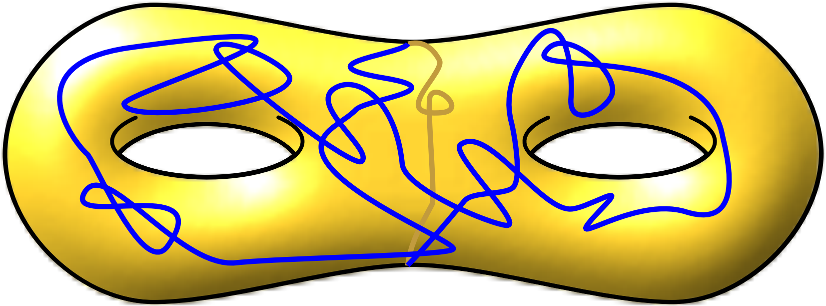
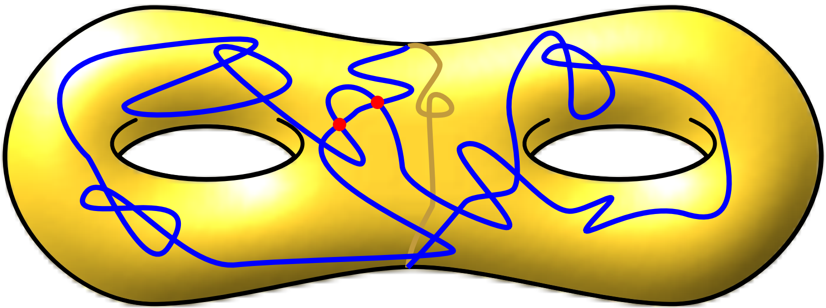
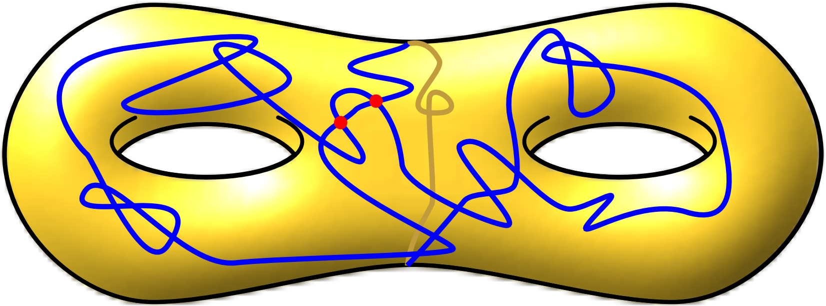
Calcul du nombre géométrique d'intersection

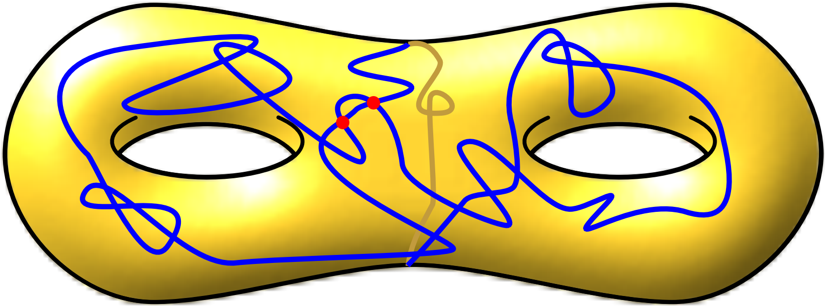
Francis Lazarus (joint work with Vincent Despré)
GIPSA-Lab, CNRS, Grenoble

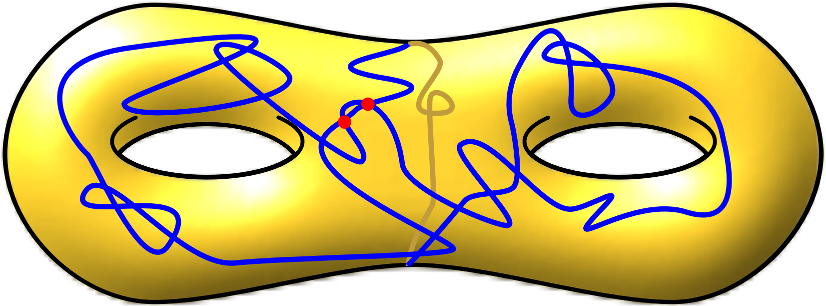
Jackson Pollock, Mural on Indian Red Ground, 1950

