Impossible Constructions with Straight edge and Compass

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***Abstract*—The paper dives into the realm of ancient Indian mathematical texts, particularly the Sulbhasutras. Although many geometric constructions are achievable through the use of a sulbha (rope), but there are limitations. Some problems remained unsolved for more than 2000 years and later found to be impossible to construct. The study primarily focuses on three famous problems outlined in the Sulbhasutras:**
(i) Constructing the side of a larger cube, with double the volume of a given cube.
(ii) Given a specified angle and constructing an angle one-third of its measurement. (iii) Constructing a regular heptagon using only straight edge and compass.
The goal of this paper is to use extended group theory to demonstrate the impossibility of these particular constructions solely through the use of a straight edge and compass.

***Index Terms*—Sulbhasutras, Geometric Construction, Doubling Cube, Trisecting angle, Constructing Regular Heptagon**

I. Introduction

The ancient Greeks were looking for a way of using straightedge and compass to trisect an arbitrary angle and to draw a segment of length $\sqrt[3]{2}$ etc. They also struggled to draw many n-gonals, such as 7, 9, 11, and so on. Over the course of 2000 years, they struggled to find the solution, but they failed since it was impossible to do so. In the 19th century, it was finally proven that these constructions were impossible to construct. In 1796, Gauss discovered a straightedge and compass construction for the regular 17-sided polygon. It was this discovery, the first advance on construction problems in 2000 years, that motivated Gauss to devote himself to mathematics. There are many ways to prove these impossible constructions, but the most popular is using group theory.

II. Methodology

A. Straight Edge and Compass

Straightedge: It's a ruler-like tool that is a straight line with no markings, allowing the drawing of perfectly straight lines between any two points. A straightedge has no units or measurements, which is crucial for geometric constructions where precision and accuracy are essential. It helps in drawing lines, extending existing lines, and verifying the straightness of other lines.

Compass: It has two arms— one with a pointed end and the other with a pencil. The pointed end is fixed on the paper, while the other end can be adjusted to different length. It's primarily used to draw circles and arcs of various sizes and radii. By fixing the pointed end at a center point and extending the other end to a desired radius, circles and arcs can be accurately drawn.

Limitations

Straightedge and compass constructions rely solely on geometric principles so they do not have measurements or numerical values. As a result, these tools cannot be used for tasks requiring specific measurements or numerical precision.

B. Geometric Constructions in Sulbhasutras

The Sulbhasutras, which are texts dating back, to 800 BCE to 200 BCE hold a special place in Vedic literature as they extensively explore the world of geometric constructions. These manuscripts provide instructions for creating structures using basic tools such as ropes (sulbha) pegs and stakes. They cover a range of topics with emphasis on altar construction, home building and ceremonial rituals. Among their teachings the Sulbhasutras offer methods for constructing altars of shapes like squares, rectangles and circles. Additionally they delve into principles related to symmetry, angles and proportions. These texts also discuss operations such, as dividing lines into sections or establishing right angles through angle bisecting.

C. Group Theory and Geometric Constructions

Group theory is a branch of mathematics that focuses on studying groups which're structures representing symmetries and transformations. These groups have implications, in the field of constructions. Specifically group theory helps us determine the limitations of tools like the straightedge and compass. By exploring groups we gain insights into which geometric constructions can or cannot be achieved using these methods alone. For instance through group theory we have discovered that certain ancient challenges such as trisecting angles doubling the cube or squaring the circle cannot be accomplished with these tools due to constraints. Group theory is good, for understanding these limitations in constructions and sheds light on both unsolvable problems and those that can be solved using other approaches. Further extensions of group theory are really helpful when we have to use field theory and Galois theory.

D. Definitions and Postulates

Euclid 3's Postulate only for Ruler & Compass:

1) A Straight Line Segment can always be drawn joining any two points using a ruler.
2) Any Straight Line Segment can be extended indefinitely in any straight line.
3) A Circle can be drawn with any center and any radius.

