

# Experiment6:Laplace Transform

Abhishek Sekar  
EE18B067

March 10, 2020

## 0.1 Introduction

- We analyse the LTI systems in continuous time using Laplace transform to find the solutions to the equations governing the system with the help of python tools such as Signal toolbox
- *system.impulse* → Computes the impulse response of the transfer function
- *sp.lsim* → This simulates  $y = u(t) * h(t)$  taking  $u(t)$  and  $\mathcal{H}(s)$  as arguments
- *sp.lti* → defines a transfer function from polynomial coefficients of numerator and denominator as inputs.
- *bode()* → It's used to find the magnitude and phase response of transfer function
- We use following method to find the Laplace transform of a time domain signal, here we use these methods to find laplace of system governing differential coefficients
- Some of the equations to follow while finding laplace transform

$$(1) \quad \mathcal{L}\{x(t)\} \rightarrow \mathcal{X}(s)$$

$$(2) \quad \mathcal{L}\left\{dx(t) \frac{d}{dt}\right\} \rightarrow s\mathcal{X}(s) - x(0^-)$$

$$\mathcal{L}\left\{d^2x(t) \frac{d^2}{dt^2}\right\} \rightarrow s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-)$$

(3)

- Combining the above equations above, we find the laplace transform of  
a differential equation and analyse them.

## 0.2 Python Code

### 0.2.1 Question 1 & 2:

- To solve for the time response of the spring mass system, whose driving force varies as  $f(t)$  given as

$$f(t) = \cos(1.5t)e^{-0.5t}u_0(t)$$

(4)

- Laplace transform of  $f(t)$  using equations (1),(2) & (3) given above

$$F(s) = \frac{s+0.5}{(s+0.5)^2+2.25}$$

- Spring satisfies the below equation with  $x(0) = 0$  and  $\dot{x}(0) = 0$  for  $0 \leq t \leq 50s$ .

$$\ddot{x} + 2.25x = f(t)$$

- So we take laplace transform of the equation given above with given initial conditions and in generalised form considering  $w_0^2 = 2.25$  in the differential equation above, natural frequency of the system is  $w_0 = 1.5\text{rads}^{-1}$ , and decay factor of  $f(t)$  as  $d = 0.5$  and frequency of the input as  $\omega = 1.5\text{rads}^{-1}$  in this question.
- In general we get

$$(5) \quad X(s) = s + d \frac{1}{((s+d)^2 + w^2)(s^2 + w_0^2)}$$

- In question 1 we get with given values and  $d = 0.5$

$$(6) \quad X(s) = s + 0.5 \frac{1}{((s+0.5)^2 + 2.25)(s^2 + 2.25)}$$

- Solve the above problem with much smaller decay with same initial conditions, now  $f(t)$  is as follows

$$(7) \quad f(t) = \cos(1.5t)e^{-0.05t}u_0(t)$$

- So in question 2 we get with  $d = 0.05$

$$(8) \quad X(s) = s + 0.05 \frac{1}{((s+0.05)^2 + 2.25)(s^2 + 2.25)}$$

- To solve for  $x(t)$  displacement for each of the cases using Laplace transform with python tools such as *system.impulse* and plot them.
- To analyse the plots obtained and discuss the effect of decay on  $x(t)$ .

---

```

1  #importing necessary header files
2  import scipy
3  import scipy.signal as sp
4  import matplotlib.pyplot as plt
5  import numpy as np
6
7
8  #obtaining the x signals using a function
9  def x_solver(decay,freq):
10     num = np.poly1d([1, decay])
11     denom = np.polymul([1, 0, 2.25], [1, 2*decay, (pow(freq, 2)+pow(decay, 2))])
12     # Computes the impulse response of the transfer function
13     X = sp.lti(num, denom)
14     t, x = sp.impulse(X, None, np.linspace(0, 50, 100))
15     return t,x
16
17  #obtaining the x signals for Q1 and Q2
18  t1,x1= x_solver(0.5,1.5)
19  t2,x2= x_solver(0.05,1.5)
20
21  #plotting the x signals
22  #plot for Q1
23  plt.plot(t1,x1,label='decay=0.5',color='black')
24  plt.title('Fig 1:Decay=0.5')
25  plt.xlabel('time ->')
26  plt.ylabel('x1(t)->')
27  plt.ylim((-1,1))
28  plt.grid()
29  plt.show()
30
31
32  #plot for Q2
33  plt.plot(t2,x2,label='decay=0.05',color='black')
34  plt.title('Fig 2:Decay=0.05')