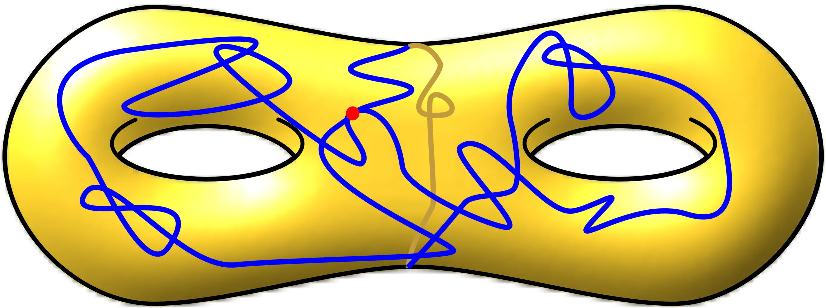


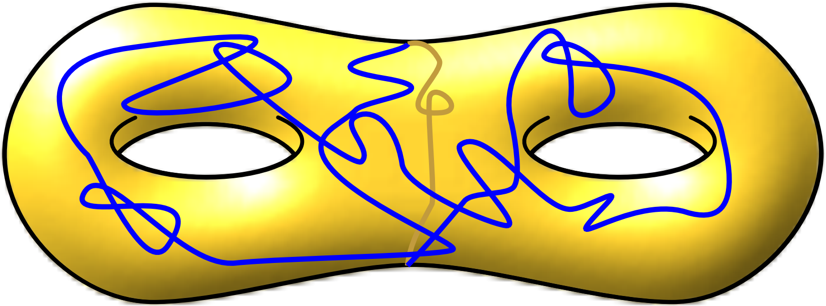


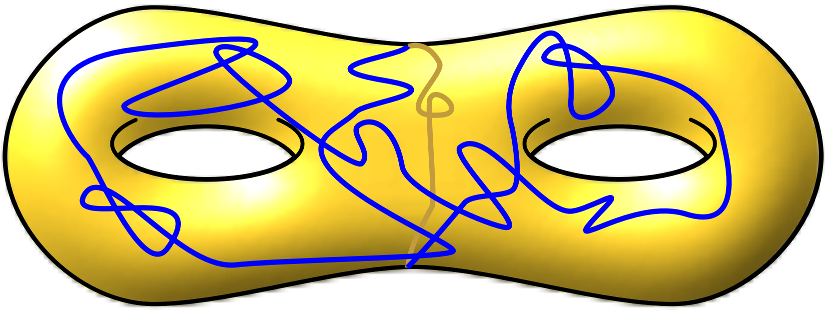




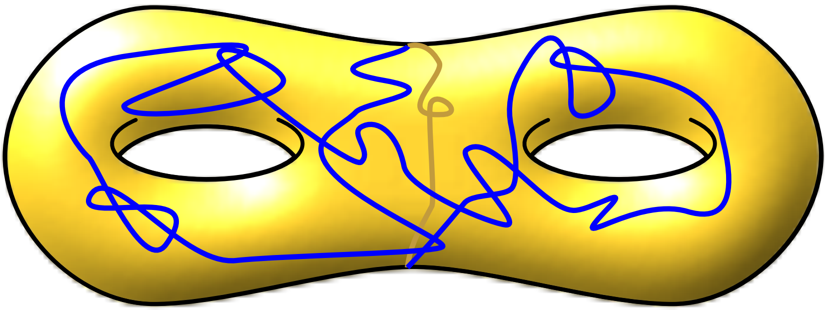








$$i(c) = \min_{d \sim c} \left(\#\text{crossings of } d\right)$$



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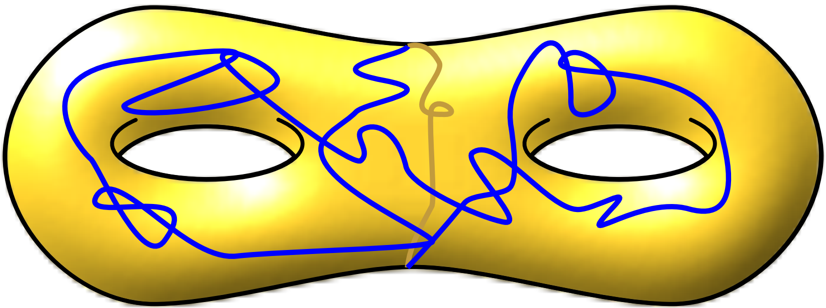