Definition 1 (Irreducible Polynomial): Let G be a field. A polynomial $g(y) \in G[y]$ is said to be irreducible over G if there do not exist non-constant polynomials $k(y)$ and $m(y)$ in $G[y]$, other than the trivial cases of $k(y) = 1$ or $m(y) = 1$, satisfying:

1) Each polynomial has a degree less than that of $g(y)$, and
2) $g(y) = k(y) \cdot m(y)$.

Definition 2 (Fermat Numbers): The Fermat numbers are defined as integers of the form:

$$F_n = 2^{2^n} + 1, \text{ where } n \geq 0.$$

Eisenstein's Irreducibility Criterion: Let F be an integral domain and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial in $F[x]$ with coefficients a_i in F for $0 \leq i \leq n$ such that:

1) There exists a prime element $p \in F$ such that p divides all coefficients a_i for $i \neq n$ (except a_n).
2) p does not divide a_n.
3) p^2 does not divide a_0.

Then the polynomial $f(x)$ is irreducible over the field of fractions of F $(F(x))$.

E. Theorems and Corollary

Around 300 B.C., the ancient Greek mathematician Euclid wrote a series of thirteen books that he called The Elements.

It is a collection of definitions, postulates (axioms), and theorems & proofs, covering geometry, elementary number theory, and the Greeks' "geometric algebra."

Book 1 contained Euclid's famous 10 postulates and other basic propositions of geometry.

- **Theorem 1: (Degree of a Constructible Number Theorem)** The set $K \subseteq \mathbb{C}$ of constructible numbers is a field. Moreover, if $a \in K$, then $[\mathbb{Q}(a) : \mathbb{Q}] = 2^n$ for some integer n.
- **Corollary 1.1:** If a real number x satisfies an irreducible polynomial over $\mathbb{Q}$ of degree n, and if n is not a power of 2, then x is not constructible.
- **Theorem 2:** The regular polygon of n sides is constructible by an unmarked ruler and compass if and only if n is a number of the form

$$n = 2^r \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_s,$$

where $r, s \geq 0$ and the p_i are distinct odd primes, each of which is a fermat prime.

III. Result

Using the above Theorems and Corollary now we are ready to prove the following problems

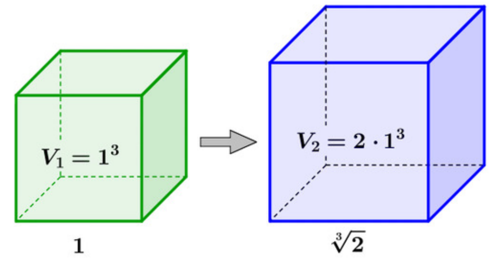

Fig. 1. Doubling The Given Cube.

A. Doubling The Given Cube

Given the edge of the cube, Let the given cube has an edge of length a so the volume will be a^3. Doubling the volume equals to $2a^3$, which means the edge of the cube will be $\sqrt[3]{2a^3}$ = $a\sqrt[3]{2}$. So to double the cube volume we have to draw $a\sqrt[3]{2}$. However. $\sqrt[3]{2}$ is a zero of the irreducible polynomial x^3 - 2 over $\mathbb{Q}$ that can be shown using the Eisenstein's Irreducibility Criterion. i.e
$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$
here 3 is not a power of 2 and from Corollary 1.1 we can thus say $\sqrt[3]{2}$is not constructible with straight edge and compass. And since we cannot draw the edge obviously we cannot draw the cube.

B. Trisecting a given angle

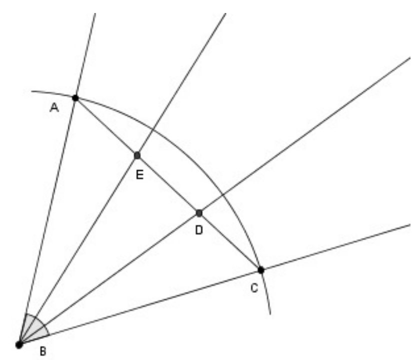

Fig. 2. Trisecting an angle.

Given any angle θ, in order to trisect that angle we should be able to construct $\theta/3$ or equivalently, construct $cos(\theta/3)$ from $cos(\theta)$.

