Example of Elliptic moment sequence which has a measure, but no flat extension.

First, we generate a generic moment matrix with 10 general atoms satisfying the relation $X^3 = Y^2 + A\,X + B\,1$.

```
monList[vars_List, m_Integer?NonNegative] := Flatten[Map[Apply[Times, vars^#] &,
   Table[FrobeniusSolve[ConstantArray[1, Length[vars]], k], {k, 0, m}], {2}]]
(* NOTE: the above function produces things in reverse lex order *)

f[i_, j_, monomials1_, monomials2_] := Module[{ans},
   ans = monomials1[[i]] * monomials2[[j]];
   Return[ans];
  ];
m[X_, Y_, n_] := Module[{mat, x, y, monomials, k, momentMat},
   monomials = monList[{y, x}, n];
   k = Length[monomials];
   mat = ConstantArray[0, {k, k}];
   For[i = 1, i < k + 1, i++,
    For[j = 1, j < k + 1, j++,
      mat[[i, j]] = f[i, j, monomials, monomials];
     ];
   ];
   momentMat = mat /. {x → X, y → Y};
   Return[momentMat];
  ];
b[X_, Y_, n_] :=
  Module[{monomials1, monomials2, x, y, k1, k2, mat, bmat, rmon, cmon, cmon2},
   rmon = monList[{y, x}, (n - 1)];
   cmon = monList[{y, x}, n];
   cmon2 = Complement[cmon, rmon];
   k1 = Length[rmon];
   k2 = Length[cmon2];
   mat = ConstantArray[0, {k1, k2}];
   For[i = 1, i < k1 + 1, i++,
    For[j = 1, j < k2 + 1, j++,
      mat[[i, j]] = f[i, j, rmon, cmon2];
     ];
   ];
   bmat = mat /. {x → X, y → Y};
   Return[bmat];
  ];
```

```
(*m[x,y,3]/.{x^3→y^2+A x+B, x^4→x y^2+A x^2+B x,
  x^5→x^2y^2+A y^2+B x^2+A^2x+A B,x^6→y^4+A^2x^2+B^2+2A x y^2+2B y^2+ 2A B x,
  x^7→ x y^4+B^2x+2A x^2y^2+2B x y^2 +2A B x^2+A^2y^2+A^3x+A^2B ,x^2y^5→θ, x y^6→ϕ, y^7→ψ}*)
m3 = m[x1, y1, 3] /. {x1^3 → y1^2 + A x1 + B, x1^4 → x1 y1^2 + A x1^2 + B x1,
    x1^5 → x1^2 y1^2 + A y1^2 + B x1^2 + A^2 x1 + A B, x1^6 → y1^4 + A^2 x1^2 + B^2 + 2 A x1 y1^2 + 2 B y1^2 + 2 A B x1,
    x1^7 → x1 y1^4 + B^2 x1 + 2 A x1^2 y1^2 + 2 B x1 y1^2 + 2 A B x1^2 + A^2 y1^2 + A^3 x1 + A^2 B ,
    x1^2 y1^5 → θ, x1 y1^6 → ϕ, y1^7 → ψ};
