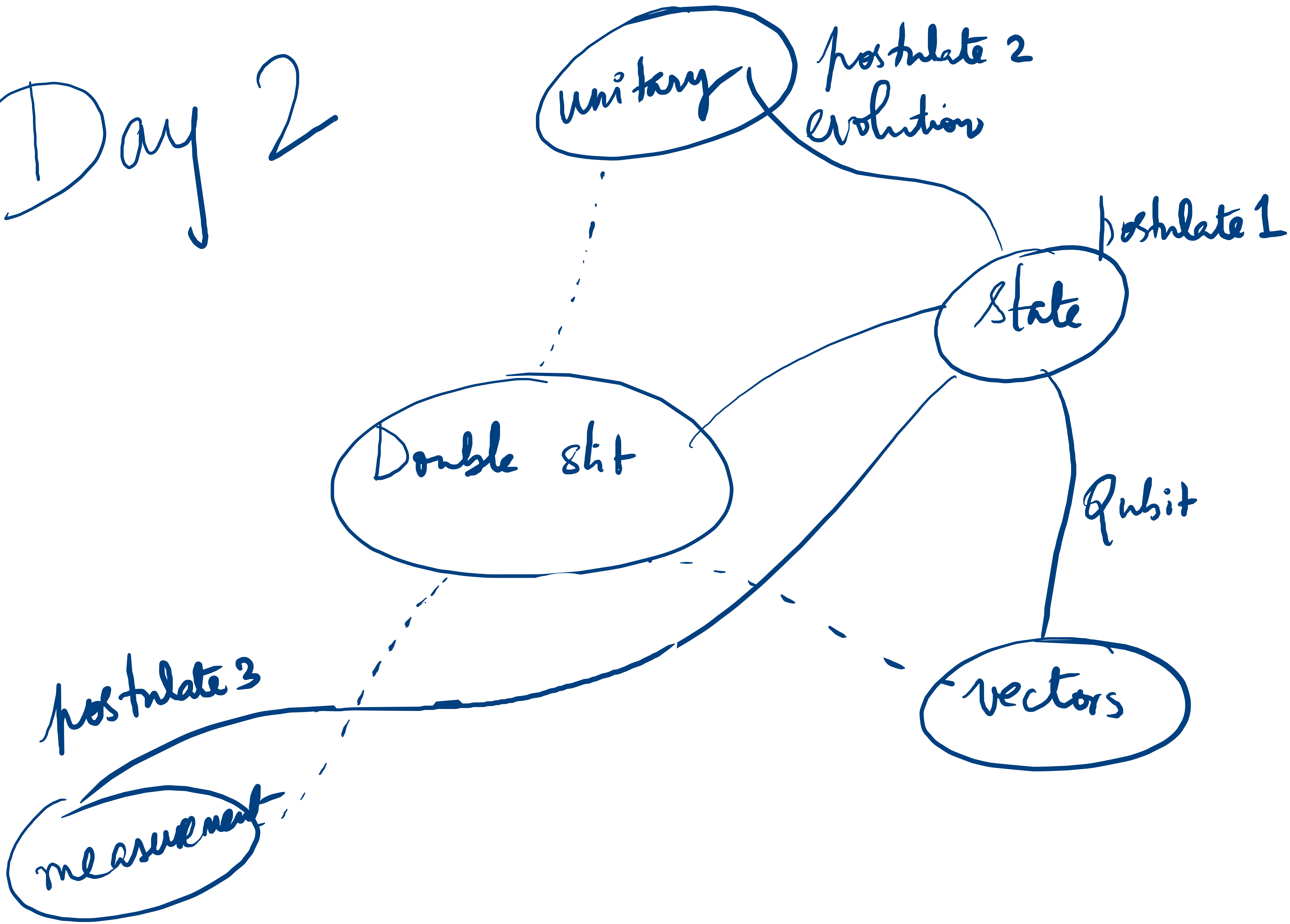
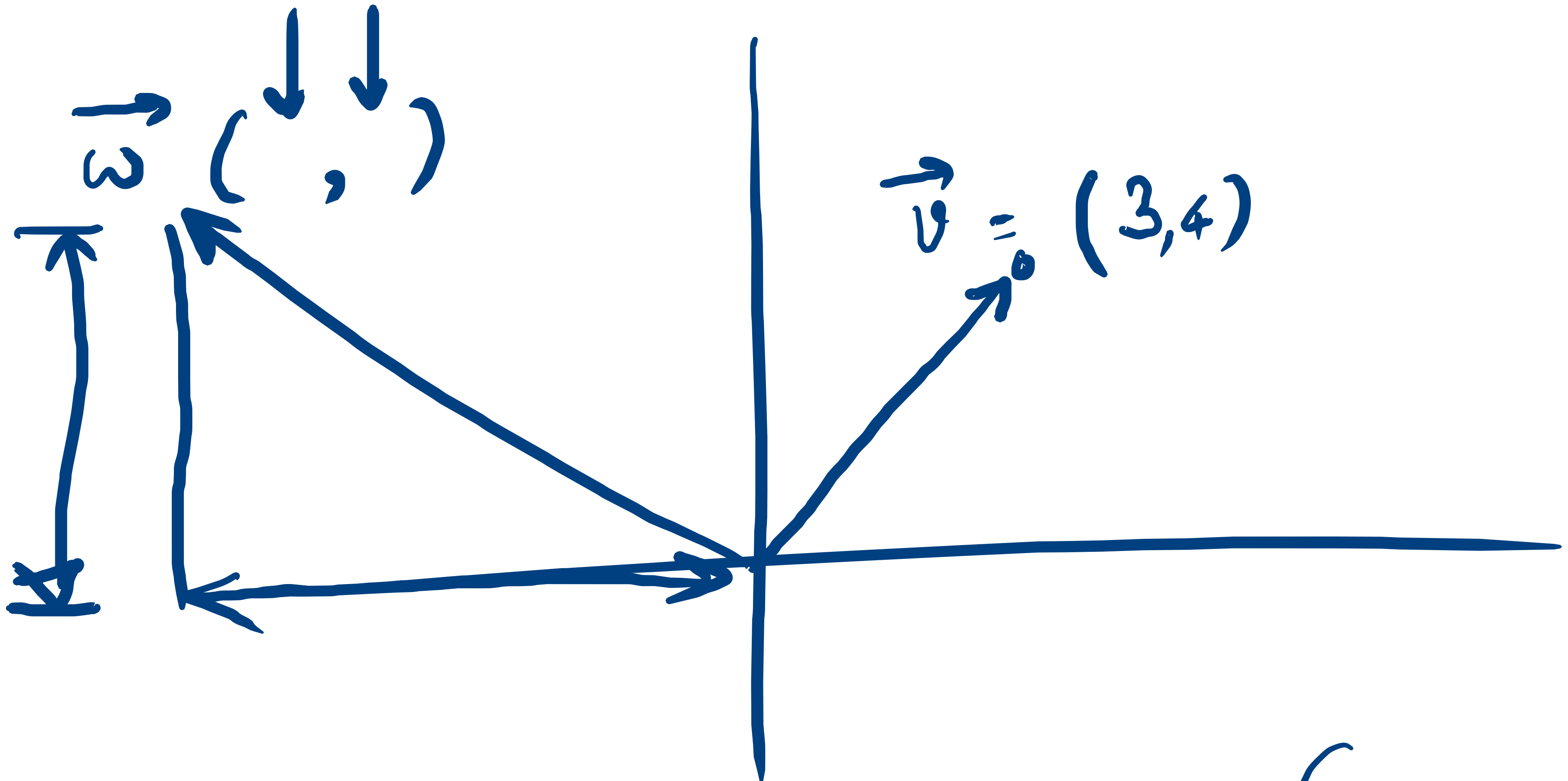
Day 2


unitary
postulate 2
evolution
state
postulate 1
Double slit
Qubit
vectors
postulate 3
measurement



$$\text{length } \vec{v} = \|\vec{v}\|$$
$$= \sqrt{3^2+4^2}$$
$$= 5$$

Euclidean norm

$$\vec{v} = \begin{pmatrix}3\\4\end{pmatrix}$$

column vector

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad |a|^2+|b|^2 = 1$$

vector space — Inner product space

$$\langle \ , \ \rangle$$

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$$

$$\langle\psi| = \begin{pmatrix} a^* & b^* \end{pmatrix} = a^*\langle 0| + b^*\langle 1|$$

$$\langle\psi|\psi\rangle = \underbrace{\begin{pmatrix} a^* & b^* \end{pmatrix}}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}} = |a|^2 + |b|^2 = 1$$

States are of unit norm

$$\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$$

$$|\psi\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle$$

unitary preserves length, preserves angle

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle = 1$$

$$\Big(|\psi_1\rangle, |\psi_2\rangle\Big) = \langle\psi_1|\psi_2\rangle = \Big(U|\psi_1\rangle, U|\psi_2\rangle\Big)$$

