A Maximum-Likelihood TDOA Localization Algorithm Using Difference-of-Convex Programming

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***Abstract*—A popular approach to estimate a source location using time difference of arrival (TDOA) measurements is to construct an objective function based on the maximum likelihood (ML) method. An iterative algorithm can be employed to minimize that objective function. The main challenge in this optimization process is the non-convexity of the objective function, which precludes the use of many standard convex optimization tools. Usually, approximations, such as convex relaxation, are applied, resulting in performance loss. In this work, we take advantage of difference-of-convex (DC) programming tools to develop an efficient solution to the ML TDOA localization problem. We show that, by using a simple trick, the objective function can be modified into an exact difference of two convex functions. Hence, tools from DC programming can be leveraged to carry out the optimization task, which guarantees convergence to a stationary point of the objective function. Simulation results show that, when initialized within the convex hull of the anchors, the proposed TDOA localization algorithm outperforms a number of benchmark methods, behaves as an exact ML estimator, and indeed achieves the Cramér-Rao lower bound.**

***Index Terms*—Localization, TDOA, non-convex, optimization, CCCP, DC programming, difference of convex, convex-concave.**

I. Introduction

SOURCE localization has been an active research topic for many years due to its wide range of applications in data communications, navigation, sensor networks, and so on [1], [2]. Among a diverse arsenal of localization techniques, time difference of arrival (TDOA) based methods drew significant attention since they simplify the setup by eliminating the need for the *anchors*' clocks to be synchronized with the clock of the *source*. TDOA localization algorithms can be divided into two broad categories: (a) Simple closed-form algebraic solutions [3], [4] that avoid convergence problems and achieve global optima, but are usually sensitive to TDOA measurement errors; (b) Iterative methods [5], [6], where an optimum of an objective function (constructed using TDOA measurements and the anchor positions) is sought using multiple iterations.

A number of objective functions have been proposed in the context of TDOA based localization [7]. A popular formulation is based on the maximum likelihood (ML) concept [8], [9]. The popularity of ML estimation stems from the fact that ML estimators are known to be *efficient* in the sense that they achieve the Cramér-Rao lower bound (CRLB). However, objective functions based on ML are usually not convex; hence, many standard convex optimization tools cannot be used. To circumvent the non-convexity issue, the literature proposes using convex relaxation methods, such as SDP [6], [10], and combined methods, as in [11], [12].

Other (non ML based) methods have also been proposed for source localization, e.g., steered-response power [13], least-mean absolute error [5], weighted least squares [14], [15], and deep learning [16] based methods. All these localization techniques either involve approximations that compromise their accuracy, or they do not offer convergence guarantees.

According to [17], an optimization involving a real-valued objective function on a convex set is called a *difference-of-convex* (DC) program if the objective function can be represented as a difference of two convex functions. A DC problem can be solved globally by methods such as branch-and-bound, outer-approximation, or a combination of both (see [18] and the references therein). The convex-concave procedure (CCCP) [19] is a heuristic algorithm for finding a local optimum of a DC function, which is guaranteed to converge to a stationary point [20], [21].

In this letter, we show that ML TDOA localization can be formulated as a DC program in an exact manner. We propose a simple way to manipulate the ML TDOA localization problem into a DC program. Then, we apply the CCCP to obtain a solution. CCCP iterations are guaranteed to converge to a stationary point of the objective function. With a carefully crafted initialization, a global optimum can be reached. Hence, since no approximation is used in our formulation, the proposed algorithm can achieve the CRLB. Simulation results confirm the ability of the proposed algorithm to globally converge, given that it is initialized within the convex hull of the anchors.

The remainder of this letter is organized as follows. In Section II, we formulate the problem of source localization using TDOA measurements. The details of the proposed solution are given in Section III. Section IV presents simulation results followed by brief conclusive remarks in Section V.

