Day 3

Recap

Qubits

Superposition (state)

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} a_x |x\rangle$$

measurement

Basis

Observable

evolution

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$|x\rangle$ state

$x \in \{0,1\}^n$ $|x\rangle$ vector

$|00\rangle$ vector ? ✓

$x = 00$

$x = 0100$ $\underline{\underline{|x\rangle = |0100\rangle}}$

$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$

$|\psi\rangle = \sum_{x \in \{0,1\}^2} a_x |x\rangle$ 2 qubit.

$|\psi\rangle = \sum_{x \in \{0,1\}^n} a_x |x\rangle$ n qubit

$a_x \in \mathbb{C}$ $\qquad x \in \{0,1\}^n = n$ bit tuple

n bit string

$\{0,1\}$ 1 or 0

$\{0,1\}^2 = \{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$

$\{0,1\}^n = \{0,1\} \times \{0,1\} \times \{0,1\} \times \dots \times \{0,1\}$

$= \{000\dots0,\ 000\dots1,\ 00\dots10,\ \dots,\ 111\dots1\}$

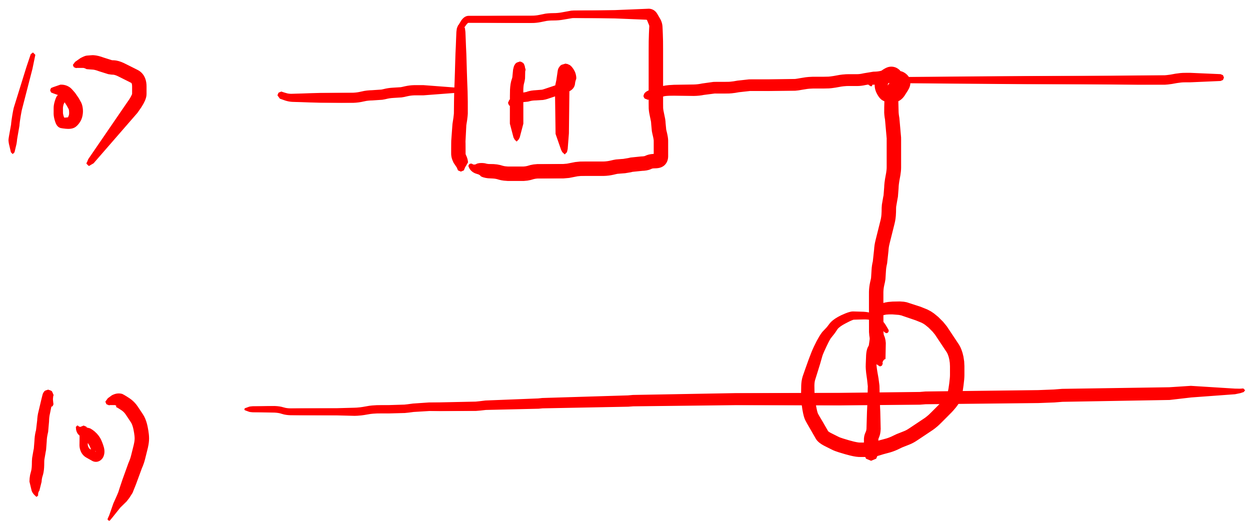
entanglement

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\neq |\psi_1\rangle \otimes |\psi_2\rangle$$

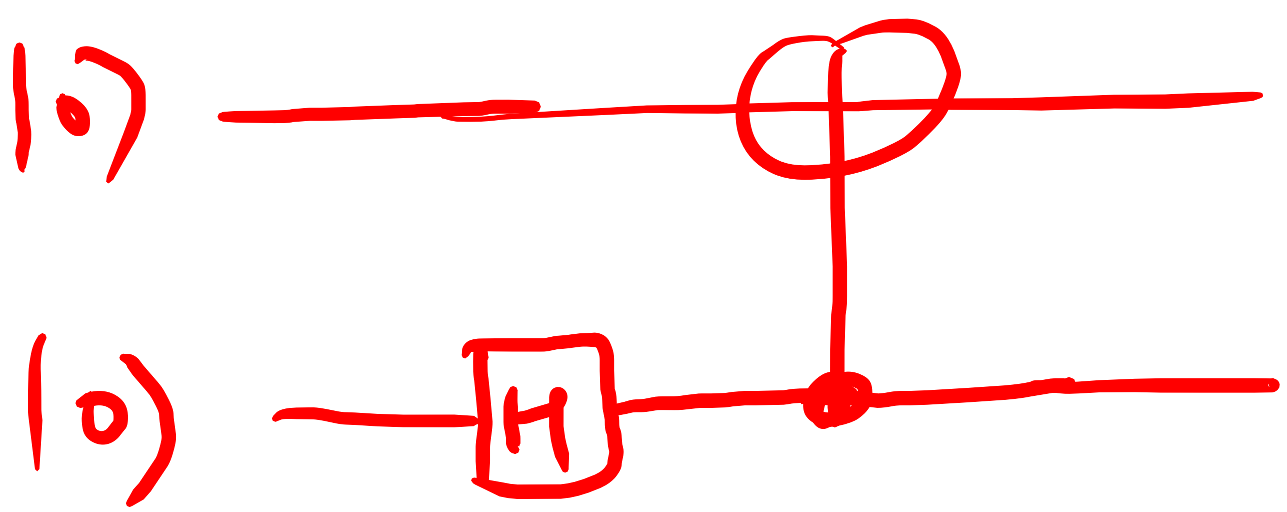
$$\begin{tikzcd}
 & {|0\rangle} \\
{|\psi\rangle} \arrow[ur, "|0\rangle"] \arrow[dr, "|1\rangle"'] & \\
 & {|1\rangle}
\end{tikzcd}$$

Superdense coding

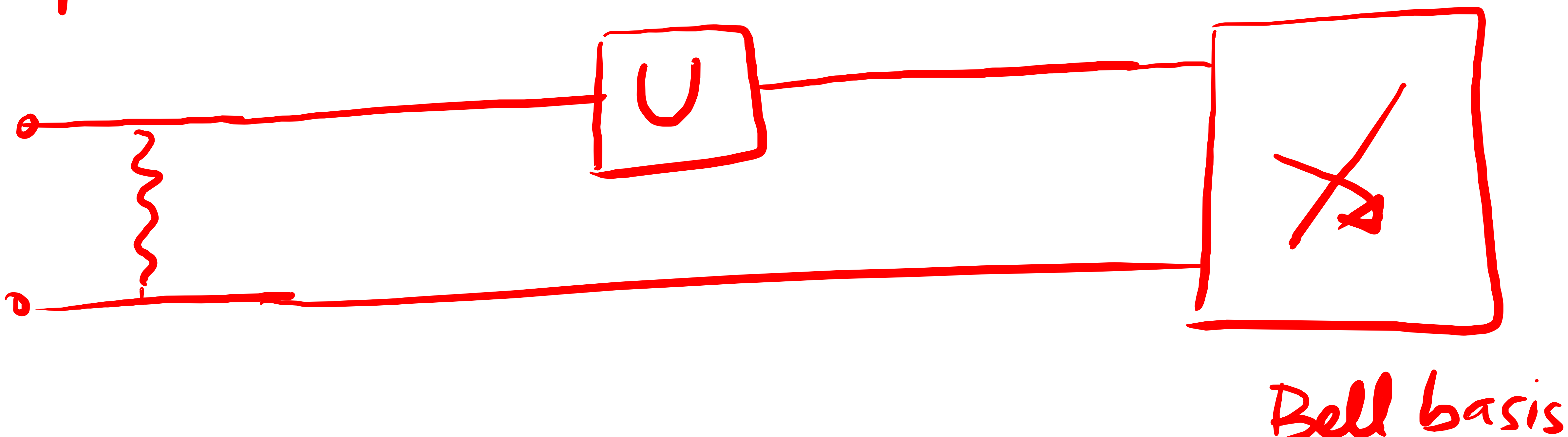


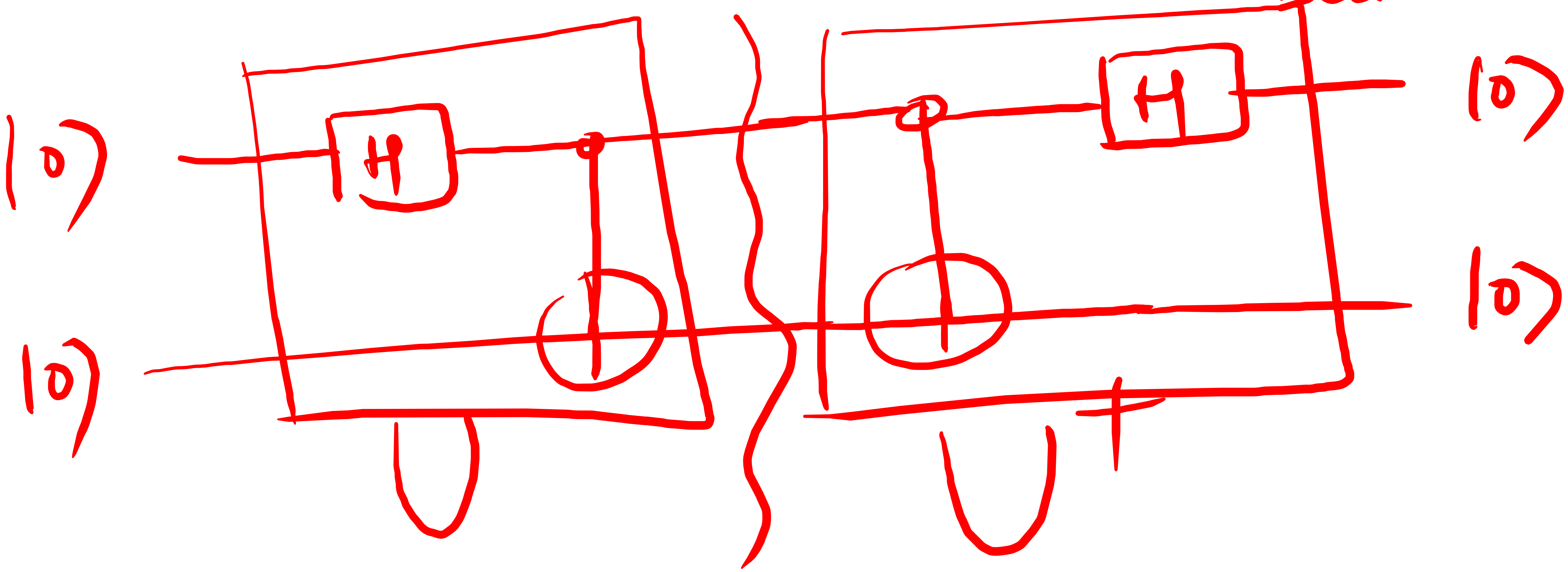
$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Bell state

