Understanding and Applying Black-Scholes Equation

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III.1 Modeling change in the world around us: Partial Differential Equations

Certificate of Originality

The work embodied in this report entitled "***Understanding and Applying Black-Scholes Equation***" has been carried out by ***Abhishek Kumar*** for the paper"Modeling change in the world around us: Partial Differential Equations". We declare that the work and language included in this project report is free from any kind of plagiarism.

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Abstract

Understanding and Applying Black-Scholes Equation

By

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Cluster Innovation Centre, 2022

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The Black-Scholes model, also known as the Black-Scholes-Merton (BSM) model, is one of the most important concepts in modern financial theory. This mathematical equation estimates the theoretical value of derivatives based on other investment instruments, taking into account the impact of time and other risk factors.

This paper strives to explore the solution of the Black-Scholes Equation which is used in mathematical finance. It will derive the solution to the Black-Scholes equation, using the solution of the Heat Equation. This solution can then be used to find the fair price of an international call option. It also includes various examples and applications of Black-Scholes Equation by using current stocks.

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Introduction

The Black-Scholes equation is a partial differential equation (PDE) in mathematical finance that governs the price evolution of a European call or European put in the Black-Scholes model. In general, the term may refer to a similar PDE that can be derived for a variety of options, or derivatives more broadly.

The Black-Scholes-Merton (BSM) model is a financial pricing model. It is used to calculate the value of stock options. The BSM model calculates the fair value of stock options based on six variables: volatility, type, underlying stock price, strike price, time, and risk-free rate. It is founded on the hedging principle and aims to eliminate the risks associated with the volatility of underlying assets and stock options.

The Black-Scholes-Merton model can be described as a second order partial differential equation, known as the Black-Scholes equation – which describes the price of stock options over time.

Our Aim

Our main objective is to understand the Black-Scholes Equation, derive its solution, and to apply the equation to find the fair price of a European call option, including examples using current stocks.

General Assumptions

There are assumptions that must be made to use the Black-Scholes equation. They include:

1. The option can only be exercised at the expiration date, as it is a European option.
2. The direction of the stock's price can't be consistently predicted and is completely random.
3. Constant composition returns are normally distributed.
4. The future stock price at a given point of time must be lognormally distributed.
5. Volatility is known and constant.
6. There are efficient markets.
7. There are no dividends during the life of the option.
8. The risk-free rate is known and constant.
9. There are no taxes or transaction costs involved.
10. The returns on the risky asset are normally distributed.

Assumption of Variables

Each variable that is used describes a particular entity:

• S = Stock price at the beginning of the time period of the option.

• K = Strike price, a price set between the buyer and seller of the option.

• $(T – t)$ = Expiration date minus start date, the total amount of time until the option is exercised (in years).

• r = Risk-free interest rate.

• σ = Volatility of the stock.

Taylor’s Theorem

First off, one must understand the significance of Taylor’s theorem. It is used for the expansion of the infinite series such as sin x, log x etc. so that one can approximate the values of these functions or polynomials. Taylor’s theorem is used for approximation of k-time differentiable function.

Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor series,

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2\frac{f''(a)}{2!} + (x-a)^3\frac{f'''(a)}{3!} + \ldots + (x-a)^k\frac{f^k(a)}{k!}$$

Ito’s Theorem

Ito's Lemma is a key component in the Ito Calculus, used to determine the derivative of a time-dependent function of a stochastic process.

$$\int_0^t X_s \, dB_s = \lim_{n\to\infty} \sum_{k=0}^{nt} X_{k/n} (B_{(k+1)/n} - B_{k/n})$$

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds$$

Explanation of Black-Scholes Equation

The Black-Scholes option pricing formula was developed in 1973 to price the European put or call options on a stock that does not pay a dividend or make other distributions. It assumes the underlying stock price follows a Brownian motion.

$$C_{call} = S\phi(d_1) - Xe^{-rT}\phi(d_2)$$
$$P_{put} = Xe^{-rT}\phi(-d_2) - S\phi(-d_1)$$

Here, C is the call price and P is the put price.

$$d_1 = \frac{\log(S/X) + (r + \sigma^2/2)}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

All other variables have been discussed earlier in the report.

