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1. [TRAPEZOIDAL RULE]
Compute $\int_0^1 e^{x^2} dx$ using the trapezoidal rule and the trapezoidal rule with end corrections using the first and third derivatives. Perform this by subdividing $[0,1]$ into $N \in \{2,5,10,20,50,100,200,500,1000\}$ panels and plot the decay of the absolute error using the **three methods**. The value of the integral accurate upto 16 digits is 1.4626517459071816.

Solution:
The methodologies:
For trapezoidal rule, we divide the interval $[0,1]$ into N panels with the ith panel denoted by x_i and the absolute difference between two successive panels denoted by h. Then, the required variants of trapezoidal rule are given as follows.

- Trapezoidal Rule:

$$\int_0^1 e^{x^2} dx = \frac{h}{2} \cdot \left(e^{(x_0)^2} + e^{(x_N)^2} \right) + h \cdot \sum_{i=1}^{N-1} e^{(x_i)^2} + \mathcal{O}(h^2)$$

Where, the final term signifies the error made by the trapezoidal rule.

- Trapezoidal Rule with endpoint corrections using the first derivative:
The generalized trapezoidal rule with endpoint correction can be derived using the Euler Maclaurin sum(shown in the next subdivision) and the final expression makes use of Bernoulli numbers. The expression which corresponds to using the first derivative at the endpoints is given as,

$$\int_0^1 e^{x^2} dx = \frac{h}{2} \cdot \left(e^{(x_0)^2} + e^{(x_N)^2} \right) + h \cdot \sum_{i=1}^{N-1} e^{(x_i)^2} - \frac{b_2}{2!} h^2 \cdot \left(2x e^{x^2} |_{x=0}^{x=1} \right) + \mathcal{O}(h^4)$$

$$\Rightarrow \frac{h}{2} \cdot \left(e^{(x_0)^2} + e^{(x_N)^2} \right) + h \cdot \sum_{i=1}^{N-1} e^{(x_i)^2} - \frac{1}{6} h^2 \cdot e + \mathcal{O}(h^4)$$

Where we've used the fact that the second Bernoulli number b_2 is $\frac{1}{6}$ and that the first derivative of e^{x^2} is $2x e^{x^2}$

- Trapezoidal Rule with endpoint corrections using both the first and third derivatives:

$$\int_0^1 e^{x^2} dx = \frac{h}{2} \cdot \left(e^{(x_0)^2} + e^{(x_N)^2} \right) + h \cdot \sum_{i=1}^{N-1} e^{(x_i)^2} - \frac{b_2}{2!} h^2 \cdot \left(2x e^{x^2} |_{x=0}^{x=1} \right) - \frac{b_4}{4!} h^4 \cdot \left((8x^3 + 12x) e^{x^2} |_{x=0}^{x=1} \right) + \mathcal{O}(h^6)$$

$$\Rightarrow \frac{h}{2} \cdot \left(e^{(x_0)^2} + e^{(x_N)^2} \right) + h \cdot \sum_{i=1}^{N-1} e^{(x_i)^2} - \frac{1}{6} h^2 \cdot e + \frac{1}{36} h^4 \cdot e + \mathcal{O}(h^6)$$

Where we've used the fact that the fourth Bernoulli number b_4 is $-\frac{1}{30}$ and that the third derivative of e^{x^2} is $(8x^3 + 12x) e^{x^2}$

The Plots:
The loglog plots of the decay of the absolute error is given below for the three methods.
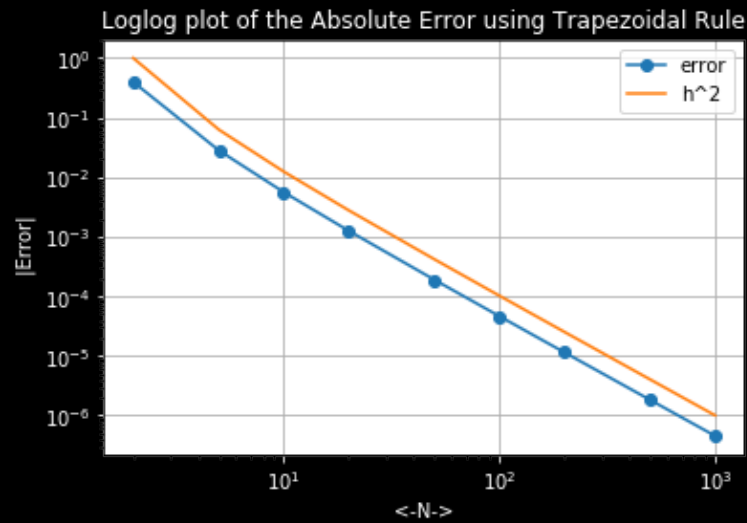