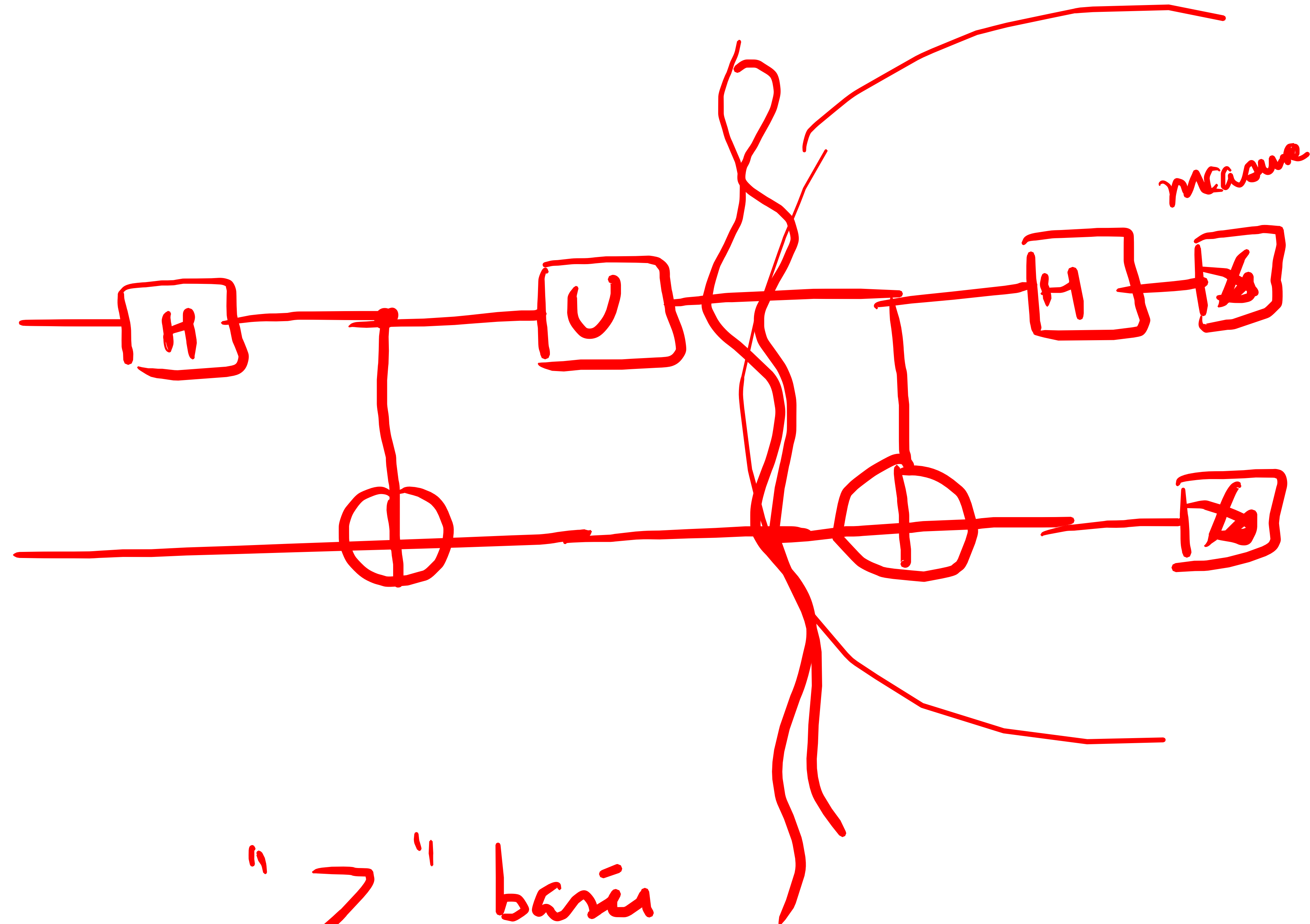


Superdense





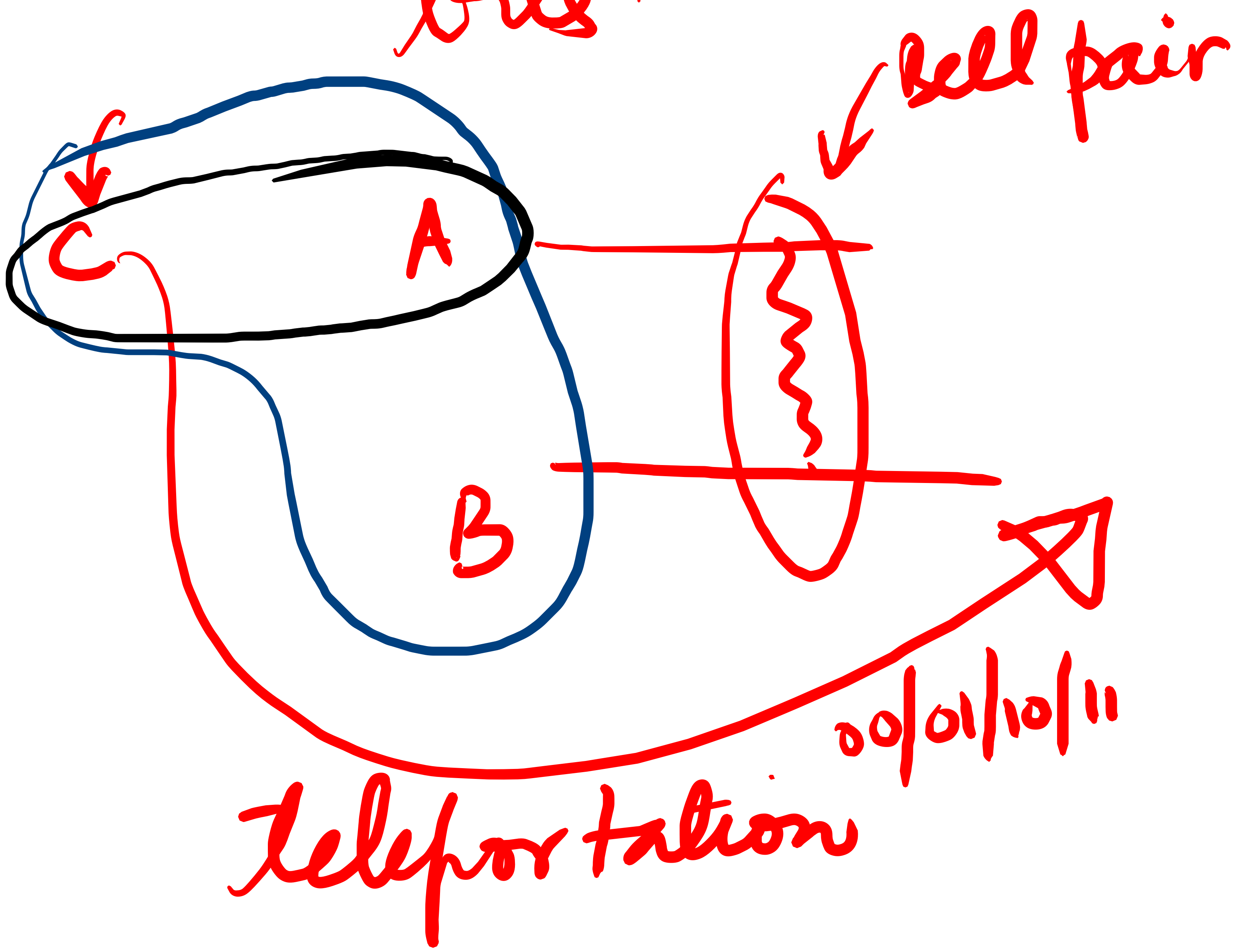
0	0	I
0	1	X
1	0	Z
1	1	ZX

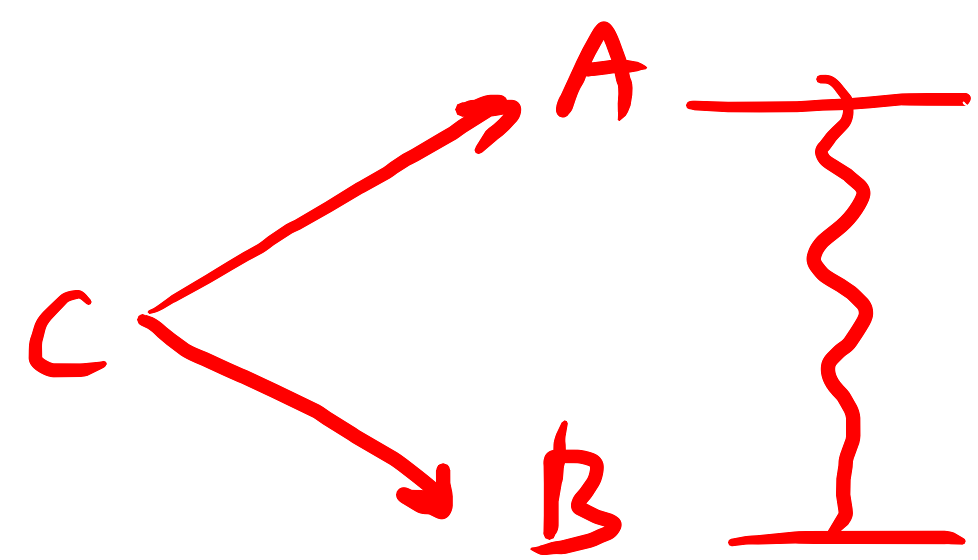


"Z" basis

Teleportation

Transmit/Teleport one qubit using one Bell pair & 2 classical bits.





$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

$\boxed{00}$

$\boxed{01}$

$\boxed{10}$

$\boxed{11}$

Bell states

Q. Are they orthogonal?

Yes!

Distinguish orth state

$$|\chi\rangle_C \, |\phi^+\rangle_{AB} = \left(a|0\rangle_C + b|1\rangle_C\right) \otimes \left(\frac{1}{\sqrt{2}}|00\rangle_{AB} + \frac{1}{\sqrt{2}}|11\rangle_{AB}\right)$$

Combine C & A

$$= \frac{1}{\sqrt{2}} a\,|00\rangle_{CA}|0\rangle_B + \frac{1}{\sqrt{2}} a\,|01\rangle_{CA}|1\rangle_B$$

$$+ \frac{1}{\sqrt{2}} b\,|10\rangle_{CA}|0\rangle_B + \frac{1}{\sqrt{2}} b\,|11\rangle_{CA}|1\rangle_B$$

$$= \frac{1}{\sqrt{2}} a \left(\frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}}\right)|0\rangle_B + \frac{1}{\sqrt{2}} a \left(\frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}}\right)|1\rangle_B$$

$$+ \frac{1}{\sqrt{2}} b \left(\frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}}\right)|0\rangle_B + \frac{1}{\sqrt{2}} b \left(\frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}}\right)|1\rangle_B$$

Joint state $= \frac{1}{2}|\phi^+\rangle_{CA}\left(a|0\rangle_B + b|1\rangle_B\right)$

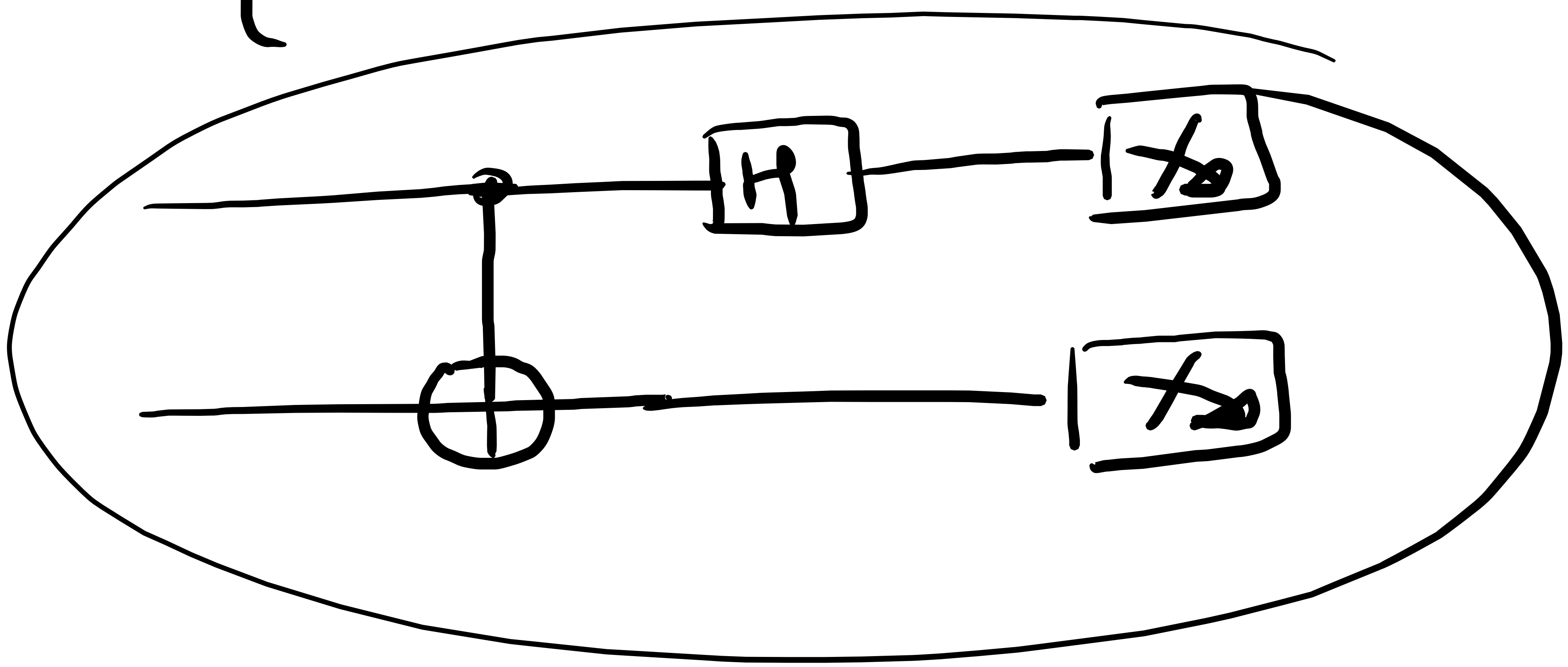
$+\frac{1}{2}|\phi^-\rangle_{CA}\left(a|0\rangle_B - b|1\rangle_B\right)$

$\frac{1}{4}$

$+\frac{1}{2}|\psi^+\rangle_{CA}\left(a|1\rangle_B + b|0\rangle_B\right)$

$+\frac{1}{2}|\psi^-\rangle_{CA}\left(a|1\rangle_B - b|0\rangle_B\right)$

Bell basis $\{ |\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle \}$



$\{|0\rangle, |1\rangle\}$

$$|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-| = I_4$$

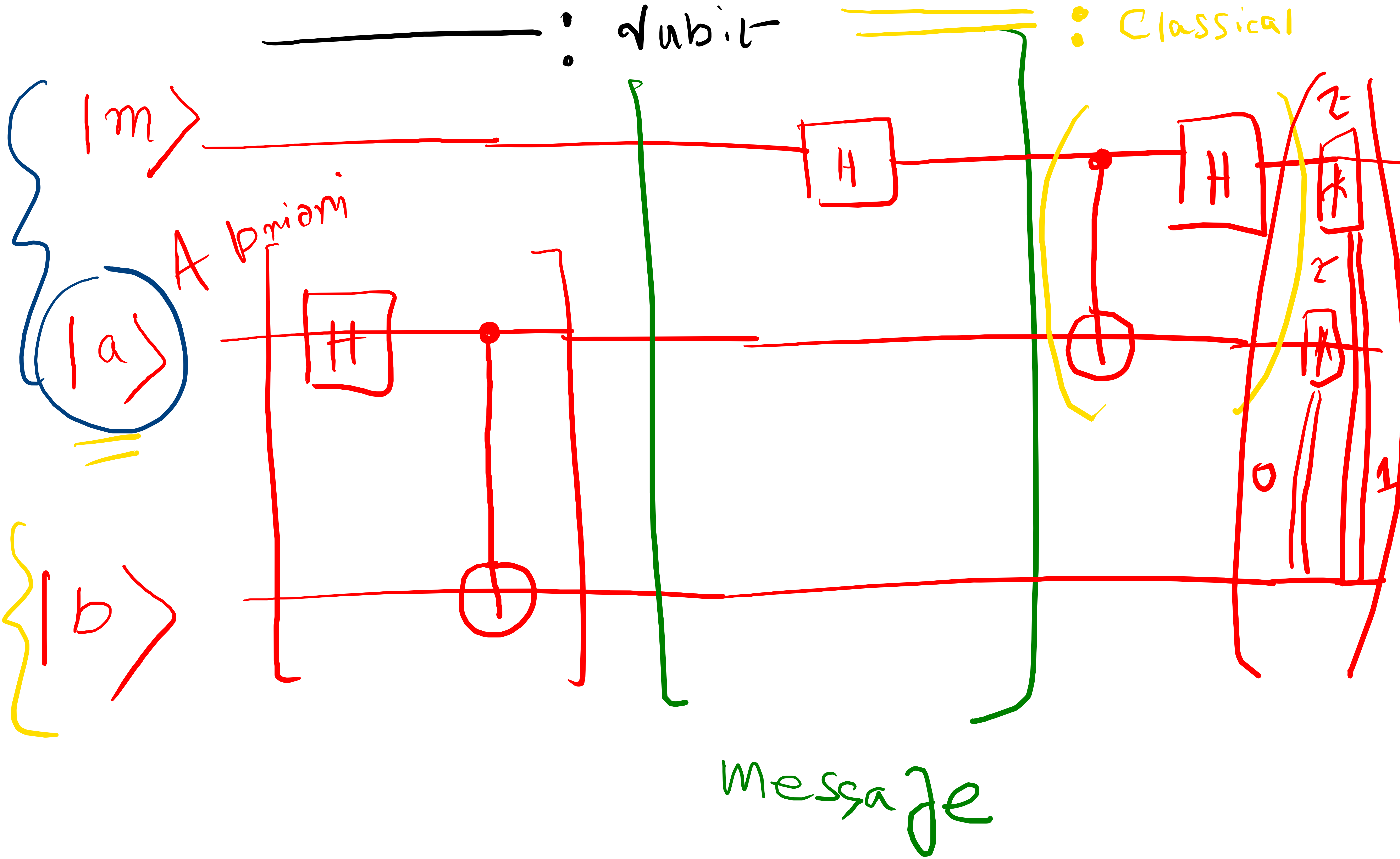
CA	B	Classical	gate to be applied
$\vert\phi^+\rangle$	$a\vert 0\rangle + b\vert 1\rangle$	00 ✓	I
$\vert\phi^-\rangle$	$a\vert 0\rangle - b\vert 1\rangle$	10	Z ✓
$\vert\psi^+\rangle$	$a\vert 1\rangle + b\vert 0\rangle$	01	X ✓
$\vert\psi^-\rangle$	$a\vert 1\rangle - b\vert 0\rangle$	11	ZX ✓ ✓

XZ ✓

$$a|0\rangle + b|1\rangle$$

$$\ominus \left(a|0\rangle + b|1\rangle\right)$$

$$|a|^2 \qquad |b|^2$$


: qubit
: classical
|m⟩
|a⟩
|b⟩
A priori
H
H
H
message
0
1

density_matrix

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$

$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow |+\rangle\langle +| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= |0\rangle \times |0\rangle$

$|0\rangle \otimes |0\rangle$

clone

$= |1\rangle \otimes |0\rangle$

copy

$|1\rangle \otimes |1\rangle$

$00 \to 00$

$01 \to 01$

$\alpha|0\rangle + \beta|1\rangle$

U

copy?

General?

