

Cryptographic Hash Functions

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Cryptographic Hash Functions

- Important building block in cryptography
- Provide data integrity by construction of a short fingerprint or *message digest*
- Map arbitrary length inputs to fixed length outputs
 - For example, output length can be 256 bits
- Applications
 - Password hashing
 - Digital signatures on arbitrary length data
 - Commitment schemes

Properties

- Let $H : \mathcal{X} \mapsto \mathcal{Y}$ denote a cryptographic hash function
- \mathcal{X} and \mathcal{Y} are subsets of $\{0, 1\}^*$
- $H(x)$ can be computed efficiently for all $x \in \mathcal{X}$
- If H is considered secure, three problems are difficult to solve
 - Preimage
 - Given $y \in \mathcal{Y}$, find $x \in \mathcal{X}$ such that $H(x) = y$
 - Second Preimage
 - Given $x \in \mathcal{X}$, find $x' \in \mathcal{X}$ such that $x' \neq x$ and $H(x) = H(x')$
 - Collision
 - Find $x, x' \in \mathcal{X}$ such that $x' \neq x$ and $H(x) = H(x')$
- If $|\mathcal{X}| \geq 2|\mathcal{Y}|$, then we have

Collision resistance \implies Second preimage resistance \implies Preimage resistance

(Proof in Section 4.2, Stinson, 3rd edition)

SHA-256

- SHA = Secure Hash Algorithm, 256-bit output length
- Accepts bit strings of length upto $2^{64} - 1$
- Announced in 2001 by NIST, US Department of Commerce
- Output calculation has two stages
 - Preprocessing
 - Hash Computation
- Preprocessing
 1. The input M is padded to a length which is a multiple of 512
 2. A 256-bit state variable $H^{(0)}$ is set to

$$H_0^{(0)} = \mathbf{0x6A09E667}, \quad H_1^{(0)} = \mathbf{0xBB67AE85},$$

$$H_2^{(0)} = \mathbf{0x3C6EF372}, \quad H_3^{(0)} = \mathbf{0xA54FF53A},$$

$$H_4^{(0)} = \mathbf{0x510E527F}, \quad H_5^{(0)} = \mathbf{0x9B05688C},$$

$$H_6^{(0)} = \mathbf{0x1F83D9AB}, \quad H_7^{(0)} = \mathbf{0x5BE0CD19}.$$

SHA-256 Input Padding

- Let input M be l bits long
 - Find smallest non-negative k such that

$$k + l + 65 = 0 \bmod 512$$

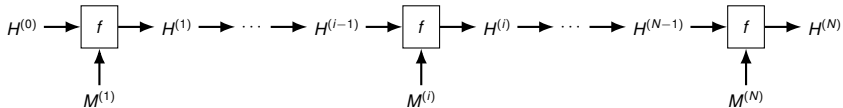
- Append $k + 1$ bits consisting of single 1 and k zeros
 - Append 64-bit representation of l
- Example: $M = 101010$ with $l = 6$
 - $k = 441$
 - 64-bit representation of 6 is $000 \dots 00110$
 - 512-bit padded message

$$\underbrace{101010}_M \ 1 \ \underbrace{00000 \dots 00000}_{441 \text{ zeros}} \ \underbrace{00 \dots 00110}_l .$$

SHA-256 Hash Computation

1. Padded input is split into N 512-bit blocks $M^{(1)}, M^{(2)}, \dots, M^{(N)}$
2. Given $H^{(i-1)}$, the next $H^{(i)}$ is calculated using a function f

$$H^{(i)} = f(M^{(i)}, H^{(i-1)}), \quad 1 \leq i \leq N.$$



3. f is called a *compression function*
4. $H^{(N)}$ is the output of SHA-256 for input M

SHA-256 Compression Function Building Blocks

- U, V, W are 32-bit words
- $U \wedge V, U \vee V, U \oplus V$ denote bitwise AND, OR, XOR
- $U + V$ denotes integer sum modulo 2^{32}
- $\neg U$ denotes bitwise complement
- For $1 \leq n \leq 32$, the shift right and rotate right operations

$$\text{SHR}^n(U) = \underbrace{000 \cdots 000}_n u_0 u_1 \cdots u_{30-n} u_{31-n},$$

$$\text{ROTR}^n(U) = u_{31-n+1} u_{31-n+2} \cdots u_{30} u_{31} u_0 u_1 \cdots u_{30-n} u_{31-n},$$

- Bitwise choice and majority functions

$$\text{Ch}(U, V, W) = (U \wedge V) \oplus (\neg U \wedge W),$$

$$\text{Maj}(U, V, W) = (U \wedge V) \oplus (U \wedge W) \oplus (V \wedge W),$$

- Let

$$\Sigma_0(U) = \text{ROTR}^2(U) \oplus \text{ROTR}^{13}(U) \oplus \text{ROTR}^{22}(U)$$

$$\Sigma_1(U) = \text{ROTR}^6(U) \oplus \text{ROTR}^{11}(U) \oplus \text{ROTR}^{25}(U)$$

$$\sigma_0(U) = \text{ROTR}^7(U) \oplus \text{ROTR}^{18}(U) \oplus \text{SHR}^3(U)$$

$$\sigma_1(U) = \text{ROTR}^{17}(U) \oplus \text{ROTR}^{19}(U) \oplus \text{SHR}^{10}(U)$$

