On the Product of Two Correlated Complex Gaussian Random Variables

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***Abstract*—In this letter, we derive the exact joint probability density function (pdf) of the amplitude and phase of the product of two correlated non-zero mean complex Gaussian random variables with arbitrary variances. This distribution is useful in many problems, for example radar and communication systems. We determine the joint pdf in terms of an infinite summation of modified Bessel functions of the first and second kinds, which generalizes the existing results. The truncation error is also studied when a truncated sum is employed. Finally, we evaluate the derived expressions through numerical experiments.**

***Index Terms*—Product of correlated complex Gaussian random variables, joint probability density function, non-zero means and arbitrary variances.**

I. Introduction

WE CONSIDER the following complex random variable (RV)

$$Z = XY^*, \tag{1}$$

where X and Y are two correlated complex Gaussian RVs with non-zero means and arbitrary variances. This RV is useful in many applications. For example, in a single channel M-ary phase-shift-keying (MPSK) communication system [1], the linear combiner output can be characterized by the product of two complex random RVs as in (1). In time reversal detection [2], [3], the aggregate random channel can be modeled by the RV in (1). For radar applications, in the case of multipath scattering, for over-the-horizon (OTH) radar for example, the RV is useful to characterize the overall reflection coefficients [4].

The distribution for the product of two random variable has been investigated intensively. The distribution for the product of two classic RVs (Gaussian $\times$ Gaussian, Rayleigh $\times$ Rayleigh, Rice $\times$ Rayleigh, and Rice $\times$ Rice) under certain conditions is studied in [5]. The work in [6] studied the probability density function (pdf) for the product of two correlated exponential random variables. The pdf of the product of two independent κ-μ random variables has been been derived in [7], where the class of κ-μ random variables includes Rayleigh, Rice, and Nakagami-m random variable, as special cases. Numerical methods for computing the pdf of the product of two RVs is studied in [8].

Gaussian random variables play a central role in signal processing. The pdf for the product of two correlated real Gaussian RVs with zero means has been derived in [9]. Then, the pdf for the product of two correlated non-zero mean real Gaussian RVs with arbitrary variances has been derived in [10]. However, complex Gaussian RVs are used more extensively in signal processing. In [1], the joint characteristic function of the real and imaginary parts is derived for the inner product of two independent complex Gaussian vectors with arbitrary mean vectors and covariance matrices being scaled identity matrices. However, inversion of the characteristic function to obtain the joint pdf, must be computed numerically. In [3], the joint pdf of the amplitude and phase is derived for the product of independent complex Gaussian RVs with non-zero means. In [11], the authors derived the joint characteristic function of the real and imaginary parts of the RV presented in (1) in closed-form. They also derived the joint pdf for the zero mean case along with an approximate joint pdf for the case where the mean-to-standard-deviation ratios are large enough. However, to the best of our knowledge, the exact pdf for the product of two correlated complex Gaussian random variables with arbitrary means has never appeared in the open literatures.

In this letter, we derive an exact expression for the joint pdf of the product of two correlated complex Gaussian RVs with non-zero means and arbitrary variances. We show that the joint pdf of the amplitude and phase is an infinite summation of modified Bessel functions of the first and second kinds. The results for two special cases where the two random variables are independent or zero mean are presented. The approximation error of the truncated series is studied. Finally, we verify the derived results through numerical experiments.

Notation: The symbol $\mathbb{E}\{\cdot\}$ denotes expectation, $(\cdot)^*$ denotes conjugate, and $(\cdot)^T$ and $(\cdot)^H$ denote transpose and conjugate transpose, respectively. The real and imaginary parts of a complex number are denoted by $\Re(\cdot)$ and $\Im(\cdot)$, respectively. The functions $I_\mu(\cdot)$ and $K_\mu(\cdot)$ are the modified Bessel functions of the first and second kinds with order μ, respectively.

II. Derivation of Joint PDF

The following theorem provides a general expression for the product of two correlated complex Gaussian random variables.