$|0\rangle$

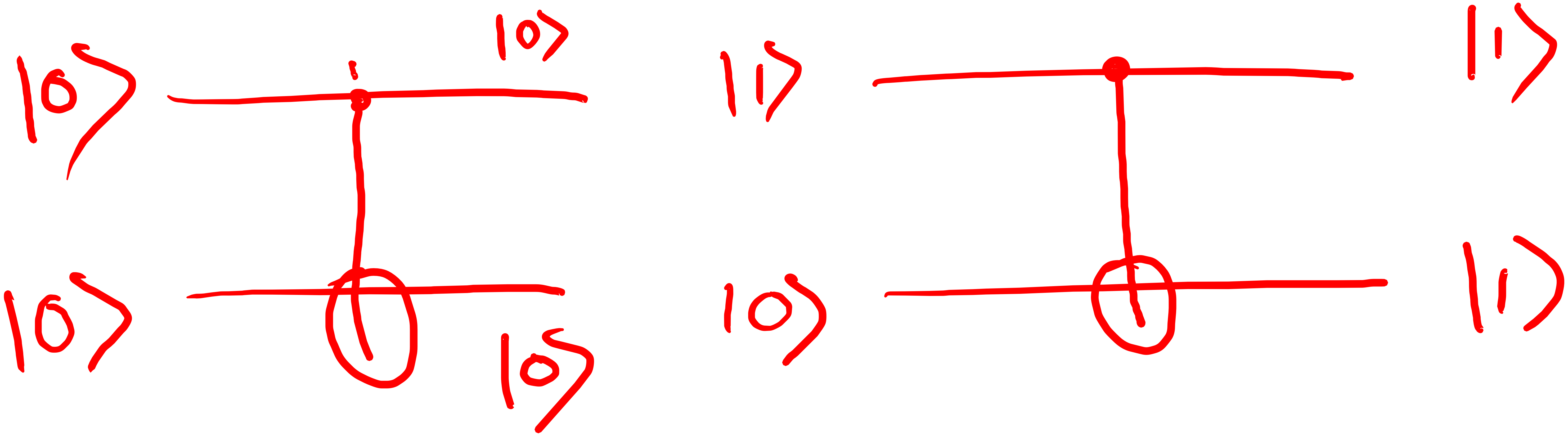
$$\Big(|\alpha\rangle + |\beta\rangle\Big) \otimes |\phi\rangle \longrightarrow \left.\begin{array}{l} |\alpha\rangle|\phi\rangle + |\beta\rangle|\phi\rangle \\ = |\alpha\rangle|\alpha\rangle + |\beta\rangle|\beta\rangle \end{array}\right\}$$

$\longrightarrow$ Blank state

$$\longrightarrow \Big(|\alpha\rangle + |\beta\rangle\Big) \otimes \Big(|\alpha\rangle + |\beta\rangle\Big)$$

$$= \left.\begin{array}{l} |\alpha\rangle|\alpha\rangle + |\alpha\rangle|\beta\rangle + |\beta\rangle|\alpha\rangle \\ \qquad + |\beta\rangle|\beta\rangle \end{array}\right\}$$

$$\left.\begin{array}{l} |\alpha\rangle \otimes |\phi\rangle \xrightarrow{U} |\alpha\rangle|\alpha\rangle \\ |\beta\rangle \otimes |\phi\rangle \xrightarrow{U} |\beta\rangle|\beta\rangle \end{array}\right\}$$ Definition

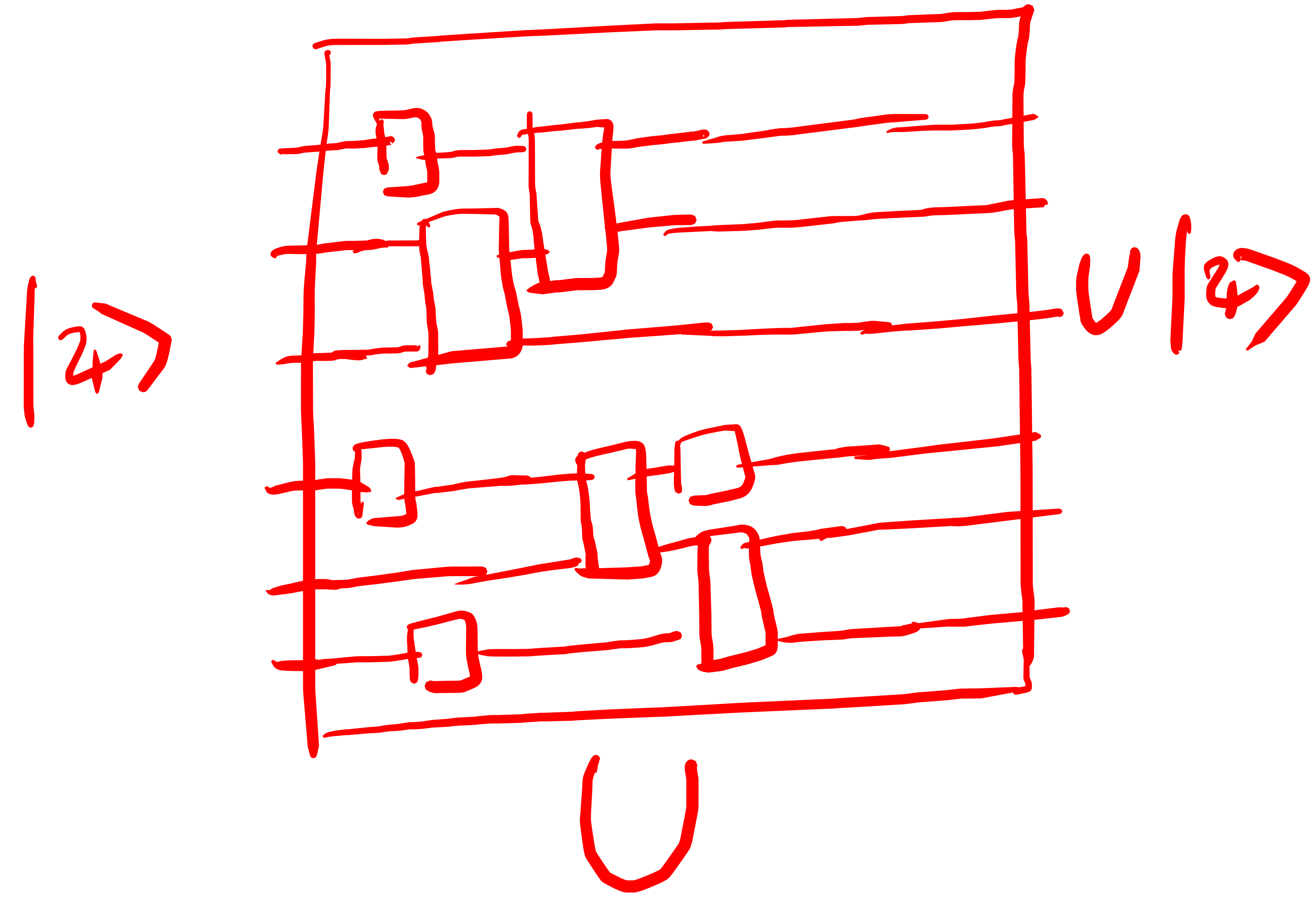


$|0\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|1\rangle$
$|0\rangle$
$|1\rangle$
$|1\rangle$

Summary

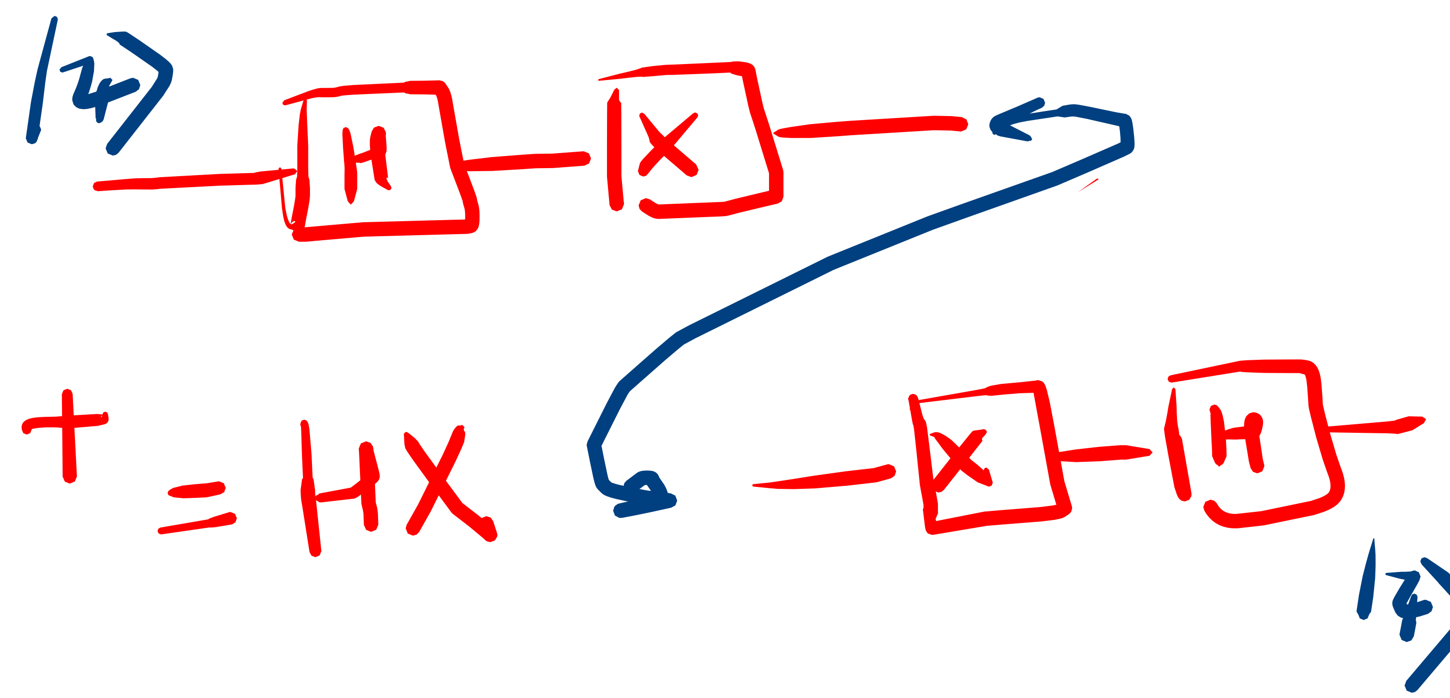
$\lvert \chi_c \rangle$ $a\lvert 0\rangle + b\lvert 1\rangle$ $\left.\begin{matrix} A \\ B \end{matrix}\right\} \lvert \phi^+ \rangle$	Bell basis on C&A	Send measurement result on a classical channel	Apply a gate dependent on the classical message

Quantum circuits and reversibility



for eg $U = XH$

$U^\dagger = HX$

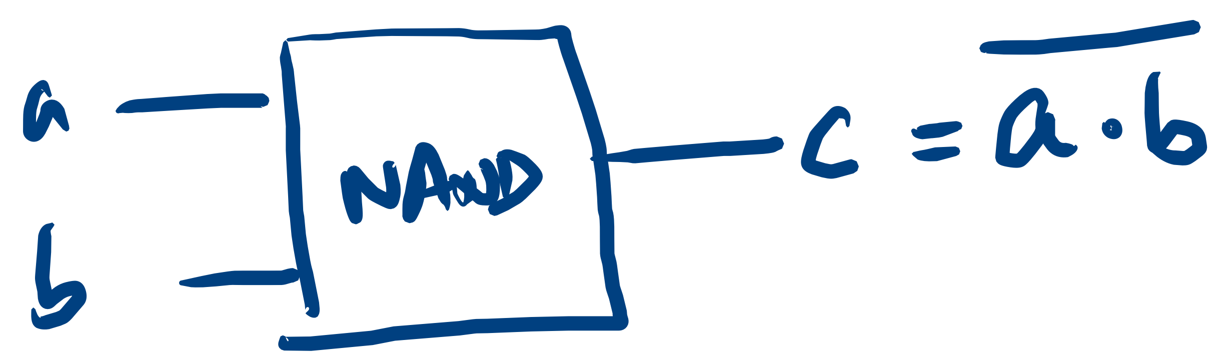


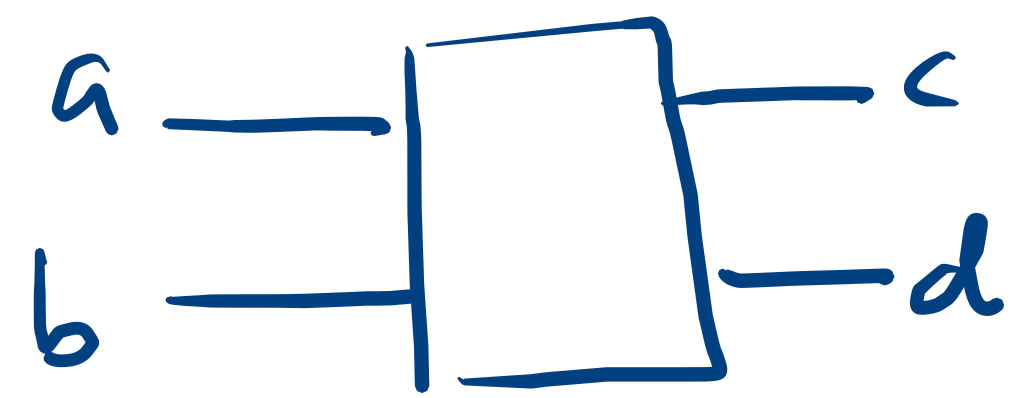
$U \qquad U^\dagger$

$$|\psi\rangle \xrightarrow{\;U\;} U|\psi\rangle \xrightarrow{\;U^\dagger\;} U^\dagger U|\psi\rangle = |\psi\rangle$$

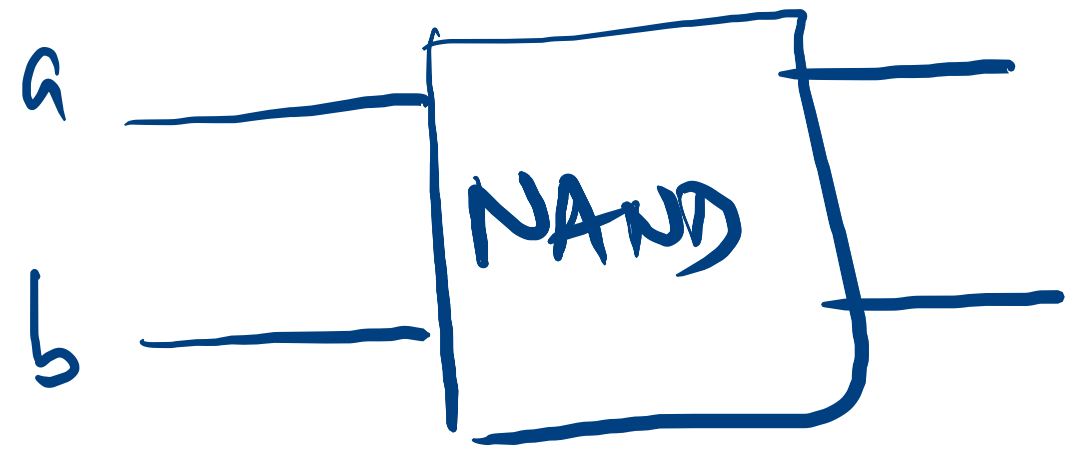
universal — NAND ⟶ <u>Is it reversible</u>

$\{H, T\}$;





Can you make NAND reversible?