We will take an angle $\theta = 60°$ and show that it cannot be trisected. In other words, that $\alpha = cos(20°)$ cannot be constructed from $cos(60°)$.
Using trigonometric formula of cosine angles:
$cos(\theta) = 4cos^3(\theta/3)$ - 3 $cos(\theta/3)$.
Putting $\theta = 60°$ in the equation above, we get
$cos(60°) = 4cos^3(20°)$ - 3 $cos(20°)$ Now , $cos(\theta) = 1/2$ and let $\alpha = cos(20°)$ gives
$4\alpha^3 - 3\alpha - \frac{1}{2} = 0.$

Changing variables by $u = 2\alpha$ and then multiplying through by 2:

$u^3 - 3u - 1 = 0$

Using Eisenstein's Irreducibility Criterion we can see that this polynomial is irreducible in Q and [Q(u): Q] = 3, which is not a power of 2.

Hence, $u = 2\ cos(20°)$ is not constructible, so neither is $\alpha = cos(20°)$.

C. Constructing a regular Heptagon

We can construct many regular figures using only by the use of Straight Edge and Compass only but there is limitation to this also, we tried to draw some basic regular n-gonals shapes using only the above two, below are some examples to it

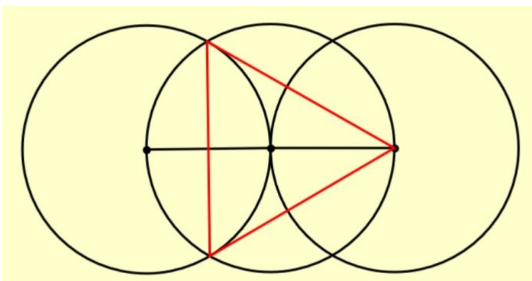

Fig. 3. Triangle.

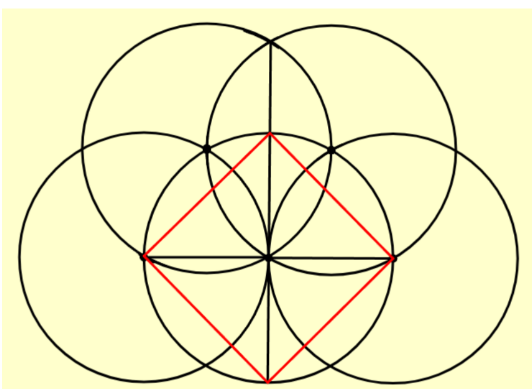

Fig. 4. Square.

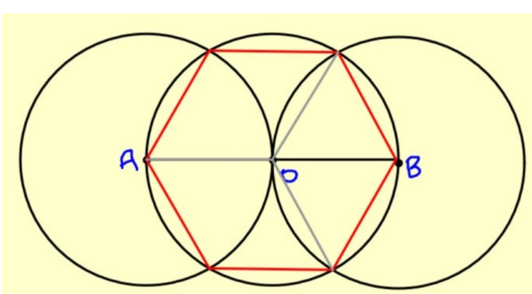

Fig. 5. Cube.

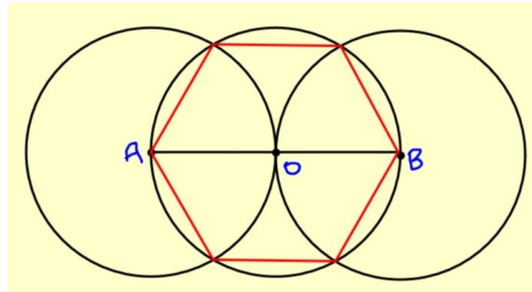

Fig. 6. Hexagon.

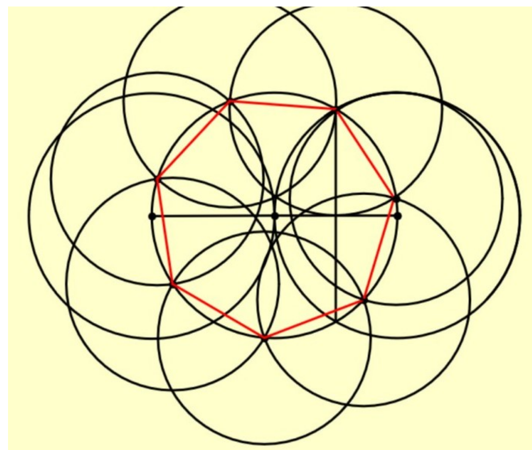

Fig. 7. Heptagon (Not Regular).