$U|\psi_1\rangle \quad U|\psi_2\rangle$

Measurement

Measurement in a certain basis

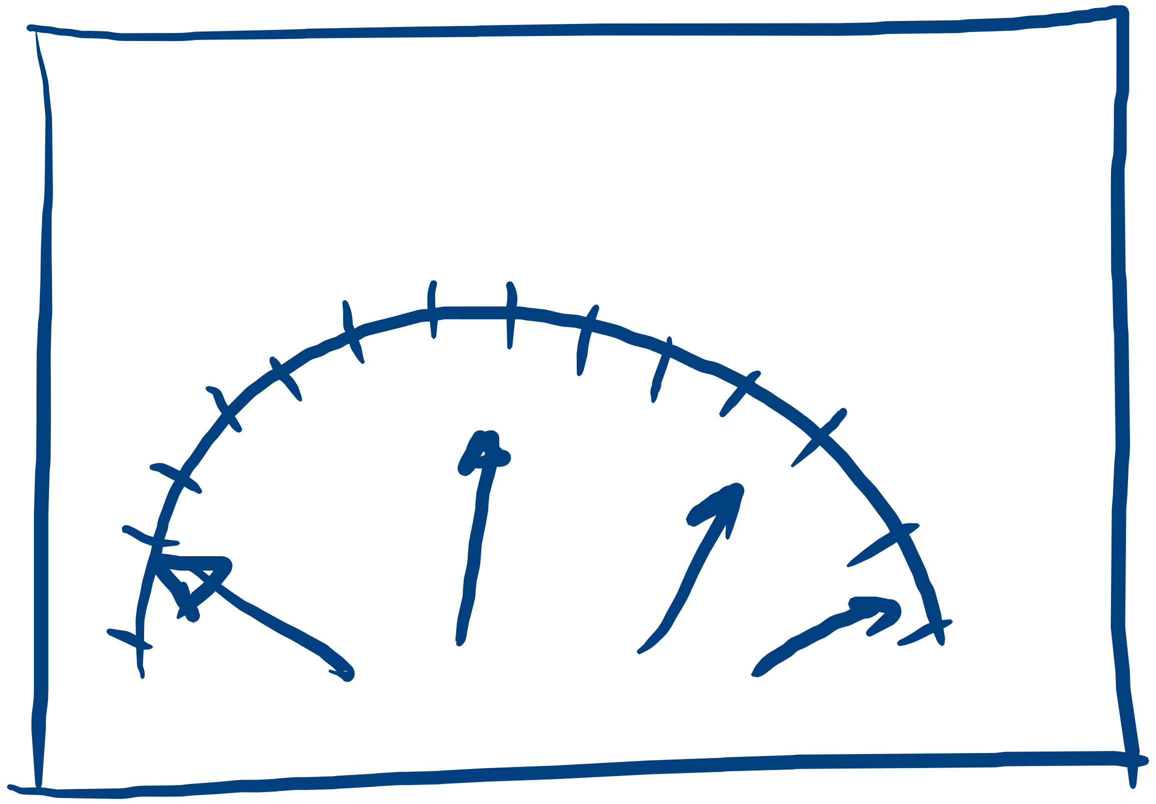
$\{|0\rangle, |1\rangle\}$ basis

$$a|0\rangle + b|1\rangle \xrightarrow{|a|^2} |0\rangle$$

$$a|0\rangle + b|1\rangle \xrightarrow{|b|^2} |1\rangle$$

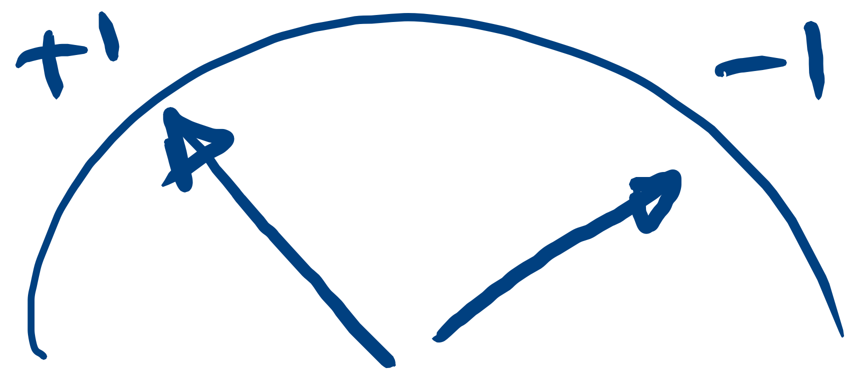
Observable matrix — Hermitian $A^\dagger = A$

eigen values



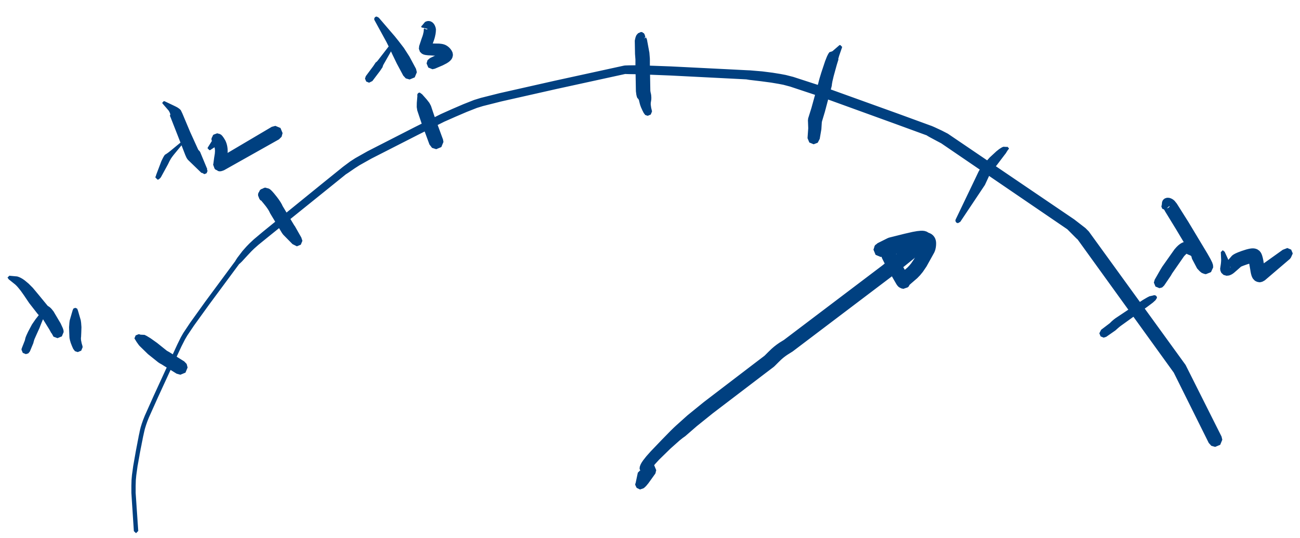
Measuring an observable A

real quantities — eigenvalues of A



$$Z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$$

$$A = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2| + \cdots + \lambda_n |e_n\rangle\langle e_n|$$

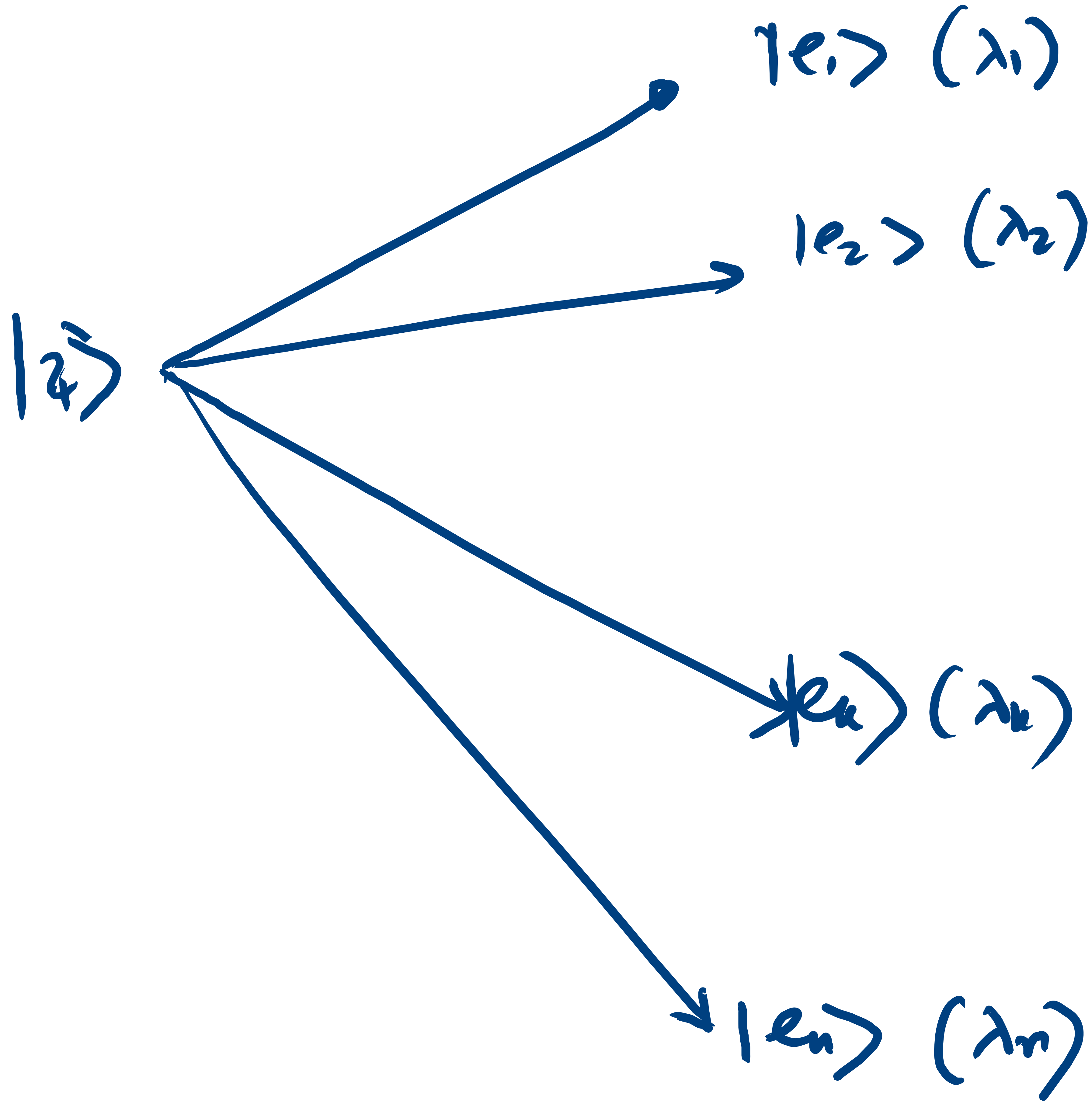


$$\{ |e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle \}$$

$$\lambda_k \qquad |e_k\rangle$$

$$A = \sum_{j=1}^{n} \lambda_j |e_j\rangle\langle e_j|$$

$$|\psi\rangle = \sum_{k=1}^{n} a_k \, |f_k\rangle$$



$$\boxed{|\psi\rangle = a|0\rangle + b|1\rangle}$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$\{|0\rangle, |1\rangle\}$ basis

$$|\langle 0|\psi\rangle|^2$$

$$\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$|\vec{v}\rangle = 3|0\rangle + 4|1\rangle$$

$$\langle 0|\vec{v}\rangle = 3\underline{\langle 0|0\rangle} + 4\underbrace{\langle 0|1\rangle}$$

$$= 3$$

$$\langle 1|\vec{v}\rangle = 3\langle 1|0\rangle + 4\langle 1|1\rangle$$

$$= 4$$

$$|\psi\rangle = a|0'\rangle + b|1'\rangle$$

$$\langle 0|\psi\rangle = \langle 0|\left[a|0'\rangle + b|1'\rangle\right]$$

$$= a\langle 0|0'\rangle + b\langle 0|1'\rangle$$

$$= \frac{a+b}{\sqrt{2}}$$

$$|\langle 0|\psi\rangle|^2 = \frac{|a+b|^2}{2} \qquad |\langle 1|\psi\rangle|^2 = \frac{|a-b|^2}{2}$$

$$\left|\frac{a+b}{2}\right|^2 + \left|\frac{c-b}{2}\right|^2 = 1$$

$$|\langle 0|\psi\rangle|^2 + |\langle 1|\psi\rangle|^2 = 1$$

$$\langle 0|\psi\rangle^* \langle 0|\psi\rangle + \langle 1|\psi\rangle^* \langle 1|\psi\rangle = 1$$

$$\langle \psi|0\rangle\langle 0|\psi\rangle + \langle \psi|1\rangle\langle 1|\psi\rangle = 1$$

$$\langle \psi| \big[\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|} \big] |\psi\rangle = 1 \quad \forall |\psi\rangle$$

