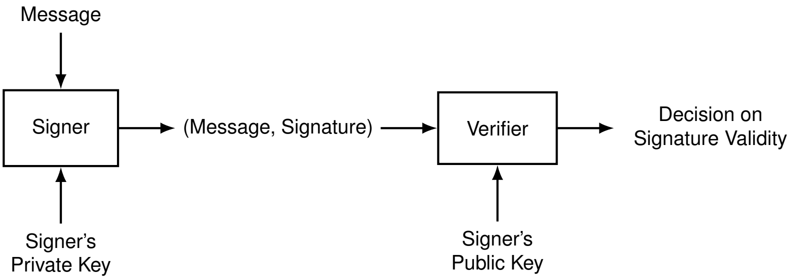

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  2. Let $F : G \mapsto \mathbb{Z}_q$ be a function, not necessarily CHF
  3. **Signer:**
     - 3.1 On input $m \in \{0,1\}^*$, chooses $k \leftarrow \mathbb{Z}_q^*$ and sets $r := F(g^k)$
     - 3.2 Computes $s := \left[k^{-1} \cdot (H(m) + xr)\right] \bmod q$
     - 3.3 If $r = 0$ or $s = 0$, choose k again
     - 3.4 Outputs (r, s) as signature for m
  4. **Verifier**

Digital Signature Algorithm

- Part of the Digital Signature Standard issued by NIST in 1994
- Based on the following identification protocol
  1. Suppose prover knows $x \in \mathbb{Z}_q$ such that public key $h = g^x$
  2. Prover chooses $k \leftarrow \mathbb{Z}_q^*$ and sends $I := g^k$
  3. Verifier chooses uniform $\alpha, r \in \mathbb{Z}_q$ and sends them
  4. Prover sends $s := \left[k^{-1} \cdot (\alpha + xr) \bmod q\right]$ as response
  5. Verifier accepts if $s \neq 0$ and

$$g^{\alpha s^{-1}} \cdot h^{rs^{-1}} \stackrel{?}{=} I$$

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  4. **Verifier**
     - 4.1 On input m and (r, s) with $r \neq 0, s \neq 0$ checks

$$F\left(g^{H(m)s^{-1}} h^{rs^{-1}}\right) \stackrel{?}{=} r$$

ECDSA in Bitcoin

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- As j is randomly chosen, ECDSA output is random for same m

References

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