```

```
35 plt.xlabel('time ->')
36 plt.ylabel('x2(t)->')
37 plt.grid()
38 plt.show()
```

---

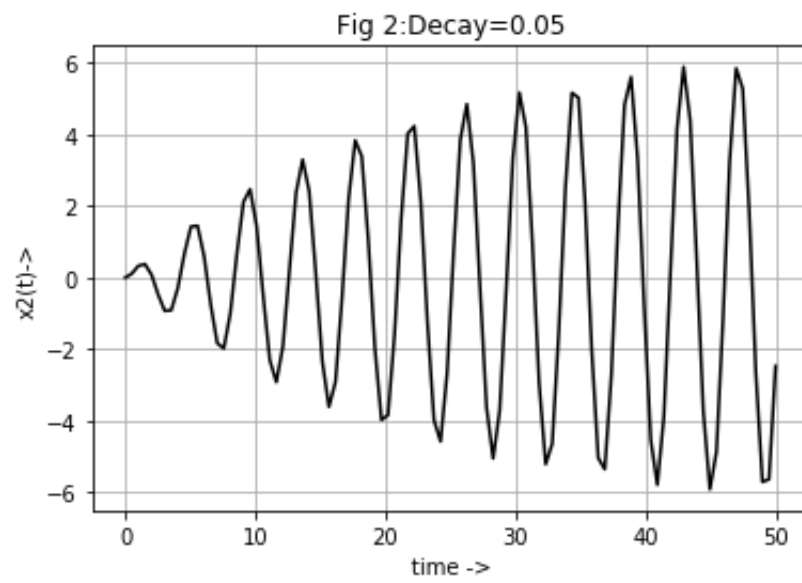
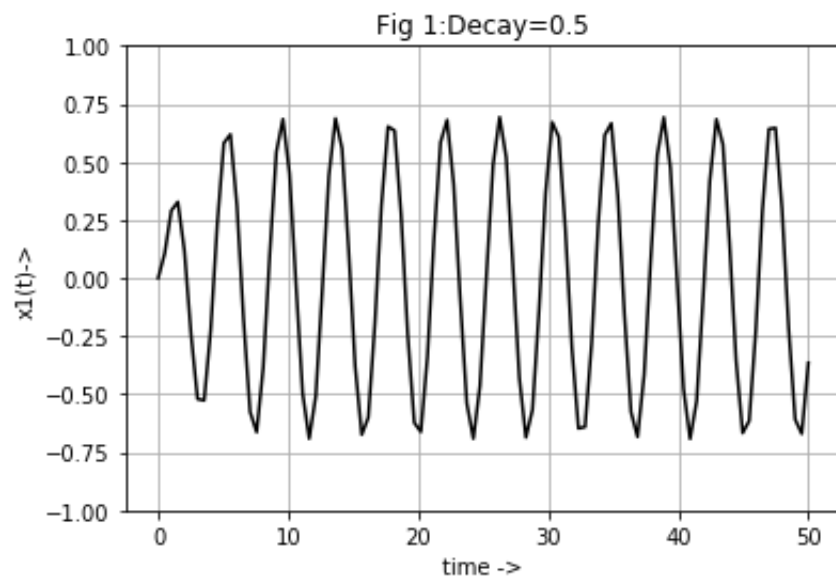