Measurements

$$Z = (+1)\,|0\rangle\langle 0| + (-1)\,|1\rangle\langle 1| \quad \checkmark \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$X = (+1)\,|+\rangle\langle +| + (-1)\,|-\rangle\langle -| \quad \checkmark \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|0'\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1'\rangle = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|i+\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$Y = (+1)\,|i+\rangle\langle i+| + (-1)\,|i-\rangle\langle i-|$$

$$|i-\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \checkmark$$

$$|i+\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix} \checkmark$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$|0\rangle \longrightarrow |+\rangle$

$|1\rangle \longrightarrow |-\rangle$

HW $|0\rangle \longrightarrow |i+\rangle$

$|1\rangle \longrightarrow |i-\rangle$

$$\frac{X+Z}{\sqrt{2}}$$

$$|2_+\rangle \longrightarrow H|2_+\rangle$$

$$\frac{X-Z}{\sqrt{2}}$$

$$A = \sum_k \lambda_k |e_k\rangle\langle e_k| \qquad \text{matrix}(A)?$$

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$|\psi\rangle \;\text{——}\; \boxed{U} \;\text{——}\; |\psi'\rangle = U|\psi\rangle$$

Classical gates OR, AND, NOT, NAND, NOR

Quantum gates?

Generate unitaries

Inventory : $\boxed{X, Y, Z, H, S, T, R_x, R_y, R_z}$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

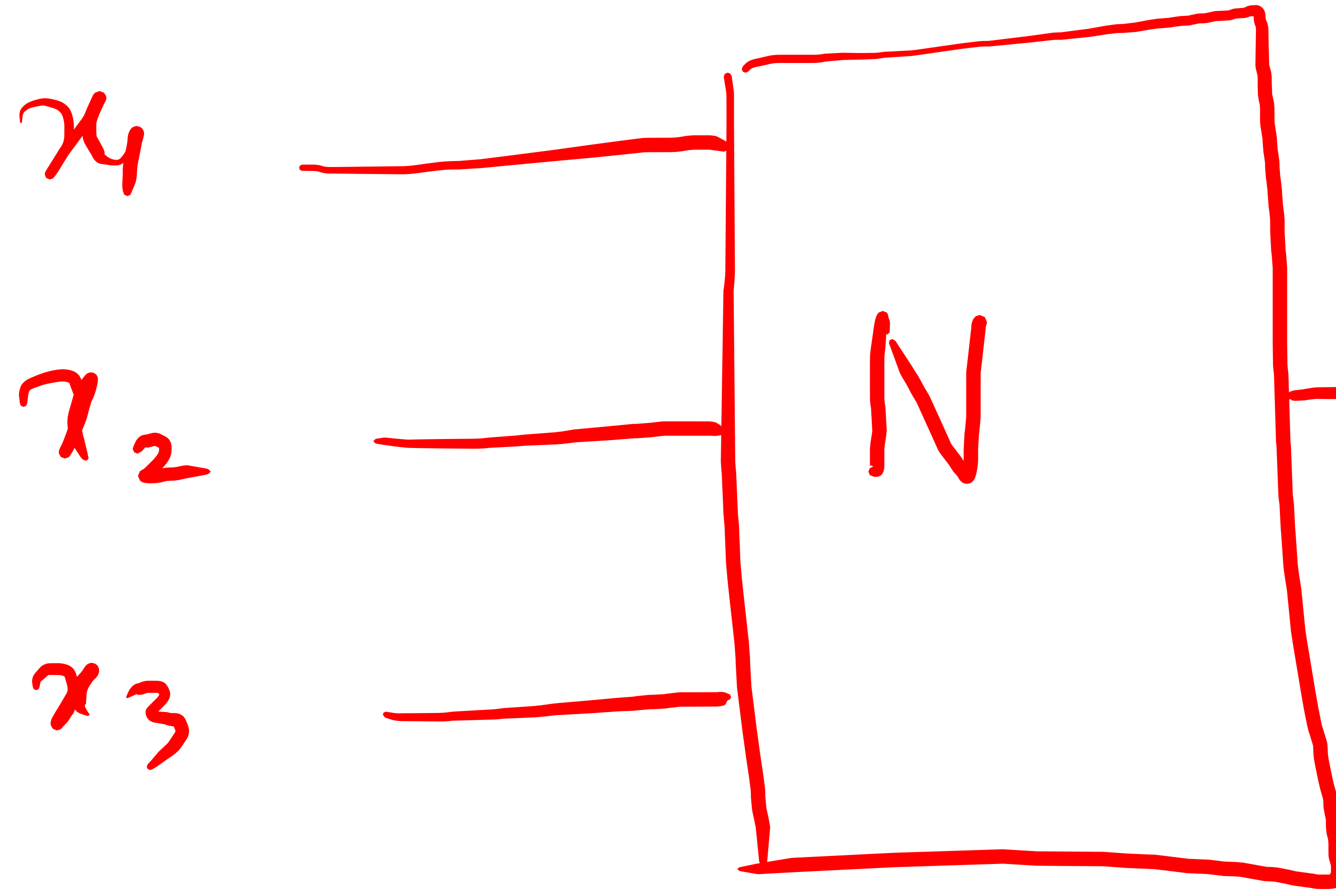
<u>universality</u>

Two or more qubits.

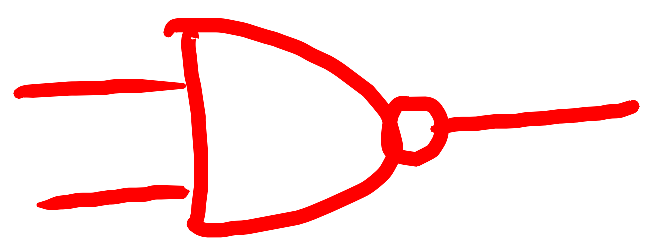
classical

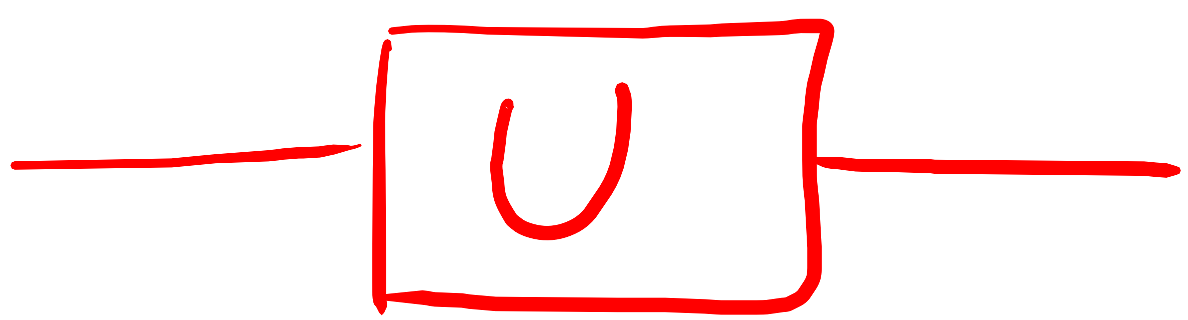
$f(x_1, x_2, x_3)$

$= x_1 \oplus x_2 \oplus x_3$



$$f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$$





$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

What gates to use?

How many?

What sequence?

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

$$|\psi\rangle = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} \in \mathbb{C}^4$$

$$a_{00}, a_{01}, a_{10}, a_{11} \in \mathbb{C}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|0\rangle} + b\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle}$$

$$\begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} = a_{00}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_{01}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_{10}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_{11}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \otimes |0\rangle = |00\rangle$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle \otimes |0\rangle = |10\rangle$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle \otimes |1\rangle = |01\rangle$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \otimes |1\rangle = |11\rangle$$

n qubit $\qquad \underline{2^n \text{ numbers}}$

$|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle = \underline{\underline{|00\cdots0\rangle}}$

$|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots \otimes |1\rangle = \underline{\underline{|00\cdots1\rangle}}$

$|00\rangle \longrightarrow \text{vector} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$|11\rangle \longrightarrow \text{vector} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$|000\ldots0\rangle$ ✓