SHA-256 Compression Function Calculation

- Maintains internal state of 64 32-bit words $\{W_j \mid j = 0, 1, \dots, 63\}$
- Also uses 64 constant 32-bit words K_0, K_1, \dots, K_{63} derived from the first 64 prime numbers $2, 3, 5, \dots, 307, 311$
- $f(M^{(i)}, H^{(i-1)})$ proceeds as follows

1. Internal state initialization

$$W_j = \begin{cases} M_j^{(i)} & 0 \leq j \leq 15, \\ \sigma_1(W_{j-2}) + W_{j-7} + \sigma_0(W_{j-15}) + W_{j-16} & 16 \leq j \leq 63. \end{cases}$$

2. Initialize eight 32-bit words

$$(A, B, C, D, E, F, G, H) = (H_0^{(i-1)}, H_1^{(i-1)}, \dots, H_6^{(i-1)}, H_7^{(i-1)}).$$

3. For $j = 0, 1, \dots, 63$, iteratively update A, B, \dots, H

$$T_1 = H + \Sigma_1(E) + \text{Ch}(E, F, G) + K_j + W_j$$

$$T_2 = \Sigma_0(A) + \text{Maj}(A, B, C)$$

$$(A, B, C, D, E, F, G, H) = (T_1 + T_2, A, B, C, D + T_1, E, F, G)$$

4. Calculate $H^{(i)}$ from $H^{(i-1)}$

$$(H_0^{(i)}, H_1^{(i)}, \dots, H_7^{(i)}) = (A + H_0^{(i-1)}, B + H_1^{(i-1)}, \dots, H + H_7^{(i-1)}).$$

The Merkle-Damgård Transform

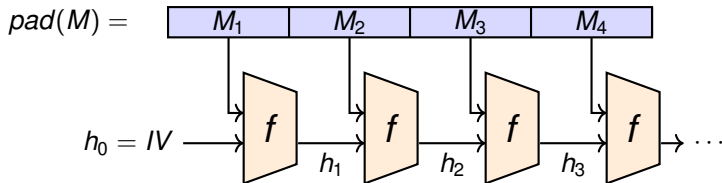


Figure source: <https://www.iacr.org/authors/tikz/>

- The SHA-256 construction is an example of the MD transform
- Typical hash function design
 - Construct collision-resistant compression function
 - Extend the domain using MDT to get collision-resistant hash function

Birthday Attacks for Finding Collisions

- Birthday Problem: Given Q people, what is the probability of two of them having the same birthday?
- Suppose the size of \mathcal{Y} is M . For SHA-256, $M = 2^{256}$.
- If we calculate H for Q inputs, the probability of a collision is

$$1 - \left(1 - \frac{1}{M}\right) \left(1 - \frac{2}{M}\right) \cdots \left(1 - \frac{Q-1}{M}\right) \approx 1 - \exp \frac{-Q(Q-1)}{2M}$$

- For success probability ε , the number of “queries” is

$$Q \approx \sqrt{2M \ln \frac{1}{1-\varepsilon}}$$

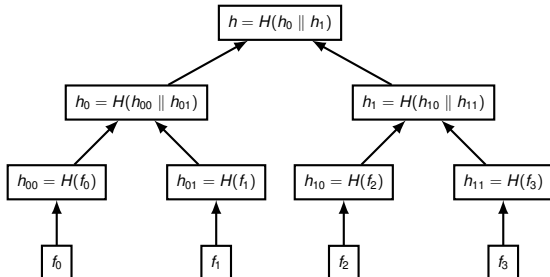
- For $\varepsilon = 0.5$, $Q \approx 1.17\sqrt{M}$
- For SHA-256, $Q \approx 2^{128}$

Applications

- Virus fingerprinting
- Data deduplication
- Digital signatures on arbitrary length data
- Password hashing
- Commitment schemes
 - A kind of digital envelope
 - Allows one party to “commit” to a message m by sending a commitment c to the counterparty
 - Set $c = H(m||r)$ where r is a random n -bit string
 - **Hiding**: c reveals nothing about m
 - **Binding**: Infeasible for c to be opened to a different message m'

Merkle Trees

- Alternative to Merkle-Damgård transform for domain extension
- Suppose a client uploads multiple files to server
- Client wants to ensure file integrity at a later retrieval



- For N files, $\mathcal{O}(\log N)$ communication from server ensures integrity
- The communication is called a *Merkle proof*

References

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- Chapter 4 of *Cryptography: Theory and Practice*, Douglas R. Stinson, 3rd edition
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- Chapter 3 of *An Introduction to Bitcoin*, S. Vijayakumaran, www.ee.iitb.ac.in/~sarva/bitcoin.html
- *Hashcash - A Denial of Service Counter-Measure*, A. Back, <http://hashcash.org/papers/hashcash.pdf>