Theorem 1: Let $\boldsymbol{v} = [X, Y]^T \sim \mathcal{CN}(\boldsymbol{m}, \boldsymbol{\Sigma})$ be a 2×1 complex Gaussian vector, where

$$\boldsymbol{m} = \mathbb{E}\{\boldsymbol{v}\} = [m_x, m_y]^T \tag{2}$$

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and

$$\boldsymbol{\Sigma} = \mathbb{E}\{(\boldsymbol{v}-\boldsymbol{m})(\boldsymbol{v}-\boldsymbol{m})^H\} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho^*\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \tag{3}$$

are the mean vector and covariance matrix of $\boldsymbol{v}$, respectively. Let

$$Z = XY^* = Z_I + jZ_Q = Re^{j\Theta}. \tag{4}$$

The joint pdf of (R,Θ) is given by

$$f_{R,\Theta}(r,\theta) = \frac{2re^{-g}}{c\pi\sigma_y^2}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\beta_1^n\beta_2^p}{n!p!\beta_3^{\frac{n+p}{2}}}\left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right)I_{n+p}(2\sqrt{\beta_3}), \tag{5}$$

where

$$\alpha = \frac{|b|^2}{c} + \frac{1}{\sigma_y^2}, \tag{6}$$

$$g = \frac{|a|^2 - 2\Re\{re^{-j\theta}b\}}{c} + \frac{|m_y|^2}{\sigma_y^2}, \tag{7}$$

$$\beta_1 = \frac{r^2|a|^2}{c^2}, \tag{8}$$

$$\beta_2 = \left|-\frac{b^*a}{c} + \frac{m_y}{\sigma_y^2}\right|^2, \tag{9}$$

and

$$\beta_3 = 2\Re\left\{\frac{re^{-j\theta}a}{c}\left(\frac{-ba^*}{c} + \frac{m_y^*}{\sigma_y^2}\right)\right\}, \tag{10}$$

in which

$$a = m_x - \frac{\rho\sigma_x}{\sigma_y}m_y, \tag{11}$$

$$b = \frac{\rho\sigma_x}{\sigma_y}, \tag{12}$$

and

$$c = \sigma_x^2(1-|\rho|^2). \tag{13}$$

Proof: We first derive the joint pdf $f_Z(z_I,z_Q)$ of the real part Z_I and imaginary part Z_Q. Then, the joint pdf for the amplitude R and phase Θ can be obtained by [12]

$$f_{R,\Theta}(r,\theta) = rf_Z(r\cos\theta, r\sin\theta). \tag{14}$$

Denote by X_I and X_Q the real part and imaginary part of X, respectively. The notation Y_I and Y_Q are similarly defined. The joint pdf for Z_I and Z_Q can be computed as

$$f_Z(z_I,z_Q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Z|Y}(z_I,z_Q)f_Y(y_I,y_Q)\mathrm{d}y_I\mathrm{d}y_Q, \tag{15}$$

where $f_{Z|Y}(z_I,z_Q)$ is the pdf of Z conditioned on Y and $f_Y(y_I,y_Q)$ is the pdf of Y. The conditional distribution of any complex RV X conditioned on a complex RV Y is given by [13], [14]

$$X|Y \sim \mathcal{CN}\left(m_x + \frac{\rho\sigma_x}{\sigma_y}(Y-m_y), \sigma_x^2(1-|\rho|^2)\right). \tag{16}$$

Hence, the conditional distribution of Z conditioned on Y is

$$Z|Y = XY^*|Y \sim \mathcal{CN}\left(aY^* + b|Y|^2, c|Y|^2\right), \tag{17}$$

where $a = m_x - \frac{\rho\sigma_x}{\sigma_y}m_y$, $b = \frac{\rho\sigma_x}{\sigma_y}$, and $c = \sigma_x^2(1-|\rho|^2)$. From (2) and (3),