First, start with the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Then set $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$ and solve for τ:

$$\frac{\tau}{\frac{1}{2}\sigma^2} = T - t$$

$$\tau = (T - t)\frac{1}{2}\sigma^2$$

Next set $S = Ke^x$ and solve for x:

$$e^x = \frac{S}{K}$$

$$x = ln\frac{S}{K}$$

With both of these equations, set:

$$V(S, t) = Kv(x, \tau) \tag{2}$$

The next step is to take the first and second derivatives of V with respect to stock price and the first derivative with respect to time:

$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial \tau} * \frac{\partial \tau}{\partial t} = K\frac{\partial v}{\partial \tau}\left((T - t)\frac{1}{2}\sigma^2\frac{\partial}{\partial t}\right) = K\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}$$

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x} * \frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x}\left(ln\frac{S}{K}\frac{\partial}{\partial S}\right) = K\frac{\partial v}{\partial x} * \frac{1}{S}$$

Using $\frac{\partial x}{\partial S} = \frac{1}{\frac{S}{K}} * \frac{1}{K} = \frac{1}{S}$:

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x} * \frac{1}{S}\right)$$
$$= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\frac{1}{S}$$
$$= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial S} * \frac{1}{S}$$
$$= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}$$

With these equations, the terminal equation is set to:

$$V(S, T) = max(S - K, 0) = max(Ke^x - K, 0)$$

$$V(S, T) = Kv(x, 0) \text{ and } v(x, 0) = max(e^x - 1, 0)$$

Take the derivatives and plug them back into equation (1):

$$K\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2} + \frac{\sigma^2}{2}S^2\left(K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\right) + rS\left(K\frac{\partial v}{\partial x} * \frac{1}{S}\right) - rKv = 0$$

Simplify the equation by factoring out the K values, canceling out S and S^2:

$$\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2} + \frac{\sigma^2}{2}S^2\left(\frac{\partial v}{\partial x} * \frac{-1}{S^2} + \frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\right) + rS\left(\frac{\partial v}{\partial x} * \frac{1}{S}\right) - rv = 0$$

$$\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2} + \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\left(\frac{\partial v}{\partial x}\right) - rv = 0$$

Solve for $\frac{\partial v}{\partial \tau}$:

$$\frac{\partial v}{\partial \tau} * \frac{\sigma^2}{2} = \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\frac{\partial v}{\partial x} - rv$$

Factor out $\frac{\sigma^2}{2}$, let $k = \frac{r}{\frac{\sigma^2}{2}}$ to substitute, and combine like terms:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} + \frac{r}{\frac{\sigma^2}{2}} * \frac{\partial v}{\partial x} - \frac{r}{\frac{\sigma^2}{2}} v$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} + k\frac{\partial v}{\partial x} - kv$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \qquad (3)$$

This leaves one parameter, k, that has no dimension. From this, rescale the v equation so that:

$$v = e^{\alpha x+\beta\tau}u(x,\tau) \qquad (4)$$

Derive according to x and τ:

$$\frac{\partial v}{\partial \tau} = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}$$

$$\frac{\partial v}{\partial x} = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}$$

Plug these derivatives into equation (3):

$$\beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right) - ke^{\alpha x+\beta\tau}u$$

Divide by $e^{\alpha x+\beta\tau}$ and combine like terms:

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku$$

$$\frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku - \beta u$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(k-1+2\alpha) + u(\alpha^2 + k\alpha - \alpha - k - \beta) \qquad (5)$$

The coefficients should be equal to zero, meaning that $u = 0$ and $\frac{\partial u}{\partial x} = 0$. Choose $\alpha = \frac{-(k-1)}{2}$ and $\beta = \alpha^2 + (k-1)\alpha - k = \frac{-(k+1)^2}{4}$ then plug into equation (5). This will lead to the basis of the Heat Equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left(k-1+2\left(-\frac{k-1}{2}\right)\right) + u\left(\left(-\frac{k-1}{2}\right)^2 + k\left(-\frac{k-1}{2}\right) - \left(-\frac{k-1}{2}\right) - k - \frac{-(k+1)^2}{4}\right)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[k-1-(k-1)] + u\left(\frac{k^2-2k+1}{4} - \frac{k^2-k}{2} + \frac{k-1}{2} - k + \frac{k^2+2k+1}{4}\right)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[0] + u\left(\frac{k^2-2k+1}{4} - \frac{2k^2-2k}{4} + \frac{2k-2}{4} - \frac{4k}{4} + \frac{k^2+2k+1}{4}\right)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + u\left(\frac{k^2-2k+1-2k^2+2k+2k-2-4k+k^2+2k+1}{4}\right)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + u[0]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$u_\tau = u_{xx}$$