Figure 1: Plots of displacement of block for two different decays

### Results and Discussion :

- As we observe the plot that for smaller decay of  $e^{-0.05t}$ ,  $x(t)$  has large amplitude and its growing as time increases and oscillates.
- Whereas the for higher decay value the amplitude of  $x(t)$  is very small but since its second order system it oscillates.
- And If we observe the Figure 1a, amplitude growth stopped and settles quicker than for smaller decay value whereas for small decay value the  $x(t)$  amplitude settling time is larger because as given below:
- Our input  $f(t)$  to the system has natural frequency that is  $w = w_0$ , so its a resonant case,so the solution of differentail equation for sinusoidal inputs from observing the plot can be of the form  $te^{-dt} \cos(w_0t)$  so for smaller decay value the graph takes more time neutralise the growing effect of  $t$  in the solution.
- So to conclude for small decay ,  $x(t)$  has large amplitude and the time required for it settle or saturate to a certain maximum amplitude is higher compared to large decay case



### 0.2.2 Question 3:

- In an LTI system.  $f(t)$  is the input, and  $x(t)$  is the output.
- To Obtain the system transfer function  
$$\frac{\mathcal{X}(s)}{\mathcal{F}(s)}$$
- we use *signal.lsim* to simulate the problem.
- Here we try to plot the system response for different values of excitation frequencies i.e input frequencies with natural frequency of the system as  $w_0 = 1.5 \text{rads}^{-1}$
- So using a for loop, we sweep the frequency  $w$  of the  $f(t)$  from 1.4 to 1.6 in steps of 0.05 keeping the exponent as  $e^{0.05t}$  that is  $d = 0.05$  and plot the resulting responses.
- So with above conditions laplace transform of  $x(t)$  is

$$X(s) = \frac{s+0.05}{((s+0.05)^2+w^2)(s^2+2.25)} \quad (9)$$

- So the transfer function of the system is

$$H(s) = \frac{s+0.05}{((s+0.05)^2+w^2)(s^2+2.25)} \quad (10)$$

- Using this we analyse the results.

---

```
1  #Q3
2  terms=linspace(1.4,1.6,4)
3  for i in (terms):
4      t,x=x_solver(0.05,i)
5      plt.plot(t,x)
```

```
6
7
8 plt.title('Fig3: Variation with frequency')
9 plt.xlabel('time ->')
10 plt.ylabel('$x_{i}(t)$ ->')#subscript
11 plt.grid()
12 plt.show()
```