$|000\ldots1\rangle$ ✓

$\vdots$

$|10\ldots0\rangle$ ✓

$\vdots$

$|111\ldots1\rangle$ ✓

Basis elements for $\mathbb{C}^{2^n}$

$n=1$ $\quad \mathbb{C}^2 \quad |0\rangle, |1\rangle$

$n=2$ $\quad \mathbb{C}^4 \quad |00\rangle, |01\rangle, |10\rangle, |11\rangle$

$n=3$ $\quad \mathbb{C}^8 \quad |000\rangle, |001\rangle$

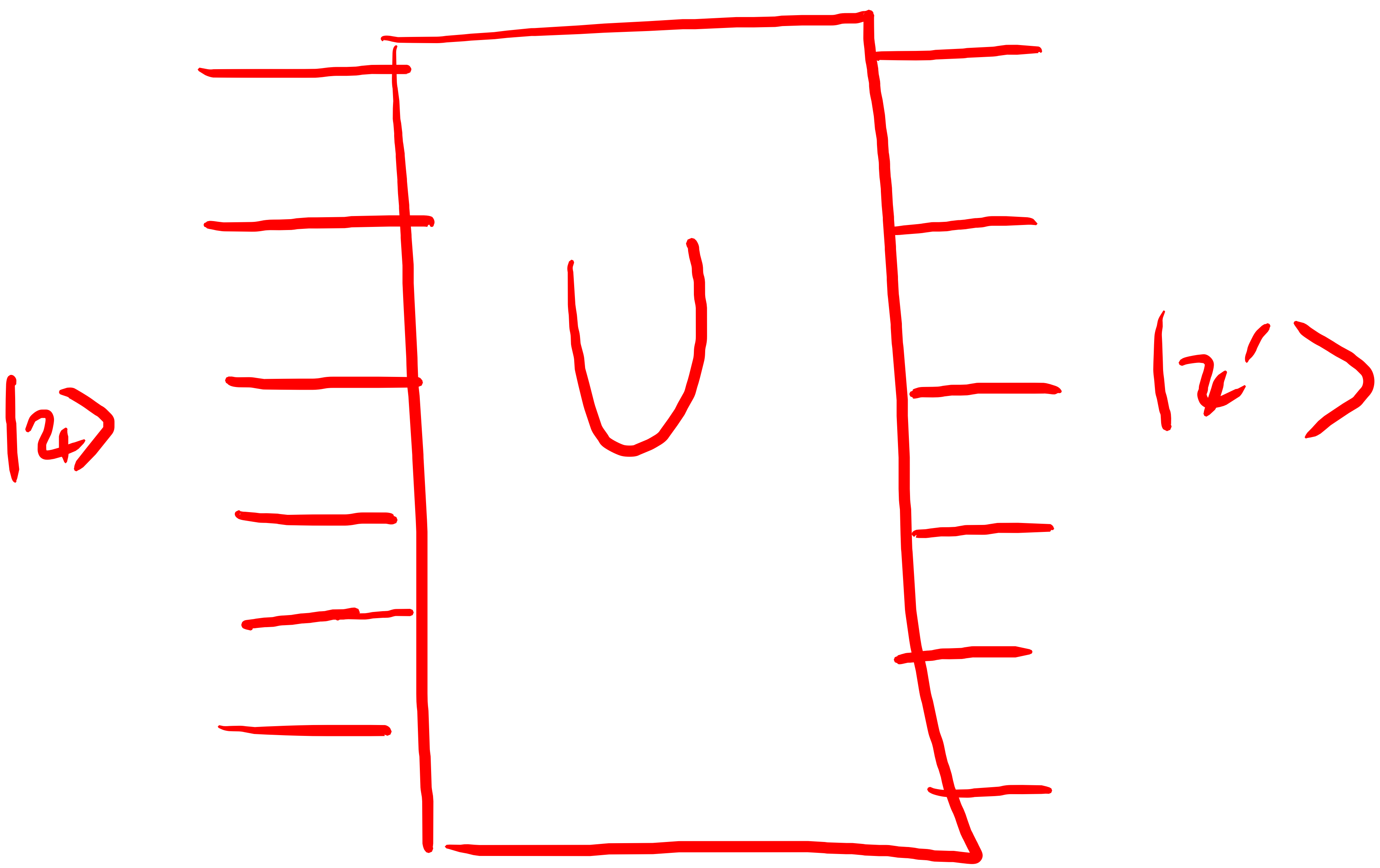
$\vdots$

$|111\rangle$

$|\psi\rangle \in \mathbb{C}^{2^n}$

How many qubit state?

n qubit system



Single qubit gates

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad |0\rangle \longrightarrow |1\rangle \quad |1\rangle \longrightarrow |0\rangle$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad |0\rangle \longrightarrow |0\rangle \quad |1\rangle \longrightarrow -|1\rangle$

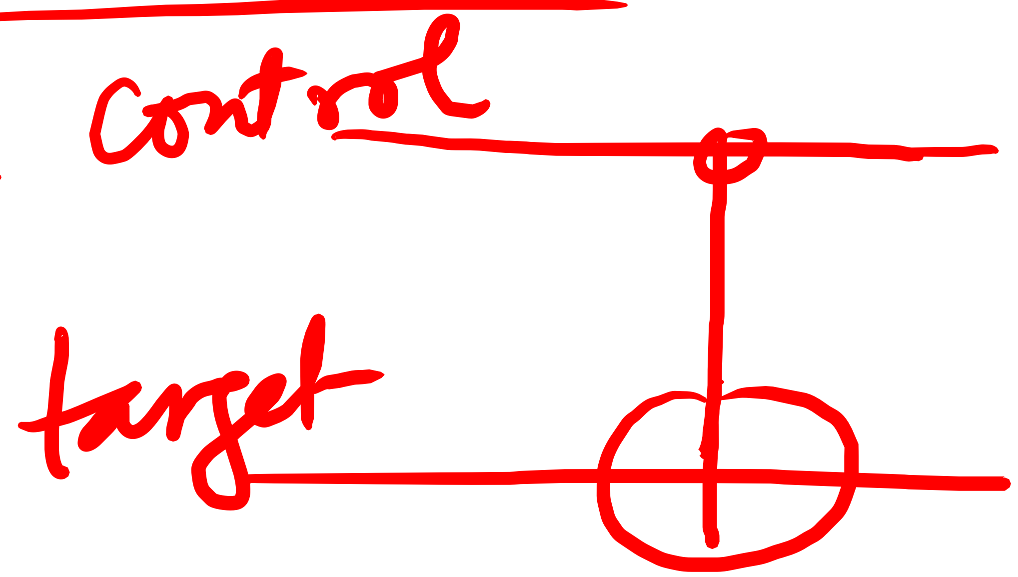
$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad |0\rangle \longrightarrow |+\rangle \quad |1\rangle \longrightarrow |-\rangle$

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $\quad |0\rangle \longrightarrow |0\rangle \quad |1\rangle \longrightarrow i|1\rangle$

$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$ $\quad |0\rangle \longrightarrow |0\rangle \quad |1\rangle \longrightarrow e^{i\pi/4}|1\rangle$

Two qubit gates

CNOT



$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |11\rangle$

$|11\rangle \longrightarrow |10\rangle$

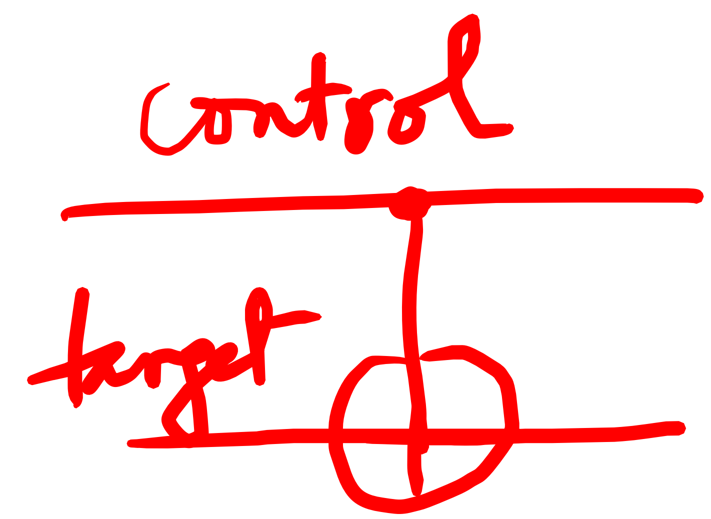
$$|\psi\rangle = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} \xrightarrow{\text{CNOT}} \begin{pmatrix} a_{00} \\ a_{01} \\ a_{11} \\ a_{10} \end{pmatrix}$$

CZ controlled Z gate

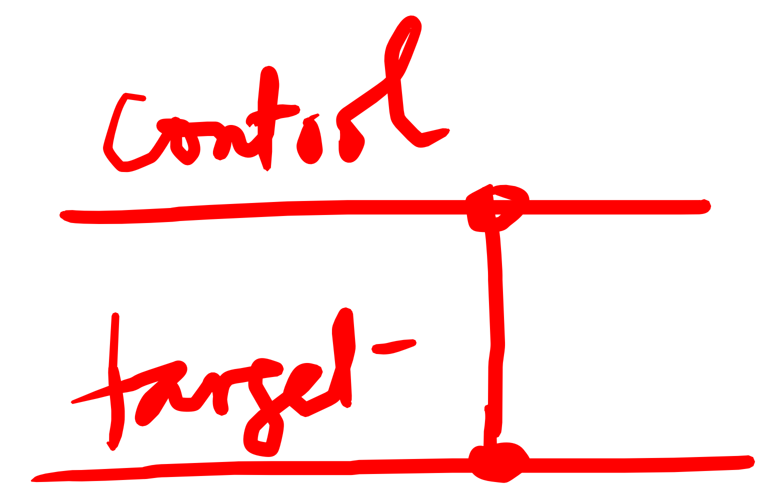
$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$