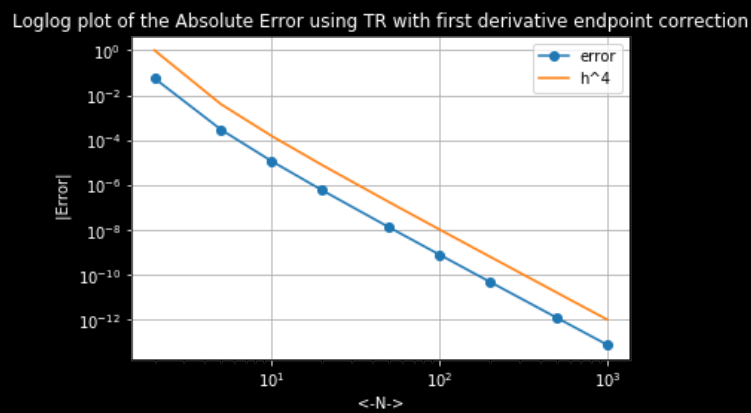Figure 1: Loglog plot of error using Trapezoidal Rule

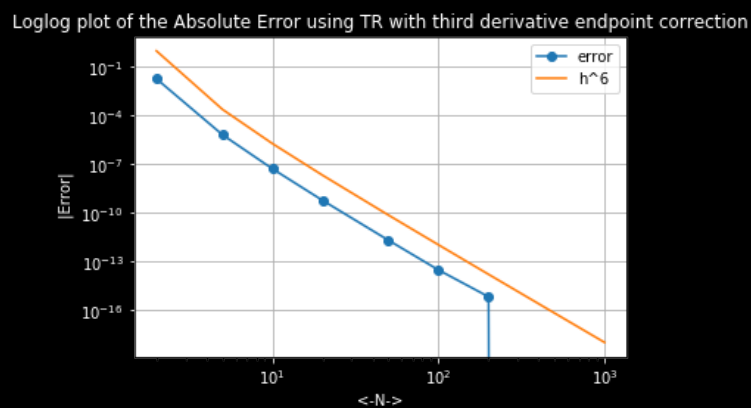Figure 2: Loglog plot of error using Trapezoidal Rule with end corrections using just the first derivative

Figure 3: Loglog plot of error using Trapezoidal Rule with end corrections using the first derivative and the third derivatives

Observations

- As we'd expect, the last method is the best, while plain Trapezoidal Rule is the worst.

- We see that the rate of decay of the absolute error follow the same trends derived above.

- Thus we can conclude that the three methods do a good job at estimating the given integral even for moderately large values of N.

2. [EULER-MACLUARIN APPROXIMATION]
Use the Euler-Macluarin to obtain

$$\log(n!) = \log\left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \mathcal{O}\left(\frac{1}{n}\right)$$

where C is some constant.

Solution:
The Euler Maclaurin sum:
Let g(x) be a function defined in $[c, d]$ where c,d $\in \mathcal{Z}$ and let it have (p+1) derivatives.
Then the Euler Maclaurin sum is written as follows,

$$\sum_{n=c}^{d} g(n) = \int_c^d g(x)dx + \left(\frac{g(c) + g(d)}{2}\right) + \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!}\left(g^{2k-1}(d) - g^{2k-1}(c)\right) - \int_c^d \frac{B_{2p}(\{t\})}{(2p)!}g^{2p}(t)dt \tag{1}$$