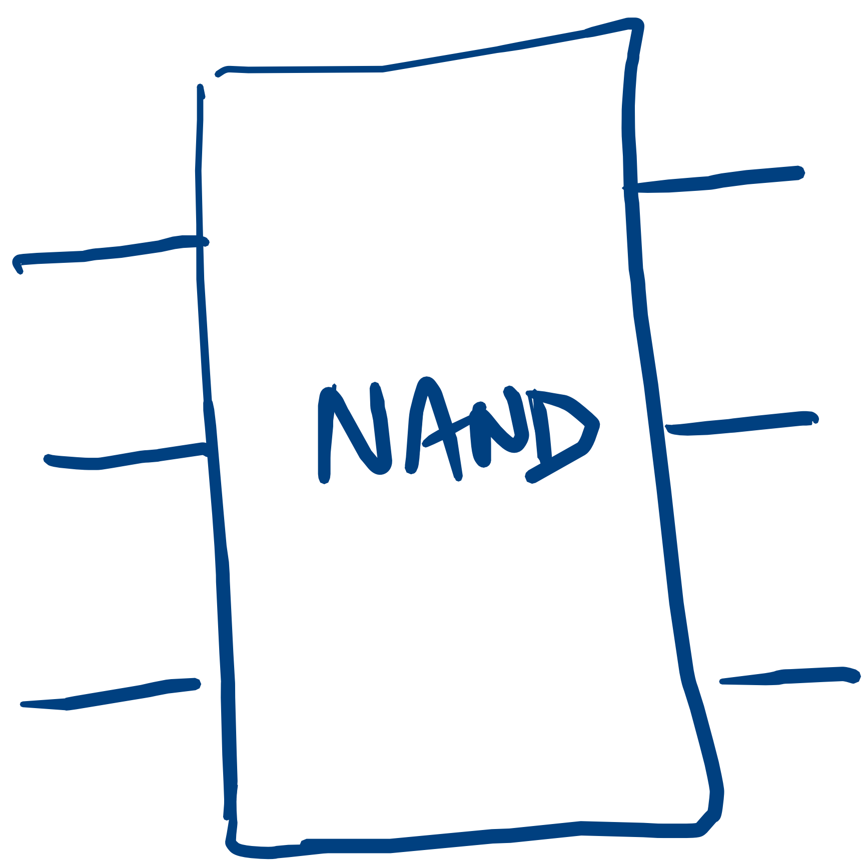


$c = \overline{a \cdot b}$

$d = b$

Truth table

a	b	c	d
0	0	1	0
0	1	1	1
1	0	1	0
1	1	0	1



Q. Can you make NAND reversible

2 - 2

Make the following gate reversible

Q: NOR, OR, AND, XOR,

NOT gate

input	output
0	1
1	0

$$X = \begin{array}{cc} 0 & 1 \\ \hline \multicolumn{2}{c}{\begin{bmatrix} |1\rangle & |0\rangle \end{bmatrix}} \end{array} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$U = \begin{array}{cccc} 000 & 001 & & 111 \\ \hline \multicolumn{4}{c}{\begin{bmatrix} |1\rangle & |1\rangle & \cdots & |1\rangle \end{bmatrix}} \end{array}$$

Truth table

$\downarrow$

U

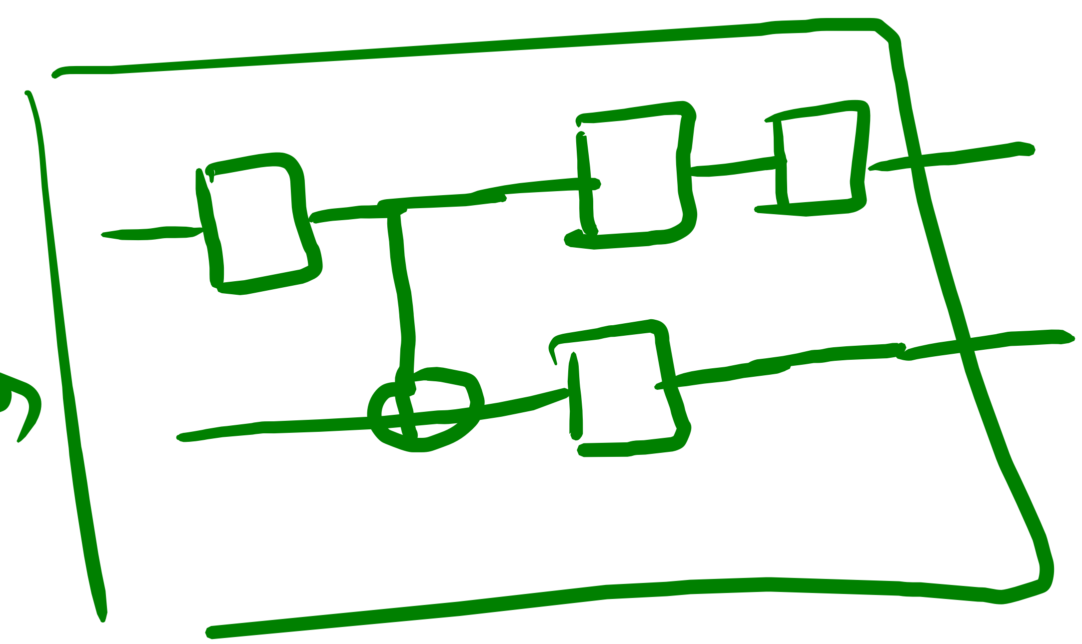
sequence of gates

U

Classical gate — reversible

$$U = \begin{bmatrix} | & | & \cdots & | \end{bmatrix}$$

Can you come up the hardware?



$$\begin{array}{ccc|ccc} a & b & c & a' & b' & c' \\ 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ \vdots & & & & & \\ 1 & 1 & 1 & & & \end{array}$$

$$U = \begin{array}{c} \begin{array}{cccc} 000 & 001 & \cdots & 111 \end{array} \\ \begin{bmatrix} \underbrace{|\ \rangle\,|\ \rangle}_{} \end{bmatrix} \end{array}$$

Truth table

$\boxed{n=4}$

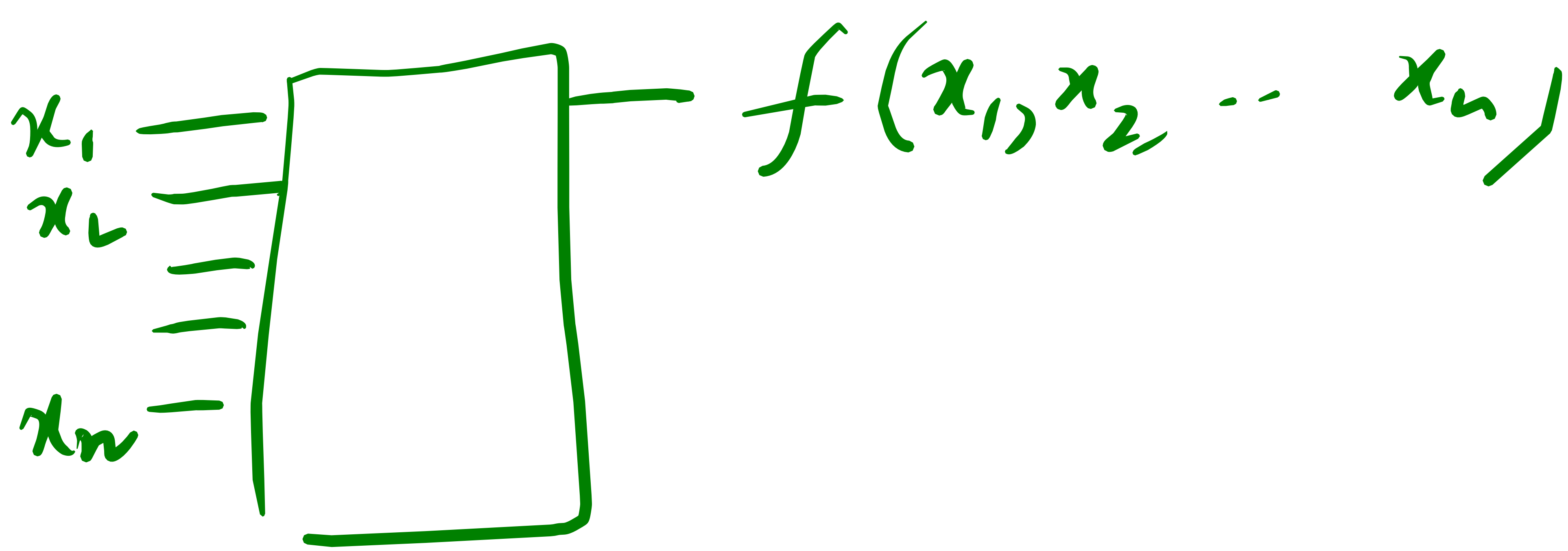
AND

$$f(x_1, x_2, x_3, \ldots, x_n) = (x_1 \vee x_2) \cdot (x_3 \vee x_4) \cdot \ldots (x_{n-1} \vee x_n)$$

$$= (x_1 \vee x_2) \cdot (x_3 \vee x_4)$$

$x_1, x_2, \ldots \quad x_n \in \{0,1\}$

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$



$n=4$

$f(x_1, x_2, x_3, x_4)$

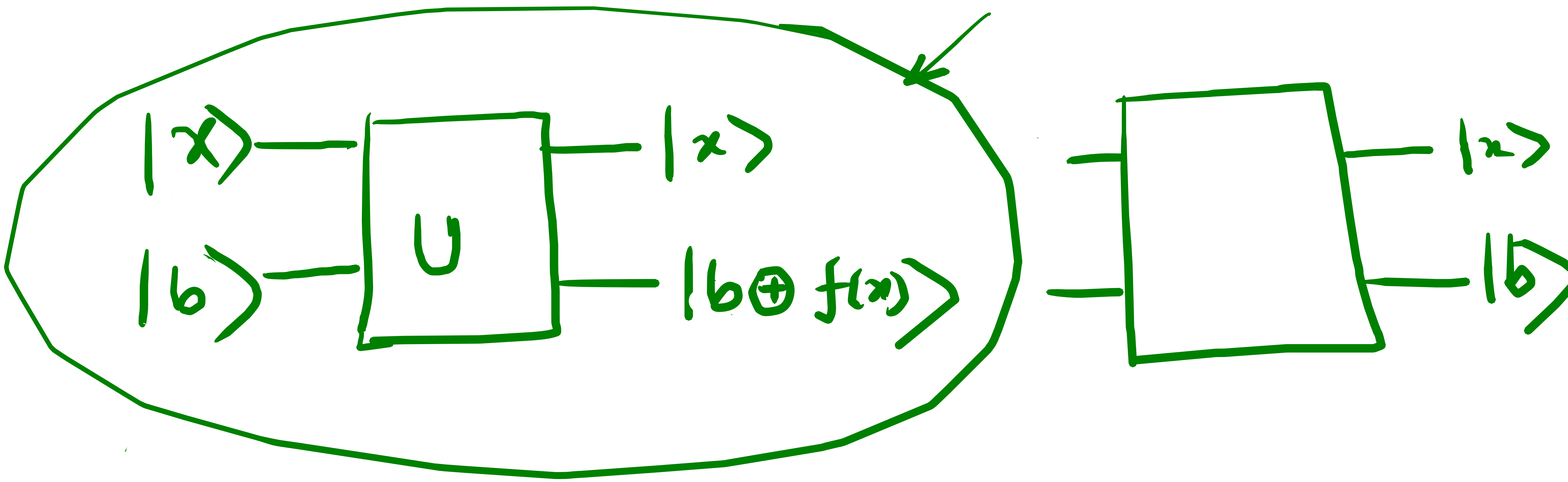
Q: Make f reversible?

$$U = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

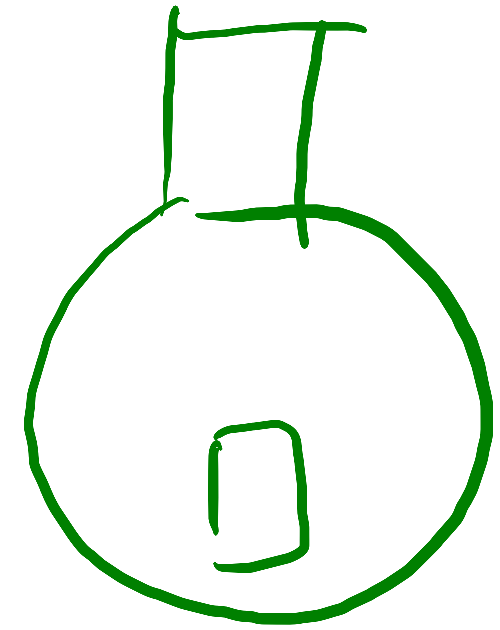
oracle — function $f: \{0,1\}^n \to \{0,1\}^m$

Given f , design U_f .

$$|x\rangle = |x_1, x_2, \ldots, x_n\rangle$$



Intro Quantum algos.

Key → 

1 unique key

n bits

2^n

↓

1 key

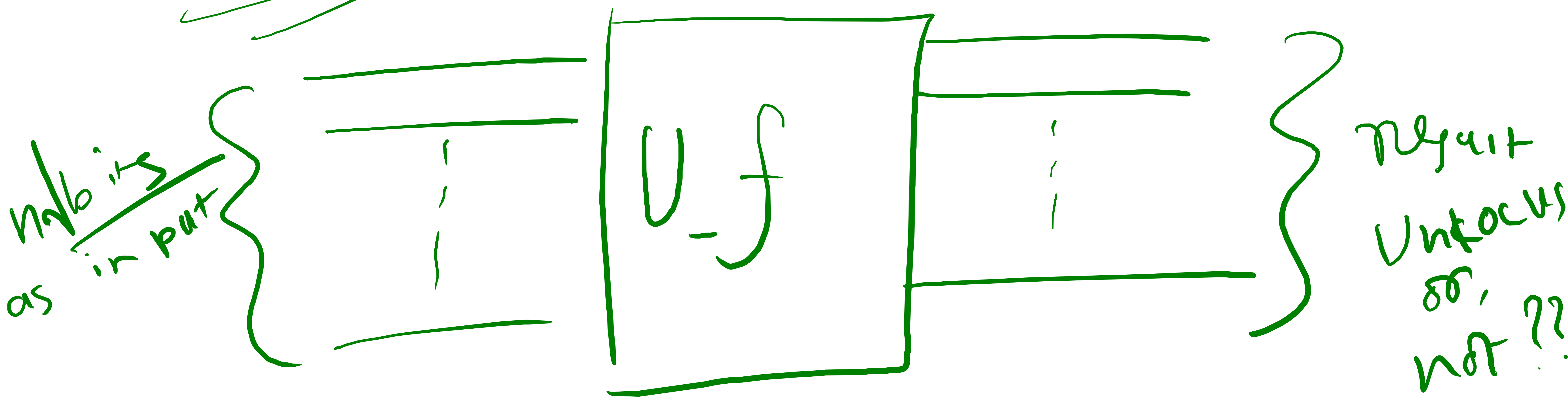
$n = 3$

8 → time complexity

no. of bits = n

Worst case: 2^n

Quantum Algorithms:



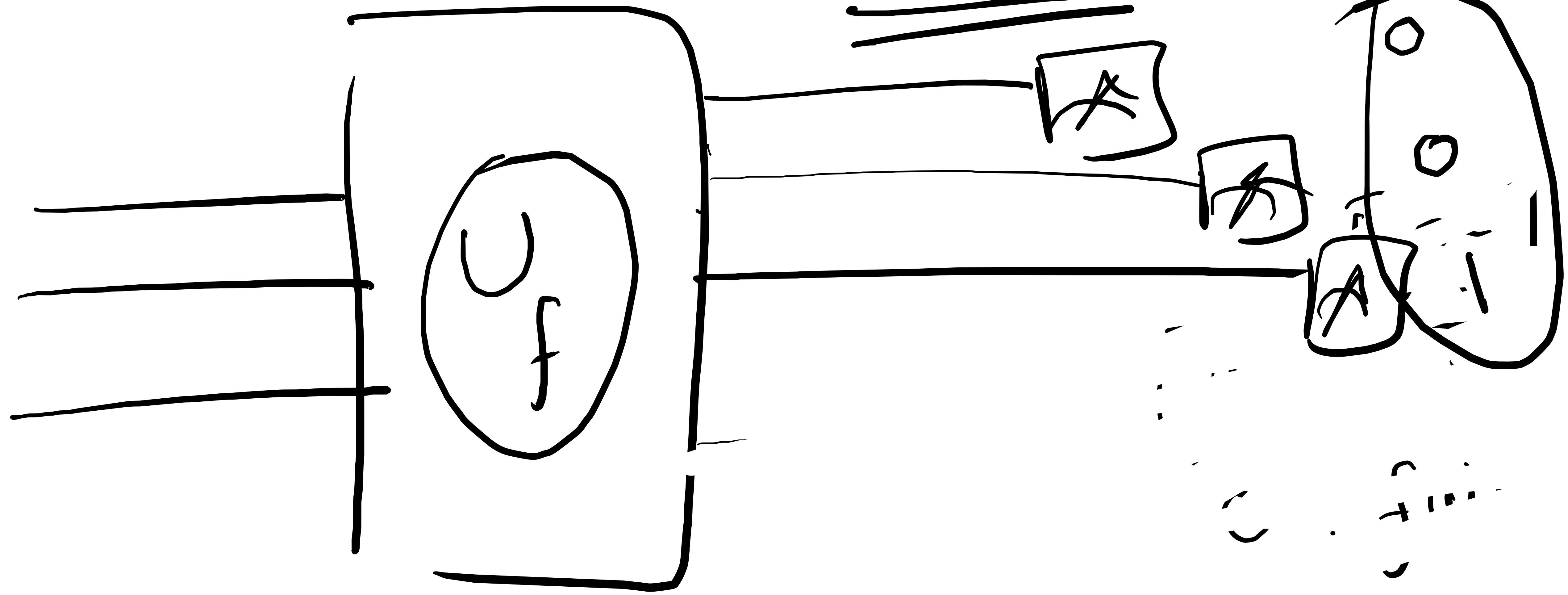
Blackbox

$\boxed{U_f}$

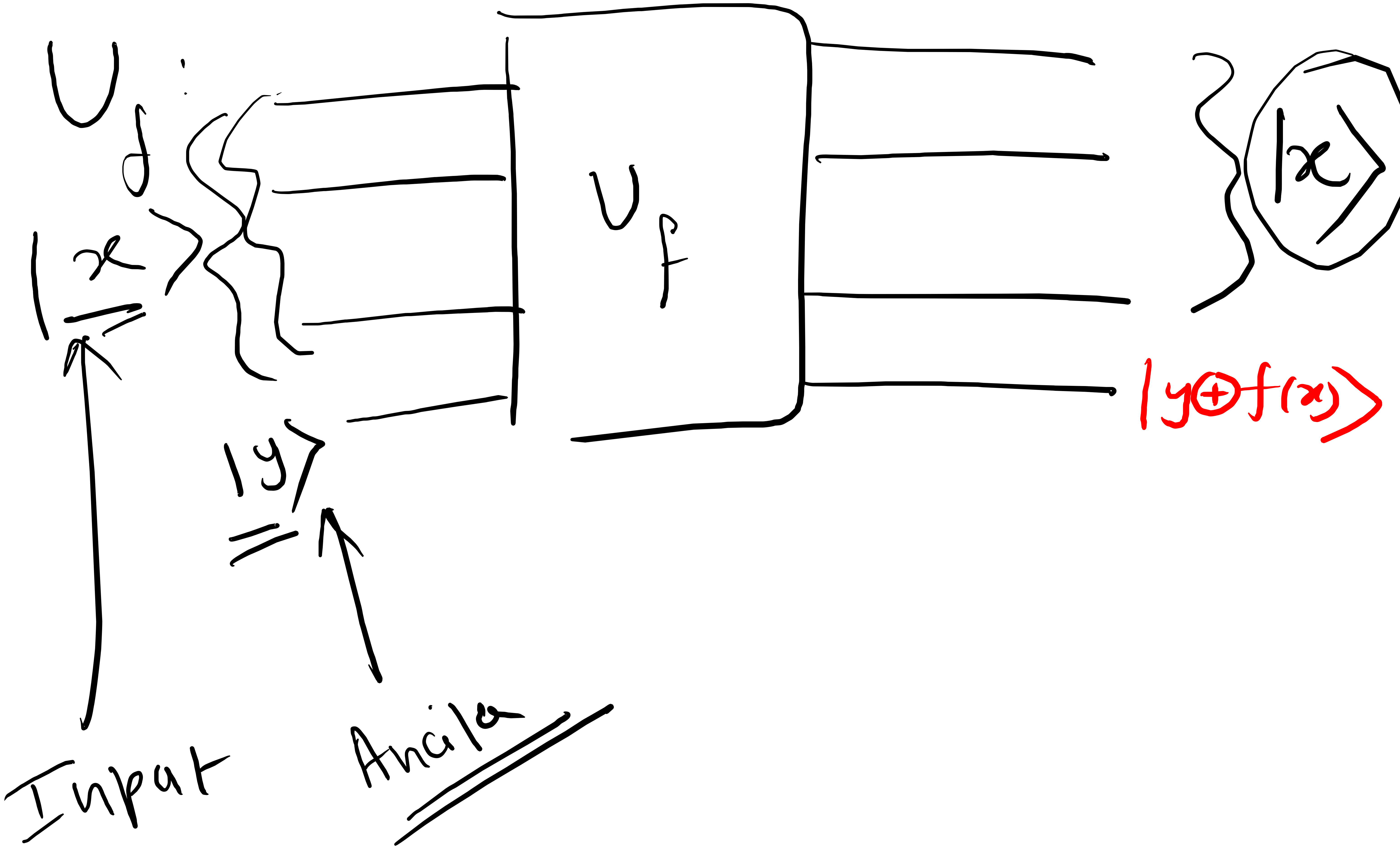
Oracles

Input

$n = 3$

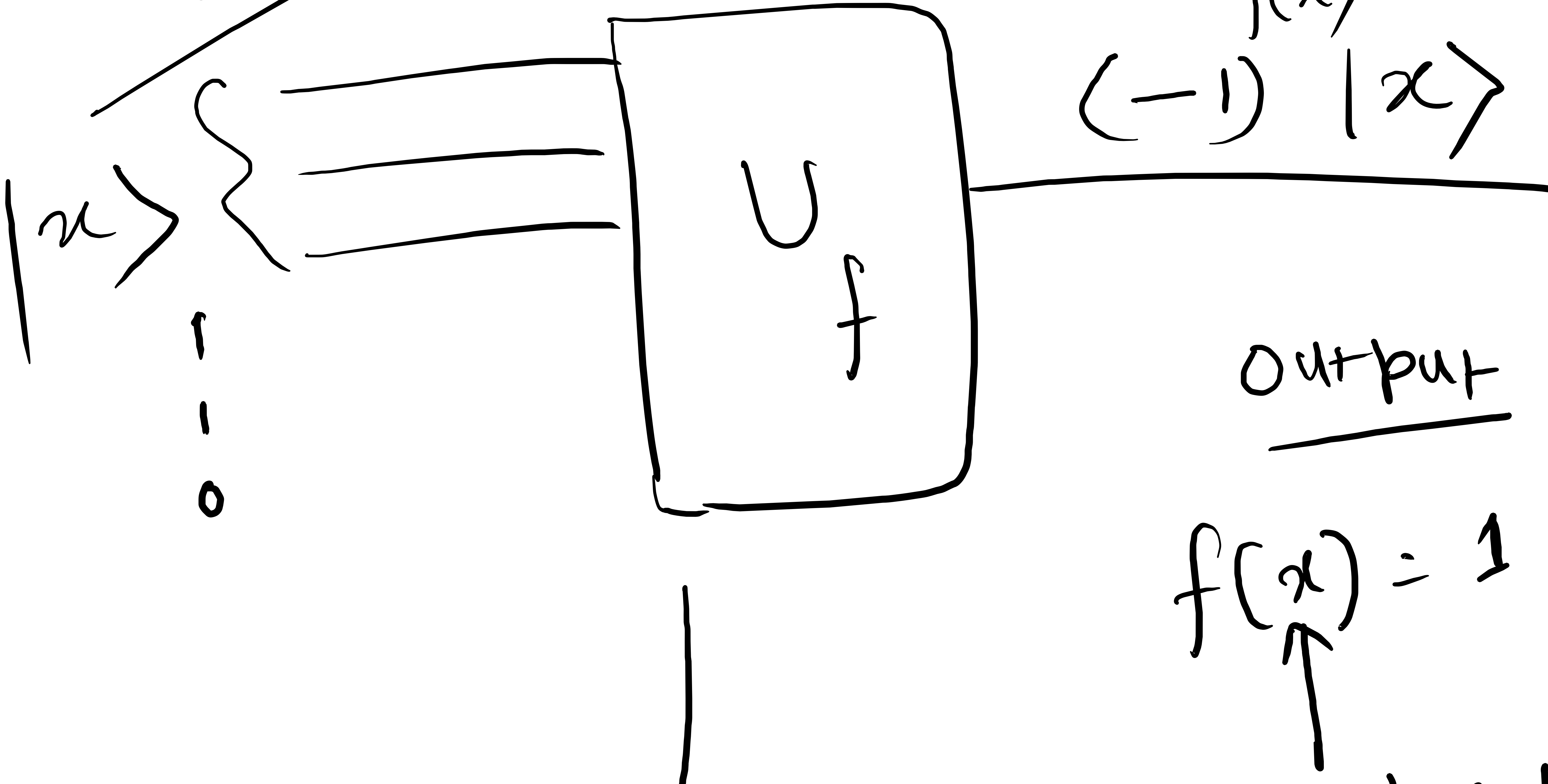


$[000 + 001 + 010 + 100 + 011 + 110 + 101 + 111]$



U_f :
$|x\rangle$
$|y\rangle$
U_f
$|x\rangle$
$|y \oplus f(x)\rangle$
Input
Ancila

Phase oracle



$\underline{\underline{-|x\rangle}}$

Output

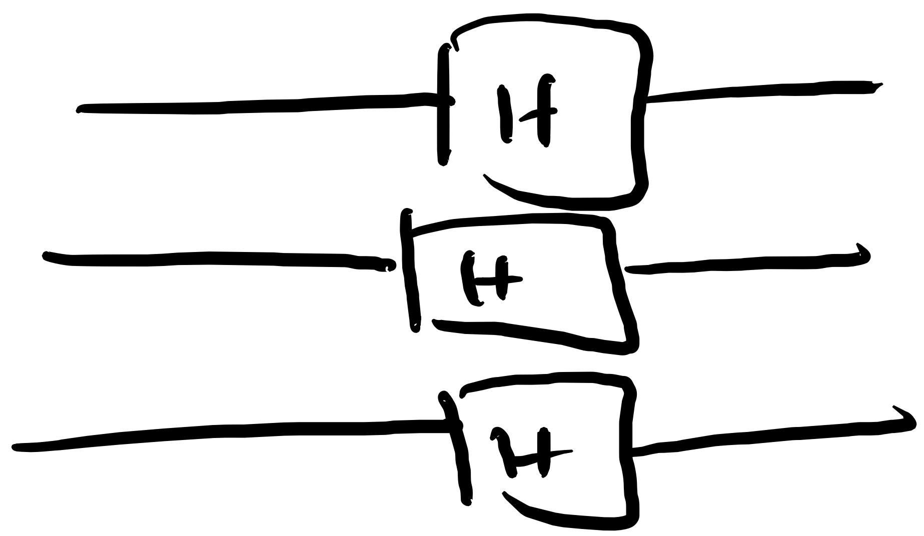
$f(x) = 1$

x has key

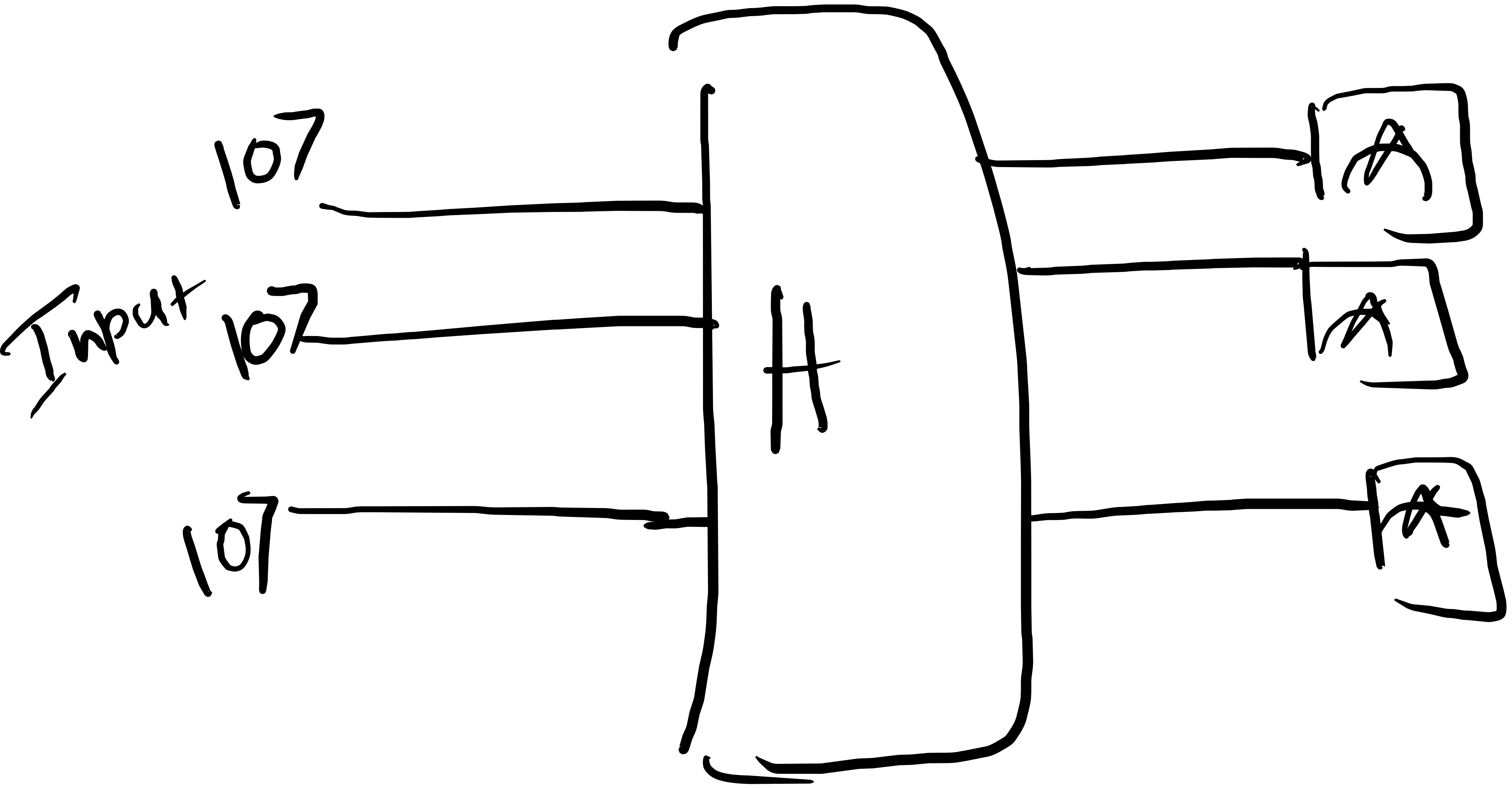
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Super-key