m3 = m3 + m[x2, y2, 3] /. {x2^3 → y2^2 + A x2 + B, x2^4 → x2 y2^2 + A x2^2 + B x2, x2^5 → x2^2 y2^2 + A y2^2 +
      B x2^2 + A^2 x2 + A B, x2^6 → y2^4 + A^2 x2^2 + B^2 + 2 A x2 y2^2 + 2 B y2^2 + 2 A B x2, x2^7 → x2 y2^4 + B^2 x2 +
      2 A x2^2 y2^2 + 2 B x2 y2^2 + 2 A B x2^2 + A^2 y2^2 + A^3 x2 + A^2 B , x2^2 y2^5 → θ, x2 y2^6 → ϕ, y2^7 → ψ};
m3 = m3 + m[x3, y3, 3] /. {x3^3 → y3^2 + A x3 + B, x3^4 → x3 y3^2 + A x3^2 + B x3,
     x3^5 → x3^2 y3^2 + A y3^2 + B x3^2 + A^2 x3 + A B, x3^6 → y3^4 + A^2 x3^2 + B^2 + 2 A x3 y3^2 + 2 B y3^2 + 2 A B x3,
     x3^7 → x3 y3^4 + B^2 x3 + 2 A x3^2 y3^2 + 2 B x3 y3^2 + 2 A B x3^2 + A^2 y3^2 + A^3 x3 + A^2 B ,
     x3^2 y3^5 → θ, x3 y3^6 → ϕ, y3^7 → ψ};
m3 = m3 + m[x4, y4, 3] /. {x4^3 → y4^2 + A x4 + B, x4^4 → x4 y4^2 + A x4^2 + B x4,
     x4^5 → x4^2 y4^2 + A y4^2 + B x4^2 + A^2 x4 + A B, x4^6 → y4^4 + A^2 x4^2 + B^2 + 2 A x4 y4^2 + 2 B y4^2 + 2 A B x4,
     x4^7 → x4 y4^4 + B^2 x4 + 2 A x4^2 y4^2 + 2 B x4 y4^2 + 2 A B x4^2 + A^2 y4^2 + A^3 x4 + A^2 B ,
     x4^2 y4^5 → θ, x4 y4^6 → ϕ, y4^7 → ψ};
m3 = m3 + m[x5, y5, 3] /. {x5^3 → y5^2 + A x5 + B, x5^4 → x5 y5^2 + A x5^2 + B x5,
     x5^5 → x5^2 y5^2 + A y5^2 + B x5^2 + A^2 x5 + A B, x5^6 → y5^4 + A^2 x5^2 + B^2 + 2 A x5 y5^2 + 2 B y5^2 + 2 A B x5,
     x5^7 → x5 y5^4 + B^2 x5 + 2 A x5^2 y5^2 + 2 B x5 y5^2 + 2 A B x5^2 + A^2 y5^2 + A^3 x5 + A^2 B ,
     x5^2 y5^5 → θ, x5 y5^6 → ϕ, y5^7 → ψ};
m3 = m3 + m[x6, y6, 3] /. {x6^3 → y6^2 + A x6 + B, x6^4 → x6 y6^2 + A x6^2 + B x6,
     x6^5 → x6^2 y6^2 + A y6^2 + B x6^2 + A^2 x6 + A B, x6^6 → y6^4 + A^2 x6^2 + B^2 + 2 A x6 y6^2 + 2 B y6^2 + 2 A B x6,
     x6^7 → x6 y6^4 + B^2 x6 + 2 A x6^2 y6^2 + 2 B x6 y6^2 + 2 A B x6^2 + A^2 y6^2 + A^3 x6 + A^2 B ,
     x6^2 y6^5 → θ, x6 y6^6 → ϕ, y6^7 → ψ};
m3 = m3 + m[x7, y7, 3] /. {x7^3 → y7^2 + A x7 + B, x7^4 → x7 y7^2 + A x7^2 + B x7,
     x7^5 → x7^2 y7^2 + A y7^2 + B x7^2 + A^2 x7 + A B, x7^6 → y7^4 + A^2 x7^2 + B^2 + 2 A x7 y7^2 + 2 B y7^2 + 2 A B x7,
     x7^7 → x7 y7^4 + B^2 x7 + 2 A x7^2 y7^2 + 2 B x7 y7^2 + 2 A B x7^2 + A^2 y7^2 + A^3 x7 + A^2 B ,