Where $\{t\}$ refers to the fractional part of t. **Applying it to log(n!):**
In the equation 1, let us use g(x) = log(x), c = 1 and d = n and p = 1. This construction is valid as the second derivative of log(x) exists. Here, we're making use of the fact that log(n!) $= \sum_{k=1}^{n} \log k$. So with this in our hand, applying the Euler Maclaurin sum yields,

$$\sum_{k=1}^{n} \log n = \int_1^n \log x dx + \left(\frac{\log 1 + \log n}{2}\right) + \frac{b_2}{2!}\left(\frac{1}{x}\big|_{x=1}^{x=n}\right) - \int_1^n \frac{B_2(\{t\})}{2!}\cdot\frac{1}{t^2}dt$$

$$\Rightarrow \int_1^n \log x dx + \left(\frac{\log n}{2}\right) + \frac{b_2}{2!}\left(\frac{1}{n} - 1\right) - \int_1^n \frac{B_2(\{t\})}{2!}\cdot\frac{1}{t^2}dt \qquad (\text{As }\log(1) = 0)$$

Now, using the fact that $\int \log x dx = x(\log x - 1)$ and $B_2(x) = x^2 - x + \frac{1}{6}$, we have,

$$\log n! = x(\log x - 1)\big|_{x=1}^{x=n} + \left(\frac{\log n}{2}\right) + \frac{1}{12}\left(\frac{1}{n} - 1\right) - \int_1^n \frac{B_2(\{t\})}{2!}\cdot\frac{1}{t^2}dt \qquad (\text{As }b_2 = \frac{1}{6})$$

$$\Rightarrow n\log n - n + 1 + \left(\frac{\log n}{2}\right) + \frac{1}{12}\left(\frac{1}{n} - 1\right) - \int_1^n \frac{B_2(\{t\})}{2t^2}dt$$

Analyzing the final integral, we see that we can impose a trivial bound on $B_2(\{t\})$. We know that $0 \leq \{t\} < 1$. Therefore, we can assert that, $B_2(\{t\}) < B_2(1)$ as $B_2(t) = t^2 - t + \frac{1}{6}$ is an increasing function in the interval $[0.5, \infty]$ which can be observed by taking a look at it's first derivative $2t - 1$ which is always positive in the interval. Besides this, we also see that $B_2(0) = B_2(1) = \frac{1}{6}$, therefore using the previous inference, we see that, $B_2(\{t\}) < \frac{1}{6} < 2$. Using this, we can find an upper bound for the integral $\int_1^n \frac{B_2(\{t\})}{2t^2}dt$.

$$\int_1^n \frac{B_2(\{t\})}{2t^2}dt < \int_1^n \frac{2}{2t^2}dt$$
$$< \frac{-1}{t}\big|_{t=1}^{t=n}$$
$$< 1 - \frac{1}{n}$$

Therefore, with this, we can say that $\int_1^n \frac{B_2(\{t\})}{2t^2}dt = 1 - \frac{1}{n} - k$ where k is a positive constant which appropriately converts the above inequality to an equality.

Using this in the earlier equation we derived, we have,

$$\log n! = n \log n - n + 1 + \left(\frac{\log n}{2}\right) + \frac{1}{12}\left(\frac{1}{n} - 1\right) - \int_1^n \frac{B_2(\{t\})}{2t^2}dt$$

$$\Rightarrow n \log n - n + 1 + \left(\frac{\log n}{2}\right) + \frac{1}{12}\left(\frac{1}{n} - 1\right) - \left(1 - \frac{1}{n} - k\right)$$

Clubbing all the constants into a single constant log(C) and clubbing all the $\frac{1}{n}$ terms together, we have

$$\Rightarrow n \log \frac{n}{e} + \log \sqrt{n} + \mathcal{O}\left(\frac{1}{n}\right) + \log C$$

$$\Rightarrow \log \left(C \left(\frac{n}{e}\right)^n \cdot \sqrt{n}\right) + \mathcal{O}\left(\frac{1}{n}\right)$$

Hence, we've obtained that, $\log(n!) = \log\left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \mathcal{O}\left(\frac{1}{n}\right)$ for an appropriate constant C.

3. [STIRLING FORMULA]
   We will now determine C in the above question as follows.