$$|\psi\rangle = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix}$$

CNOT

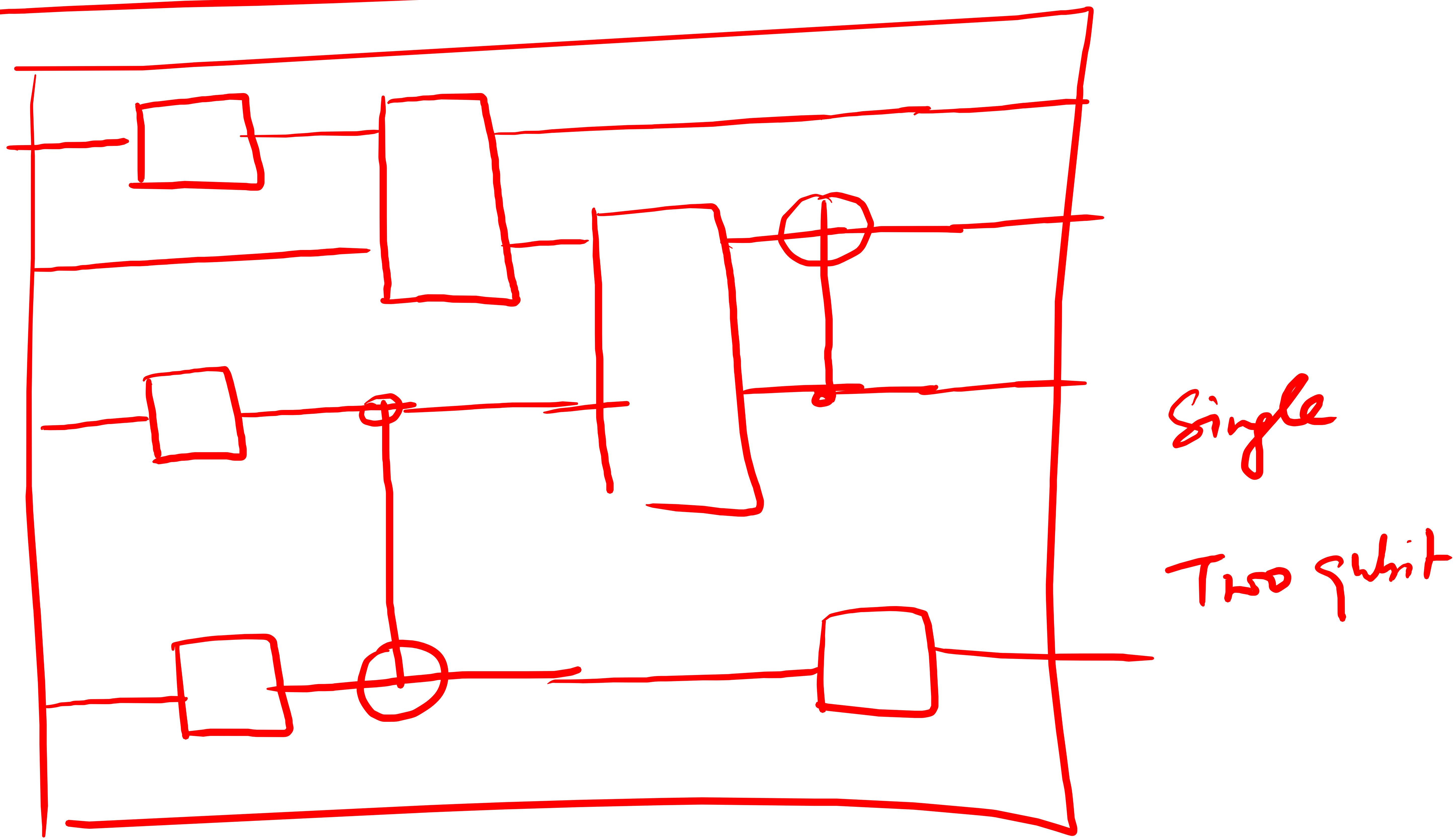


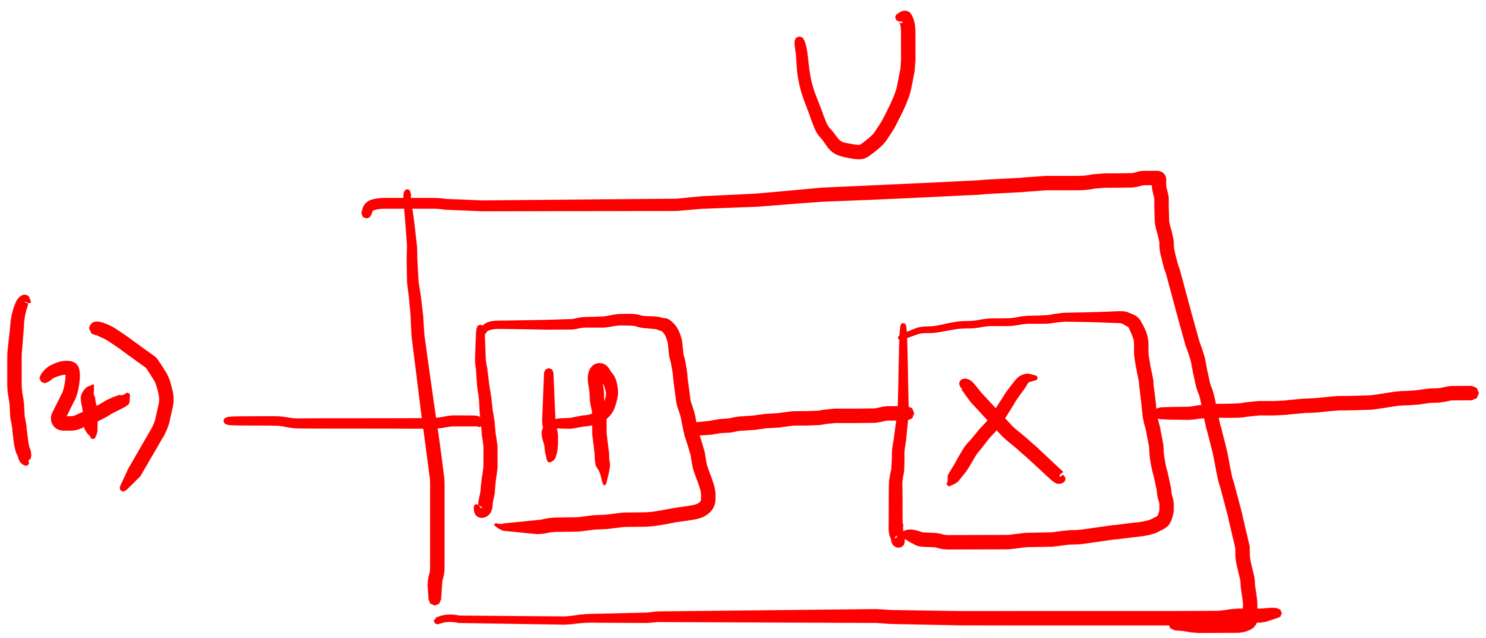
CZ



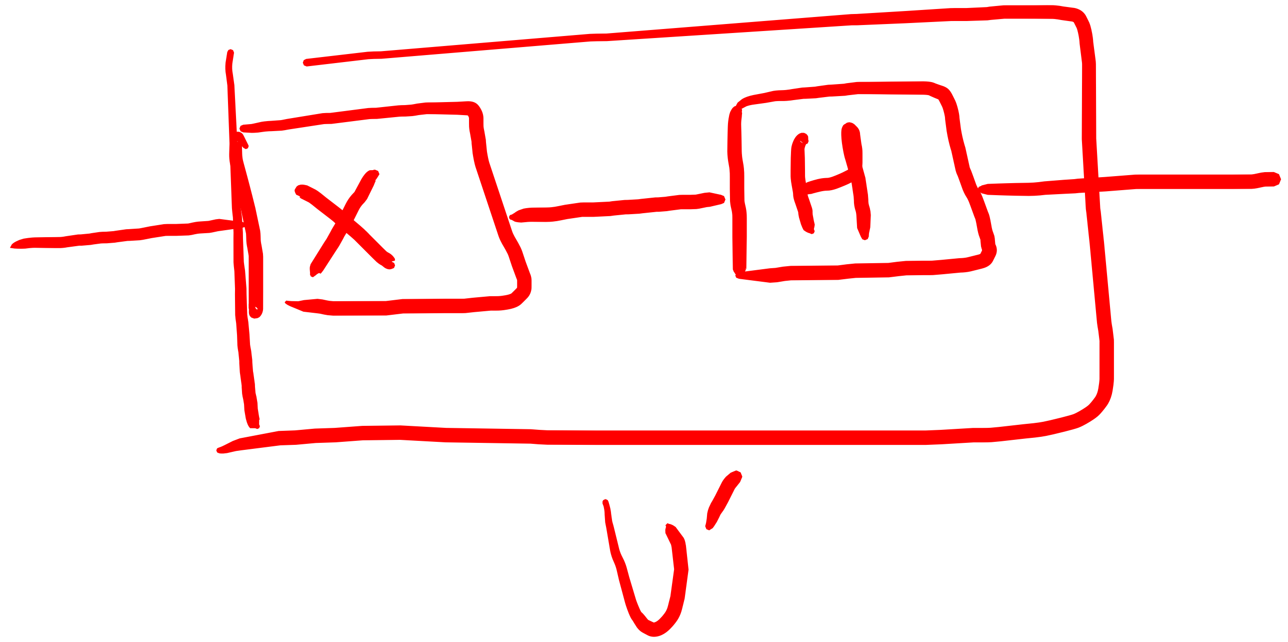
$$|\psi\rangle = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ -a_{11} \end{pmatrix}$$

n qubit unitary



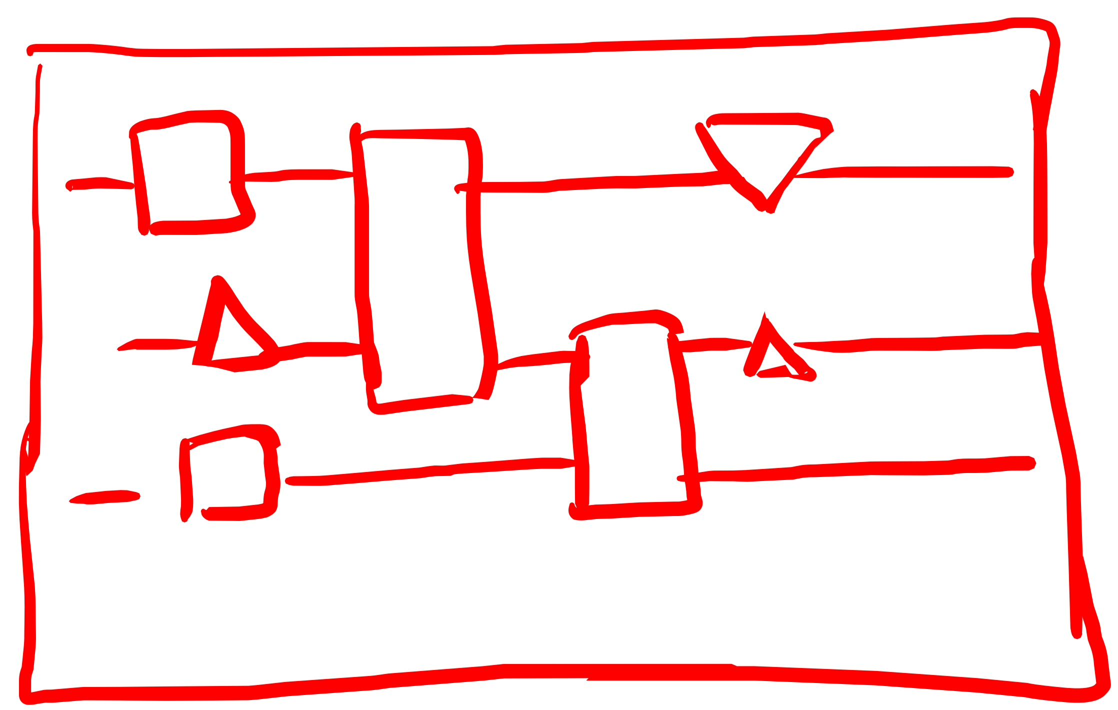


$U|\psi\rangle = X\,H|\psi\rangle$



$U'|\psi\rangle = H\,X|\psi\rangle$

U



Single

Two qubit

Sequence

Quantum entanglement

$$|00\rangle \quad |01\rangle \quad |10\rangle \quad |11\rangle$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