$$Y \sim \mathcal{CN}\left(m_y, \sigma_y^2\right). \tag{18}$$

Applying (17) and (18) to (15), we have

$$f_Z(z_I,z_Q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi c|y|^2}e^{-\frac{|z-ay^*-b|y|^2|^2}{c|y|^2}} \times \frac{1}{\pi\sigma_y^2}e^{-\frac{|y-m_y|^2}{\sigma_y^2}}\mathrm{d}y_I\mathrm{d}y_Q, \tag{19}$$

where $y = y_I + jy_Q$ and $z = z_I + jz_Q$. In (19),

$$\begin{aligned}\left|z - ay^* - b|y|^2\right|^2 &= |z|^2 + |a|^2|y|^2 + |b|^2|y|^4 - 2\Re\{z^*ay^*\} \\ &\quad - 2\Re\{z^*b\}|y|^2 + 2\Re\{b^*ay^*\}|y|^2 \\ &= |z|^2 + |a|^2|y|^2 + |b|^2|y|^4 - 2\Re\{z^*a\}y_I - 2\Im\{z^*a\}y_Q \\ &\quad - 2\Re\{z^*b\}|y|^2 + 2\Re\{b^*a\}|y|^2y_I + 2\Im\{b^*a\}|y|^2y_Q\end{aligned} \tag{20}$$

and

$$|y-m_y|^2 = |y|^2 + |m_y|^2 - 2\Re\{m_y\}y_I - 2\Im\{m_y\}y_Q. \tag{21}$$

Substituting (20) and (21) into (19) and letting $y_I = t\cos\psi$ and $y_Q = t\sin\psi$, we obtain

$$\begin{aligned}&f_Z(z_I,z_Q) \\ &= \frac{e^{-g}}{c\pi^2\sigma_y^2}\int_{-\infty}^{\infty}\frac{1}{t}e^{-\alpha t^2 - \frac{|z|^2}{ct^2}}\int_0^{2\pi}e^{\lambda_1(t,z)\cos\psi + \lambda_2(t,z)\sin\psi}\mathrm{d}\psi\mathrm{d}t \\ &= \frac{2e^{-g}}{c\pi\sigma_y^2}\int_{-\infty}^{\infty}\frac{1}{t}e^{-\alpha t^2 - \frac{|z|^2}{ct^2}}I_0(\lambda(z,t))\mathrm{d}t,\end{aligned} \tag{22}$$

where $\alpha = \frac{|b|^2}{c} + \frac{1}{\sigma_y^2}$, $g = \frac{|a|^2 - 2\Re\{z^*b\}}{c} + \frac{|m_y|^2}{\sigma_y^2}$,

$$\lambda_1(t,z) = \frac{2\Re\{z^*a\}}{ct} - \frac{2\Re\{b^*a\}t}{c} + \frac{2\Re\{m_y\}t}{\sigma_y^2}, \tag{23}$$

$$\lambda_2(t,z) = \frac{2\Im\{z^*a\}}{ct} - \frac{2\Im\{b^*a\}t}{c} + \frac{2\Im\{m_y\}t}{\sigma_y^2}, \tag{24}$$

and

$$\lambda(t,z) = \sqrt{\lambda_1(t,z)^2 + \lambda_2(t,z)^2} \tag{25}$$

$$= 2\sqrt{\beta_1 t^{-2} + \beta_2 t^2 + \beta_3}, \tag{26}$$

in which $\beta_1 = \frac{|z|^2|a|^2}{c^2}$, $\beta_2 = |-\frac{b^*a}{c} + \frac{m_y}{\sigma_y^2}|^2$, and $\beta_3 = 2\Re\{\frac{z^*a}{c}(\frac{-ba^*}{c} + \frac{m_y^*}{\sigma_y^2})\}$. Recall the identity [15, 8.445-1]

$$I_\mu(u) = \left(\frac{u}{2}\right)^\mu\sum_{m=0}^{\infty}\frac{\left(\frac{u}{2}\right)^{2\,m}}{m!(m+\mu)!} \tag{27}$$

where μ is an integer. Using (27), the modified Bessel function $I_0(\lambda(t,z))$ in (22) can be written as

$$I_0(\lambda(t,z)) = \sum_{m=0}^{\infty} \frac{\left(\beta_1 t^{-2} + \beta_2 t^2 + \beta_3\right)^m}{(m!)^2}. \quad (28)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n} \frac{(\beta_1 t^{-2})^n (\beta_2 t^2)^p \beta_3^{m-n-p}}{m!n!p!(m-n-p)!}. \quad (29)$$