   (a) Use integration by parts to obtain an expression for $I_k = \int_0^{\frac{\pi}{2}} \sin^k(x)dx$.
       (Hint: It might be easier to look at the even and the odd cases separately)

   Solution:
   Deriving a recurrence relation for I_k
   We have $I_k = \int_0^{\frac{\pi}{2}} \sin^k(x)dx$. Inspecting I_k and substituting k = 0, we have $I_0 = \frac{\pi}{2}$ while substituting k = 1, yields $\int_0^{\frac{\pi}{2}} \sin(x)dx = 1$.

   With this information in hand, we can integrate I_k by parts, i.e., $\int u \, dv = uv - \int v \, du$, by taking u = $\sin^{k-1}(x)$ and v = -cos(x).

   $$\int_0^{\frac{\pi}{2}} \sin^k(x)dx = -\sin^{k-1}(x)\cos(x)\Big|_{x=0}^{x=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (k-1)\sin^{k-2}(x)\cos^2(x)dx$$

   The first term evaluates to 0, so we're left with

   $$\Rightarrow \int_0^{\frac{\pi}{2}} (k-1)\sin^{k-2}(x)\left(1 - \sin^2(x)\right)dx \qquad \text{(As } \cos^2(x) = 1 - \sin^2(x)\text{)}$$

   $$\Rightarrow \int_0^{\frac{\pi}{2}} (k-1)\sin^{k-2}(x)dx - \int_0^{\frac{\pi}{2}} (k-1)\sin^k(x)dx$$

   $$\Rightarrow (k-1)I_{k-2} - (k-1)I_k$$

   Therefore, from this, we have,

   $$I_k = \frac{k-1}{k}I_{k-2} \qquad (2)$$

   Using this recurrence relation and the values of I_0 and I_1 we derived earlier, we can find the value of any I_k as follows,

- Case 1: k = 2n,i.e., k is even

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$
$$\Rightarrow \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot I_{2n-4}$$

Repeating the above step till we hit I_0, we have

$$\Rightarrow \prod_{k=1}^{n} \left(\frac{2k-1}{2k} \right) I_0$$
$$\Rightarrow \frac{\pi}{2} \cdot \prod_{k=1}^{n} \left(\frac{2k-1}{2k} \right)$$

- Case 2: k = 2n+1, i.e., k is odd

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$$

Following the same steps as the prev case, we have

$$\Rightarrow \prod_{k=1}^{n} \left(\frac{2k}{2k+1} \right) I_1$$
$$\Rightarrow \prod_{k=1}^{n} \left(\frac{2k}{2k+1} \right)$$

Thus, we have the following expressions for I_k

$$I_k = \begin{cases} \frac{\pi}{2} \cdot \prod_{k=1}^{n} \left(\frac{2k-1}{2k} \right) & \text{k = 2n} \\ \prod_{k=1}^{n} \left(\frac{2k}{2k+1} \right) & \text{k = 2n+1} \end{cases} \tag{3}$$

(b) Prove that I_k is a monotone decreasing sequence.

Solution:
In the given interval $\left[0, \frac{\pi}{2} \right]$, sin(x) is a monotonically increasing function with $0 \le \sin(x) \le 1$ with equality attained at the endpoints.
Therefore, using this, we can assert that $\sin^k(x) \le \sin^{k-1}(x)$ in the given interval, $\forall$ x $\in \left(0, \frac{\pi}{2} \right)$ with equality attained at the endpoints.
Therefore, by this, as the above inequality holds $\forall$ x in the interval, we can extend this to the integrals I_k, I_{k-1}.
Thus, we have $I_k \le I_{k-1} \forall k \in \mathbb{N}$ making I_k a monotonically decreasing sequence.

(c) Show that

$$\lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}} = 1$$

Solution:
From the result in the previous subdivision, we have,

$$\frac{I_{2m-1}}{I_{2m+1}} \ge 1$$

as I_k is a monotonically decreasing sequence $\forall k \in \mathbb{N}$.