Bell state

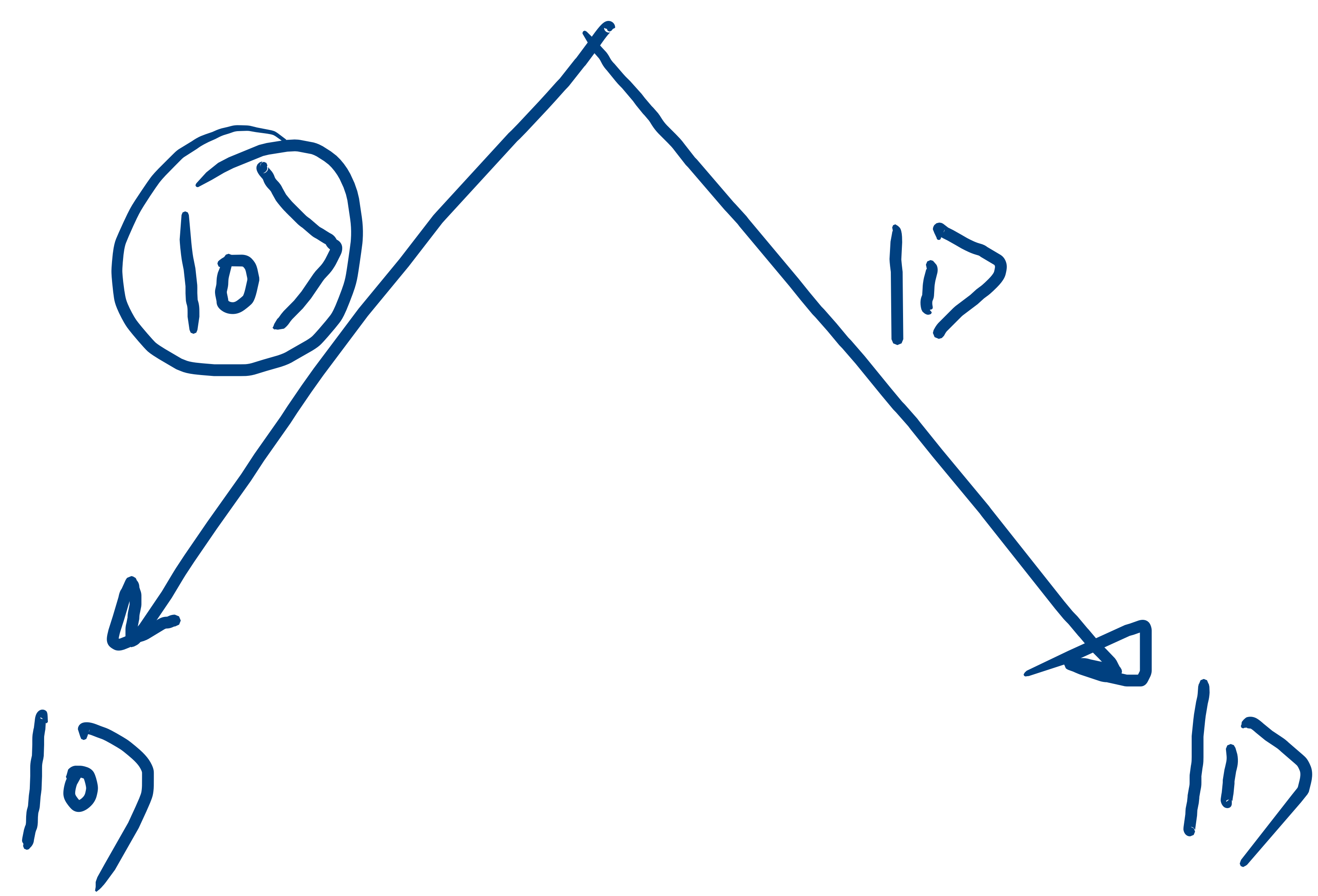
$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