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II. Problem Formulation

Consider a system consisting of N anchors installed at known positions $\mathbf{r}_i, i = 1, 2, \ldots N$. These anchors collaborate to localize an unknown source located at $\mathbf{p}$. For any signal emitted by the source, the time difference of arrival (TDOA) observed between two anchors at $\mathbf{r}_i$ and $\mathbf{r}_j$ is given by

$$\tau_{ij}(\mathbf{p}) = \frac{1}{v}(||\mathbf{p} - \mathbf{r}_i||_2 - ||\mathbf{p} - \mathbf{r}_j||_2), \tag{1}$$

where v is the signal propagation speed, and $||\cdot||_2$ denotes the ℓ_2 norm. In practice, the actual TDOA values are unknown, only measurements of the form $\hat{\tau}_{ij} = \tau_{ij}(\mathbf{p}) + n_{ij}$ are available, where n_{ij} is additive noise. Assuming that the additive noise terms n_{ij} are i.i.d. Gaussian, ML estimation of the source location boils down to the minimization

$$\hat{\mathbf{p}} = \arg\min_{\mathbf{p}} c(\mathbf{p}) = \arg\min_{\mathbf{p}} \sum_{i \neq j} [\hat{\tau}_{ij} - \tau_{ij}(\mathbf{p})]^2. \tag{2}$$

The objective function $c(\mathbf{p})$ can be expanded as

$$c(\mathbf{p}) = 2\sum_{i\neq j}\Bigg[\underbrace{\frac{\hat{\tau}_{ij}^2}{2} + \frac{1}{2v^2}||\mathbf{p} - \mathbf{r}_i||_2^2 + \frac{1}{2v^2}||\mathbf{p} - \mathbf{r}_j||_2^2}_{f_{ij}(\mathbf{p})}$$
$$\underbrace{-\frac{1}{v^2}||\mathbf{p} - \mathbf{r}_i||_2||\mathbf{p} - \mathbf{r}_j||_2}_{g_{ij}(\mathbf{p})} - \underbrace{\frac{\hat{\tau}_{ij}}{v}(||\mathbf{p} - \mathbf{r}_i||_2 - ||\mathbf{p} - \mathbf{r}_j||_2)}_{q_{ij}(\mathbf{p})}\Bigg]. \tag{3}$$

It is easy to see that, in the general case, (3) is a non-convex function. As such, the presence of local minima and convergence issues make it difficult to realize an ML estimator in a precise way. To tackle these issues, in the following section, we develop an efficient solution to the ML TDOA localization problem where convergence is guaranteed.

III. The Proposed Algorithm

A. DC Programming

DC programming requires that a continuous function $E(\mathbf{p})$ be written as a sum of a convex function $v(\mathbf{p})$ and a concave function $a(\mathbf{p})$ over a convex domain $\mathcal{D}$ [17], [18], i.e.,

$$E(\mathbf{p}) = v(\mathbf{p}) + a(\mathbf{p}) = v(\mathbf{p}) - (-a(\mathbf{p})).$$

By definition, for any $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{D}$,

$$v(\mathbf{p}_2) \leq v(\mathbf{p}_1) - (\mathbf{p}_1 - \mathbf{p}_2)^T \nabla v(\mathbf{p}_2), \tag{4}$$

$$a(\mathbf{p}_2) \leq a(\mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1)^T \nabla a(\mathbf{p}_1), \tag{5}$$

where ∇ is the gradient operator. Now, consider, an iterative algorithm that seeks to minimize $E(\mathbf{p})$. Let $\mathbf{p}^t$ and $\mathbf{p}^{t+1}$ be the results of any two consecutive iterations. By setting $\mathbf{p}_1 = \mathbf{p}^t, \mathbf{p}_2 = \mathbf{p}^{t+1}$, substituting in (4) and (5), and adding the results, we obtain

$$E(\mathbf{p}^{t+1}) \leq E(\mathbf{p}^t) + (\Delta\mathbf{p})^T(\nabla v(\mathbf{p}^{t+1}) + \nabla a(\mathbf{p}^t)), \tag{6}$$

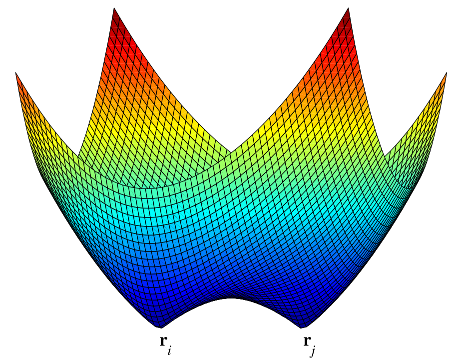

Fig. 1. An example plot of the non-convex component $g_{ij}(\mathbf{p})$ in the 2-D case. Note that $g_{ij}(\mathbf{p})$ has two minima.