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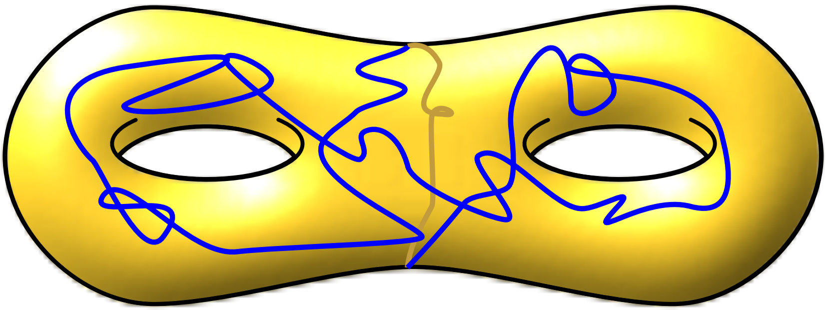

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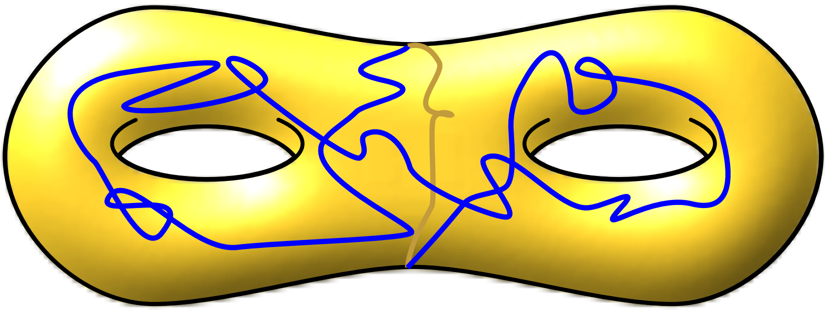

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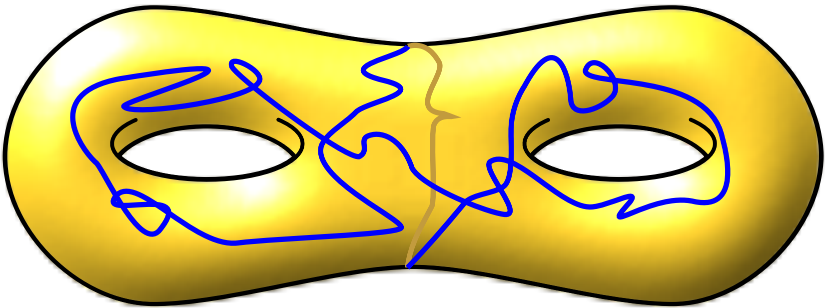

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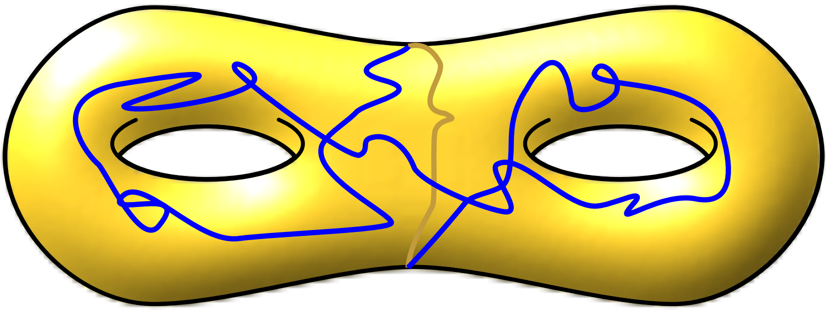