From 2, we have,

$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}$$

Thus, we can devise an inequality from either side for $\frac{I_{2m-1}}{I_{2m+1}}$.

$$1 \le \frac{I_{2m-1}}{I_{2m+1}} \le \frac{2m+1}{2m}$$

$$\lim_{m \to \infty} 1 \le \lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}} \le \lim_{m \to \infty} \frac{2m+1}{2m}$$

Invoking the Sandwich Theorem, we see that $\lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}} = 1$.

(d) Conclude that

$$\lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} = 1$$

Solution:
Proceeding as we did with the previous case, since I_k is a monotonically decreasing sequence, we have,

$$I_{2m+1} \le I_{2m} \le I_{2m-1}$$

Dividing throughout by I_{2m+1}

$$1 \le \frac{I_{2m}}{I_{2m+1}} \le \frac{I_{2m-1}}{I_{2m+1}} \qquad \text{(As } I_k > 0 \ \forall k \in \mathbb{N})$$

$$\lim_{m \to \infty} 1 \le \lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} \le \lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}}$$

Invoking the Sandwich Theorem yet again, we see that $\lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} = 1$ as $\lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}} = 1$.

(e) Hence, infer that the central binomial coefficient is asymptotically given by

$$^{2m}C_m \sim \frac{4^m}{\sqrt{m\pi}}$$

where $f(m) \sim g(m) \Rightarrow \lim_{m \to \infty} \frac{f(m)}{g(m)} = 1$

Solution:
Expanding $^{2m}C_m$
Let us try expanding $^{2m}C_m$.

$$^{2m}C_m = \frac{(2m)!}{(m!) \cdot (m!)}$$

$$\Rightarrow \frac{2m \cdot (2m-1) \cdot (2(m-1)) \ldots 3 \cdot 2 \cdot 1}{m \cdot (m-1) \ldots 1 \cdot (m!)}$$

$$\Rightarrow \frac{2m \cdot (2(m-1)) \ldots 2 \cdot 1}{m \cdot (m-1) \cdot 1} \cdot \frac{(2m-1) \cdot (2m-3) \ldots 3 \cdot 1}{m!}$$

$$\Rightarrow 2^m \cdot \frac{(2m-1) \cdot (2m-3) \ldots 3 \cdot 1}{m!} \qquad \text{(Cancelling common terms)}$$

$$\Rightarrow 2^m \prod_{k=1}^{m} \left(\frac{2k-1}{k} \right)$$

Alternative way to arrive at a similar expression

Using the results of the previous two subdivisions, we have, $\lim\limits_{m\to\infty} \frac{I_{2m}}{I_{2m-1}} = \lim\limits_{m\to\infty} \frac{\frac{I_{2m}}{I_{2m+1}}}{\frac{I_{2m-1}}{I_{2m+1}}} = \frac{\lim\limits_{m\to\infty} \frac{I_{2m}}{I_{2m+1}}}{\lim\limits_{m\to\infty} \frac{I_{2m-1}}{I_{2m+1}}} = 1$

From 3, we have, $I_{2m} = \frac{\pi}{2} \cdot \prod\limits_{k=1}^{m} \left(\frac{2k-1}{2k}\right)$ and $I_{2m-1} = \prod\limits_{k=1}^{m-1} \left(\frac{2k}{2k+1}\right)$. Multiplying and dividing I_{2m-1} by 2m and changing

the indices of the product to match that of I_{2m}, we have, $I_{2m-1} = \frac{1}{2m} \cdot \prod\limits_{k=1}^{m} \left(\frac{2k}{2k-1}\right)$.

Using this, we obtain,

$$\frac{I_{2m}}{I_{2m-1}} = \frac{\frac{\pi}{2} \cdot \prod\limits_{k=1}^{m} \left(\frac{2k-1}{2k}\right)}{\frac{1}{2m} \cdot \prod\limits_{k=1}^{m} \left(\frac{2k}{2k-1}\right)}$$

$$\Rightarrow m\pi \cdot \left(\prod_{k=1}^{m} \left(\frac{2k-1}{2k}\right)\right)^2$$

$$\Rightarrow \frac{m\pi}{4^m} \cdot \left(\prod_{k=1}^{m} \left(\frac{2k-1}{k}\right)\right)^2$$