     x7^2 y7^5 → θ, x7 y7^6 → ϕ, y7^7 → ψ};
m3 = m3 + m[x8, y8, 3] /. {x8^3 → y8^2 + A x8 + B, x8^4 → x8 y8^2 + A x8^2 + B x8,
     x8^5 → x8^2 y8^2 + A y8^2 + B x8^2 + A^2 x8 + A B, x8^6 → y8^4 + A^2 x8^2 + B^2 + 2 A x8 y8^2 + 2 B y8^2 + 2 A B x8,
     x8^7 → x8 y8^4 + B^2 x8 + 2 A x8^2 y8^2 + 2 B x8 y8^2 + 2 A B x8^2 + A^2 y8^2 + A^3 x8 + A^2 B ,
     x8^2 y8^5 → θ, x8 y8^6 → ϕ, y8^7 → ψ};
m3 = m3 + m[x9, y9, 3] /. {x9^3 → y9^2 + A x9 + B, x9^4 → x9 y9^2 + A x9^2 + B x9,
     x9^5 → x9^2 y9^2 + A y9^2 + B x9^2 + A^2 x9 + A B, x9^6 → y9^4 + A^2 x9^2 + B^2 + 2 A x9 y9^2 + 2 B y9^2 + 2 A B x9,
     x9^7 → x9 y9^4 + B^2 x9 + 2 A x9^2 y9^2 + 2 B x9 y9^2 + 2 A B x9^2 + A^2 y9^2 + A^3 x9 + A^2 B ,
     x9^2 y9^5 → θ, x9 y9^6 → ϕ, y9^7 → ψ};
m3 = m3 + m[x10, y10, 3] /. {x10^3 → y10^2 + A x10 + B, x10^4 → x10 y10^2 + A x10^2 + B x10,
     x10^5 → x10^2 y10^2 + A y10^2 + B x10^2 + A^2 x10 + A B,
     x10^6 → y10^4 + A^2 x10^2 + B^2 + 2 A x10 y10^2 + 2 B y10^2 + 2 A B x10,
     x10^7 → x10 y10^4 + B^2 x10 + 2 A x10^2 y10^2 + 2 B x10 y10^2 + 2 A B x10^2 + A^2 y10^2 + A^3 x10 + A^2 B ,
     x10^2 y10^5 → θ, x10 y10^6 → ϕ, y9^7 → ψ};


b4 = b[x1, y1, 4] /. {x1^3 → y1^2 + A x1 + B, x1^4 → x1 y1^2 + A x1^2 + B x1,
    x1^5 → x1^2 y1^2 + A y1^2 + B x1^2 + A^2 x1 + A B, x1^6 → y1^4 + A^2 x1^2 + B^2 + 2 A x1 y1^2 + 2 B y1^2 + 2 A B x1,
    x1^7 → x1 y1^4 + B^2 x1 + 2 A x1^2 y1^2 + 2 B x1 y1^2 + 2 A B x1^2 + A^2 y1^2 + A^3 x1 + A^2 B ,
    x1^2 y1^5 → θ, x1 y1^6 → ϕ, y1^7 → ψ};
b4 = b4 + b[x2, y2, 4] /. {x2^3 → y2^2 + A x2 + B, x2^4 → x2 y2^2 + A x2^2 + B x2, x2^5 → x2^2 y2^2 + A y2^2 +
      B x2^2 + A^2 x2 + A B, x2^6 → y2^4 + A^2 x2^2 + B^2 + 2 A x2 y2^2 + 2 B y2^2 + 2 A B x2, x2^7 → x2 y2^4 + B^2 x2 +
      2 A x2^2 y2^2 + 2 B x2 y2^2 + 2 A B x2^2 + A^2 y2^2 + A^3 x2 + A^2 B , x2^2 y2^5 → θ, x2 y2^6 → ϕ, y2^7 → ψ};
b4 = b4 + b[x3, y3, 4] /. {x3^3 → y3^2 + A x3 + B, x3^4 → x3 y3^2 + A x3^2 + B x3,
```