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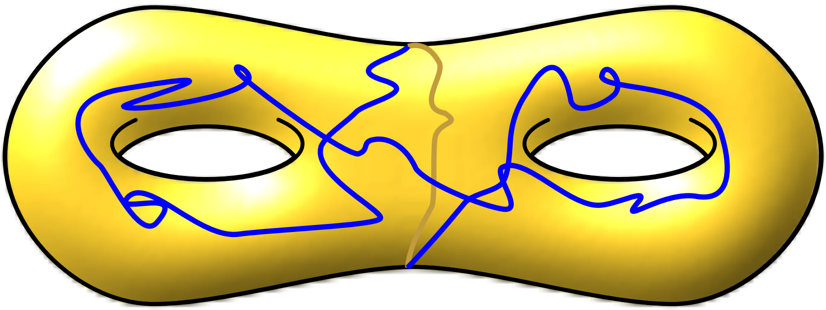

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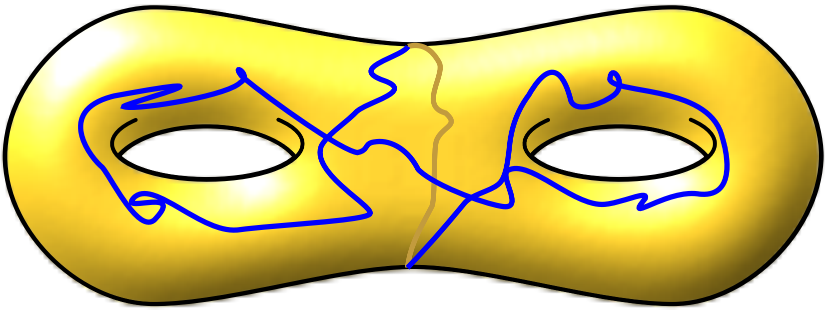

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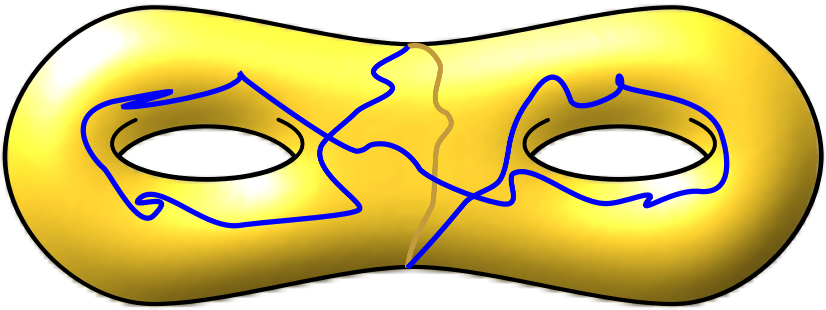

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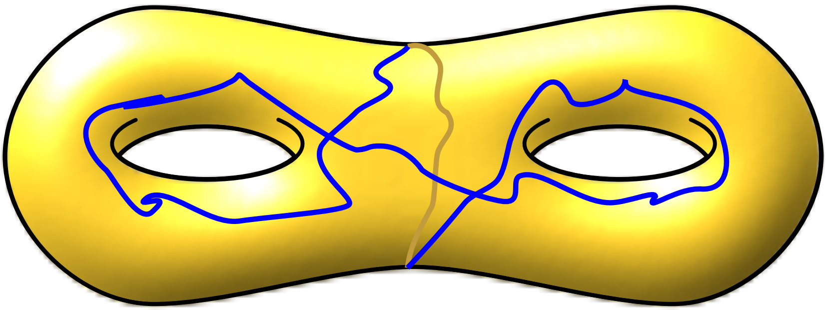

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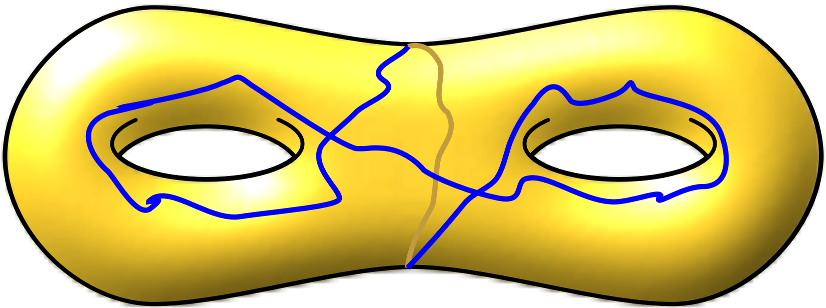

