A Fast Method for Estimating Frequencies of Multiple Sinusoidals

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***Abstract*—Estimating the frequencies of a multi-component sinusoidal signal is a fundamental problem in signal processing that has very wide and diverse application areas. Most of these applications require very fast and accurate estimation of the sinusoidal frequencies. This letter presents a fast, accurate, simple, and powerful DFT-based algorithm that estimates the frequencies of sums of multiple exponential sinusoidal signals. We show that the presented algorithm requires much less computational cost compared to conventional parametric and non-parametric methods. The simulation results show that the proposed method nearly reaches the Crámer-Rao lower limit after a certain SNR threshold.**

***Index Terms*—Multiple frequency estimation, fourier transform, fast frequency estimation, direction of arrival.**

I. Introduction

ESTIMATING the frequencies of a multi-component sinusoidal is a fundamental problem in signal processing [1], [2], modelling [3], system identification [4], radar applications and wireless communications. Huge application areas vary from line spectrum estimation to direction-of-arrival information retrieval. The demand for increased data rates has increased the use of MIMO and massive MIMO antennas, especially in 5G and 6G. These networks may need to estimate the frequencies of the sums of multiple sinusoidal for direction-of-arrival and beam forming purposes [5]. Furthermore, direction-of-arrival problems are also encountered in military and biomedical problems, where frequencies of multiple sinusoidals need to be estimated as quickly as possible. Thus, it is necessary to have fast and accurate multiple frequency estimation algorithms.

There have been numerous multiple sinusoidal frequency estimators. Well-known non-parametric spectral estimators are Capon [6], APES [7], and IAA [8]. These algorithms have good frequency resolution and therefore can estimate frequencies of closely separated components. Their more computationally efficient versions have also been proposed in many works, such as [9], [10] and [11]. However, even the computationally efficient versions of them have large computational complexities.

Well known subspace-based parametric methods include MUSIC [12] and ESPRIT [13], which have an overall good asymptotic performance when the signal-to-noise ratio (SNR) is not low. However, they require higher computational costs due to the need for performing an eigenvalue decomposition. Even their computationally efficient versions [14], [15] have a computational cost only slightly less than the original version. Both non-parametric and parametric methods suffer high computational cost and degraded performance when the signal length is low or the SNR is small.

CLEAN [16] and RELAX [17] algorithms utilize the maximum of the discrete periodogram to estimate the parameters of multiple sinusoidals sequentially. These algorithms employ an iterative estimation-subtraction procedure to estimate the parameters of the sinusoidals. However, CLEAN and RELAX algorithms suffer from the bias resulting from the interference of the components with one another. To overcome this problem, they employ zero-padding to increase the periodogram spectral resolution to enhance the estimation accuracy, which results in an increased computational cost.

In a recent letter [18], a new algorithm that estimates the frequency of a single-component complex sinusoidal by interpolating the DFT by $1/|q|$ times has been proposed, where $|q| \ll 1$. Interpolation is performed without zero-padding. By selecting an appropriate value of q, the estimator performance has been shown to achieve the Crámer-Rao lower bound (CRLB).

In this letter, we propose a new, fast, easily employable, and powerful DFT-based multiple frequency estimator based on the recently proposed HAQSE algorithm [18]. The proposed HAQSE-based algorithm first estimates the coarse frequencies of the multi-component sinusoidal followed by a fine estimation procedure. Therefore, we call it coarse-to-fine HAQSE (CFH). It has been shown in this letter that the computational cost of the proposed CFH algorithm is much lower than both parametric and non-parametric methods. The method is similar to the RELAX algorithm. Furthermore, the proposed method nearly achieves the CRLB after an SNR threshold. Since the proposed method is DFT-based, it can be quickly and easily realizable by digital signal processors, GPUs or FPGAs.

This letter does not focus on estimating the number of frequency components, where estimating the number of components is a separate problem. Estimating the number of frequency components can be achieved by employing a generalized Akaike information criterion [17] or the minimum description length [19] algorithms. Furthermore in [12], [13] and [20], methods for estimating the number of components are also presented and can be employed to estimate the number of components.

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Algorithm 1: HAQSE Algorithm.