---

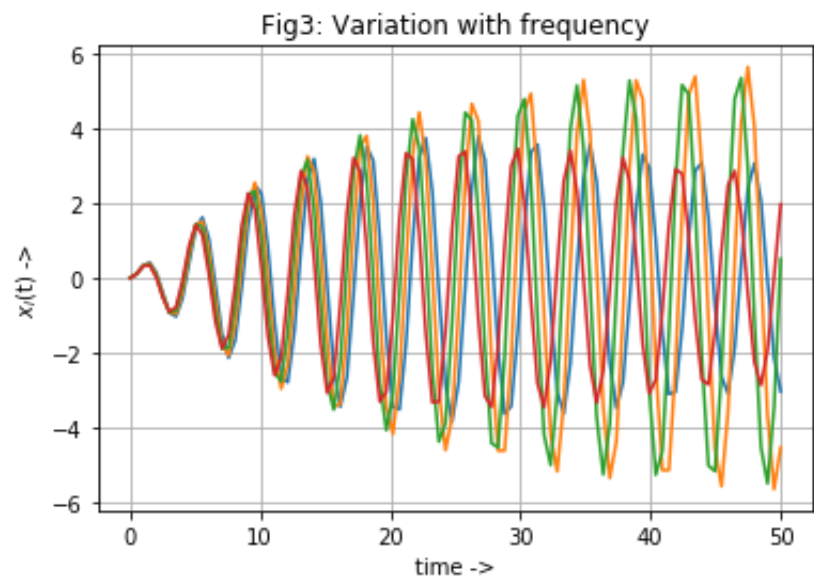


Figure 2: Plots of displacement of block for different frequencies

### Results and Discussion:

- We see that for frequency  $\omega = 1.5 \text{ rad s}^{-1}$ , the amplitude is the largest since it is the resonance frequency of the system.
- As we observe in Figure 2 that when we force (we are basically exciting the spring mass system with  $f(t)$ ) the system with  $w \neq w_0$  where  $w_0 = 1.5$  and  $w \approx w_0$  i.e around  $1.5 \text{ rad s}^{-1}$ , since initially the system is at rest, so when  $f(t)$  is forced upon it, they are in phase, so constructive superposition occurs so the amplitude of  $x(t)$  increases in same fashion for  $w$  very close to  $w_0$  at starting, after that since the forced and natural frequency are not same, it is detuned so amplitude after peak amplitude starts falling slightly but at steady state all forced response dies out, so basically  $x(t)$  will vary only with natural mode since forced response died out eventually.
- Whereas for  $w = w_0$  case resonant occurs, so system and forcing input have same frequency so, the forced response adds up to natural response basically 'tuned', so the amplitude is very high in steady state.

### 0.2.3 Question 4:

- To Solve for a coupled spring problem:
- System satisfies the below equation with  $x(0) = 1$  and  $\dot{x}(0) = y(0) = \dot{y}(0) = 0$ .

$$+ (x-y) = 0$$

(11)

$$+ 2(y-x) = 0$$

(12)

- Solve for  $x(t)$  and  $y(t)$  for  $0 \leq t \leq 20s$  by taking laplace transform of both equations given above and solve for  $\mathcal{X}(s)$  and  $\mathcal{Y}(s)$  using substitution method.
- Now from  $\mathcal{X}(s)$  and  $\mathcal{Y}(s)$  find  $x(t)$  and  $y(t)$  using *system.impulse*.
- Plot  $x(t)$  and  $y(t)$  in the same graph and analyse them

**Code:**

---

```
1  #Q4
2  #solving the coupled differential equations we get fourth order equations fo
3  #which reduce to the following transfer function
4  X=sp.lti([1,0,2],[1,0,3,0])
5  Y=sp.lti([2],[1,0,3,0])
6  t,x=sp.impulse(X, None, np.linspace(0,20,200))
7  t,y=sp.impulse(Y, None, np.linspace(0,20,200))
8
9  #plotting x and y
```

```

10
11 plt.plot(t,x,label='x(t)')
12 plt.legend()
13 plt.title('Fig4a:Solution to X in the coupled differential equations')
14 plt.xlabel('time ->')
15 plt.ylabel('x(t)->')
16 plt.grid()
17 plt.show()
18
19 plt.plot(t,y,label='y(t)')
20 plt.legend()
21 plt.title('Fig4b:Solution to Y in the coupled differential equations')
22 plt.xlabel('time ->')
23 plt.ylabel('y(t)->')
24 plt.grid()
25 plt.show()
26
27 #phase space y vs x plot
28 plt.plot(x,y)
29 plt.title('Fig4c:Phase Space plot of y vs x')
30 plt.xlabel('x ->')
31 plt.ylabel('y ->')
32 plt.grid()
33 plt.show()