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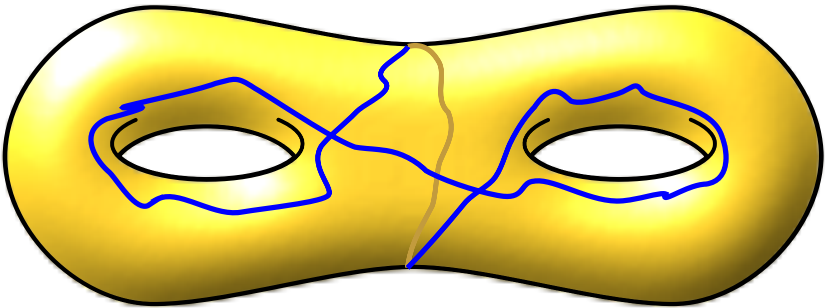

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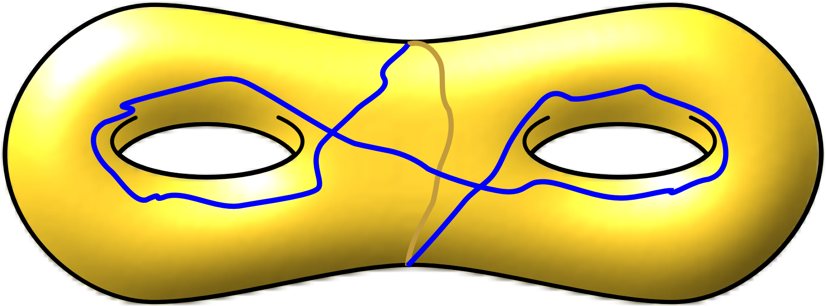

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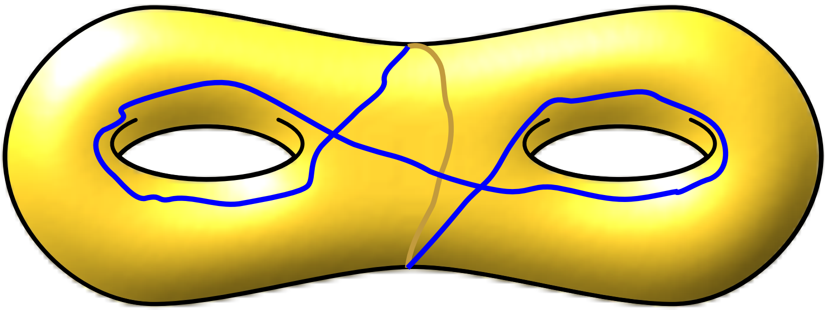

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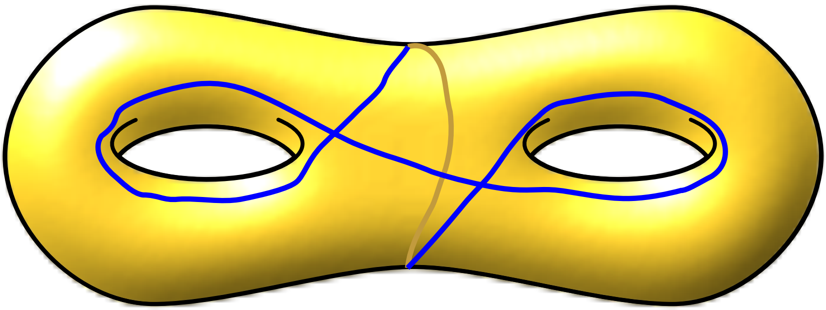

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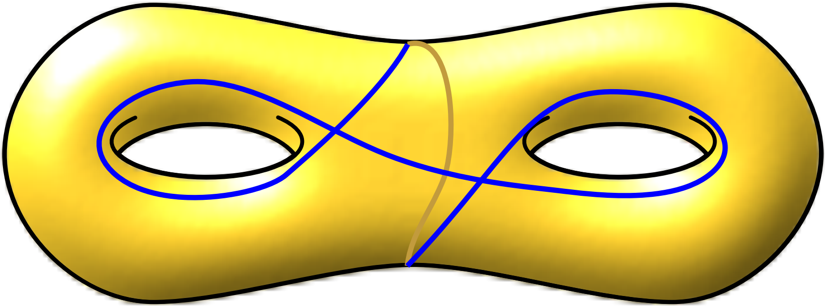