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    x3^5 -> x3^2 y3^2 + A y3^2 + B x3^2 + A^2 x3 + A B, x3^6 -> y3^4 + A^2 x3^2 + B^2 + 2 A x3 y3^2 + 2 B y3^2 + 2 A B x3,
    x3^7 -> x3 y3^4 + B^2 x3 + 2 A x3^2 y3^2 + 2 B x3 y3^2 + 2 A B x3^2 + A^2 y3^2 + A^3 x3 + A^2 B ,
    x3^2 y3^5 -> θ, x3 y3^6 -> ϕ, y3^7 -> ψ};
b4 = b4 + b[x4, y4, 4] /. {x4^3 -> y4^2 + A x4 + B, x4^4 -> x4 y4^2 + A x4^2 + B x4,
    x4^5 -> x4^2 y4^2 + A y4^2 + B x4^2 + A^2 x4 + A B, x4^6 -> y4^4 + A^2 x4^2 + B^2 + 2 A x4 y4^2 + 2 B y4^2 + 2 A B x4,
    x4^7 -> x4 y4^4 + B^2 x4 + 2 A x4^2 y4^2 + 2 B x4 y4^2 + 2 A B x4^2 + A^2 y4^2 + A^3 x4 + A^2 B ,
    x4^2 y4^5 -> θ, x4 y4^6 -> ϕ, y4^7 -> ψ};
b4 = b4 + b[x5, y5, 4] /. {x5^3 -> y5^2 + A x5 + B, x5^4 -> x5 y5^2 + A x5^2 + B x5,
    x5^5 -> x5^2 y5^2 + A y5^2 + B x5^2 + A^2 x5 + A B, x5^6 -> y5^4 + A^2 x5^2 + B^2 + 2 A x5 y5^2 + 2 B y5^2 + 2 A B x5,
    x5^7 -> x5 y5^4 + B^2 x5 + 2 A x5^2 y5^2 + 2 B x5 y5^2 + 2 A B x5^2 + A^2 y5^2 + A^3 x5 + A^2 B ,
    x5^2 y5^5 -> θ, x5 y5^6 -> ϕ, y5^7 -> ψ};
b4 = b4 + b[x6, y6, 4] /. {x6^3 -> y6^2 + A x6 + B, x6^4 -> x6 y6^2 + A x6^2 + B x6,
    x6^5 -> x6^2 y6^2 + A y6^2 + B x6^2 + A^2 x6 + A B, x6^6 -> y6^4 + A^2 x6^2 + B^2 + 2 A x6 y6^2 + 2 B y6^2 + 2 A B x6,
    x6^7 -> x6 y6^4 + B^2 x6 + 2 A x6^2 y6^2 + 2 B x6 y6^2 + 2 A B x6^2 + A^2 y6^2 + A^3 x6 + A^2 B ,
    x6^2 y6^5 -> θ, x6 y6^6 -> ϕ, y6^7 -> ψ};
b4 = b4 + b[x7, y7, 4] /. {x7^3 -> y7^2 + A x7 + B, x7^4 -> x7 y7^2 + A x7^2 + B x7,
    x7^5 -> x7^2 y7^2 + A y7^2 + B x7^2 + A^2 x7 + A B, x7^6 -> y7^4 + A^2 x7^2 + B^2 + 2 A x7 y7^2 + 2 B y7^2 + 2 A B x7,
    x7^7 -> x7 y7^4 + B^2 x7 + 2 A x7^2 y7^2 + 2 B x7 y7^2 + 2 A B x7^2 + A^2 y7^2 + A^3 x7 + A^2 B ,
    x7^2 y7^5 -> θ, x7 y7^6 -> ϕ, y7^7 -> ψ};
b4 = b4 + b[x8, y8, 4] /. {x8^3 -> y8^2 + A x8 + B, x8^4 -> x8 y8^2 + A x8^2 + B x8,
    x8^5 -> x8^2 y8^2 + A y8^2 + B x8^2 + A^2 x8 + A B, x8^6 -> y8^4 + A^2 x8^2 + B^2 + 2 A x8 y8^2 + 2 B y8^2 + 2 A B x8,
    x8^7 -> x8 y8^4 + B^2 x8 + 2 A x8^2 y8^2 + 2 B x8 y8^2 + 2 A B x8^2 + A^2 y8^2 + A^3 x8 + A^2 B ,
    x8^2 y8^5 -> θ, x8 y8^6 -> ϕ, y8^7 -> ψ};
b4 = b4 + b[x9, y9, 4] /. {x9^3 -> y9^2 + A x9 + B, x9^4 -> x9 y9^2 + A x9^2 + B x9,
    x9^5 -> x9^2 y9^2 + A y9^2 + B x9^2 + A^2 x9 + A B, x9^6 -> y9^4 + A^2 x9^2 + B^2 + 2 A x9 y9^2 + 2 B y9^2 + 2 A B x9,
    x9^7 -> x9 y9^4 + B^2 x9 + 2 A x9^2 y9^2 + 2 B x9 y9^2 + 2 A B x9^2 + A^2 y9^2 + A^3 x9 + A^2 B ,
    x9^2 y9^5 -> θ, x9 y9^6 -> ϕ, y9^7 -> ψ};
b4 = b4 + b[x10, y10, 4] /. {x10^3 -> y10^2 + A x10 + B, x10^4 -> x10 y10^2 + A x10^2 + B x10,
    x10^5 -> x10^2 y10^2 + A y10^2 + B x10^2 + A^2 x10 + A B,
    x10^6 -> y10^4 + A^2 x10^2 + B^2 + 2 A x10 y10^2 + 2 B y10^2 + 2 A B x10,
    x10^7 -> x10 y10^4 + B^2 x10 + 2 A x10^2 y10^2 + 2 B x10 y10^2 + 2 A B x10^2 + A^2 y10^2 + A^3 x10 + A^2 B ,
    x10^2 y10^5 -> θ, x10 y10^6 -> ϕ, y10^7 -> ψ};
```