```

---

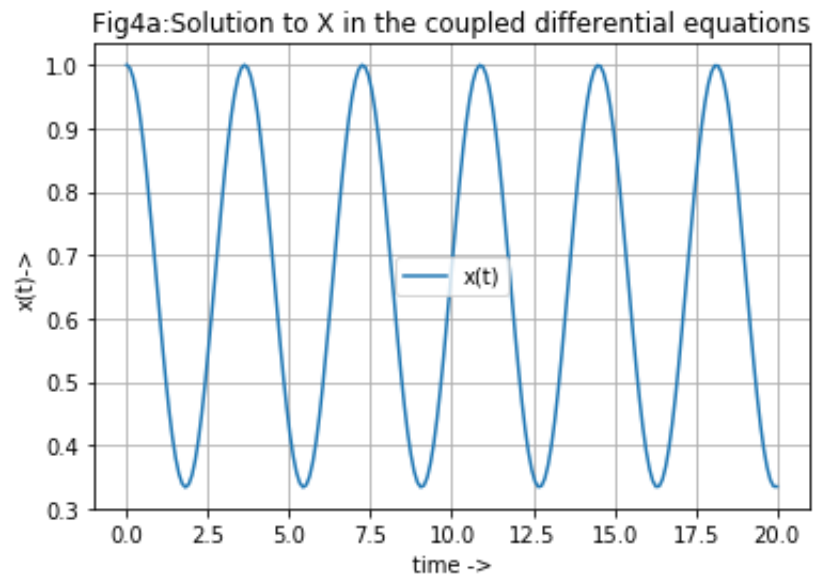


Figure 3: Plot of displacement  $x$  of two coupled blocks

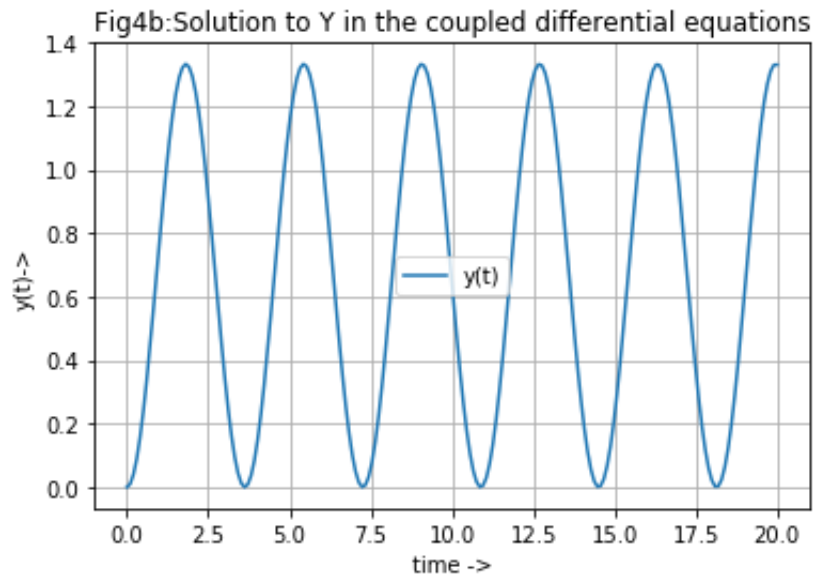


Figure 4: Plot of displacement  $y$  of two coupled blocks

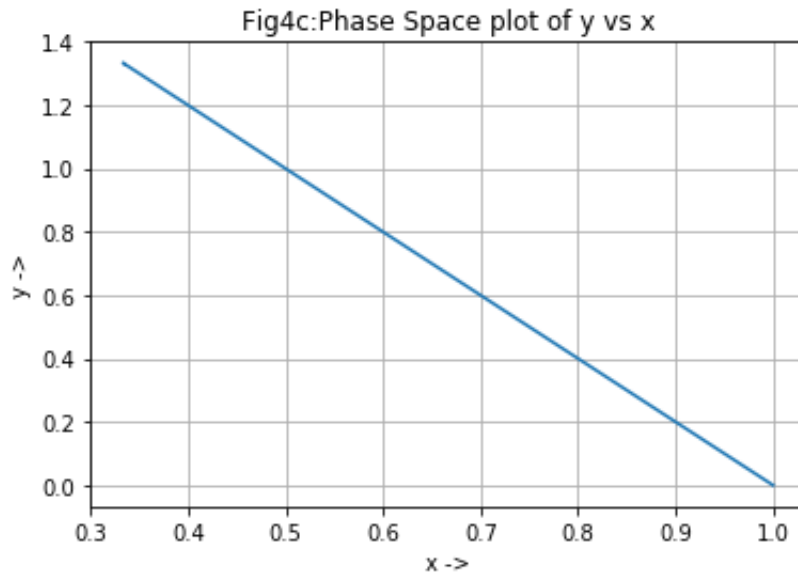


Figure 5: Plot of y vs x of two coupled blocks

#### Results and Discussion:

- As we observe figure 4 that, the  $x(t)$  and  $y(t)$  obtained satisfies the given initial conditions, and oscillating sinusoidally with  $180^\circ$  out of phase reiterated by the third plot above.