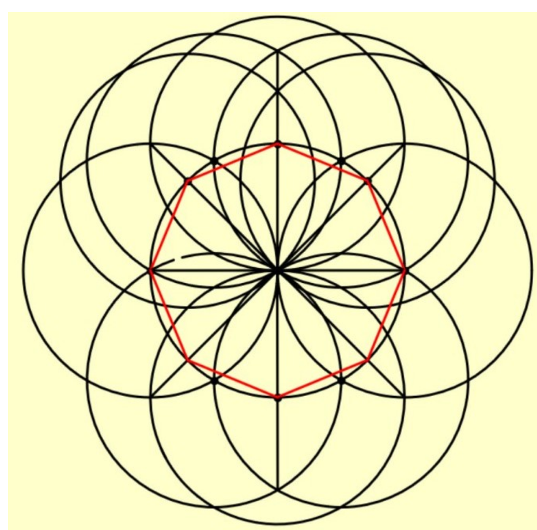

Fig. 8. Octagon.

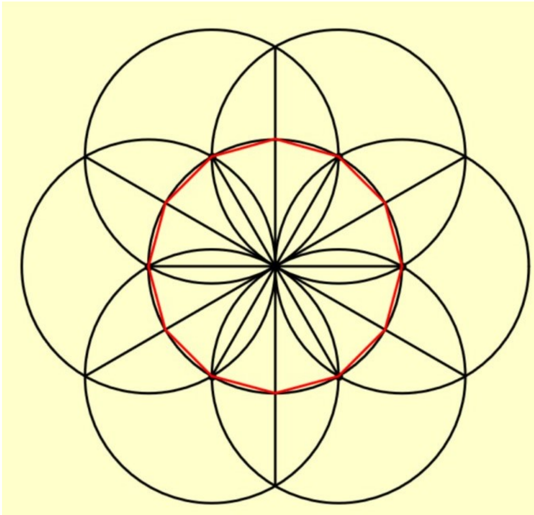

Fig. 9. Dodecagon.

From the above figures, it can be seen that a regular polygon can be formed by dividing a circle into equal arcs and then joining the successive points of division by chords. The Greeks were able to construct a regular polygon of 3, 5, and all $2n$-sided polygons where n is an integer greater than 1. The problem arises when they tried to construct other polygons such as a heptagon. Furthermore, if a regular polygon of p and q sides can be constructed and p and q are relatively prime, then a regular polygon of $p \cdot q$ sides can also be constructed. For example, 3 and 5 are relatively prime, so a regular polygon of 15 sides can be constructed, as well as regular polygons of $15 \cdot 2^n$ sides, where n is a positive integer.

For the construction of a regular heptagon using Theorem 2, we can show that for $n = 7$, no value for r and p exists such that

$$7 = 2^r \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_s,$$

So we can say that it is not possible to construct a regular heptagon using only a straightedge and compass. Overall, an n-gon is constructible if $n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, \ldots$ while an n-gon is not constructible with a compass and straightedge if $n = 7, 9, 11, 13, 14, 18, 19, 21, 22, 23, 25, \ldots$

IV. Conclusion

Three problems that had remained unsolved for more than a thousand years can be proven to be impossible using the extension of group theory and field theory. The first two problems, that is, the inability to double the given cube and trisect a given angle, can be proven by forming a polynomial equation and then using Eisenstein's Irreducibility Criterion to show that the polynomial is irreducible over Q. Lastly, from the theorem, it can be proven to be an impossible construction. For the third problem, which is constructing a regular heptagon, can also be proven with the help of a theorem. There is no value for r and p such that $7 = 2^r \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_s$.

If we had a marked ruler, then there is an advantage, as the first two problems can be solved using a marked ruler and compass. However, the last problem remains unsolvable.

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