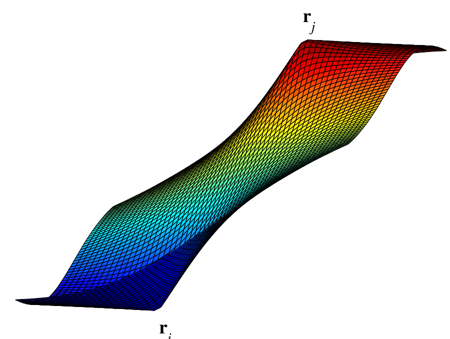

Fig. 2. An example plot of the non-convex component $q_{ij}(\mathbf{p})$ in the 2-D case for a positive $\hat{\tau}_{ij}$.

where $\Delta\mathbf{p} = \mathbf{p}^{t+1} - \mathbf{p}^t$. Based on (6),

$$\nabla v(\mathbf{p}^{t+1}) = -\nabla a(\mathbf{p}^t) \tag{7}$$

is a sufficient condition to guarantee that $E(\mathbf{p}^{t+1}) \leq E(\mathbf{p}^t)$. The CCCP applies (7) as a rule to determine $\mathbf{p}^{t+1}$ given $\nabla a(\mathbf{p}^t)$, which can be formulated as a minimization of a time sequence of convex update functions [19]

$$E_0(\mathbf{p}^{t+1}) = v(\mathbf{p}^{t+1}) + (\mathbf{p}^{t+1})^T \nabla a(\mathbf{p}^t). \tag{8}$$

B. Analyzing the Objective Function

From the previous discussion, we see that if our objective function (3) is expressed in the form of a difference of two convex functions, then we can create a non-increasing sequence of iterations according to (6) and (7) to reach a local minimum (or a saddle point). The objective function (3) has the form

$$c(\mathbf{p}) = 2\sum_{i\neq j} c_{ij}(\mathbf{p}) = 2\sum_{i\neq j}[f_{ij}(\mathbf{p}) - g_{ij}(\mathbf{p}) - q_{ij}(\mathbf{p})]. \tag{9}$$

Obviously, $f_{ij}(\mathbf{p})$ is convex, $g_{ij}(\mathbf{p})$ and $q_{ij}(\mathbf{p})$ are non-convex since they respectively consist of the non-convex norm products and norm differences. Fig. 1 and Fig. 2 depict example visualizations of $g_{ij}(\mathbf{p})$ and $q_{ij}(\mathbf{p})$.

C. DC Decomposition of the Objective Function

To put the objective function (3) in a DC form, it is sufficient to express each of the summands, $c_{ij}(\mathbf{p}) = f_{ij}(\mathbf{p}) - g_{ij}(\mathbf{p}) -$

$q_{ij}(\mathbf{p})$, in (9) as difference of convex functions. To this end, we introduce $\alpha_{ij}(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$ to compensate the non-convex parts of $g_{ij}(\mathbf{p})$ and $q_{ij}(\mathbf{p})$. That is, we construct appropriate functions, $\alpha_{ij}(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$, such that $f_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p})$, $g_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p})$ and $q_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p})$ are all convex functions. This allows us to convert $c_{ij}(\mathbf{p})$ to a difference of two convex functions,

$$c_{ij}(\mathbf{p}) = c_{ij,1}(\mathbf{p}) - c_{ij,2}(\mathbf{p}), \tag{10}$$

where $c_{ij,1}(\mathbf{p}) = f_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p})$ and $c_{ij,2}(\mathbf{p}) = g_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p}) + q_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p})$.

As for how to construct $\alpha_{ij}(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$, we just need to ensure that these functions are more concave than the most concave part of $g_{ij}(\mathbf{p})$ and $q_{ij}(\mathbf{p})$. Specifically, we set

$$\alpha_{ij}(\mathbf{p}) = -\frac{1}{v^2}(\mathbf{p} - \mathbf{r}_i)^T(\mathbf{p} - \mathbf{r}_j), \tag{11}$$

$$\beta_{ij}(\mathbf{p}) = -b\frac{\hat{\tau}_{ij}}{v}||\mathbf{p} - \mathbf{r}_j||_2 + (1-b)\frac{\hat{\tau}_{ij}}{v}||\mathbf{p} - \mathbf{r}_i||_2, \tag{12}$$