#### 0.2.4 Question 5:

- To Obtain the magnitude and phase response of the Steady State Transfer function of the following two-port network
- Transfer function of the RLC network from input to voltage at capacitor in general for given Network is

$$V_o(s) \frac{1}{V_i(s)=\mathcal{H}(s)=\frac{1}{s^2 LC+sRC+1}} \quad (13)$$

- For the given values of  $R = 100\Omega$  ,  $L = 1\mu H$ ,  $C = 1\mu F$
- We get

$$H(s) = \frac{1}{s^2 10^{-12} + s 10^{-4} + 1} \quad (14)$$

- can be written as

$$H(s) = \frac{1}{(1+\frac{s}{10^8})(1+\frac{s}{10^4})} \quad (15)$$

- So system has poles on left half s plane, that too real poles with  $s = -10^4, -10^8 \text{ rads}^{-1}$ , we will observe the effect of poles on magnitude and phase response by analysing plots of them
- Magnitude Response is  $|H(s)|$  evaluated at any point on imaginary axis so basically  $|H(j\omega)|$

- Phase response is  $\angle H(j\omega)$
- Using  $\mathcal{H}(s)$  we calculate magnitude and phase response of it using *Bode()* and plot them and analyse.

---

```

1  #Q5 RLC circuit
2  H=sp.lti([1],[10**(-12),10**(-4),1])
3  w,S,phi=H.bode() #gives bode parameters
4  plt.subplot(2,1,1)
5  plt.semilogx(w,S)
6  plt.title('Fig5a: Bode Magnitude Plot')
7  plt.ylabel('|h(t)|->')
8  plt.xlabel('w')
9  plt.show()
10 plt.subplot(2,1,2)
11 plt.semilogx(w,phi)
12 plt.title('Fig5b: Bode Phase Plot')
13 plt.ylabel('phase(h(t))->')
14 plt.xlabel('w')
15 plt.show()