Applying (29) to (22), we obtain

$$f_Z(z_I, z_Q) = \frac{2e^{-g}}{c\pi\sigma_y^2} \sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n} \frac{\beta_1^n \beta_2^p \beta_3^{m-n-p}}{m!n!p!(m-n-p)!} \times \int_0^{\infty} \frac{t^{2p-2n}}{t} e^{-\alpha t^2 - \frac{|z|^2}{ct^2}} \mathrm{d}t. \quad (30)$$

Rewriting the integral in (30) as follows [15, 3.478-4]

$$\int_0^{\infty} \frac{t^{2p-2n}}{t} e^{-\alpha t^2 - \frac{|z|^2}{ct^2}} \mathrm{d}t = \left(\frac{|z|^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2|z|\sqrt{\frac{\alpha}{c}}\right), \quad (31)$$

then (30) is reduced to

$$f_Z(z_I, z_Q) = \frac{2e^{-g}}{c\pi\sigma_y^2} \sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n} \frac{\beta_1^n \beta_2^p \beta_3^{m-n-p}}{m!n!p!(m-n-p)!} \times \left(\frac{|z|^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2|z|\sqrt{\frac{\alpha}{c}}\right). \quad (32)$$

Reordering the summation gives [3]

$$f_Z(z_I, z_Q) = \frac{2e^{-g}}{c\pi\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\beta_1^n \beta_2^p \beta_3^{-n-p}}{n!p!} \left(\frac{|z|^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2|z|\sqrt{\frac{\alpha}{c}}\right) \sum_{m=n+p}^{\infty} \frac{\beta_3^m}{m!(m-n-p)!} \quad (33)$$

Applying the identity

$$\sum_{m=n+p}^{\infty} \frac{\beta_3^m}{m!(m-n-p)!} = \beta_3^{\frac{n+p}{2}} I_{n+p}\left(2\sqrt{\beta_3}\right), \quad (34)$$

and (14) to (33), the result in (5) can be obtained. ■

A. Special Cases

In the following, we look at some special cases of Theorem 1.

1) Independent RVs: In this case, $\rho = 0$, the expression (5) can be recast as

$$f_{R,\Theta}(r,\theta) = \frac{2re^{-(k_x^2+k_y^2)}}{\pi\sigma_x^2\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{n!p!} \left(\frac{k_x}{k_y}\right)^{n-p} \times \left(\frac{|\varphi(r,\theta)|}{2\sqrt{\Re\{\varphi(r,\theta)\}}}\right)^{n+p} K_{p-n}\left(\frac{2r}{\sigma_x\sigma_y}\right) I_{n+p}\left(2\sqrt{\Re\{\varphi(r,\theta)\}}\right), \quad (35)$$

where $k_x = |m_x|/\sigma_x$, $k_y = |m_y|/\sigma_y$, and $\varphi(z) = 2\frac{\Re\{re^{-j\theta} m_x m_y^*\}}{\sigma_x^2\sigma_y^2}$, which agrees with the result in [3].

2) Zero Means: As the means approaches zero, i.e. $m_x, m_y \to 0$, the limits of the terms in (5) become

$$\lim_{m_x,m_y\to 0} g = \frac{-2\Re\{re^{-j\theta}b\}}{c} \triangleq g_0, \quad (36)$$

$$\lim_{m_x,m_y\to 0} \beta_i = 0, \quad i = 1,2, \text{ or } 3, \quad (37)$$

and

$$\lim_{\beta_3\to 0} \frac{I_{p+n}(2\sqrt{\beta_3})}{(\beta_3)^{\frac{p+n}{2}}} = \lim_{\beta_3\to 0} \sum_{m=0}^{\infty} \frac{\beta_3^m}{m!(m+n+p)!} = \frac{1}{(n+p)!}. \quad (38)$$