$n = 3$



$$|\psi\rangle = \left(\frac{1}{\sqrt{2^3}}\right)\left[|000\rangle + |001\rangle + \cdots\right]$$



$$H|0\rangle = \frac{1}{\sqrt{2}}\left[(-1)^{0\cdot 0}|0\rangle\right]$$

$n=1$

$$S = s_0 s_1 s_2$$
$$x = x_0 x_1 x_2$$

$|001\rangle$

Hadamard_Tranform

$$\boxed{H^{\otimes n}|s\rangle = \frac{1}{\sqrt{2^n}}\sum_{x_{\text{on},s}}(-1)^{s\cdot x}|x\rangle}$$

$$s\cdot x = s_0\cdot x_0 \oplus s_1\cdot x_1 \oplus s_2\cdot x_2$$

$$H^{\otimes n}|s\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{s \cdot x} |x\rangle$$

n = 1

$$H|s\rangle = \frac{1}{\sqrt{2^1}} \left[(-1)^{s \cdot 0} |0\rangle + (-1)^{s \cdot 1} |1\rangle \right]$$

$$|s\rangle = |0\rangle \to \frac{\left[|0\rangle + |1\rangle \right]}{\sqrt{2}}$$

$$H^{\otimes 3} |000\rangle =$$

$$H^{\otimes 3}|000\rangle = \frac{1}{\sqrt{2^3}} \sum_{x \in \{0,1\}^3} (-1)^{s.x} |x\rangle$$

$0 \oplus 0 = 0$
$0 \oplus 1 = 1$
$1 \oplus 0 = 1$
$1 \oplus 1 = 0$

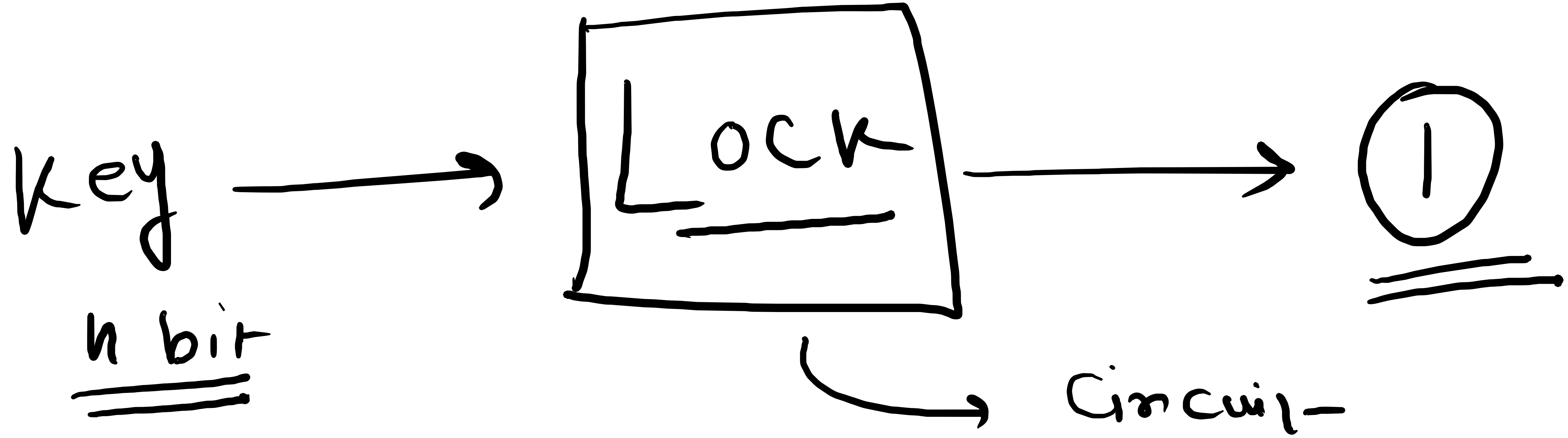
s

Summation goes over all the possibilities of 3 bits.

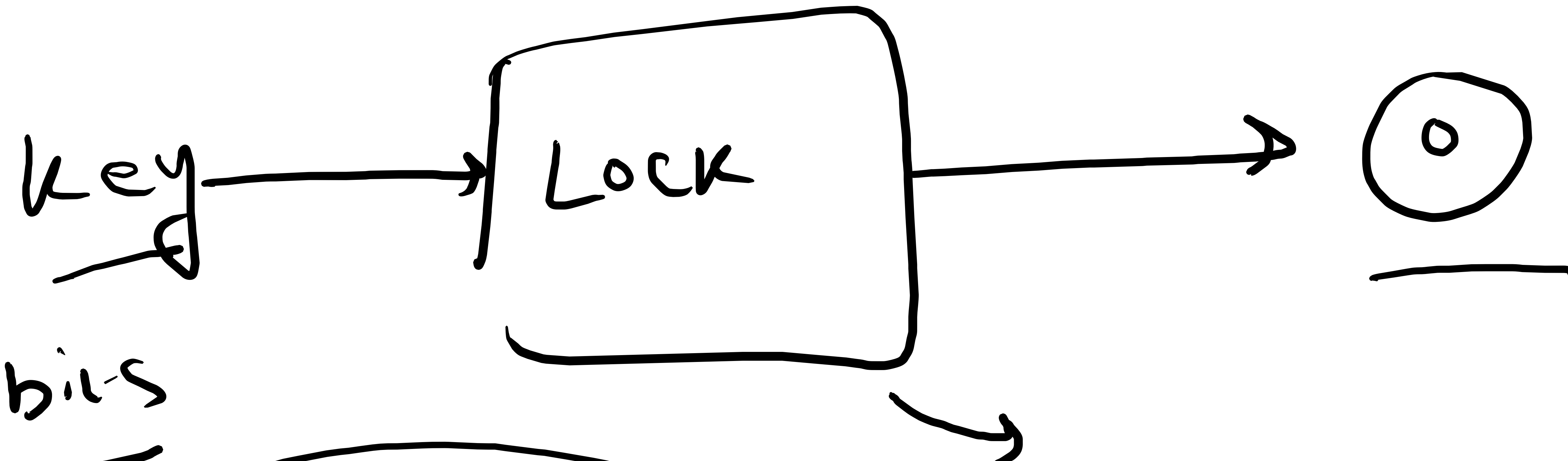
$$s.x = s_0 \cdot x_0 \oplus s_1 \cdot x_1 \oplus s_2 \cdot x_2$$

$s_0 = 0, \quad s_1 = 0, \quad s_2 = 0$

Success



failure



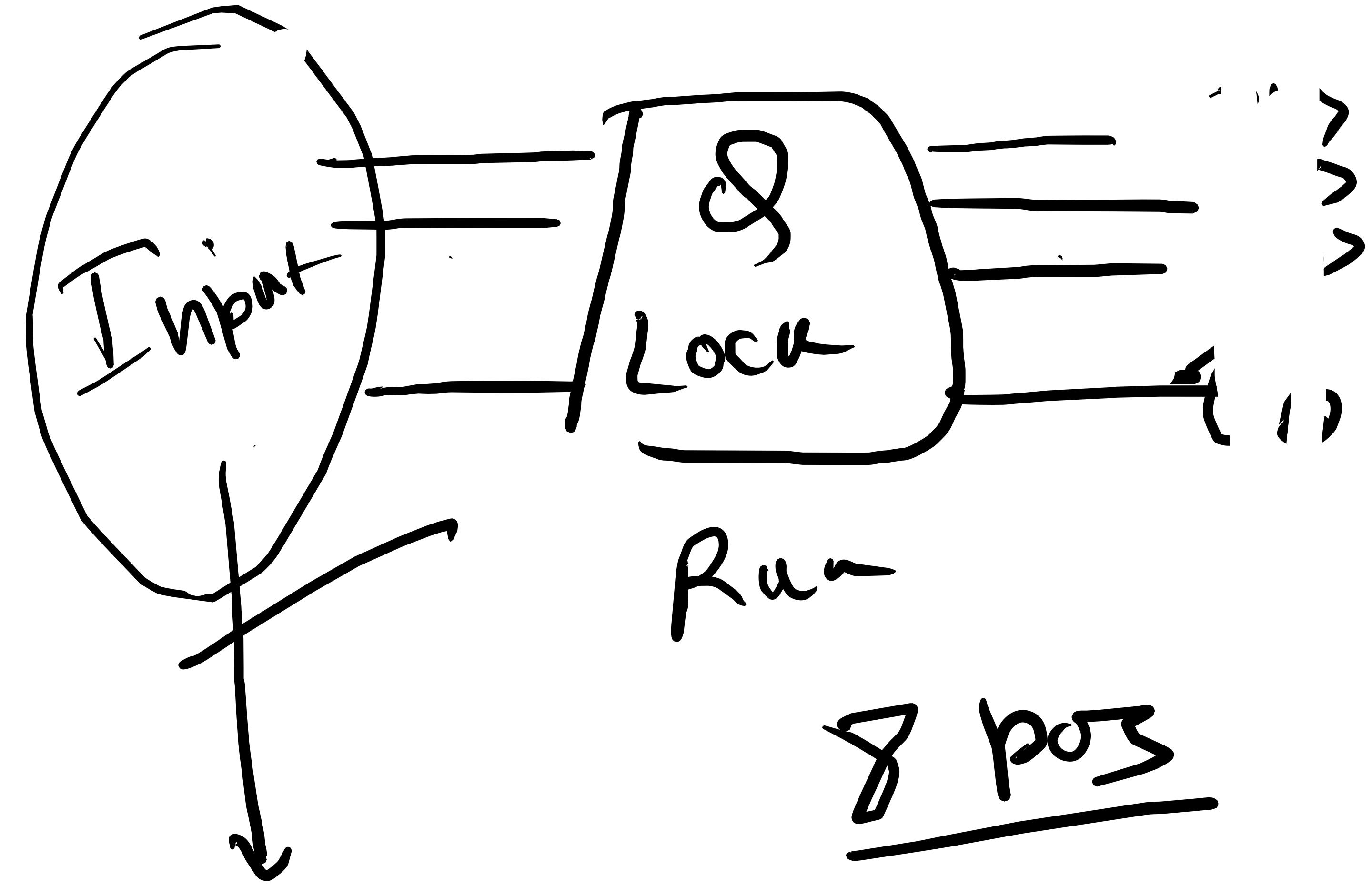
key of 3 bits

{0,1}

Problem

Quantum Algorithm:

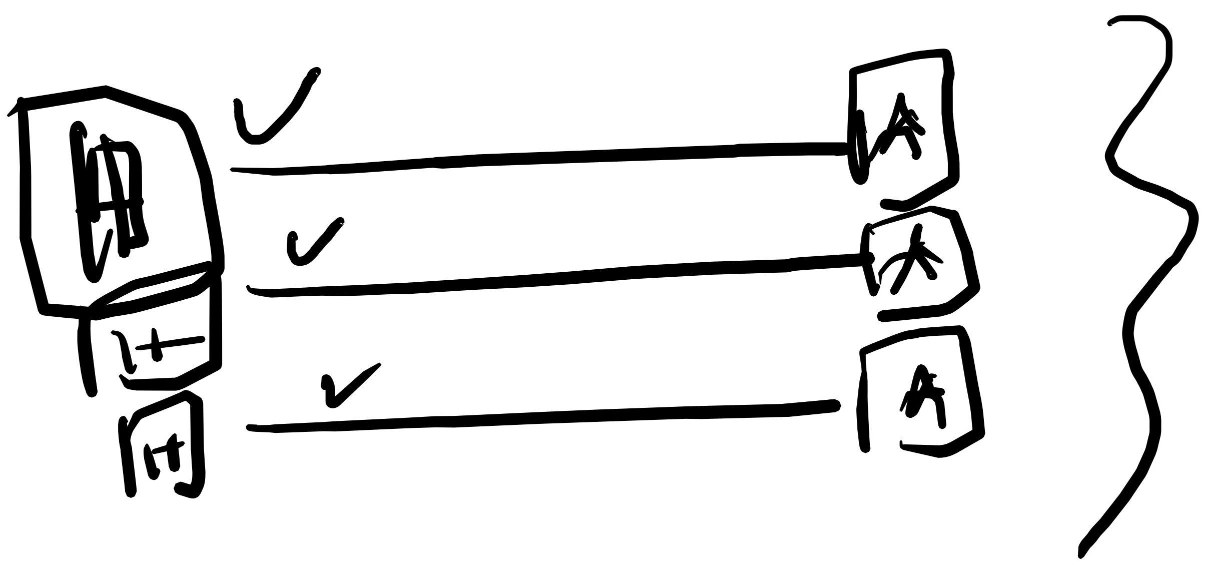
1. Prepare input:



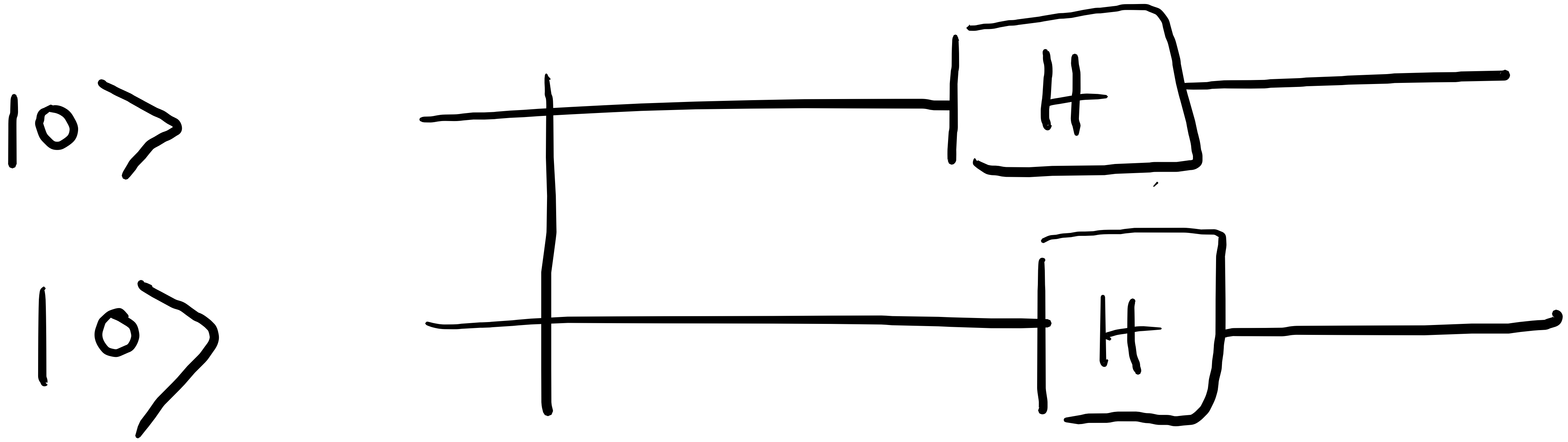
8 pos

$$|\psi\rangle = \left[|000\rangle + |001\rangle + \dots + |111\rangle \right]$$

2.