where $b = 1$ for $\hat{\tau}_{ij} \geq 0$, $b = 0$ for $\hat{\tau}_{ij} < 0$. Based on (11) and (12), we can write

$$\begin{aligned} q_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p}) &= \frac{\hat{\tau}_{ij}}{v}(||\mathbf{p} - \mathbf{r}_i||_2 - ||\mathbf{p} - \mathbf{r}_j||_2) \\ &- \left(-b\frac{\hat{\tau}_{ij}}{v}||\mathbf{p} - \mathbf{r}_j||_2 + (1-b)\frac{\hat{\tau}_{ij}}{v}||\mathbf{p} - \mathbf{r}_i||_2\right), \end{aligned} \tag{13}$$

and

$$\begin{aligned} g_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p}) = \frac{1}{v^2}[&||\mathbf{p} - \mathbf{r}_i||_2||\mathbf{p} - \mathbf{r}_j||_2 \\ &+ (\mathbf{p} - \mathbf{r}_i)^T(\mathbf{p} - \mathbf{r}_j)]. \end{aligned} \tag{14}$$

Theorem 1: The function $q_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p})$ in (13) is always convex regardless of the value of $\hat{\tau}_{ij}$.

Proof: The proof of Theorem 1 is straightforward. Using (13) and the definition of the binary variable b we have

$$\begin{aligned} q_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p}) &= \frac{\hat{\tau}_{ij}}{v}||\mathbf{p} - \mathbf{r}_i||_2, \text{for } \hat{\tau}_{ij} \geq 0, \text{and} \\ q_{ij}(\mathbf{p}) - \beta_{ij}(\mathbf{p}) &= -\frac{\hat{\tau}_{ij}}{v}||\mathbf{p} - \mathbf{r}_j||_2, \text{for } \hat{\tau}_{ij} < 0, \end{aligned} \tag{15}$$

both of which are non-negative ℓ_2 norms and hence convex. ■

Theorem 2: The function $g_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p})$ in (14) is convex.

Proof: The proof of Theorem 2 is given in Appendix A. ■

Based on Theorem 1 and Theorem 2, we can readily see that both $c_{ij,1}(\mathbf{p})$ and $c_{ij,2}(\mathbf{p})$ are convex. Hence, (10) is a DC function. Converting our original objective function (3) to the DC form (10) allows us to utilize the CCCP optimization tool for DC programming, which is guaranteed to converge to a stationary point of (3). We denote the corresponding algorithm as CCCP-ML.

D. Summary of the Proposed CCCP-ML Algorithm

i) Initialize: $t = 0$, $\mathbf{p}^t = \mathbf{p}^0$.
ii) Repeat (CCCP/outer loop):
- Compute: $\mathbf{g}_t = \sum_{i \neq j} \nabla c_{ij,2}(\mathbf{p}^t)$.
- Compute (gradient descent/inner loop): $\mathbf{p}^{t+1} = \arg\min_{\mathbf{p}} \sum_{i \neq j} c_{ij,1}(\mathbf{p}) - \mathbf{p}^T \mathbf{g}_t$,
- $t = t + 1$,
- Quit when the value of the objective function is sufficiently small.

E. Computational Complexity Analysis

Based on the steps listed in the previous subsection, the proposed algorithm consists of the CCCP iterations (outer loops), and gradient descent iterations used to minimize a convex function (inner loops). We denote the number of iterations of the inner and outer loops by K_i and K_o, respectively. For a system with N anchors and $M = N(N-1)/2$ pairs of TDOA measurements, the proposed method can be implemented using $K_\mathrm{o}K_\mathrm{i}[(6d-1)M] + K_\mathrm{o}[(9d-2)M - d]$ additions and $K_\mathrm{o}K_\mathrm{i}(3dM + d) + K_\mathrm{o}(5dM)$ multiplications, where $d \in \{2, 3\}$ is the dimension of $\mathbf{p}$.

IV. Simulation Results

The proposed method can be utilized for both 2-D and 3-D localization. To simplify the presentation of the results, we focus our simulations on 2-D scenarios with different number of anchors, anchor placements and source locations.

When the source is located outside the anchors' convex hull, the objective function is flat near the source and remains so as we move away from the convex hull. This makes estimating the source location infeasible, regardless of the method applied. Therefore, we evaluate performance only in scenarios involving source locations inside the anchors' convex hull. In such scenarios, it is observed that initialization outside the convex hull does not guarantee convergence to the correct solution. Therefore, we initialize the proposed method, and all the tested iterative methods, randomly within the convex hull.