Comparing this with the expression for $^{2m}C_m$, we see that,

$$\frac{I_{2m}}{I_{2m-1}} = \frac{m\pi}{4^m} \cdot \left(\prod_{k=1}^{m} \left(\frac{2k-1}{k}\right)\right)^2$$

$$\Rightarrow \frac{m\pi}{4^m} \cdot \frac{\left(^{2m}C_m\right)^2}{4^m}$$

$$\Rightarrow \left(\frac{\left(^{2m}C_m\right)}{\frac{4^m}{\sqrt{m\pi}}}\right)^2$$

We see that, $\lim\limits_{m\to\infty} \frac{I_{2m}}{I_{2m-1}} = 1$ and therefore, $\lim\limits_{m\to\infty} \left(\frac{\left(^{2m}C_m\right)}{\frac{4^m}{\sqrt{m\pi}}}\right)^2 = 1$ too. For this to happen, $\lim\limits_{m\to\infty} \left(\frac{\left(^{2m}C_m\right)}{\frac{4^m}{\sqrt{m\pi}}}\right)$ must be 1
as the above limit can be expressed as the limit of the square of a function which in this case is also the square of the
limit of the same function as the limit is finite.
Therefore, by the definition of asymptotic behaviour, we have shown that, $^{2m}C_m \sim \frac{4^m}{\sqrt{m\pi}}$.

(f) Hence conclude that C in the above question is $\sqrt{2\pi}$.

Solution:
Using the expression derived in the second question, we have,

$$\log(n!) = \log\left(C \left(\frac{n}{e}\right)^n \sqrt{n}\right) + \mathcal{O}\left(\frac{1}{n}\right)$$

$$n! = \left(C \left(\frac{n}{e}\right)^n \sqrt{n}\right) \cdot e^{\mathcal{O}\left(\frac{1}{n}\right)}$$

Plugging this into the expression for $^{2m}C_m$, we have,

$$^{2m}C_m = \frac{(2m)!}{(m!) \cdot (m!)}$$

$$\Rightarrow \frac{\left(C \left(\frac{2m}{e}\right)^{2m} \sqrt{2m}\right) \cdot e^{\mathcal{O}\left(\frac{1}{2m}\right)}}{\left(C \left(\frac{m}{e}\right)^{m} \sqrt{m}\right)^{2} \cdot e^{2\mathcal{O}\left(\frac{1}{m}\right)}}$$

$$\Rightarrow \frac{4^m \sqrt{2m}}{Cm} \cdot e^{\mathcal{O}\left(\frac{1}{m}\right)}$$

$$\Rightarrow \frac{4^m \sqrt{2}}{C\sqrt{m}} \cdot e^{\mathcal{O}\left(\frac{1}{m}\right)}$$

In the previous subdivision, we saw that, $\lim\limits_{m\to\infty} \left(\frac{\left(^{2m}C_m\right)}{\frac{4^m}{\sqrt{m\pi}}}\right) = 1$.

Substituting the above expression for $^{2m}C_m$, we get, $\lim\limits_{m\to\infty} \left(\frac{\frac{4^m \sqrt{2}}{C\sqrt{m}} \cdot e^{\mathcal{O}\left(\frac{1}{m}\right)}}{\frac{4^m}{\sqrt{m\pi}}}\right) = 1$.

As m $\to \infty$, $e^{\mathcal{O}\left(\frac{1}{m}\right)} \to 1$. Therefore, we have $\lim\limits_{m\to\infty} \frac{\sqrt{2\pi}}{C} = 1$. Hence C $= \sqrt{2\pi}$.

(g) Hence, obtain the Stirling Formula:

$$n! \sim \left(\frac{n}{e}\right)^{n} \sqrt{2\pi n}$$

Solution:
As we saw,as m $\to \infty$, $e^{\mathcal{O}\left(\frac{1}{m}\right)} \to 1$. Therefore, we have, $n! \sim \left(C \left(\frac{n}{e}\right)^{n} \sqrt{n}\right)$. Substituting for C derived in the previous subdivision, we obtain the Stirling Formula, $n! \sim \left(\frac{n}{e}\right)^{n} \sqrt{2\pi n}$.