Thus, for the case of zero means, (5) can be written as

$$\lim_{m_x,m_y\to 0} f_{R,\Theta}(r,\theta) = \lim_{\substack{\beta_1,\beta_2,\beta_3\to 0\\ g\to g_0}} f_{R,\Theta}(r,\theta)$$

$$= \lim_{\beta_1,\beta_2\to 0} \frac{2re^{-g_0}}{c\pi\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\beta_1^n\beta_2^p}{n!p!} \left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) \frac{1}{(n+p)!}. \quad (39)$$

Only for $n = p = 0$, the limit of general summation term in (39) is non-zero. Hence, (39) can be recast as

$$f_{R,\Theta}(r,\theta) = \frac{2re^{-g_0}}{c\pi\sigma_y^2} K_0\left(2r\sqrt{\frac{\alpha}{c}}\right) = \frac{2r}{\pi(1-|\rho|^2)\sigma_x^2\sigma_y^2} e^{\frac{2\Re\{\rho re^{-j\theta}\}}{(1-|\rho|^2)\sigma_x\sigma_y}} K_0\left(\frac{2r}{(1-|\rho|^2)\sigma_x\sigma_y}\right), \quad (40)$$

which is consistent with the result in [11].

B. Truncated Joint PDF

Since (5) involves a doubly-infinite summation, we define a truncated joint pdf approximation of (5) as

$$\bar{f}_{R,\Theta}(r,\theta;M) = \frac{2re^{-g}}{c\pi\sigma_y^2} \sum_{n=0}^{M}\sum_{p=0}^{M} \frac{\beta_1^n\beta_2^p}{n!p!\beta_3^{\frac{n+p}{2}}} \left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) I_{n+p}(2\sqrt{\beta_3}). \quad (41)$$

Thus, the truncation error is

$$\varepsilon_M = f_{R,\Theta}(r,\theta) - \bar{f}_{R,\Theta}(r,\theta;M). \quad (42)$$

Substituting (5) and (41) into (42), using inequalities and approximations in [16]–[19] and following similar steps as in [3], it can be proved that *for sufficiently large M, the truncation error (42) is bounded by*

$$|\varepsilon_M| \leqslant \frac{4re^{-g}}{c\pi\sigma_y^2} \left[\sum_{n=0}^{M+1} \psi_{n,M+1} + \sum_{p=0}^{M+1} \psi_{M+1,p}\right] \triangleq B_M, \quad (43)$$

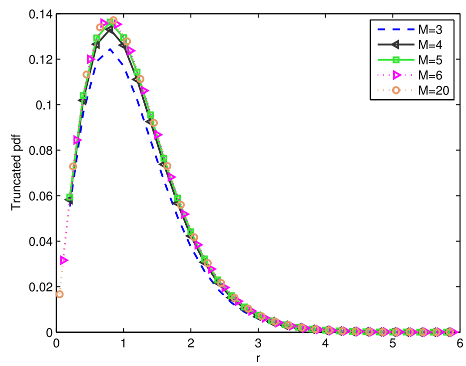

Fig. 1. $\theta = 0$ cut of the truncated pdf for different M.

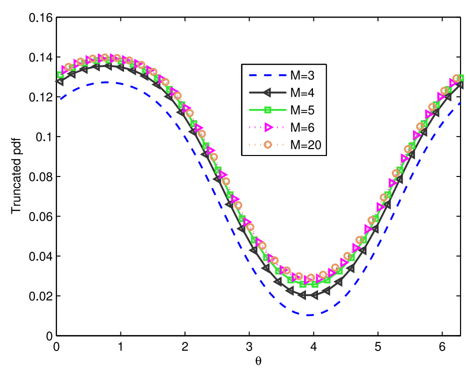

Fig. 2. $r = 1$ cut of the truncated pdf for different M.

where

$$\psi_{n,p} = \frac{\beta_1^n \beta_2^p}{n!p!|\beta_3|^{\frac{n+p}{2}}} \left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) I_{n+p}(2\sqrt{|\beta_3|}), \tag{44}$$

and that for arbitrary $C > 1$,

$$\lim_{M\to\infty} \frac{B_M}{C^{-M}} = 0, \tag{45}$$

i.e., the decay of truncation error is faster than exponential.