For each simulation scenario, TDOA across different anchor pairs is calculated using (1). Independently distributed errors are then added to simulate practical TDOA measurements. The errors are generated as Gaussian noises with zero mean and a standard deviation σ. Localization performance is evaluated using the *root mean squared error* (RMSE) calculated from 10^3 independent trials for each σ value.

The proposed method is compared with the following benchmark methods: The convex relaxation SDP [6]; the CCCP-SOCP method, which uses the ℓ_1 norm [5]; the CCCP-LP method, an approximation of the CCCP-SOCP method [5]; the I2WLS method, which requires prior knowledge of the TDOA error variance [4]; and the CRLB.

In Fig. 3 and Fig. 4 (the right columns), we plot the RMSE versus the standard deviation of the noise for different configurations (depicted in the left columns). Fig. 3 covers 4-anchor scenarios, while Fig. 4 focuses on 8-anchor setups. We can clearly observe that the proposed CCCP-ML algorithm outperforms all the benchmark methods in all scenarios, achieving or staying very close to the CRLB. Only in Fig. 4, Configuration 1, SDP appears to match the performance of the proposed algorithm in a scenario with a relatively large number of anchors and a favorable geometry.

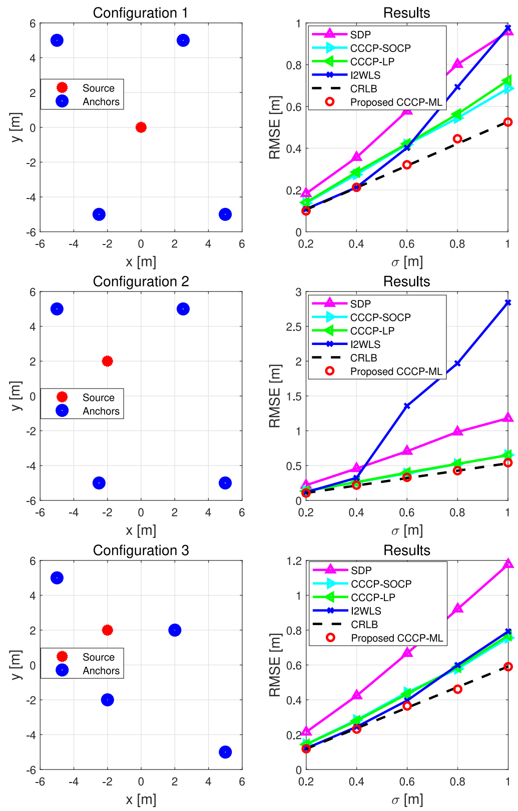

Fig. 3. Simulation results: RMSE versus TDOA error standard deviation for configurations with 4 anchors.

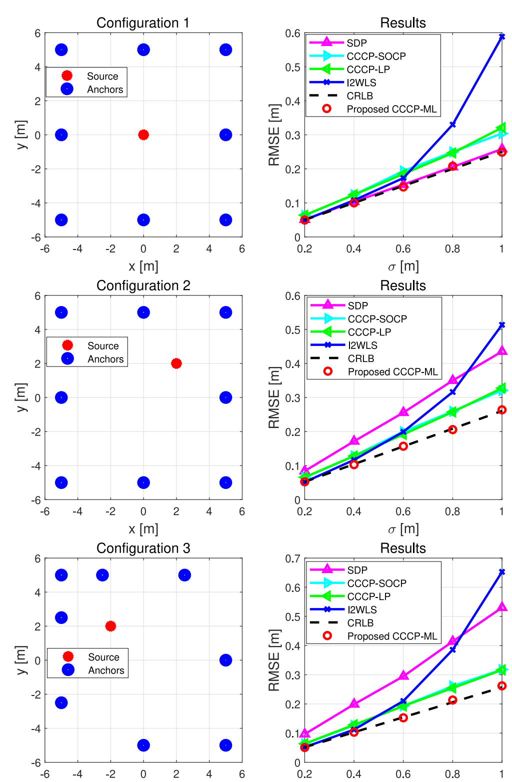

Fig. 4. Simulation results: RMSE versus TDOA error standard deviation for configurations with 8 anchors.