```

---

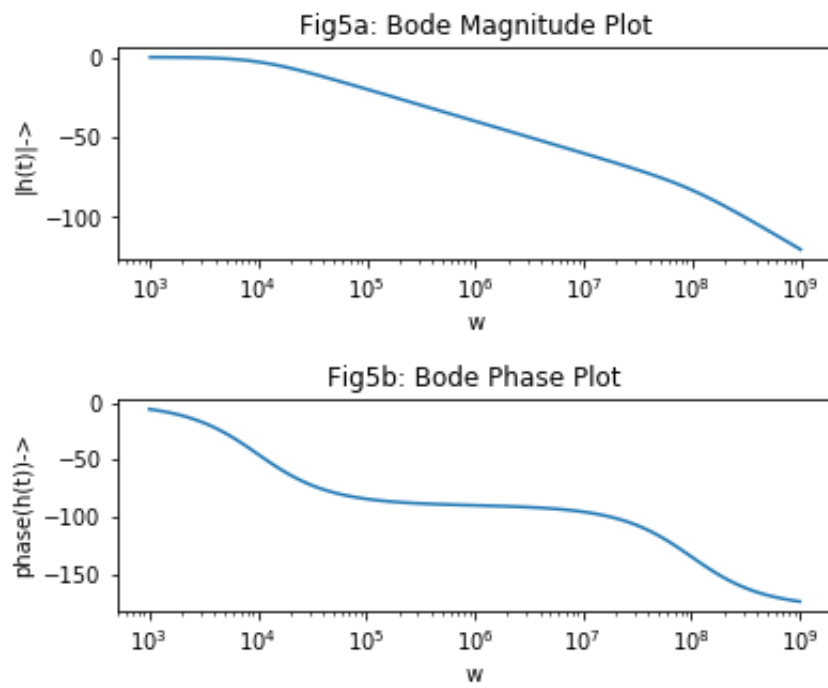


Figure 6: Bode Plots of the LCR system

**Results and Discussion:**

- As we observe figure 6 that,  $0 \leq \angle H(j\omega) < 180^\circ$ .
- So the system is unconditionally stable since phase does not go to  $180^\circ$ .
- Since each pole adds  $90^\circ$  to the phase its clear that the system is second order because it has two poles hence phase go till  $180^\circ$ .
- And since all the poles are in left half s plane RLC Network given is unconditionally stable for given values.

### 0.2.5 Question 6:

- Consider the problem in Q5. If the input signal  $v_i(t)$  is given by

$$v_i(t) = \cos(10^3 t)u(t) \cos(10^6 t)u(t) \quad (16)$$

- Obtain the output voltage  $v_0(t)$  using the transfer function of the system obtained.
- To explain the output signal for  $0 < t < 30$   
 $\mu s$
- And explain the long term response on the  $msec$  timescale

---

```
1  #Q6
2  #part6
3  #define functioning returning input
4  def V_in(t):
5      V_in=(np.cos(10**3*t)-np.cos(10**6*t))*(np.heaviside(t,1)) #heaviside(1)
6      return V_in
7
8  #Short term
9  t1=linspace(0,30e-6,1000)
10 v1=V_in(t1)
11 t1,y,svec=sp.lsim(H,v1,t1)#convolves y=h*v1
12 plt.plot(t1,y)
13 plt.title('Fig6a:Response till 30 us')
14 plt.xlabel('time ->')
15 plt.ylabel('$V_{out}(t) ->')
16 plt.grid()
17 plt.show()
18
19 #Long term
```

```
20 t2=linspace(0,1e-2,1000)
21 v2=V_in(t2)
22 t2,y,svec=sp.lsim(H,v2,t2)#convolves y=h*v2
23 plt.plot(t2,y)
24 plt.title('Fig6b:Response till 10ms')
25 plt.xlabel('time ->')
26 plt.ylabel('$V_{out}(t)$ ->')
27 plt.grid()
28 plt.show()
```