The initial condition is then transformed into:

$$u(x,0) = max\left(e^{\frac{(k+1)}{2}x} - e^{\frac{(k-1)}{2}x}, 0\right) \tag{6}$$

This leads to the Heat equation solution, which we will transform to use for the Black-Scholes equation:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_o(s)e^{\frac{-(x-s)^2}{4\tau}}\,ds$$

Make a change of variable so that $s = z\sqrt{2\tau} + x$. The goal is to get the exponent into the form of $\frac{-y^2}{2}$, which is why $z = \frac{x-s}{\sqrt{2\tau}}$, to get the equation of the standard normal deviation. This will then be used later in this derivation to find the final solution. The derivatives of these equations will then be $ds = dx$ and $dx = \sqrt{2\tau}dz$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_o(z\sqrt{2\tau}+x)e^{\frac{-z^2}{2}}\,dz \tag{7}$$

From this transformation, there is a change in equation (6) for the x value:

$$u_o = e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} \tag{8}$$

It must happen that $u_o > 0$ because the time value cannot be less than 0. So, $x > -\frac{\sqrt{x}}{2\tau}$ which transforms the base of the domain of equation (7):

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2}} dz \qquad (9)$$

After the split of the integral, take the first integral and complete the square of the exponent:

$$\frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} = -\frac{1}{2}\left(z^2 - z\sqrt{2\tau}(k+1)\right) + \frac{x(k+1)}{2}$$

$$= -\frac{1}{2}\left(z^2 - z\sqrt{2\tau}(k+1) + \frac{\tau}{2}(k+1)^2\right) + \frac{x(k+1)}{2} - - \frac{\tau(k+1)^2}{4}$$

$$= -\frac{1}{2}\left(z - \sqrt{\frac{\tau}{2}}(k+1)\right)^2 + \frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}$$

Plug this value back into the first integral of equation (9). This value is the exponent of e. The last two parts of the exponent do not have z values, so they go in front of the integral:

$$\frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\left(z-\sqrt{\frac{\tau}{2}}(k+1)\right)^2} dz$$

Set $y = z - \sqrt{\frac{\tau}{2}}(k+1)$, $dy = dz$, and $z = -\frac{x}{\sqrt{2\tau}}$ which in turn changes the domain once again:

Equation (9) then becomes:

$$\frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$u(x,\tau) = \frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k+1)} e^{-\frac{y^2}{2}} dy - \frac{e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy \qquad (10)$$

The area under normal curve formula from $-\infty \to d$ is:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

where,

$$d = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

$-\infty \to d$ is the same as $-d \to \infty$. The value of d_2 is the same as d_1 with the exception that $(k+1)$ is $(k - 1)$, so:

$$d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

$$= \frac{x}{\sqrt{2\tau}} + d_2 \sqrt{\frac{\tau}{2}}(k-1)$$

Plug N into the equation (10):

$$u(x,\tau) = e^{\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}} N(d_1) - e^{\frac{x(k-1)}{2} + \frac{\tau(k-1)^2}{4}} N(d_2) \tag{11}$$

Now plug α, β, and equation (11) into equation (4):

$$v(x,\tau) = e^{\frac{-x(k-1)}{2} - \frac{\tau(k+1)^2}{4}} u(x,\tau)$$

$$v(x,\tau) = e^{\frac{-x(k-1)}{2} - \frac{\tau(k+1)^2}{4}} * \; e^{\frac{(k+1)x}{2} + \frac{\tau(k+1)^2}{4}} N(d_1) - e^{\frac{x(k-1)}{2} + \frac{\tau(k-1)^2}{4}} N(d_2)$$

The exponents will then cancel out to get a simple equation of:

$$v(x,\tau) = e^{x} N(d_1) - e^{-k\tau} N(d_2)$$

There are two values from earlier that need to be plugged back in: $x = ln(S/K)$ and $\tau = \frac{1}{2}\sigma^2(T-t)$ to get:

$$v(x,\tau) = e^{ln(S/K)} N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)} N(d_2)$$

$$v(x,\tau) = \frac{S}{K} N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)} N(d_2) \tag{12}$$

