Topics: Normal distribution, Functions of Random Variables

1. The time required for servicing transmissions is normally distributed with μ = 45 minutes and σ = 8 minutes. The service manager plans to have work begin on the transmission of a customer's car 10 minutes after the car is dropped off and the customer is told that the car will be ready within 1 hour from drop-off. What is the probability that the service manager cannot meet his commitment?

   A. 0.3875
   B. 0.2676
   C. 0.5
   D. 0.6987

Ans: B

2. The current age (in years) of 400 clerical employees at an insurance claims processing center is normally distributed with mean μ = 38 and Standard deviation σ=6. For each statement below, please specify True/False. If false, briefly explain why.

   A. More employees at the processing center are older than 44 than between 38 and 44.

Ans: PR(X > 44) = 1 – PR(X ≤ 44)
Z = (X - 44) = (X – 38) /65
PR(X ≤ 44) = PR(Z ≤ (44 - 38) / 6) = PR(Z ≤ 1) = 84.13%
Probability that employees will be greater than age of 44 = 100 – 84.13 = 15.86%
Probability of number of employees between 38 - 44 years of age
= PR(X<44) – 0.5 = 84.13 – 0.5 = 34.13

   B. A training program for employees under the age of 30 at the center would be expected to attract about 36 employees.

Ans: Z = (X - 44) = (30 – 38) /6
PR(X ≤ 30) = PR(Z ≤ (30 - 38) / 6) = PR(Z ≤ -1.33) = 9.12%
Number of employees with probability 0.912 of them being under age 30
= 0.0912 * 400 = 36.48 i.e., 36 employees

3. If $X_1 \sim N(\mu, \sigma^2)$ and $X_2 \sim N(\mu, \sigma^2)$ are *iid* normal random variables, then what is the difference between 2 X_1 and $X_1 + X_2$? Discuss both their distributions and parameters.

Ans: We know that if X ∼ N(μ1, σ1^2), and Y ∼ N(μ2, σ2^2) are two independent random variables then X + Y ∼ N(μ1 + μ2, σ1^2 + σ2^2) , and X − Y ∼ N(μ1 − μ2, σ1^2 + σ2^2)

Similarly if Z = aX + bY , where X and Y are as defined above, i.e., Z is linear combination of X and Y , then Z ∼ N(aμ1 + bμ2, a^2σ1^2 + b^2σ2^2)

Therefore in the question

2X1~ N(2 u,4 σ^2) and

X1+X2 ~ N(μ + μ, σ^2 + σ^2) ~ N(2 u, 2σ^2)

2X1-(X1+X2) = N(4μ,6 σ^2)

4. Let X ~ N(100, 20^2). Find two values, *a* and *b*, symmetric about the mean, such that the probability of the random variable taking a value between them is 0.99.

   A. 90.5, 105.9
   B. 80.2, 119.8
   C. 22, 78
   D. 48.5, 151.5
   E. 90.1, 109.9

Ans: The Probability of getting value between a and b should be 0.99.
So the probability of not getting value will be 1 – 0.99 = 0.01
The Probability towards left from a = -0.005 (ie., 0.01/2).
The Probability towards right from b = +0.005 (ie., 0.01/2).
By finding the Standard Normal Variable Z (Z Value), we can calculate the X values.
Z=(X- μ) / σ
For Probability 0.005 the Z Value is -2.57 (from Z Table).
Z * σ + μ = X
Z(-0.005)*20+100 = -(-2.57)*20+100 = 151.5
Z(+0.005)*20+100 = (-2.57)*20+100 = 48.5
So, option D is correct.

5. Consider a company that has two different divisions. The annual profits from the two divisions are independent and have distributions $\text{Profit}_1 \sim N(5, 3^2)$ and $\text{Profit}_2 \sim N(7, 4^2)$ respectively. Both the profits are in $ Million. Answer the following questions about the total profit of the company in Rupees. Assume that $1 = Rs. 45

   A. Specify a Rupee range (centered on the mean) such that it contains 95% probability for the annual profit of the company.
   B. Specify the 5th percentile of profit (in Rupees) for the company
   C. Which of the two divisions has a larger probability of making a loss in a given year?

Ans: let profit $_{i}$. then $P_1 \sim N(\mu_1 , \sigma_2^{1})$; $P_2 \sim N(\mu_2 , \sigma_2^{2})$, with$(\mu_1, \sigma_1) = (5 , 3)$ and $(\mu_2, \sigma_2) = (7 , 4)$.

(A) Annual Profit is $P = P_1 + P_2 \sim N(\mu_1 + \mu_2 , \sigma_1^2 + \sigma_2^2)$ in $ Million.

Denote $\mu = \mu_1 + \mu_2 = 12$, $\sigma^2 = \sigma_1^2 + \sigma_2^2 = 25$

We require k_1 , k_2 such that $Pr [k_1 \leq P - 12 \leq k_2] = 0.95$

$\Longrightarrow Pr [k_1 / 5 \leq Z \leq k_2 / 5] = 0.95$; where $Z = \frac{p-12}{9} \sim N (0 , 1)$.

From the symmetry of distribution we can take $k_1 = - k_2$

$\Longrightarrow Pr [- k_2 / 5 \leq Z \leq k_2 / 5] = 0.95 \Rightarrow \phi (\frac{k2}{5}) - \phi (\frac{-k2}{5}) = 0.95$

$\Longrightarrow 2\Phi(\frac{k2}{5}) = 1 + 0.95 = 1.95 \Longrightarrow k_2 = 9.7998$. the range in Million \$ is $I =$ [12 – 9.7998, 12 + 9.7998] = [2.20018, 21.79982]

The range in rupee will be 45 × 1,000,000 × I =[99008103 , 98099187] rupees . [1 million = 1,000,000]

(B) If the 5−*th* percentile in million dollar is ξ , then, $Pr[\, P \leq \xi] = 0.05$

$\Longrightarrow Pr\, [Z \leq \frac{12-\xi}{5}] = 0.05 \Longrightarrow \xi = 12 + 5 \times \Phi^{-1}(0.05) = 3.775732$

In rupee it will be $\xi \times 45 \times 1{,}000{,}000 = 169907934$.

(C) See, $Pr\,[P1 < 0] = 0.04779035 > Pr\,[P2 < 0] = 0.04005916$.

So, first division has greater chance of losing.