I'm measuring the first qubit in the $|0\rangle, |1\rangle$ basis

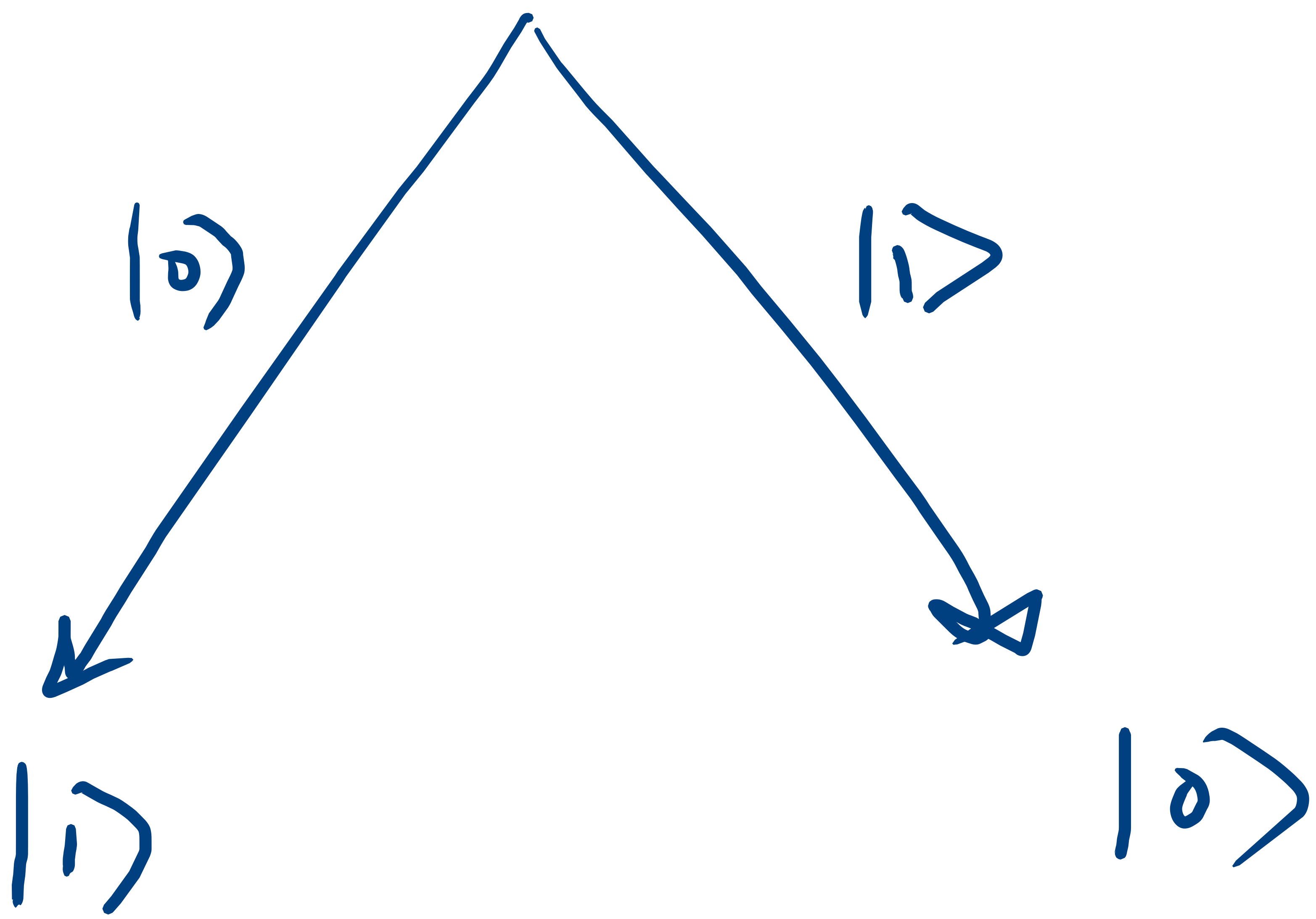
HW

Bell state

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

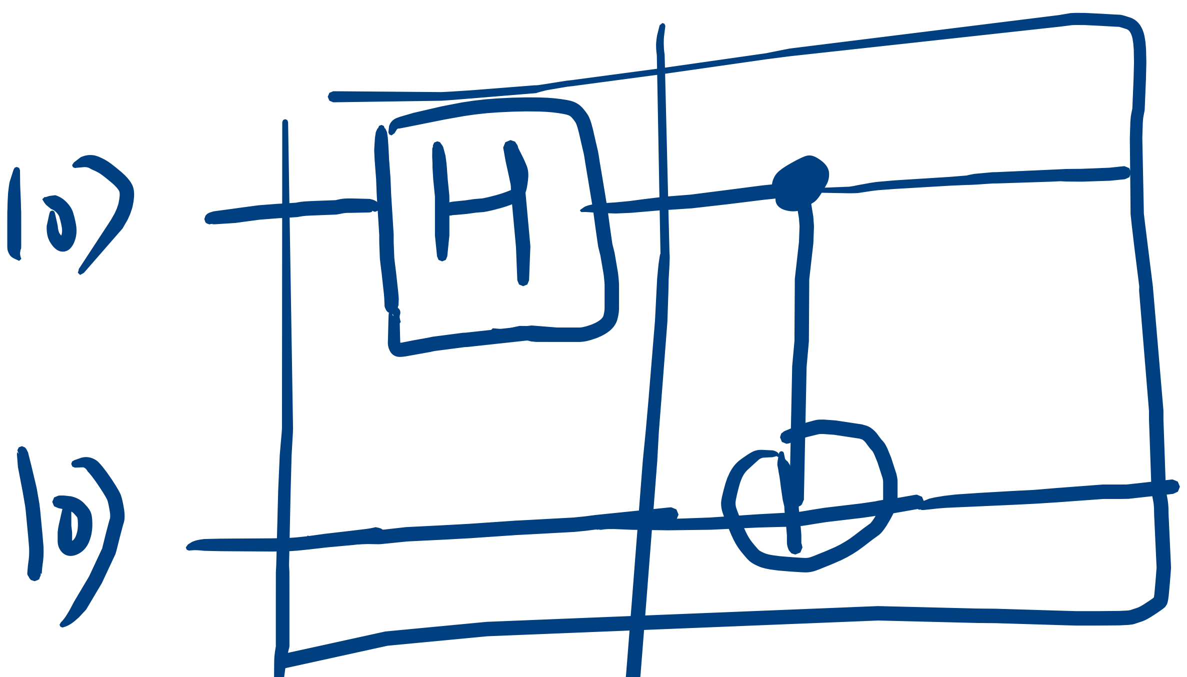


$$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$



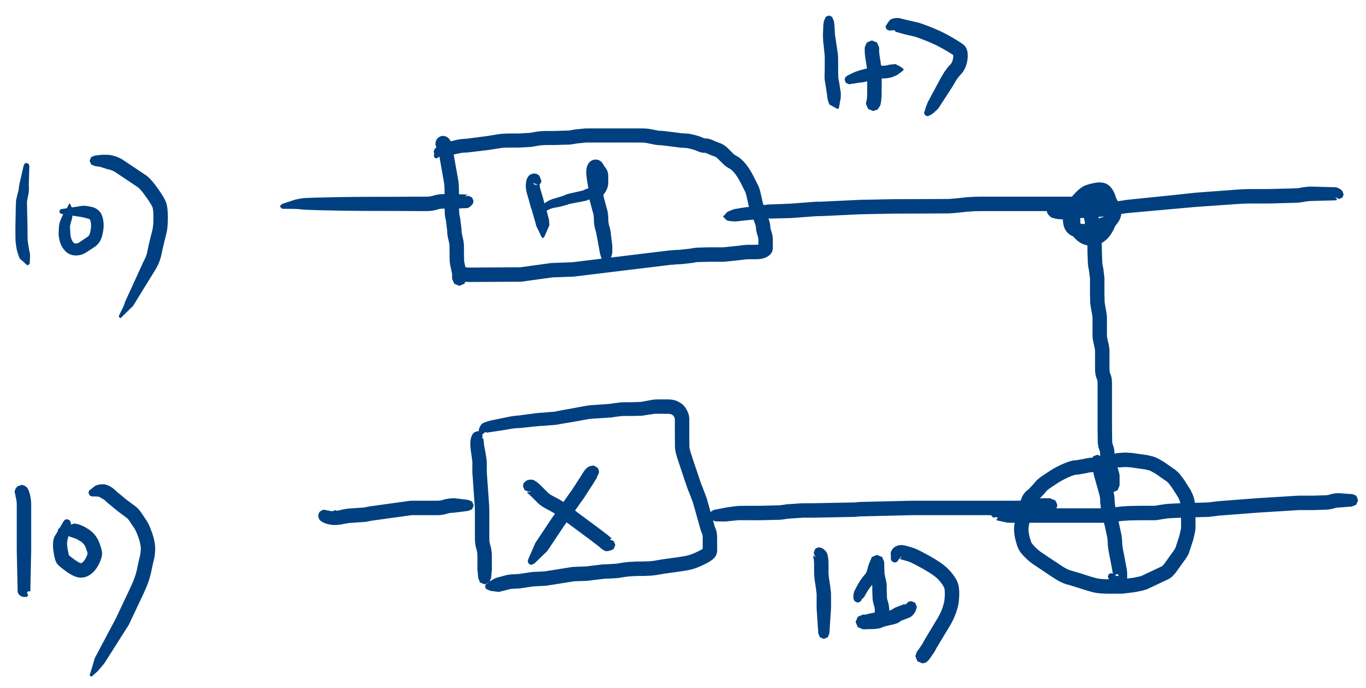
How to generate $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$?

$$|00\rangle \xrightarrow{\;U\ ?\;} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



$$|+\rangle \otimes |0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

How to generate $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$

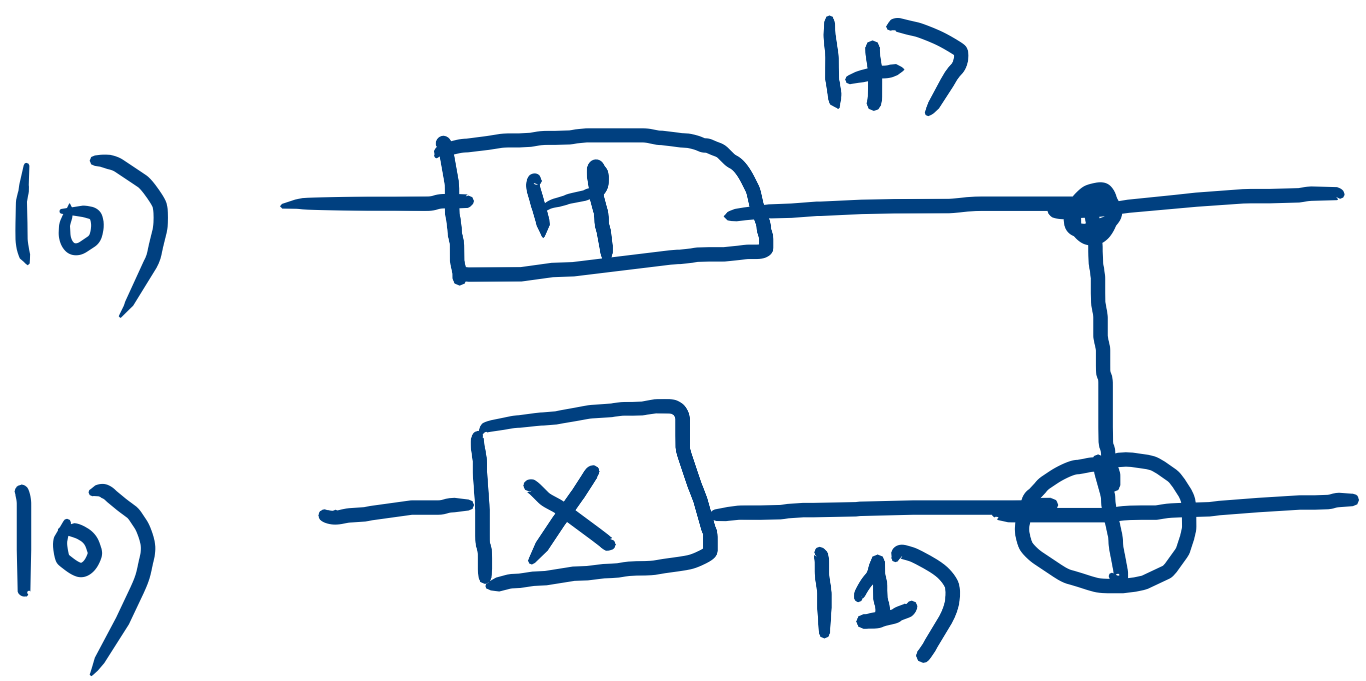


$$\mathrm{CNOT}\left(|+\rangle|1\rangle\right)$$

$$= \mathrm{CNOT}\left(\frac{|01\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

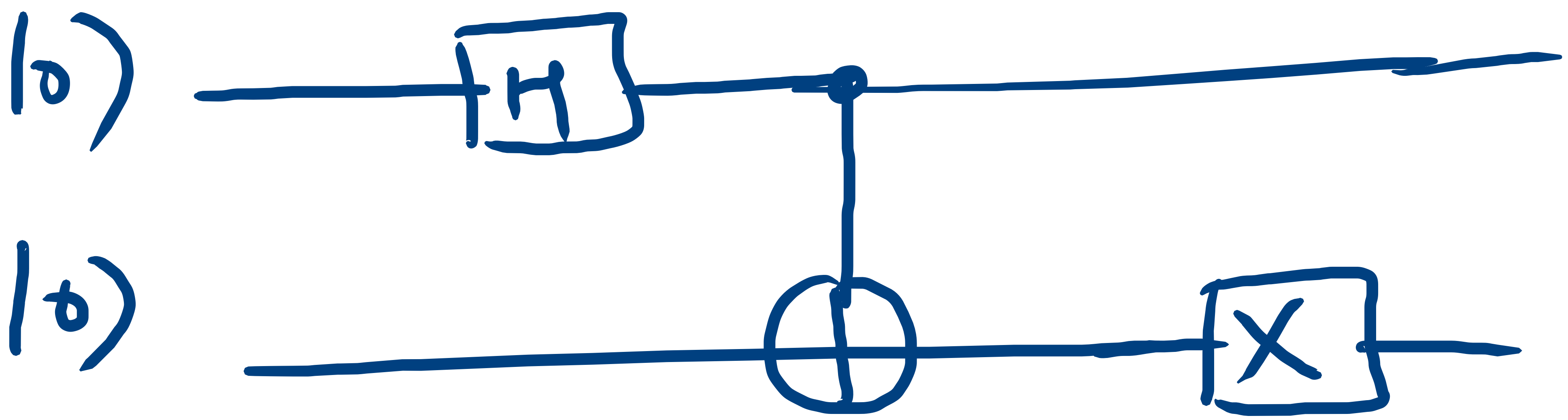
How to generate $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$