Plug these values into the d-values as well:

$$d_1 = \frac{ln\frac{S}{K}}{\sigma\sqrt{T-t}} + \frac{\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}(k+1)$$

$$= \frac{ln\frac{S}{K}}{\sigma\sqrt{T-t}} + \frac{\frac{\sigma^2}{2}(T-t)(k+1)}{\sigma\sqrt{T-t}}$$

$$= \frac{ln\frac{S}{K} + \left(\frac{\sigma^2}{2}k + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The risk-free interest rate is equal to $r = \frac{k}{2}\sigma^2$. So then equation (12) and the d-value become:

$$v(x,\tau) = \frac{S}{K} N(d_1) - e^{-r(T-t)} N(d_2) \tag{13}$$

$$d_1 = \frac{ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Equation (13) is then plugged back into equation (2):

$$V(S,t) = K\frac{S}{K}N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$= SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

We then have the solution to the Black-Scholes Equation:

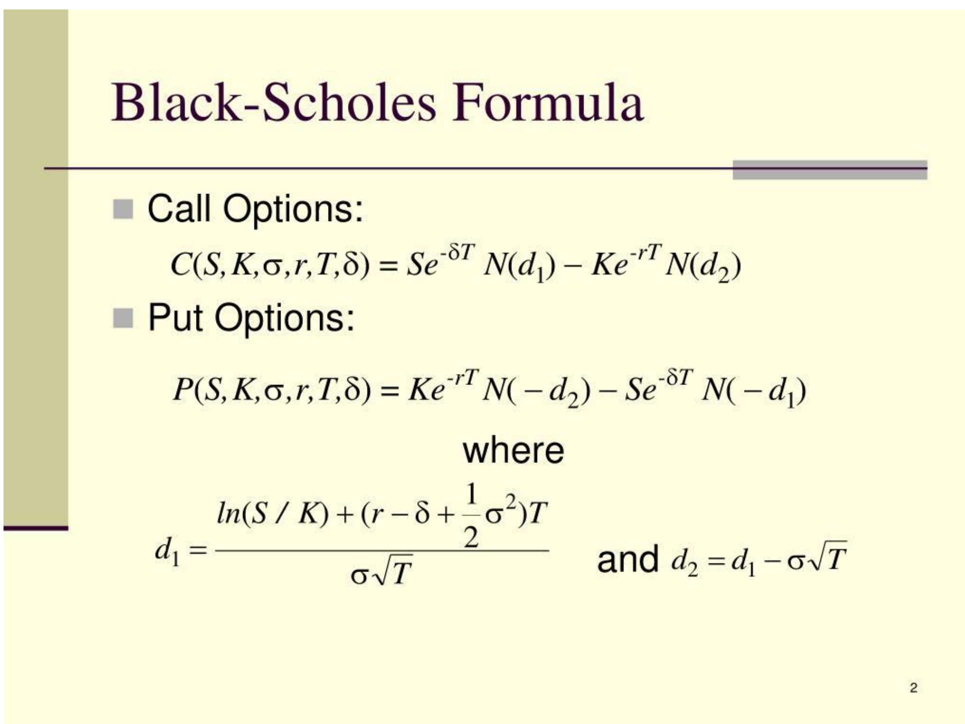
$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{14}$$

where,

$$d_1 = \frac{ln\frac{S}{K} + r + \frac{\sigma^2}{2}(T-t)}{\sqrt{}} \tag{15}$$

$$d_2 = \frac{ln\frac{S}{K} + r - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \tag{16}$$

Which can be verified as



Bitcoin Stock Evaluation

(Q) A stock we tracked was Bitcoin USD (BTC-USD). The starting stock price was \$9,355.025 and we set the strike price to \$10,000. Over the time period of 180 days, the volatility was 36.87% and an interest rate of 2.25%:

Solution :

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{ln\left(\frac{9355.025}{10000}\right) + \left(0.0225 + \frac{0.3687^2}{2}\right)\left(\frac{180}{365}\right)}{0.3687\sqrt{\frac{180}{365}}}$$

$$d_1 = -0.0852$$

$$N(d_1) = N(-0.08) = 0.4681$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_2 = -0.0852 - 0.3687\sqrt{\frac{180}{365}}$$

$$d_2 = -0.3441$$

$$N(d_2) = N(-0.34) = 0.3669$$

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$C(S,t) = 9255.025 * (0.4681) - 10000 * e^{-0.0225(\frac{180}{365})} * (0.3669)$$

$$C(S,t) = \$750.58$$

Here starting price was approximately 9,335 USD and raised to 10,000 USD – indicating a very volatile stock.

Conclusion

We have studied the Black-Scholes partial differential equation, found that it is very useful in financial engineering. We have obtained call option values of an underlying asset by Black-Scholes Equation, and observed that the call option price can be changed if we vary the parameters.

