Rank 6 analysis - Section 7

CASE: Y^2=1-X^2

We denote $\beta00 = \beta\_\{1\}$, $\beta10 = \beta\_\{X\}$, $\beta01 = \beta\_\{Y\}$, $\beta20 = \beta\_\{X\text{^}2\}$, $\beta11 = \beta\_\{XY\}$, $\beta\_\{21\} = \beta\_\{X\text{^}2\ Y\}$, $\beta30 = \beta\_\{X\text{^}3\}$, $\beta40 = \beta\_\{X\text{^}4\}$, $\beta31 = \beta\_\{X\text{^}3\ Y\}$, $\beta1111 = \beta\_\{XYXY\}$.

Form of a moment matrix M

```
M[β00_, β10_, β01_, β20_, β11_, β21_, β30_, β40_, β31_, β1111_] :=
 {{β00, β10, β01, β20, β11, β11, β00 - β20}, {β10, β20, β11, β30, β21, β21, β10 - β30},
  {β01, β11, β00 - β20, β21, β10 - β30, β10 - β30, β01 - β21},
  {β20, β30, β21, β40, β31, β31, β20 - β40},
  {β11, β21, β10 - β30, β31, β20 - β40, β1111, β11 - β31},
  {β11, β21, β10 - β30, β31, β1111, β20 - β40, β11 - β31},
  {β00 - β20, β10 - β30, β01 - β21, β20 - β40, β11 - β31, β11 - β31, β00 - 2 β20 + β40}}
```

```
MatrixForm[M[β00, β10, β01, β20, β11, β21, β30, β40, β31, β1111]]
```

$$\begin{pmatrix} \beta00 & \beta10 & \beta01 & \beta20 & \beta11 & \beta11 & \beta00-\beta20 \\ \beta10 & \beta20 & \beta11 & \beta30 & \beta21 & \beta21 & \beta10-\beta30 \\ \beta01 & \beta11 & \beta00-\beta20 & \beta21 & \beta10-\beta30 & \beta10-\beta30 & \beta01-\beta21 \\ \beta20 & \beta30 & \beta21 & \beta40 & \beta31 & \beta31 & \beta20-\beta40 \\ \beta11 & \beta21 & \beta10-\beta30 & \beta31 & \beta20-\beta40 & \beta1111 & \beta11-\beta31 \\ \beta11 & \beta21 & \beta10-\beta30 & \beta31 & \beta1111 & \beta20-\beta40 & \beta11-\beta31 \\ \beta00-\beta20 & \beta10-\beta30 & \beta01-\beta21 & \beta20-\beta40 & \beta11-\beta31 & \beta11-\beta31 & \beta00-2\,\beta20+\beta40 \end{pmatrix}$$

Existence of a measure for M

Assume $\beta10=\beta01=\beta30=\beta21=0$:

```
B[α_] := M[1, 0, 0, β20, β11, 0, 0, β40, β31, β1111] -
  α {{2, 0, 0, 2, 0, 0, 0}, {0, 2, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {2, 0, 0, 2,
     0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}}
```

```
MatrixForm[B[α]]
```

$$\begin{pmatrix} 1-2\,\alpha & 0 & 0 & -2\,\alpha+\beta20 & \beta11 & \beta11 & 1-\beta20 \\ 0 & -2\,\alpha+\beta20 & \beta11 & 0 & 0 & 0 & 0 \\ 0 & \beta11 & 1-\beta20 & 0 & 0 & 0 & 0 \\ -2\,\alpha+\beta20 & 0 & 0 & -2\,\alpha+\beta40 & \beta31 & \beta31 & \beta20-\beta40 \\ \beta11 & 0 & 0 & \beta31 & \beta20-\beta40 & \beta1111 & \beta11-\beta31 \\ \beta11 & 0 & 0 & \beta31 & \beta1111 & \beta20-\beta40 & \beta11-\beta31 \\ 1-\beta20 & 0 & 0 & \beta20-\beta40 & \beta11-\beta31 & \beta11-\beta31 & 1-2\,\beta20+\beta40 \end{pmatrix}$$