III. Numerical Results

In this section, we provide numerical results to verify the accuracy of a truncated series in (41). The parameters are set as $m_x = 1, m_y = 0.5 - j0.5, \sigma_x^2 = 1, \sigma_y^2 = 0.8$, and $\rho = -\frac{1}{2}e^{j\frac{\pi}{4}}$.

In Fig. 1 and Fig. 2, we plot the cut of the truncated pdf defined in (41) along $\theta = 0$ and $r = 1$ for $M = 3$, 4, 5, 6 and 20. In the two figures, we see that the cuts of the truncated pdf for $M = 6$ and for $M = 20$ are almost the same, which implies that for $M > 6$, the truncation error can be very small in the tested example. Other r or θ cuts of the truncated pdf are also tested and the results are similar.

Next, we compare the numerical result obtained from Monte Carlo method with the computation result from (41). In Fig. 3(a), we plot the numerical result by using 10^7 independent samples sampled from the RV in (1). In Fig. 3(b), the computation result from (41) with $M = 20$ is plotted. To evaluate the difference between Fig. 3(a) and Fig. 3(b), define

$$\epsilon(r,\theta) = |\hat{f}(r,\theta) - \overline{f}_{R,\Theta}(r,\theta;20)|, \tag{46}$$

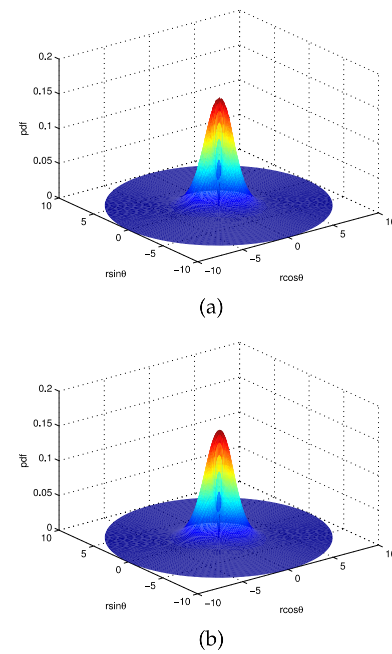

Fig. 3. Plot of the joint pdf. (a) Numerical result using Monte Carlo method. (b) Computation result from (41) with $M = 20$.

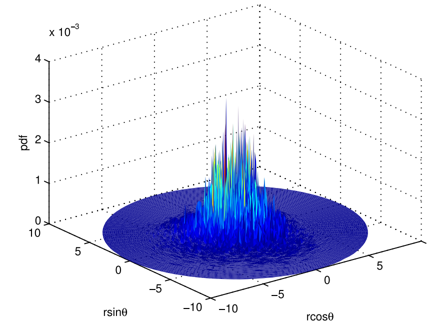

Fig. 4. Difference between numerical result and computation result.

where $\hat{f}(r,\theta)$ represents the pdf obtained from Monte Carlo method and $\overline{f}_{R,\Theta}(r,\theta;20)$ is the truncated pdf with $M = 20$. In Fig. 4, the difference $\epsilon(r,\theta)$ is plotted. We see that the peak value of $\epsilon(r,\theta)$ is no larger than 4×10^{-3}, which is very small compared with the peak values in Fig. 3(a) and Fig. 3(b). This implies that the derived pdf matches numerical result.

IV. Conclusion

In this letter, we studied the distribution of the product of two correlated complex Gaussian random variables. The joint pdf was derived, in terms of an infinite sum of modified Bessel functions of the first and second kinds. We also presented two special cases where the RVs are zero mean or independent. When a truncated sum is used to approximate the infinite sum, it was shown that as the number of summation terms increase, the truncation error decays faster than exponential. Finally, we show that the derived pdf matches numerical results.

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