Future Work

This equation can be used to advise investors with regard to stock predictions, and has scope for further research.

Appendix

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Date

Taylor's theorem.

variance

$V(S,t)$

stock price (present)

$$\partial V = V_s \delta S + V_t \delta t + \frac{1}{2!} V_{ss} \delta S^2 + \frac{1}{2!} 2V_{St} \delta S \delta t + \frac{1}{2} V_{tt} \delta t^2 + \ldots$$

unpredictable

By Using Stochastic differential equation and Itô's lemma

$v \to$ average rate of growth of ~~asset~~ stock price

$\sigma \to$ Volatility $\quad dx \to$ Normal distribution.

variation price of a financial instrument over time

$$\boxed{\frac{dS}{S} = v\,dt + \sigma\,dX}$$

$$\boxed{dS = vS\,dt + \sigma S\,dx} \to \text{single}$$

$d \to$ single

$\delta \to$ multiple

$$\delta S = vS\,\delta t + \sigma S\,\delta x$$

Now put in taylor's theorem

$$\delta V = V_S(vS\delta t + \sigma S \delta x) + V_t \delta t + \frac{1}{2!} V_{SS}(vS\delta t + \sigma S \delta x)^2 + \frac{1}{2} \ldots$$

Cancel all the unsifficient term.

$$\delta V = V_S(vS\delta t + \sigma S\delta X) + V_t \delta t + \frac{1}{2!} V_{SS}(vS\delta t + \sigma S \delta X)^2$$

derivate nikal के लिए

Now we put limit $\delta S \to 0$, $\delta t \to 0$ $\Rightarrow$ $\delta x^2 \to \delta t$ as

Ito's process (time dependent)

$$dV = V_S(vS\,dt + \sigma S\,dx) + V_t\,dt + \frac{1}{2!} V_{SS}(vS\,dt + \sigma S\,dx)^2$$

$(vSdt)^2 + (\sigma S dx)^2 + 2\sigma v S^2 dx dt$

both side squaring

$$(dS)^2 = (vS\,dt + \sigma S\,dx)^2$$

$$dS^2 = v^2 S^2 dt^2 + \sigma^2 S^2 dx^2 + 2\sigma v S^2 dx\,dt$$

Spiral

$(dS)^2 = (vSdt + \sigma Sdx)$

Substituted

Applying Ito's lemma

$$dX^2 \rightarrow dt \quad \text{as} \quad \boxed{dt \rightarrow 0}$$

$$dS^2 \rightarrow \sigma^2 S^2 dt.$$

Put in eqn

$$dV = \frac{\partial V}{\partial S}(\nu S dt + \sigma S dX) + \frac{\partial V}{\partial t} dt + \frac{1}{2!}\frac{\partial^2 V}{\partial S^2}(\sigma^2 S^2 dt)$$

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\nu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt \quad \text{— (19)}$$

(II) Portfolio → Any collection of financial ~~assest~~ stock such as stocks, bonds and cash equivalent held by an investment company.

$V(S,t)$

$$\Pi = V - \Delta S$$

Changes $\quad d\Pi = dV - \Delta dS \quad$ — (20)

$$dV = d\Pi + \Delta dS \quad \text{— (21)}$$

$$\frac{dS}{S} = \nu dt + \sigma dX \quad \text{— (10)}$$

$$dS = \nu S dt + \sigma S dX.$$

$$dV = d\Pi + \nu\Delta S dt + \sigma S \Delta dX$$

$$d\Pi = dV - \nu\Delta S dt - \sigma S \Delta dX$$

$$d\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right) dX + \left(\nu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \nu\Delta S\right)$$

$\Delta = \frac{\partial V}{\partial S}$ → put ↑

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{dt}\right) dt$$

$r\Pi dt$ → this is money change in portfolio

$$r\Pi \cancel{dt} = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial V^2}{\partial S^2} + \frac{\partial V}{\partial t}\right)\cancel{dt}$$

$$\cancel{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0}$$

Substitute ~~by~~ $\Pi = V - \Delta S$

$$r(V - \Delta S) = \frac{1}{2}\sigma^2 S^2 \frac{\partial V^2}{\partial S^2} + \frac{\partial V}{\partial t}$$

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0}$$

↳ Black Scholes Model.

$V(S,t) \rightarrow$ price for an option

$S \rightarrow$ current ~~price~~ option price of stock

$K \rightarrow$ strike price.

References

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