Input : A single-frequency sinusoidal $s(n)$
Output : Frequency estimate $\widehat{f}$
Initialize: $\widehat{p} = \text{argmax}(|FFT\{s(n)\}|^2)$
1 **Iteration 1:**
2 $\quad \widehat{\delta}_1 = \frac{N}{2\pi}\arcsin(\sin(\frac{\pi}{N})\mathcal{R}e\{\frac{S_{0.5}+S_{-0.5}}{S_{0.5}-S_{-0.5}}\});$
3 **Iteration 2:**
4 $\quad \widehat{\delta}_2 = \frac{1}{c(q)} \times \mathcal{R}e\{\frac{S_{+q}-S_{-q}}{S_{+q}+S_{-q}}\} + \widehat{\delta}_1;$
5 **return** $\widehat{f} = \frac{\widehat{p}+\widehat{\delta}_1}{F_s}N$

II. Preliminaries

A. Problem Definition

Let us consider a signal composed of a linear combination of multiple sinusoidal signals

$$s_K(n) = \sum_{k=1}^{K} A_k \exp\left[j\left(\frac{2\pi}{F_s}f_k n\right)\right] + w(n), \tag{1}$$

in which $n = 0, \ldots, N-1$, f_k and A_k are the frequency, and complex amplitude of the k-th individual sinusoidal component, respectively, and $w(n)$ is assumed to be circularly symmetric additive white Gaussian noise having a variance of σ^2. K represents the total number of sinusoidal components and N is the total signal length. Every individual frequency is composed of an integer and a residual component, which can be expressed as

$$f_k = \frac{p_k + \delta_k}{N} F_s, \tag{2}$$

where p_k are the integer coarse and $\delta_k \in [-0.5, 0.5]$ are the fractional residual frequency components of f_k.

B. HAQSE Algorithm

In [18], two iterative fast methods for estimating the frequency of a single sinusoidal was proposed, namely QSE and HAQSE algorithms. Both of these methods are shown to be asymptotically efficient i.e., they are unbiased estimators asymptotically achieving the CRLB when estimating the frequency of a single sinusoidal. Further, it was reported that HAQSE is slightly more computational efficient, compared to the QSE. Therefore, we only employ the HAQSE algorithm in this letter. However, it can also be shown that QSE would have the same overall performance if used instead of HAQSE.

HAQSE algorithm shifts the DFT of the single sinusoidal by $\pm q$, where it has been shown that selecting

$$|q| \leq N^{-1/3} \tag{3}$$

is optimum. The effect of HAQSE algorithm is equivalent to actually interpolating the DFT of the signal by $1/|q| \geq N^{1/3}$ times and employing a ratio estimator, which has been shown to achieve an estimation performance equal to the CRLB.

The HAQSE algorithm converges in only two iterations, when the A&M algorithm [21] is used in its first iteration. HAQSE algorithm is described in Algorithm 1, where

$$S_{\pm 0.5} = \sum_{n=0}^{N-1} s(n)\exp\left[-j\frac{2\pi}{N}n\left(\widehat{p} \pm 0.5\right)\right],$$
$$S_{\pm q} = \sum_{n=0}^{N-1} s(n)\exp\left[-j\frac{2\pi}{N}n\left(\widehat{p} + \widehat{\delta}_1 \pm q\right)\right], \tag{4}$$

are the shifted-DFTs of the mono-frequency sinusoidal $s(n) = A\exp[j(\frac{2\pi}{N}fn)]$ with $f = \frac{p+\delta}{F_s}N$. The symbol $\widehat{p}$ represents the estimator of p.

Even though it has been shown in [18] that when HAQSE is employed, the variance of the frequency is on the CRLB, i.e., $\text{var}(\widehat{f}) = \text{CRLB}$, making it an optimum estimator for a single-component signal, no algorithms have been shown to yield good performance when estimating the frequencies of multi-component sinusoidals.

III. Extension of HAQSE Algorithm to Multiple Frequency Estimation

In this section, we propose a new method for estimating the frequencies of multiple sinusoidals buried in Gaussian noise.

A. The Proposed Coarse-to-Fine HAQSE Algorithm

We assume that the number of sinusoidal components, K, is known. The proposed algorithm starts with estimating the coarse frequencies of the sinusoidal by employing the HAQSE algorithm, sequentially. Let the coarse estimate of this frequency component be $\widetilde{f}_1$, which can be expressed by

$$\widetilde{f}_1 = HAQSE\{s_K(n)\}, \tag{5}$$

where $s_K(n)$ is the K-component sinusoidal signal defined by (1). The first estimated coarse frequency $\widetilde{f}_1$ belongs to the component whose magnitude spectrum has the largest value. Then, it estimates the residual frequency of the signal by employing the two iterations of the HAQSE, as shown in Algorithm 1. The corresponding complex amplitude of this component is thereby estimated by projecting the signal onto the estimated subspace by

$$\widetilde{A}_1 = \frac{1}{N}\sum_{n=0}^{N-1} s_K(n)\exp\left[-j\left(\frac{2\pi}{F_s}\widetilde{f}_1 n\right)\right]. \tag{6}$$

After the first frequency $\widetilde{f}_1$ and the corresponding complex amplitude $\widetilde{A}_1$ are estimated, a coarse estimate of $(K-1)$-component sinusoidal is then obtained by

$$\widetilde{s}_{K-1}(n) = s_K(n) - \widetilde{A}_1\exp\left[j\left(\frac{2\pi}{F_s}\widetilde{f}_1 n\right)\right] \tag{7}$$

This algorithm can estimate all of the frequencies and amplitudes, coarsely, by performing (5)–(7) exhaustively until all of the K frequencies of the signal are estimated.