---

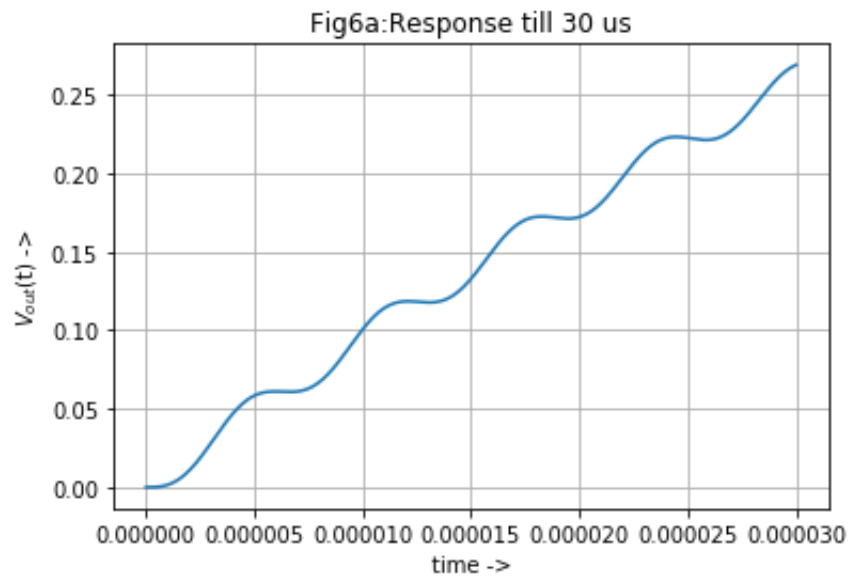


Figure 7: Output of the LCR system(LPF):short response

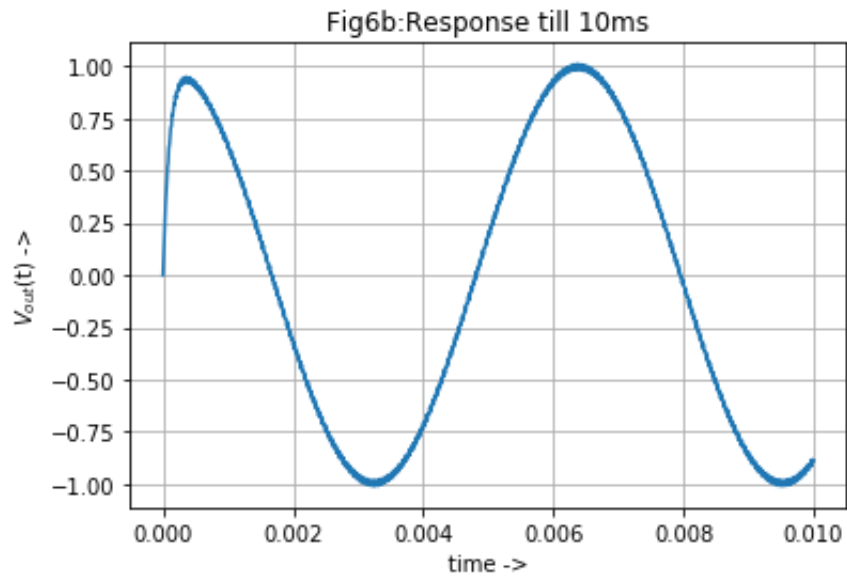


Figure 8: Output of the LCR system(LPF):long response

### Results and Discussion:

- As we observe the plot and the circuit that we know it is a Low pass filter with bandwidth  $0 < \omega < 10^4$ .
- So when the circuit will only pass input with frequencies which are in range of bandwidth only. But since its not a ideal low pass filter as its gain doesn't drop abruptly at  $10^4$  rather gradual decrease which is observed from magnitude response plot.
- So the output  $V_o(t)$  will be mainly of  $\cos(10^3 t)$  with higher frequencies riding over it in long term response i.e Steady state solution.
- The curve is very flickery or not smooth because of the high frequency components only.
- In the previous plot we've zoomed large enough to observe those components.
- The oscillatory behaviour in the graph is because of high frequency components riding over the main signal which is  $\cos(10^3 t)$  since gain of the system  $\approx 1$  for lower frequencies than  $\omega < 10^4$  and gradually decreases as 20dB/dec which is observed from magnitude response plot and for very high frequencies 40dB/dec.
- So there will be some higher frequencies but the gain will be very less since its low pass filter ,that's why we can see there are very small sinusoidal oscillations over the main output signal
- As we observe the plot of Figure 6 , for  $v_0(t)$  from  $0 < t < 30\mu sec$  increases very fast from 0 to 0.3 in just  $30\mu s$  because of transients that is we apply sinusoidal step input to the system i.e the input is zero for  $t < 0$ , so when abruptly the input becomes non zero for  $t \geq 0$ , the system output jumps or raises suddenly from 0 to non-zero values in less time.
- That's why we observe a sharp rise in output at the start.



- And the oscillatory behaviour in the graph is because of high frequency components riding over the main signal which is  $\cos(10^3 t)$  since gain of the system  $\approx 1$  for lower frequencies than  $\omega < 10^4$ .

### 0.3 Conclusion:

- So to conclude we analysed a way to find the solution of continuous time LTI systems using laplace transform with help of Python signals toolbox and got familiarised with solving of differential equations by taking laplace transform instead of doing arduous time domain analysis.