(h) Obtain the relative error in $n!$ using the Stirling formula for n $\in 20, 50$.

Solution:
Relative Error:
The expression for relative error is

$$\left|\frac{n! - \left(\frac{n}{e}\right)^{n} \sqrt{2\pi n}}{n!}\right|$$

We can however remove the modulus as stirling approximation is a lower bound.
Substituting n $\in \{20, 50\}$, we get,(rounded to 7 decimal places)

n	error
20	0.0041577
50	0.0016653

(i) Obtain a better estimate for $n!$, which is accurate upto $\mathcal{O}\left(\frac{1}{n^3}\right)$.

Solution:
Proceeding as we did in the second question, referring to 1 and plugging g(x) = log(x), c = 1 and d = n and p = 3, we

have,

$$\sum_{k=1}^{n} \log n = \int_{1}^{n} \log x\, dx + \left(\frac{\log 1 + \log n}{2}\right) + \frac{b_2}{2!}\left(\frac{1}{x}\Big|_{x=1}^{x=n}\right) + \frac{b_4}{4!}\left(\frac{-2}{x^3}\Big|_{x=1}^{x=n}\right) - \int_{1}^{n} \frac{B_4(\{t\})}{4!} \cdot \frac{-6}{t^4}\, dt$$

$$\Rightarrow n\log n - n + 1 + \left(\frac{\log n}{2}\right) + \frac{1}{12}\left(\frac{1}{n} - 1\right) - \frac{1}{720}\left(2 - \frac{2}{n^3}\right) - \int_{1}^{n} \frac{B_4(\{t\})}{4!} \cdot \frac{-6}{t^4}\, dt$$

We can see that $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$. On observing the derivative of this function, $4x^3 - 6x^2 + 2x = 4x(x-0.5)(x-1)$, we see that the function is decreasing in the interval $(0.5, 1)$ and increasing in the interval $(0, 0.5)$. Additionally, we see that x=0 and x=1 correspond to local minima and that x = 0.5 corresponds to a local maximum of this function on observing the second derivative at these values. Thus, we see that $B_4(\{t\}) \leq B_4(0.5) < 12$. Thus, as in the previous case, we see that,

$$\int_{1}^{n} \frac{B_4(\{t\})}{4!} \cdot \frac{-6}{t^4}\, dt < \int_{1}^{n} \frac{-72}{4!t^4}\, dt$$

$$< \int_{1}^{n} \frac{-3}{t^4}\, dt$$

$$< \frac{1}{t^3}\Big|_{x=1}^{x=n}$$

$$< \frac{1}{n^3} - 1$$

Therefore, for an appropriate positive constant k, we have, $\int_{1}^{n} \frac{B_4(\{t\})}{4!} \cdot \frac{-6}{t^4}\, dt = \frac{1}{n^3} - 1 - k$. Using this and as in the previous case, clubbing all $\mathcal{O}(\frac{1}{n^3})$ and the constant terms together, we get,

$$\log(n!) = n\log n - n + \frac{\log n}{2} + \frac{1}{12n} + \log C + \mathcal{O}\left(\frac{1}{n^3}\right)$$

$$\Rightarrow \log\left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) + \frac{1}{12n} + \mathcal{O}\left(\frac{1}{n^3}\right)$$

From this, taking anti-log on both sides, we have, $n! = \left(C\left(\frac{n}{e}\right)^n \sqrt{n}\right) \cdot e^{\frac{1}{12n}} \cdot e^{\mathcal{O}\left(\frac{1}{n^3}\right)}$.

As we saw before, as $n \to \infty$, $e^{\frac{1}{12n}} \cdot e^{\mathcal{O}\left(\frac{1}{n^3}\right)} \to 1$. Therefore, the problem asymptotically reduces to the one we've already solved before, the Stirling Formula. Therefore, C in the above expression is again equal to $\sqrt{2\pi}$.

Hence, a better estimate of n! accurate upto $\mathcal{O}\left(\frac{1}{n^3}\right)$ is given by $n! \sim \left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right) \cdot e^{\frac{1}{12n}}$