However, it can be asserted that the estimator performance of the last component would be better than the others. When estimating the frequency of the first component, there would be a bias-effect caused by the other components, which degrades the estimator performance. As the frequencies and amplitudes

Algorithm 2: The proposed CFH Algorithm

Input : A multi-component sinusoidal $s(n)$
Output : Frequency estimates $\widehat{\mathbf{f}} = [\widehat{f}_1, \cdots, \widehat{f}_K]^T$
Initialize: $\widetilde{s}_K(n) = s_K(n)$
/* Coarse estimation of frequencies */
for $k = 1$ **to** K **do**
  $\widetilde{f}_k = HAQSE\{\widetilde{s}_{K-k+1}(n)\}$;
  $\widetilde{A}_k = \frac{1}{N}\widetilde{s}_{K-k+1}(n)\exp\left[-j\left(\frac{2\pi}{F_s}\widetilde{f}_k n\right)\right]$;
  $\widetilde{s}_{K-k}(n) = \widetilde{s}_{K-k+1}(n) - \widetilde{A}_k \exp\left[j\left(\frac{2\pi}{F_s}\widetilde{f}_k n\right)\right]$;
end
/* Fine estimation of frequencies */
for $k = 1$ **to** K **do**
  $\widehat{s}(n) = s_K(n) - \sum_{\substack{m=1\\m\neq k}}^{M} \widehat{A}_m \exp\left[j\left(\frac{2\pi}{F_s}\widehat{f}_m n\right)\right]$;
  $\widehat{f}_k = HAQSE\{\widehat{s}(n)\}$;
end
return $\widehat{\mathbf{f}}$;

of the components are estimated and subtracted sequentially, the bias in the estimate decreases gradually. Therefore, it can be concluded that the bias-effect would be higher in the first estimated frequency, and the lowest bias effect would be encountered in the last estimated frequency.

To overcome the bias effect, we propose to first estimate all the parameters of the signal coarsely by employing (5) to (7) for all K components. Then, a fine estimation of the k-th parameter can be obtained by first removing out all of the coarsely estimated individual sinusoidals from the original signal, except for the k-th component, such that

$$\widehat{s}(n) = s_K(n) - \sum_{\substack{m=1\\m\neq k}}^{K} \widetilde{A}_m \exp\left[j\left(\frac{2\pi}{F_s}\widetilde{f}_m n\right)\right]. \tag{8}$$

Then, we apply HAQSE to $\widehat{s}(n)$ as follows to acquire the fine estimated values of frequencies. The k-th fine estimated frequency is therefore obtained by

$$\widehat{f}_k = HAQSE\{\widehat{s}(n)\}, \tag{9}$$

which is less affected by the bias effect. To estimate all the values of frequencies finely, (8)–(9) are carried out for all values of k starting from one to K.

Section IV shows that performance the proposed algorithm is nearly on the CRLB. The reason why this algorithm works so well can be explained as follows. At each step of the fine estimation procedure, we remove the all of the remaining sinusoidals whose parameters are estimated coarsely in the coarse estimation procedure. In this way, we minimize the bias effect caused by the other components. When the bias effect is minimized, fine estimation procedure produces better frequency estimates. For convenience, Algorithm 2 describes a pseudo-code of the proposed CFH algorithm.

The presented algorithm is very similar and can be considered as a fast implementation of the CLEAN [16] and RELAX [17] algorithms, in which they require zero-padding before performing DFT. Let M denote the signal length after zero-padding, which corresponds to a frequency sampling interval of $2\pi/M$. It is apparent that M should be chosen to satisfy $M \gg 2\pi/\Delta$,