Calculating $\alpha0$ from the proof of Theorem 7.5

```
Solve[Det[B[α][[{1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6}]]] == 0, α]
```

$$\left\{\left\{\alpha \to \frac{\beta 11^2 - \beta 20 + \beta 20^2}{2\,(-1 + \beta 20)}\right\},\right.$$
$$\left.\left\{\alpha \to \left(\beta 1111\, \beta 20^2 + \beta 20^3 - 4\, \beta 11\, \beta 20\, \beta 31 + 2\, \beta 31^2 + 2\, \beta 11^2\, \beta 40 - \beta 1111\, \beta 40 - \beta 20\, \beta 40 - \beta 20^2\, \beta 40 + \beta 40^2\right) \Big/ \left(2\left(2\, \beta 11^2 - \beta 1111 - \beta 20 + 2\, \beta 1111\, \beta 20 + 2\, \beta 20^2 - 4\, \beta 11\, \beta 31 + 2\, \beta 31^2 + \beta 40 - \beta 1111\, \beta 40 - 3\, \beta 20\, \beta 40 + \beta 40^2\right)\right)\right\}\right\}$$

Calculating $\alpha 2$ from the proof of Claim 1 of Theorem 7.5

```
Solve[Det[B[α][[{1, 4}, {1, 4}]]] == 0, α]
```

$$\left\{\left\{\alpha \to \frac{\beta 20^2 - \beta 40}{2\,(-1 + 2\, \beta 20 - \beta 40)}\right\}\right\}$$

Calculating $\alpha 3$ from the proof of Claim 1 of Theorem 7.5

```
Solve[Det[B[α][[{1, 5}, {1, 5}]]] == 0, α]
```

$$\left\{\left\{\alpha \to \frac{-\beta 11^2 + \beta 20 - \beta 40}{2\,(\beta 20 - \beta 40)}\right\}\right\}$$

Calculating $\alpha 4$ from the proof of Claim 1 of Theorem 7.5

```
Solve[Det[B[α][[{1, 5, 6}, {1, 5, 6}]]] == 0, α]
```

$$\left\{\left\{\alpha \to \frac{-2\, \beta 11^2 + \beta 1111 + \beta 20 - \beta 40}{2\,(\beta 1111 + \beta 20 - \beta 40)}\right\}\right\}$$

Solving the system (7.6)-(7.7) in the proof of Claim 1 of Theorem 7.5

```
Reduce[(β11^2 - β20 + β20^2)/(2 (-1 + β20)) <=
   Min[(β20^2 - β40)/(2 (-1 + 2 β20 - β40)), (-β11^2 + β20 - β40)/(2 (β20 - β40)), (-2 β11^2 + β1111 + β20 - β40)/(2 (β1111 + β20 - β40))] &&
  Det[M[1, 0, 0, β20, β11, 0, 0, β40, β31, β1111][[{3}, {3}]]] > 0 &&
  Det[M[1, 0, 0, β20, β11, 0, 0, β40, β31, β1111][[{5}, {5}]]] > 0 &&
  Det[M[1, 0, 0, β20, β11, 0, 0, β40, β31, β1111][[{2, 3}, {2, 3}]]] > 0 &&
  Det[M[1, 0, 0, β20, β11, 0, 0, β40, β31, β1111][[{1, 4}, {1, 4}]]] > 0 &&
  Det[M[1, 0, 0, β20, β11, 0, 0, β40, β31, β1111][[{1, 5, 6}, {1, 5, 6}]]] > 0]
```

```
False
```

Kernel of B(F/2G) in the notation of the proof of Theorem 7.5

```
NullSpace[
 B[(β1111 β20^2 + β20^3 - 4 β11 β20 β31 + 2 β31^2 + 2 β11^2 β40 - β1111 β40 - β20 β40 - β20^2 β40 +
     β40^2) / (2 (2 β11^2 - β1111 - β20 + 2 β1111 β20 + 2 β20^2 -
       4 β11 β31 + 2 β31^2 + β40 - β1111 β40 - 3 β20 β40 + β40^2))]]
```

$$\{\{-1, 0, 0, 1, 0, 0, 1\},$$
$$\left\{-\left(\beta 1111\, \beta 20 + \beta 20^2 - 2\, \beta 11\, \beta 31 + 2\, \beta 31^2 - \beta 1111\, \beta 40 - 2\, \beta 20\, \beta 40 + \beta 40^2\right) \Big/ (\beta 11\, \beta 20 - \beta 31 + \beta 20\, \beta 31 - \beta 11\, \beta 40),\ 0,\ 0,\right.$$
$$\left.\left. -\left(2\, \beta 11^2 - \beta 1111 - \beta 20 + \beta 1111\, \beta 20 + \beta 20^2 - 2\, \beta 11\, \beta 31 + \beta 40 - \beta 20\, \beta 40\right) \Big/ (\beta 11\, \beta 20 - \beta 31 + \beta 20\, \beta 31 - \beta 11\, \beta 40),\ 1,\ 1,\ 0\right\}\right\}$$