```
(*b4 = b[x,y,4]/.{x^3->y^2, x^4->x y^2, x^5->x^2y^2, x^6->y^4, x^7->xy^4, x^2y^5->θ, x y^6->ϕ, y^7->ψ};*)
```

Now we set the curve to be $Y^2 = (X^3) - \frac{524287}{262144} X + 1$; and choose our atoms as given in the article.

```
A = 524287/262144; B = -1;
```

```
points = {1, 1/2, 1/3, 1/4, 1/5};(*Rationalize[RandomReal[{0,1},5],1/2^6];*)
replacements =
  {x1 → points[[1]], x2 → points[[2]], x3 → points[[3]], x4 → points[[4]],
   x5 → points[[5]], x6 → points[[2]], x7 → points[[3]], x8 → points[[4]],
   x9 → points[[5]], x10 → points[[1]], y1 → Sqrt[(points[[1]])^3 - A points[[1]] - B],
   y2 → Sqrt[(points[[2]])^3 - A points[[2]] - B],
   y3 → Sqrt[(points[[3]])^3 - A points[[3]] - B],
   y4 → Sqrt[(points[[4]])^3 - A points[[4]] - B],
   y5 → Sqrt[(points[[5]])^3 - A points[[5]] - B],
   y6 → -Sqrt[(points[[2]])^3 - A points[[2]] - B],
   y7 → -Sqrt[(points[[3]])^3 - A points[[3]] - B],
   y8 → -Sqrt[(points[[4]])^3 - A points[[4]] - B],
   y9 → -Sqrt[(points[[5]])^3 - A points[[5]] - B],
   y10 → -Sqrt[(points[[1]])^3 - A points[[1]] - B]};
matM = Simplify[m3 /. replacements];
matB = Simplify[b4 /. replacements];
matJ = Simplify[matM[[{1, 2, 3, 4, 5, 6, 8, 9, 10}, {1, 2, 3, 4, 5, 6, 8, 9, 10}]]];
```

We can now solve for W such that M(3)W=B(4)

```
matW = Table[w[i, j], {i, 9}, {j, 5}];
varsW = Variables[Select[matW, # =!= 0 &]];
solW = Solve[matB[[{1, 2, 3, 4, 5, 6, 8, 9, 10}, {1, 2, 3, 4, 5}]] == matJ.matW, varsW];
Length[solW]
mW = matW /. solW[[1]];
MatrixForm[mW];
mW = Insert[mW, {0, 0, 0, 0, 0}, 7];
MatrixForm[mW];
```

```
1
```

```
MatrixForm[Simplify[matM.mW - matB]]
(* We see that Ran(B) is contained in Ran(M) and that M.W=B *)
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We generate the C block below using Smuljan’s Lemma, and specify the 3 equations we need to satisfy for a flat extension.

We have solved the first two equations for the variables ϕ and ψ, and then we are left with the last equation in θ. We see below that this equation turns out to be quadratic, even though it was a priori quartic.

```
hatC = Simplify[Transpose[mW].matM.mW];
hankelEQ = Simplify[{hatC[[4, 2]] - hatC[[3, 3]],
    hatC[[5, 2]] - hatC[[4, 3]], hatC[[5, 3]] - hatC[[4, 4]]}];
stheta = Simplify[Solve[{hankelEQ[[1]] == 0, hankelEQ[[2]] == 0}, {ϕ, ψ}, Reals]];
Length[stheta]
critpoly = Simplify[hankelEQ[[3]] /. stheta[[1]]];
```

```
1
```

```
Exponent[critpoly, θ]
```

```
2
```

The quartic polynomial Q(θ) is the following.

```
critpoly
```

$$\frac{1\,850\,617\,701\,610\,280\,004\,960\,481}{11\,427\,409\,289\,822\,154\,604\,482\,953\,216\,000}+\left(2\,518\,293\,870\,123\,022\,495\,609\,405\,302\,939\,763\,563\,092\,225\,775\,041\,011\,300\,\theta^2\right)/121\,617\,394\,571\,298\,435\,190\,879\,906\,936\,561\,845\,321\,520\,470\,769$$

We check now to see if the critical quadratic equation has any solutions in the Reals; we do this by checking the discriminant of the quadratic.

```
discA = Coefficient[critpoly, θ, 2];
discB = Coefficient[critpoly, θ, 1];
discC = Coefficient[critpoly, θ, 0];
```

```
Reduce[{discB^2 - 4 discA discC ≥ 0}, {θ}, Reals]
```

```
False
```

As we see above, this discriminant is never non-negative and so, the critical equation has no solutions θ in the Reals.

This implies that there is in fact no flat extenion to M(4) in this case, as any C block generated through Smuljan's Lemma will fail to preserve the Hankel structure for every real θ.