$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \longrightarrow$$



$$\left.\begin{matrix}\dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \\[2ex] \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\end{matrix}\right\}$$

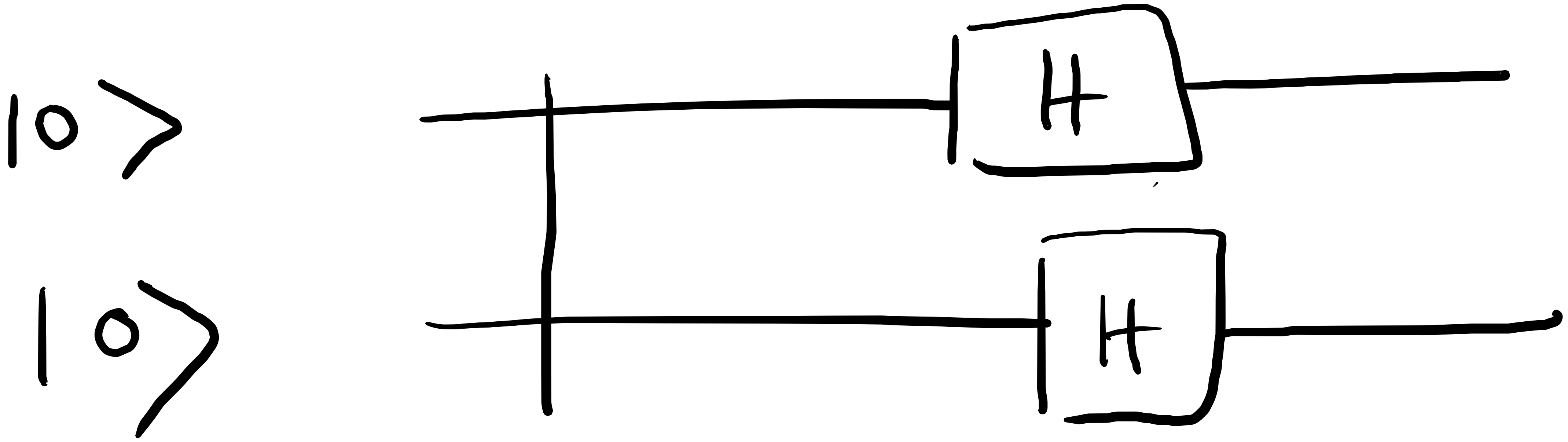
$|0\rangle \otimes |0\rangle$

$= |0\rangle |0\rangle$

1st State	2nd State
$\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$	$\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0 + |1\rangle}{\sqrt{2}}\right)$$

$$= \left[|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right] =$$



$$\left.\begin{matrix}\dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \\[2ex] \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\end{matrix}\right\}$$

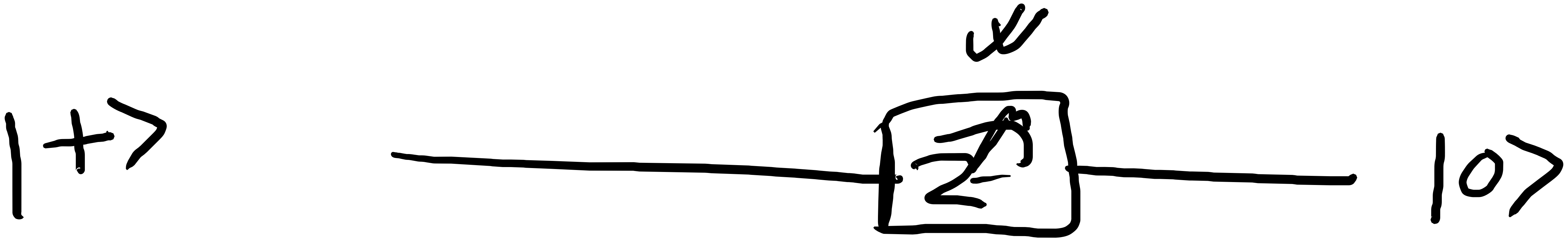
w/

$$\cancel{|0\rangle \otimes |0\rangle} = \cancel{|0\rangle|0\rangle}$$

$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0 + |1\rangle}{\sqrt{2}}\right) =$$

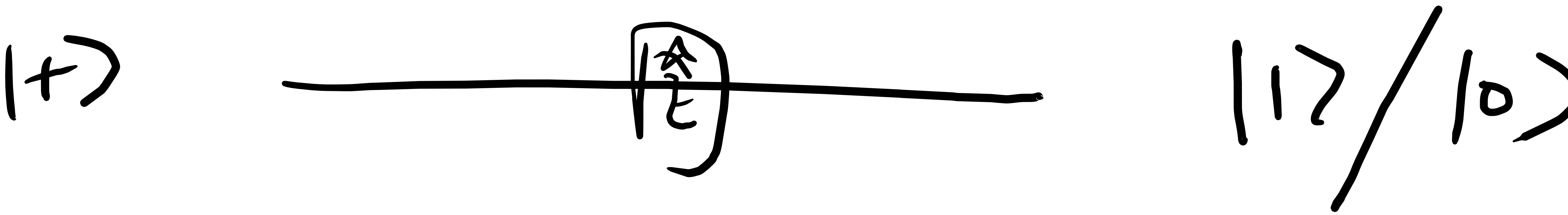
$$= \left[|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right]$$

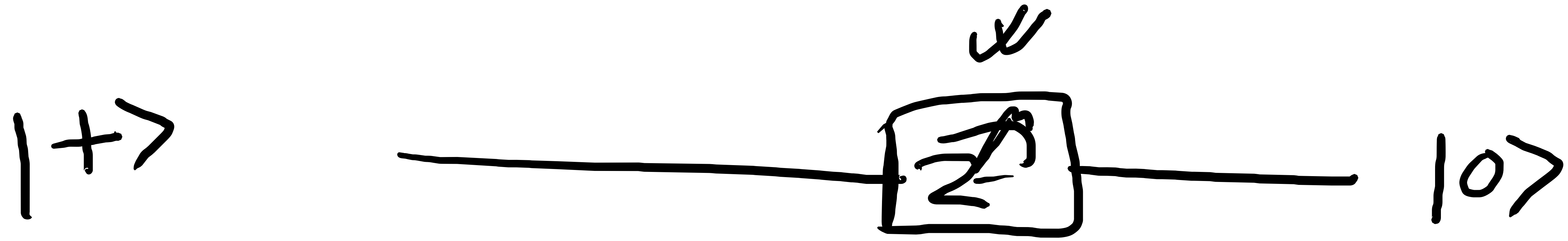
1st State	2nd State
$\frac{\vert 0\rangle + \vert 1\rangle}{\sqrt{2}}$	$\frac{\vert 0\rangle + \vert 1\rangle}{\sqrt{2}}$



$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

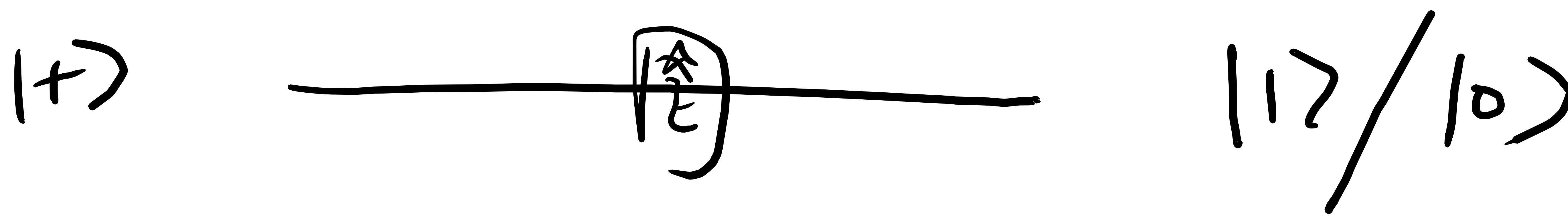
Run it 1 time





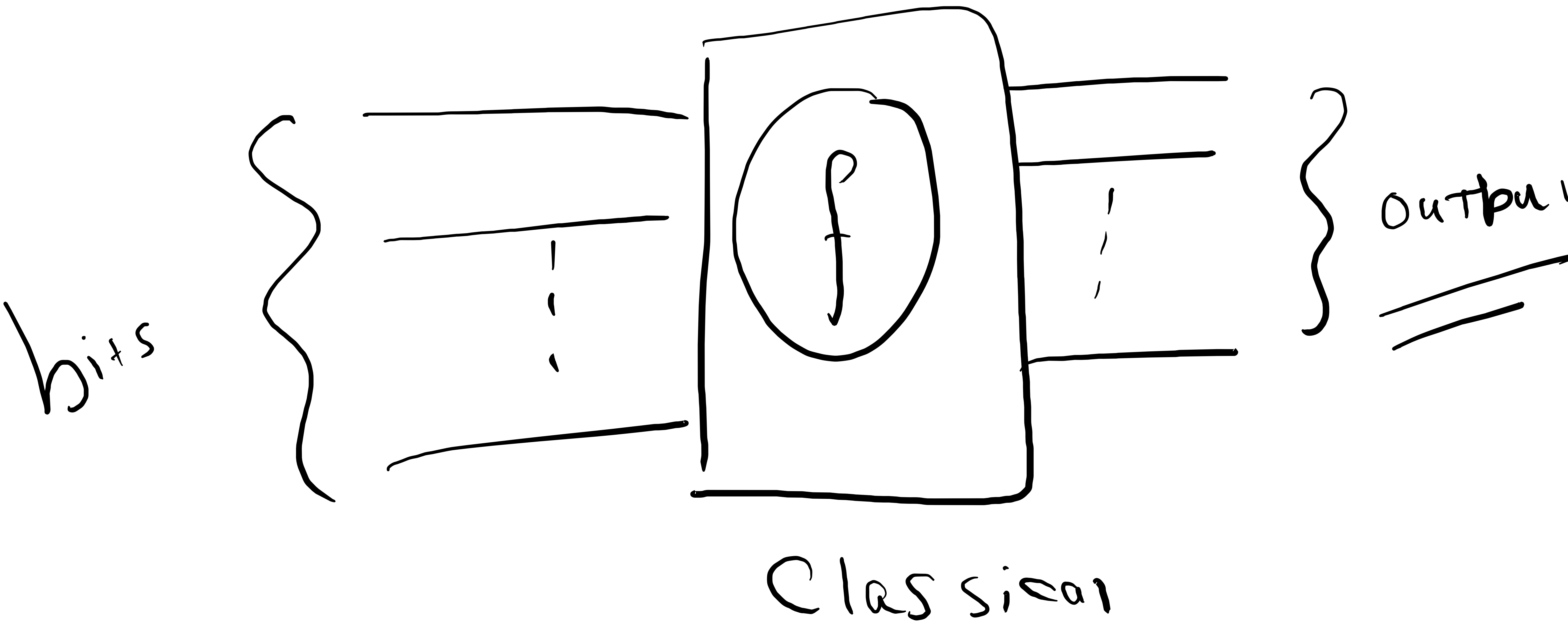
$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Run it 1 time



We have input

$$|\psi\rangle =$$



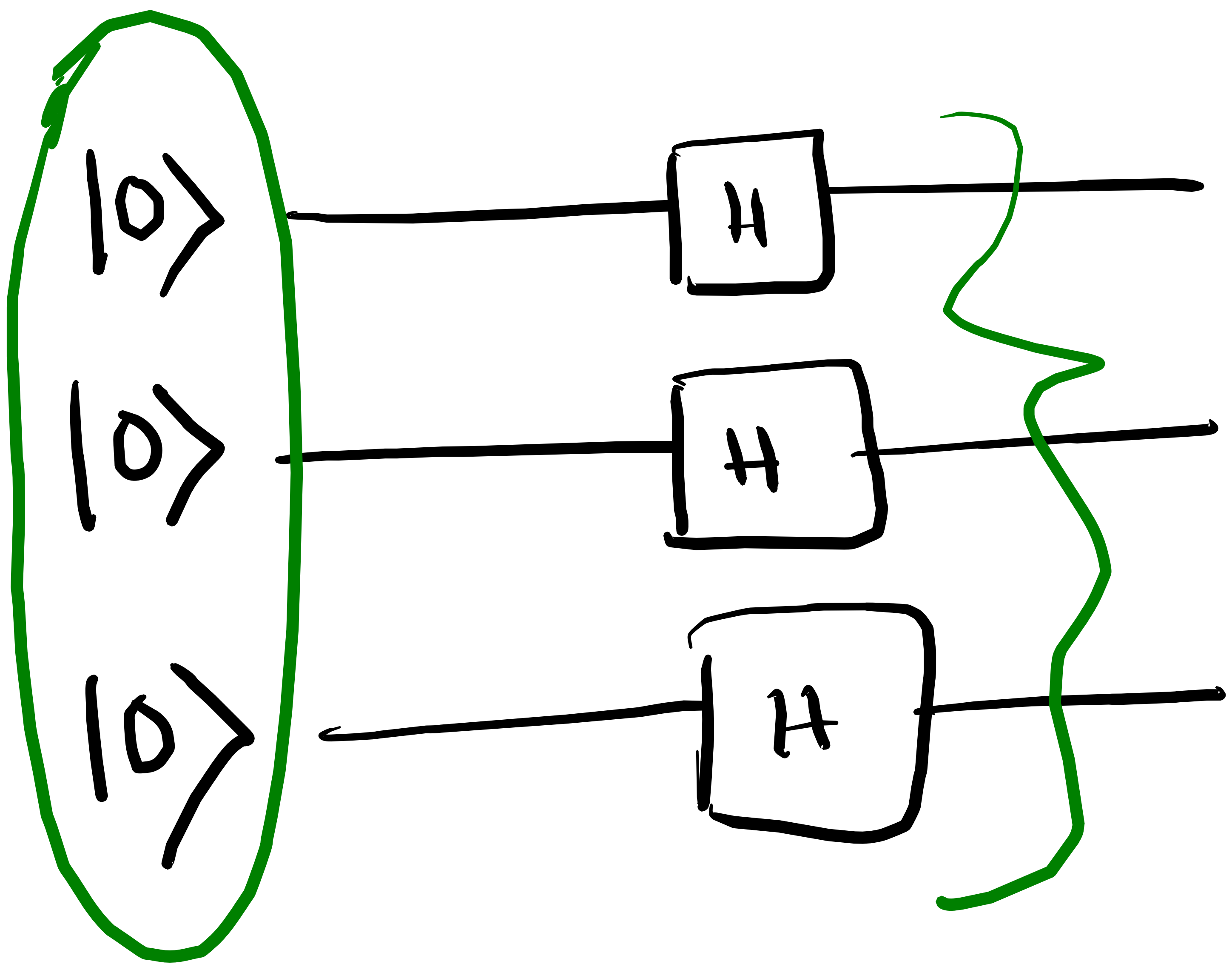
bits
f
output
Classical

$$\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] \xrightarrow{\text{Measure}} |0\rangle \text{ or } |1\rangle$$

$$H^{\otimes 3}|000\rangle = \frac{1}{\sqrt{2^3}} \sum_{x \in \{0,1\}^3} (-1)^{s \cdot x} |x\rangle$$