TABLE I
COMPARISON OF THE COMPUTATIONAL COSTS

CFH (proposed)	$\mathcal{O}(KN\log N)$, $K \ll N$
Capon [6], APES [7], and IAA [8]	$\mathcal{O}(N^2 + M\log M)$, $M \gg N$
MUSIC [12], ESPRIT [13], and MP [11]	$\mathcal{O}(N^3)$
fast Capon, fast APES, and fast IAA [9]–[11]	$\mathcal{O}(N^2 + d\log_2 d)$, $d \gg N$
fast MUSIC and fast ESPRIT [14], [15]	$\mathcal{O}(N^2 d)$, $d \ll N$
RELAX [17] and CLEAN [16]	$\mathcal{O}(KM\log M)$, $M \gg N$

where Δ is the standard deviation of the frequency estimates. To achieve a very small Δ at high SNR, M must be selected to be very large and the amount of computations required will also be very large. Therefore, the proposed algorithm can replace CLEAN and RELAX algorithms as a fast implementation, which does not require any zero-padding before the DFT. The proposed method can also be employed in the case of autoregressive noise using the same strategy presented in [17]. However, this letter does not focus on autoregressive noise condition.

B. Computational Cost

For a K component sinusoidal signal, the proposed CFH algorithm requires K HAQSE operations to estimate the frequencies of all components, coarsely. It has been shown in [18] that a HAQSE requires $\mathcal{O}(N\log N)$ complex operations. Therefore, coarse estimation procedure requires $\mathcal{O}(KN\log N)$ computational cost.

Similarly, fine estimation procedure requires an additional $\mathcal{O}(KN\log N)$ complex operations. Consequently, the total computational cost of the proposed algorithm is

$$\text{Computational Cost} = \mathcal{O}\left(KN\log N\right), \tag{10}$$

where generally $K \ll N$.

On the other hand, non-parametric methods, such as Capon, APES, and IAA require $\mathcal{O}(N^2 + M\log M)$, where $M \gg N$. Parametric methods, such as MUSIC, ESPRIT, and matrix pencil (MP) require $\mathcal{O}(N^3)$ computational cost due to the fact that they all require either singular value or eigenvalue decomposition. RELAX and CLEAN algorithms require zero-padding before performing DFT operations, therefore they require $\mathcal{O}(KM\log M)$ computational cost, where $M \gg N$.

Therefore, it can be concluded that the proposed algorithm requires the fewest calculations available in the literature as summarized in Table I.

C. Advantages of the Proposed Algorithm

The proposed algorithm has many advantages over existing algorithms. The greatest advantage of this proposed algorithm is the lower computational cost, compared to the existing algorithms. The proposed algorithm can be employed in a very easy and fast manner on digital signal processors, FPGAs and GPUs, since the algorithm is purely FFT-based. However, conventional methods, such as the MUSIC algorithm, generally require singular value decomposition, which may be very hard or expensive to develop, especially for large N. As shown in the next section, the algorithm performs nearly on the CRLB after a specific SNR threshold.

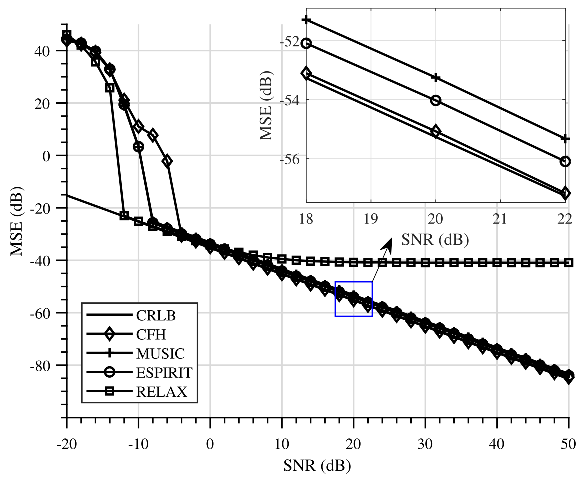

Fig. 1. Performance of the CFH algorithm compared with the MUSIC, ESPRIT, and RELAX algorithms when the number of components $K = 5$ and the frequencies are well separated.

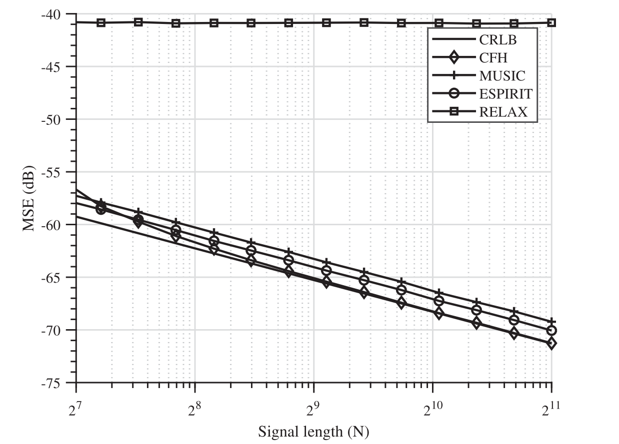

Fig. 2. Performance comparison of CFH algorithm with the MUSIC, ESPRIT, and RELAX algorithms when the number of components $K = 5$ and the frequencies are well separated.