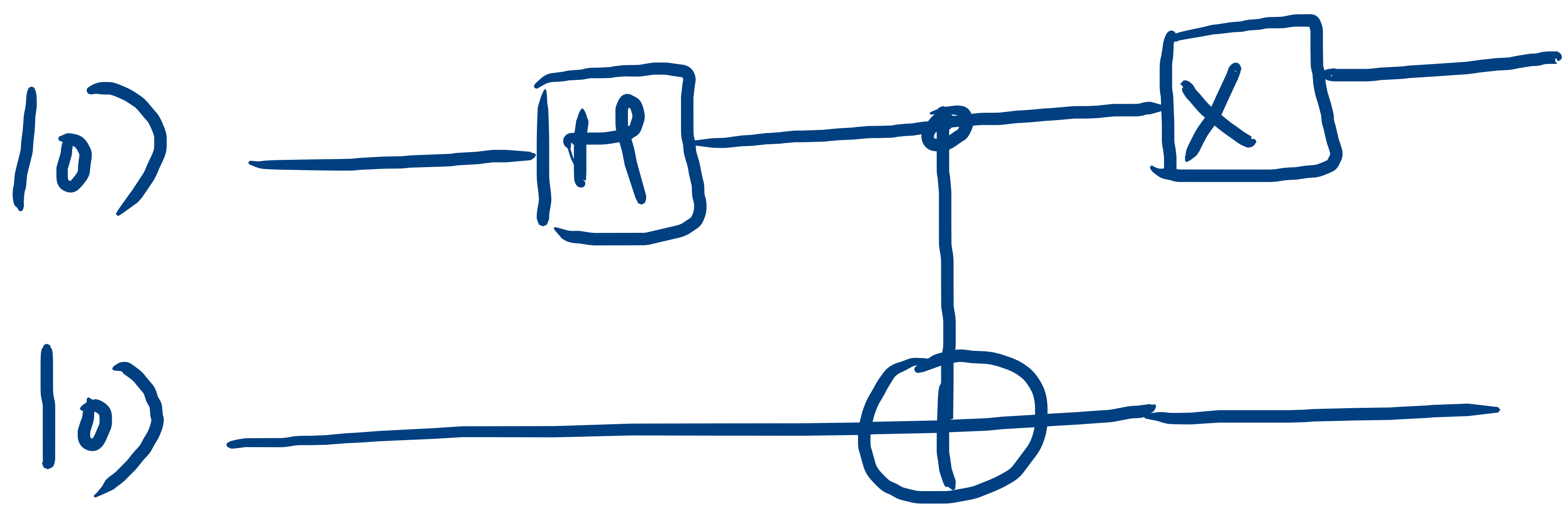


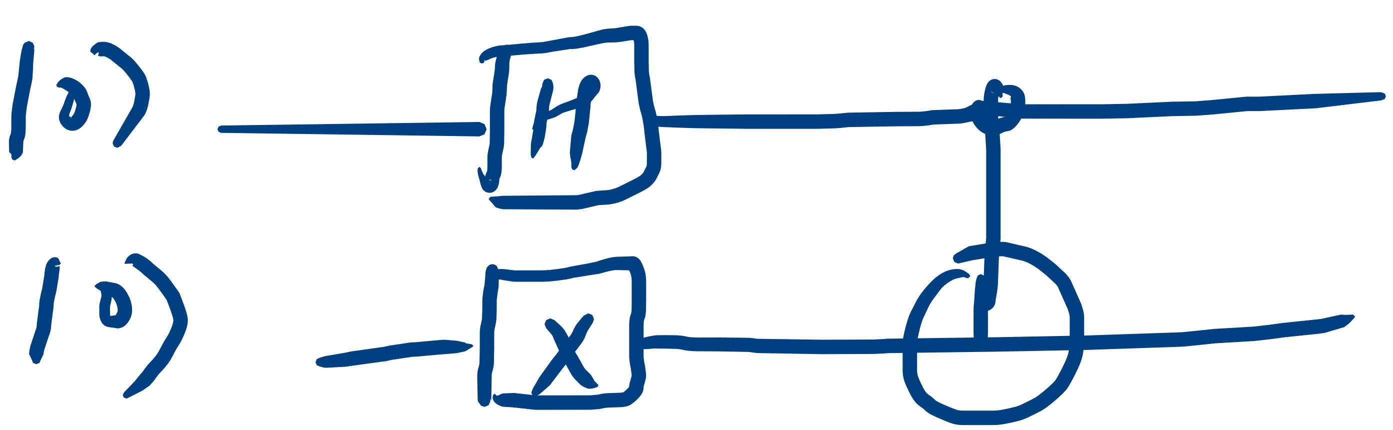
$$\text{CNOT}\left(|+\rangle|1\rangle\right)$$

$$= \text{CNOT}\left(\frac{|01\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

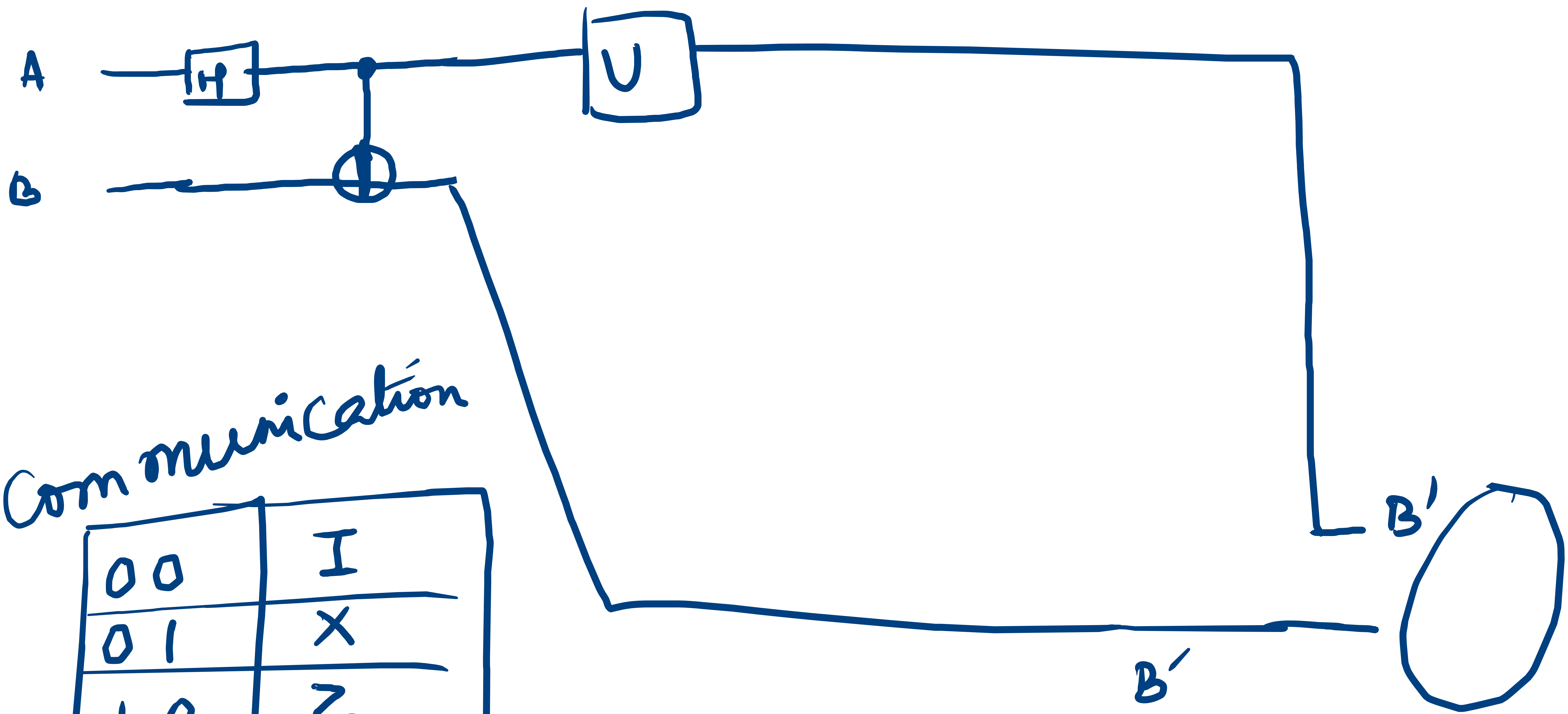

|0⟩
|0⟩
H
X


|0⟩
|0⟩
H
X


|0⟩
|0⟩
H
X

Application of a Bell state

superdense coding



Communication

00	I
01	X
10	Z
11	Y

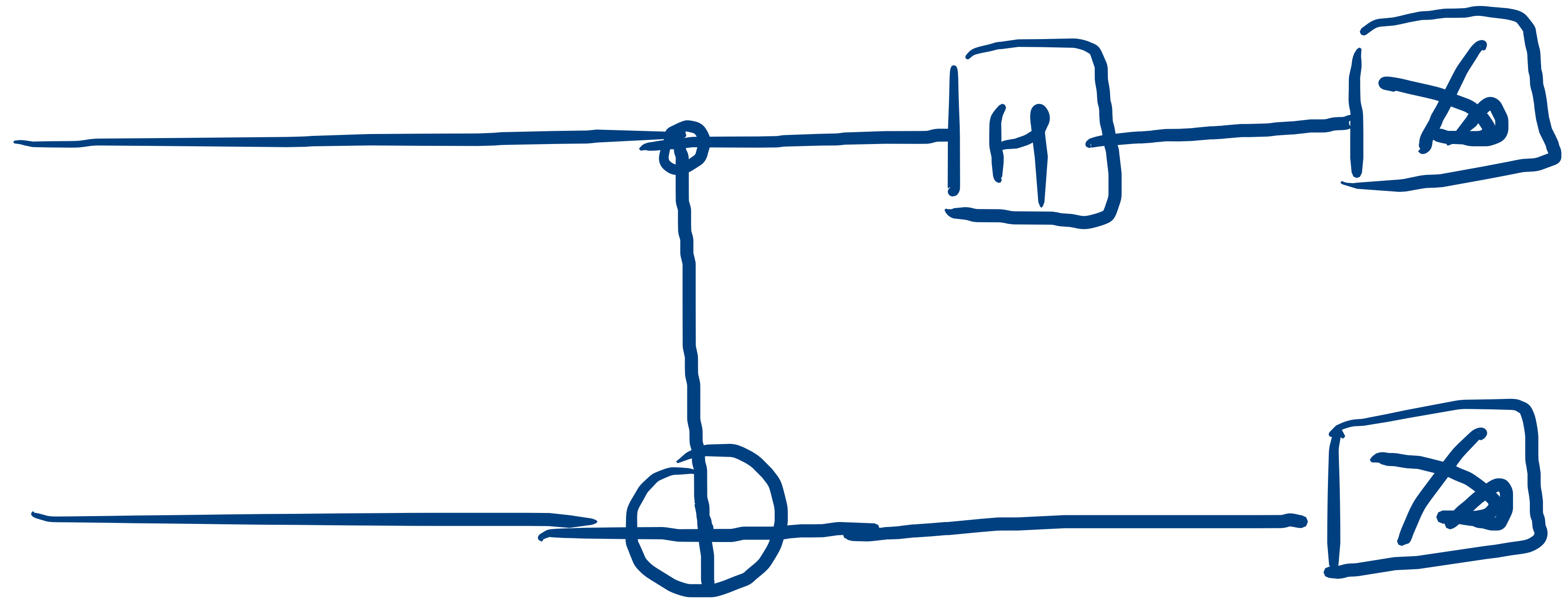
$$\frac{|00\rangle+|11\rangle}{\sqrt{2}} \xrightarrow{I\ (00)} \frac{|00\rangle+|11\rangle}{\sqrt{2}}$$

$$\frac{|00\rangle+|11\rangle}{\sqrt{2}} \xrightarrow{X\ (01)} \frac{|01\rangle+|10\rangle}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$\frac{|00\rangle+|11\rangle}{\sqrt{2}} \xrightarrow{Z\ (10)} \frac{|00\rangle-|11\rangle}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\frac{|00\rangle+|11\rangle}{\sqrt{2}} \xrightarrow{ZX\ (11)} \frac{|01\rangle-|10\rangle}{\sqrt{2}}$$

Check they are orthogonal



$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \longrightarrow |00\rangle \ (00)$$

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} \longrightarrow |10\rangle \ (10)$$

$$\frac{|01\rangle + |10\rangle}{\sqrt{2}} \longrightarrow |01\rangle \ (01)$$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} \longrightarrow |11\rangle \ (11)$$