

Probability Spaces

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

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Probability Theory

- Mathematical theory of uncertainty
- Complete information is difficult to obtain in many situations
 - Toss of a coin
 - Customer arrivals at a bank
- Applications
 - Communications and signal processing
 - Physics
 - Banking and Finance
 - Gambling

ICC World Cup 2015 Winner?

Country	Odds	Implied Probability
Australia	5/2	0.286
South Africa	4/1	0.200
India	9/2	0.182
New Zealand	7/1	0.125
Sri Lanka	8/1	0.111
England	9/1	0.100
Pakistan	10/1	0.091
West Indies	14/1	0.067
Bangladesh	66/1	0.015
Zimbabwe	200/1	0.005
Ireland	500/1	0.002
Afghanistan	750/1	0.001
Scotland	1500/1	0.001
UAE	1500/1	0.001

Source: www.skybet.com. Accessed January 8, 2015.

What is Probability?

- Informally, a method of quantifying the degree of certainty of a situation
- Classical definition: Ratio of favorable outcomes and the total number of outcomes provided all outcomes are equally likely.

$$P(A) = \frac{N_A}{N}$$

- Relative frequency definition:

$$P(A) = \lim_{N \rightarrow \infty} \frac{N_A}{N}$$

- Axiomatic definition: A countably additive function defined on the set of events with range in the interval $[0, 1]$.
- The axiomatic definition will be used in this course

Sample Space

The first step in constructing a probabilistic model for a situation is to list all the possible outcomes

Definition

The set of all possible outcomes of an experiment is called the sample space and is denoted by Ω .

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Tossing of two coins: $\Omega = \{(H, H), (T, H), (H, T), (T, T)\}$
- Coin is tossed until heads appear. What is Ω ?
- Life expectancy of a random person. $\Omega = [0, 120]$ years

Events

- An event is a subset of the sample space
- An event is said to have occurred if the outcome of the experiment belongs to it

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$.
 $E = \{\text{Heads}\}$ is the event that a head appears on the flip of a coin.
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
 $E = \{2, 4, 6\}$ is the event that an even number appears.
- Life expectancy. $\Omega = [0, 120]$.
 $E = [50, 120]$ is the event that a random person lives beyond 50 years.

Language of Events

Typical Notation	Language of Sets	Language of Events
Ω	Whole space	Certain event
ϕ	Empty set	Impossible event
A	Subset of Ω	Event that some outcome in A occurs
A^c	Complement of A	Event that no outcome in A occurs
$A \cup B$	Union	Event that an outcome in A or B or both occurs
$A \cap B$	Intersection	Event that an outcome in both A and B occurs
$A \cap B = \phi$	Disjoint sets	Mutually exclusive events

Assigning Probabilities to Events

- We want to assign probabilities to events

- Coin toss: $\Omega = \{H, T\}$

$$P(\phi) = 0, \quad P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2}, \quad P(\Omega) = 1$$

- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$P(A) = \frac{|A|}{6} \text{ for any } A \subseteq \Omega$$

- Can we always assign probabilities consistently to all the subsets of a sample space?
 - Yes, if the sample space is finite or countable
 - Not always, if the sample space is uncountable (example in next lecture)

Which subsets must be events?

- Let \mathcal{F} be a subset of the power set 2^Ω consisting of events to which we will assign probabilities
- If $\mathcal{F} \neq 2^\Omega$, which subsets of Ω must be there in \mathcal{F} ?
 - If we are interested in an event A , then A^c is also interesting

$$A \in \mathcal{F} \implies A^c \in \mathcal{F}$$

- If events A and B are interesting, then their simultaneous occurrence is also interesting

$$A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$$

- These two requirements give us the following (Why?)

$$A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$$

- They also give us $\phi \in \mathcal{F}$ if \mathcal{F} is nonempty (Why?)
- Any \mathcal{F} which satisfies these conditions is called a field
- To deal with infinite sample spaces, \mathcal{F} needs to be a σ -field

σ -fields

Definition

A collection \mathcal{F} of subsets of Ω is called a σ -field if it satisfies

- (a) $\phi \in \mathcal{F}$
- (b) if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
- (c) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Examples

- $\mathcal{F} = \{\phi, \Omega\}$ is the smallest σ -field
- If $A \subseteq \Omega$, $\mathcal{F} = \{\phi, A, A^c, \Omega\}$ is a σ -field
- 2^Ω is a σ -field

Exercises

- Is $\mathcal{F} = \left\{ A \subset \mathbb{N} \mid A \text{ is a finite set} \right\}$ a σ -field?
- Is $\mathcal{F} = \left\{ A \subset \mathbb{N} \mid \text{Either } A \text{ or } A^c \text{ is finite} \right\}$ a σ -field?

Probability Measure

Definition

Let \mathcal{F} be a σ -field of subsets of Ω . A probability measure on (Ω, \mathcal{F}) is a function $P : \mathcal{F} \mapsto [0, 1]$ satisfying

- (a) $P(\Omega) = 1$
- (b) if $A_1, A_2, \dots \in \mathcal{F}$ is a collection of disjoint members in \mathcal{F} , then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Examples

- Coin toss: $\Omega = \{H, T\}$, $\mathcal{F} = \{\phi, H, T, \Omega\}$

$$P(\phi) = 0, \quad P(H) = p, \quad P(T) = 1 - p, \quad P(\Omega) = 1$$

- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{i\}) = p_i$ for $i = 1, \dots, 6$

$$P(A) = \sum_{i \in A} p_i \text{ for any } A \subseteq \Omega$$

Probability Space

Definition

A probability space is a triple (Ω, \mathcal{F}, P) consisting of

- a set Ω ,
- a σ -field \mathcal{F} of subsets of Ω and
- a probability measure P on (Ω, \mathcal{F}) .

Summary

- Probability theory is the mathematical theory of uncertainty
- The axiomatic definition will be used in this course
- Set of all possible outcomes is called the sample space Ω
- An event is a subset of the sample space
- The set of events is a σ -field \mathcal{F}
- A probability measure is a countably additive set function $P : \mathcal{F} \mapsto [0, 1]$
- A probability space is a triple (Ω, \mathcal{F}, P)