$s = |000\rangle$

Circuit



$$H^{\otimes 2}|01\rangle = \frac{1}{\sqrt{2}}\left[\,|00\rangle + |01\rangle + |11\rangle \quad H|11\rangle\right]$$

$s \cdot x$

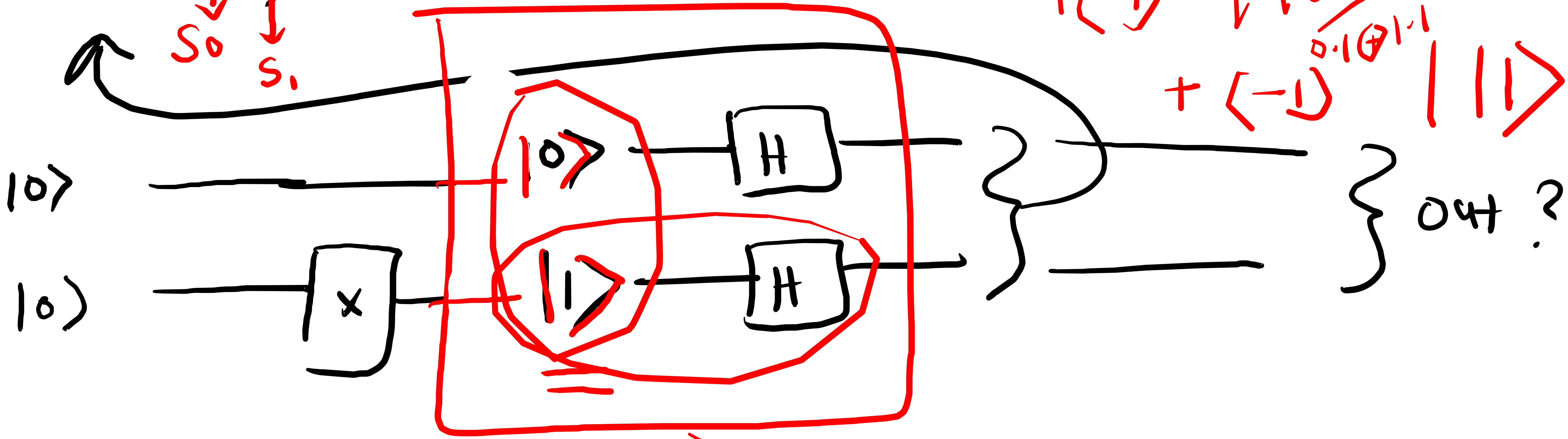
$$H^{\otimes 2}|00\rangle = \frac{1}{\sqrt{2^2}} \underline{\sum_{x\in\{00,01,10,11\}} (-1)^{s\cdot x} |x\rangle}$$

$$H^{\otimes 2}|01\rangle = \frac{1}{\sqrt{2^2}}\left[(-1)^{0\oplus 0}|00\rangle + (-1)^{0\cdot 0\oplus 0\cdot 1}|01\rangle + (-1)^{0\cdot 1\oplus 0\cdot 0}|10\rangle + (-1)^{0\cdot 1\oplus 0\cdot 1}|11\rangle\right]$$

$$s\cdot x = s_0\cdot x_0 \oplus \underline{s_1\cdot x_1}$$

$$H^{\otimes 2}|01\rangle = (-1)^{0\cdot 0 \oplus 1\cdot 0}|00\rangle + (-1)^{0\cdot 0\oplus 1\cdot 1}|01\rangle + (-1)^{0\cdot 1 + 1\cdot 0}|10\rangle + (-1)^{0\cdot 1 \oplus 1\cdot 1}|11\rangle$$

s_0 s_1 x_0 x_1



$$|1\rangle \to |-\rangle = |0\rangle - |1\rangle$$

$$= (+1)|00\rangle + (-1)|01\rangle + (+1)|10\rangle + (-1)|11\rangle$$

$$\frac{1}{\sqrt{2}^{2}}\left[\,|00\rangle - |01\rangle + |10\rangle - |11\rangle\,\right]$$

$$|11\rangle \rightarrow \left(-\frac{1}{2}\right)$$

$$\frac{1}{\sqrt{2}^2}\left[|00\rangle - |01\rangle + |10\rangle - |11\rangle \right]$$

$$|11\rangle \rightarrow \left(-\frac{1}{2} \right)^2$$

Algorithm

1. Prepare input state.

$$H^{\otimes n} \, |000\cdots 0\rangle$$

n zeros

$$|0\rangle^{\otimes n}$$

What do $\oint$ do??

$n=2$

$$|00\rangle + |01\rangle + |10\rangle + |11\rangle$$

$$\downarrow \; U_f$$

$f(x)=1$

key

$$(-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle$$

$$\downarrow$$

$$\left[\, |00\rangle - |01\rangle + |10\rangle + |11\rangle \,\right]$$

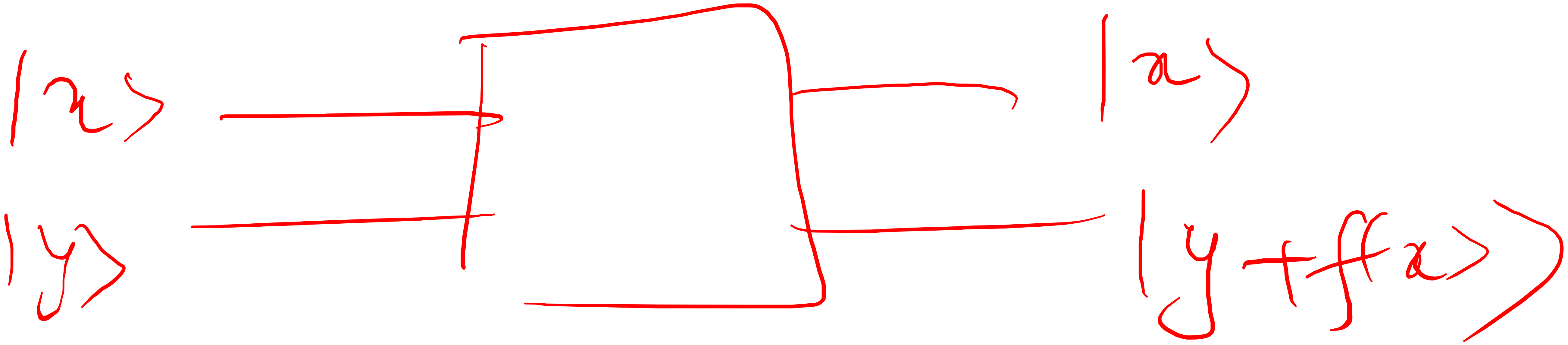
1. Prepare the input.

2. Identify the solution inside the superposition.

U_f

1. Prepare the input.
2. Identify the solution inside the superposition.

U_f



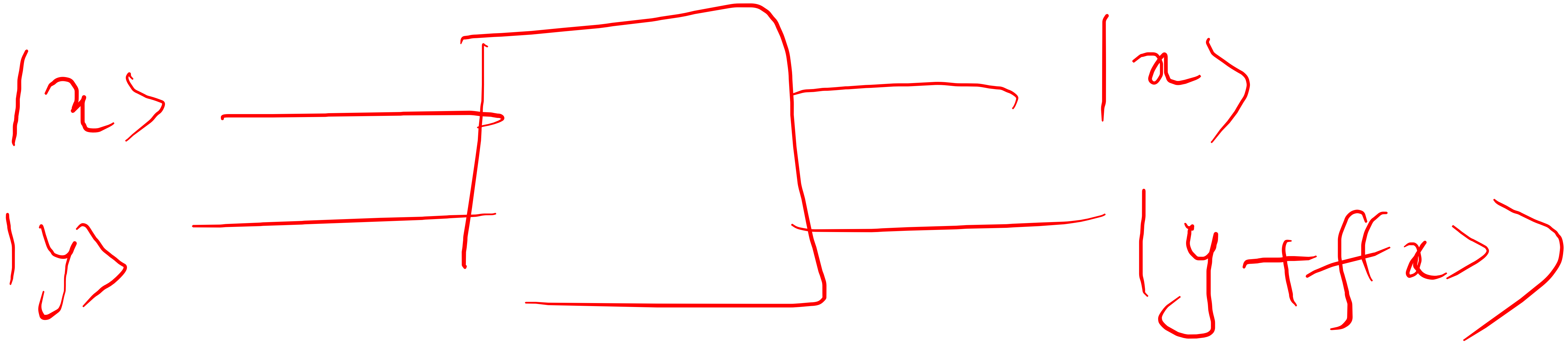
$(0\ 1)(0\ -1)$

$|+\rangle|-\rangle$

$= |0\rangle|0\rangle + |0 - 0| - 1|$

$= |0\rangle(0 + f(0)) + |1\rangle(0 + f(1))$

$$f(x) = \begin{cases} 1 \to 0 \\ 0 \to 1 \end{cases}$$



$$f(x) = \begin{cases} 1 \to 0 \\ 0 \to 1 \end{cases}$$

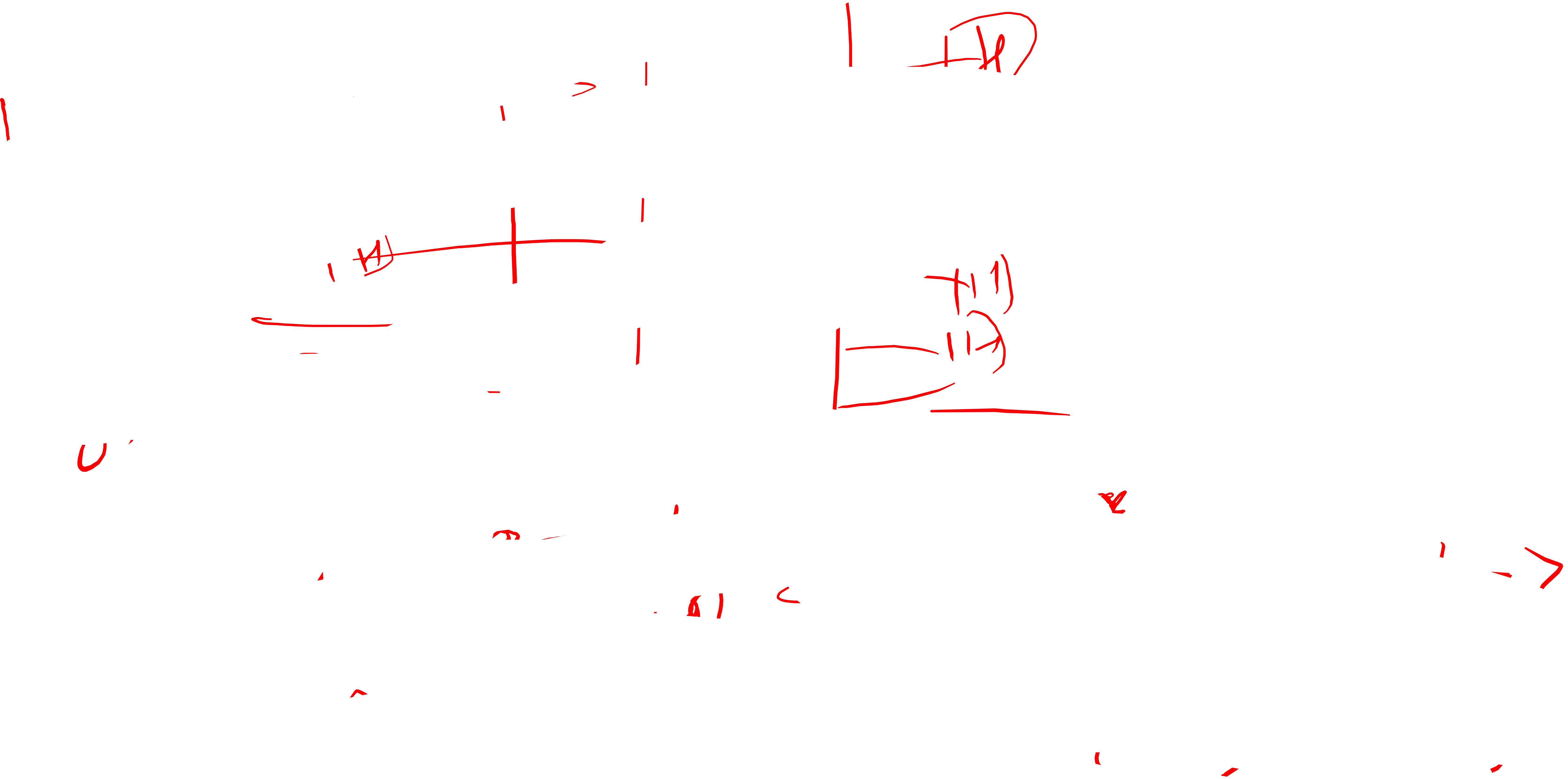
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \end{pmatrix}$$

$$|+\rangle|-\rangle$$

$$= |0\rangle|0\rangle + |0 - 0| - 1|$$

$$|-\rangle = |0\rangle(0 + f(0)) + |1\rangle(0 + f(1)) - |0\rangle(0 + f(0)\rangle - |1\rangle|1 \oplus f(1)\rangle$$

$$|x\rangle|y\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$



$$\left(U_f = I - 2|\psi\rangle\langle\psi| \right)$$

key

Oracle

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

1 qubit

$$X = \begin{bmatrix} & \end{bmatrix}$$

$$\left(U_f = I - 2|\psi\rangle\langle\psi| \right)$$

key

Oracle

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

1 qubit

$$\downarrow$$

$$X = \begin{bmatrix} \quad \end{bmatrix}$$

x $|00\rangle$ $\frac{|01|}{\;}$ $\longrightarrow [0,1]$ $|10\rangle + |11\rangle$

y

$n=2$

$$U_f\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

$$= \left[I - 2|01\rangle\langle 01|\right]\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

$$-2|01\rangle\langle 01|11\rangle$$

$$= |00\rangle + |01\rangle + |10\rangle + |11\rangle$$

$$-2|01\rangle\left[0 + 1 + 0 + 0\right]$$

1 qubit

$$X = \begin{array}{cc} \begin{array}{cc} 0 & 1 \end{array} & \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

$$X|1\rangle = |0\rangle$$

$$X|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

CNOT

$|01\rangle$

$|01\rangle$

$0 \leftarrow |00\rangle \longrightarrow |00\rangle$

$1 \leftarrow |01\rangle \longrightarrow |01\rangle$

$2 \leftarrow |10\rangle \longrightarrow |11\rangle$

$3 \leftarrow |11\rangle \longrightarrow |10\rangle$

$|01\rangle$

$$
\begin{array}{cccccc}
 & 0 & 1 & 2 & 3 & \\
\left[\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
\end{array}
$$

4×4

$n : n$

$2^n \times 2^n$

$$\text{CNOT}\ |01\rangle = |11\rangle$$

Target control

$$\begin{array}{c|cccc|c} & 0 & 1 & 2 & 3 & \\ \hline & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 0 & 2 \\ & 0 & 1 & 0 & 0 & 3 \end{array}$$

0

$|00\rangle \Rightarrow |00\rangle$

$|01\rangle \rightarrow |11\rangle$

$\text{CNOT}\ |10\rangle \rightarrow |10\rangle$

$\text{''}\ |11\rangle \rightarrow |01\rangle$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix}$$

$$|00\rangle \downarrow 0$$

CNOT

$$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\downarrow$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|10\rangle \downarrow 2$$

$$2^2 \times 1$$

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix}$$

$$|01\rangle \downarrow 1$$