IV. Simulation Results

The effectiveness of the proposed algorithm is shown by carrying out computer simulations. First, we generate a five-component complex exponential sinusoidal as

$$s(n) = \sum_{k=1}^{5} \exp\left[j\left(\frac{2\pi}{F_s}\left(p_k + \delta_k\right)n + \phi_k\right)\right] + w(n), \quad (11)$$

in which the coarse frequencies are selected as $p_k = \{50, 100, 150, 200, 250\}$. The values of δ_k and ϕ_k are selected uniformly random between $[-0.5, 0.5]$ and $[-\pi, \pi]$, respectively. The signal length was chosen as $N = 512$, where the sampling frequency was selected as $F_s = N$. All the results are averaged over 20,000 Monte-Carlo simulations. For the RELAX algorithm, the DFT size was selected as $32 \times N$, for all values of N.

Fig. 1 plots the estimation performance of the proposed CFH algorithm comparatively with the celebrated MUSIC, ESPRIT, and RELAX algorithms, with respect to the SNR. Here, SNR is defined as the ratio of power of only a *single* component to the power of the noise. Even though the SNR threshold is worse than all these methods, the figure proves that the proposed method is slightly better than both MUSIC and ESPIRIT algorithms after an SNR threshold difference of nearly -4 dB. While RELAX has the best SNR breakdown threshold, it suffers from bias effect and the proposed algorithm performs better after an SNR threshold difference of nearly -8 dB.

In order to further evaluate the performance of the proposed CFH algorithm, Fig. 2 shows the performance of the proposed method comparatively, with respect to the signal length N at a constant SNR of 30 dB. The figure clearly illustrates that the proposed CFH algorithm outperforms MUSIC, ESPRIT, and RELAX for all values of the signal length $N > 150$. For this simulation we have selected the coarse frequencies as $p_k = \{\lfloor N/10 \rfloor, \lfloor 2N/10 \rfloor, \lfloor 3N/10 \rfloor, \lfloor 4N/10 \rfloor\}$ and fine frequencies $\delta_k = 0.25$, where $\lfloor \cdot \rfloor$ is the floor operator. The phases ϕ_k are again selected as uniformly random between $[-\pi, \pi]$. The signal length N is increased from 128 to 2048, logarithmically spaced.

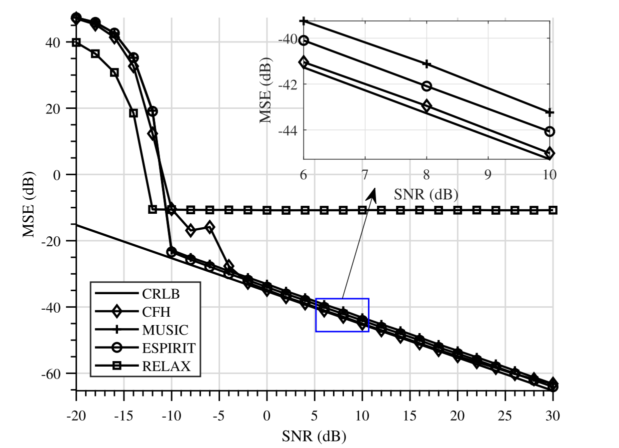

Fig. 3. Performance comparison of CFH algorithm when the number of components $K = 2$ and the frequencies are closely separated.

Now we generate a bi-component signal whose frequencies are closely separated. For this purpose, we set $K = 2, k_1 + \delta_1 = 103.25 + \delta$, and $k_2 + \delta_2 = 100.25 + \delta$, i.e., $\frac{2\pi}{F_s}(p_1 + \delta_1) - \frac{2\pi}{F_s}(p_2 + \delta_2) \approx 0.0368$, in (11), where δ uniformly random between $[-0.5, 0.5]$. Fig. 3 illustrates the performance of the proposed algorithm in comparison with MUSIC, ESPIRIT, and RELAX algorithms. The figure clearly shows that the proposed algorithm is effective in the case of closely spaced frequencies.

V. Conclusion

Estimating the frequencies of a multi-component sinusoidal has very wide application areas. In this letter, we have proposed a new, fast multi-frequency estimation algorithm based on the Fourier transform. The algorithm has a lower computation cost, when compared to the existing algorithms, such as MUSIC, ESPIRIT, and RELAX. Simulation results shows that the performance of the proposed algorithm is better than these algorithms under certain conditions, such as higher SNR and signal length.

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