The performance of all methods tends to deteriorate as TDOA noise levels increase. However, the benchmark methods exhibit greater performance deterioration compared to the proposed method. This makes the advantage of the proposed method more visible in scenarios with high noise levels. Finally, considering the performance of various algorithms across different configurations, the benchmarks methods do not show a level of consistency. This to be contrasted with the steady performance displayed by the proposed method across all scenarios.

V. Conclusion

We have presented a maximum-likelihood (ML) TDOA localization algorithm based on the convex-concave procedure (CCCP). We have shown that the ML TDOA localization problem can be posed as an exact difference-of-convex (DC) program by manipulating the maximum-likelihood objective function. Hence, the CCCP can be applied as an iterative method that guarantees convergence to a stationary point. We have shown that, by initializing the proposed algorithm inside the anchors' convex hull, the proposed CCCP-ML algorithm achieves the Cramér-Rao lower bound, hence outperforming a host of benchmark methods.

Appendix A
Proof That (14) Is Convex

Based on (14), let us define $d_{ij}(\mathbf{p})$ as

$$d_{ij}(\mathbf{p}) := v^2[g_{ij}(\mathbf{p}) - \alpha_{ij}(\mathbf{p})]. \tag{A.1}$$

We want to prove that $d_{ij}(\mathbf{p})$ is convex.

The Hessian of $d_{ij}(\mathbf{p})$ is given by

$$\begin{aligned}\nabla^2 d_{ij}(\mathbf{p}) = & \underbrace{\left[\frac{\mathbf{I}}{||\mathbf{p}-\mathbf{r}_i||_2} - \frac{(\mathbf{p}-\mathbf{r}_i)(\mathbf{p}-\mathbf{r}_i)^T}{||\mathbf{p}-\mathbf{r}_i||_2^3}\right] ||\mathbf{p}-\mathbf{r}_j||_2}_{\mathbf{D}_1(\mathbf{p})} \\ & + \underbrace{\left[\frac{\mathbf{I}}{||\mathbf{p}-\mathbf{r}_j||_2} - \frac{(\mathbf{p}-\mathbf{r}_j)(\mathbf{p}-\mathbf{r}_j)^T}{||\mathbf{p}-\mathbf{r}_j||_2^3}\right] ||\mathbf{p}-\mathbf{r}_i||_2}_{\mathbf{D}_2(\mathbf{p})} \\ & + \underbrace{\frac{(\mathbf{p}-\mathbf{r}_i)(\mathbf{p}-\mathbf{r}_j)^T}{||\mathbf{p}-\mathbf{r}_i||_2||\mathbf{p}-\mathbf{r}_j||_2} + \mathbf{I}}_{\mathbf{D}_3(\mathbf{p})} + \underbrace{\frac{(\mathbf{p}-\mathbf{r}_j)(\mathbf{p}-\mathbf{r}_i)^T}{||\mathbf{p}-\mathbf{r}_i||_2||\mathbf{p}-\mathbf{r}_j||_2} + \mathbf{I}}_{\mathbf{D}_4(\mathbf{p})},\end{aligned} \tag{A.2}$$

where $\mathbf{I}$ is the identity matrix. To prove convexity, we prove that all four terms in (A.2) are positive semidefinite.

The eigenvalues of $\mathbf{D}_1(\mathbf{p})$ are 0 and $\frac{||\mathbf{p}-\mathbf{r}_j||_2}{||\mathbf{p}-\mathbf{r}_i||_2} \geq 0$. Hence $\mathbf{D}_1(\mathbf{p})$ is positive semidefinite. Similarly $\mathbf{D}_2(\mathbf{p})$ is positive semidefinite. Considering $\mathbf{D}_3(\mathbf{p})$, we can see that for any real vector $\mathbf{x}$, the following relation holds:

$$\mathbf{x}^T\mathbf{D}_3(\mathbf{p})\mathbf{x} = ||\mathbf{x}||_2^2\,[1 + \cos(\theta)\cos(\psi)] \geq 0, \tag{A.3}$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{p} - \mathbf{r}_i$, ψ is the angle between the vectors $\mathbf{x}$ and $\mathbf{p} - \mathbf{r}_j$. The result (A.3) proves that $\mathbf{D}_3(\mathbf{p})$ is positive semidefinite. Similarly, we can show that $\mathbf{D}_4(\mathbf{p})$ is also positive semidefinite.

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