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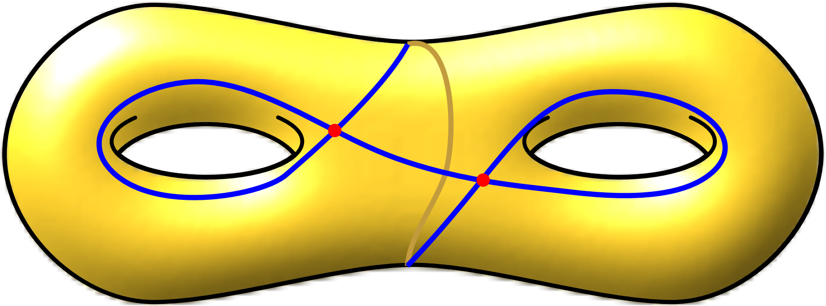

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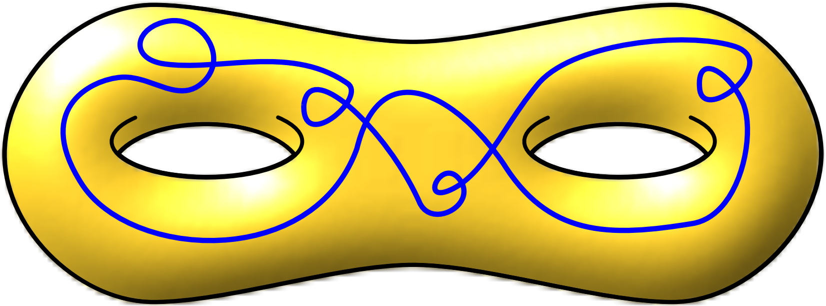

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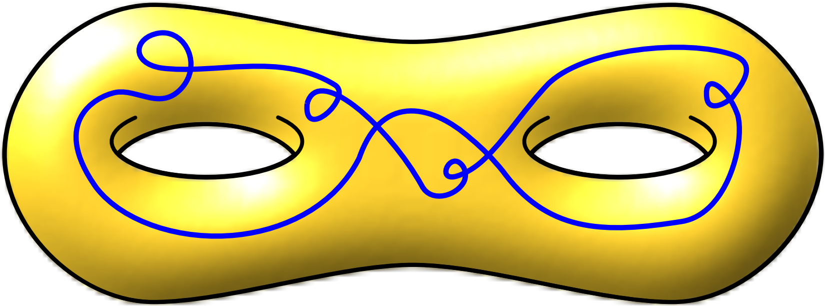

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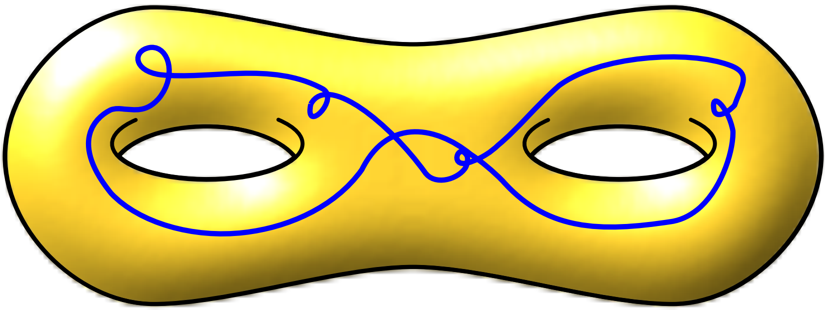

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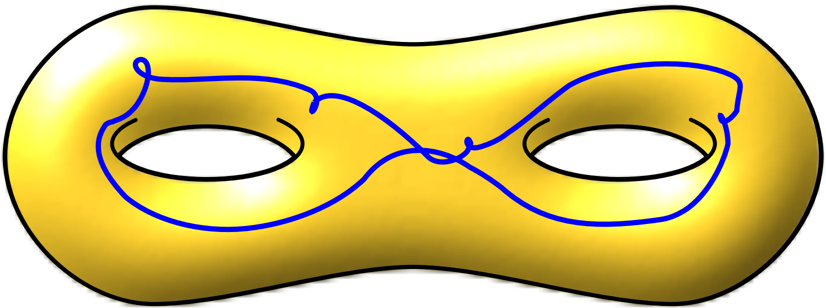

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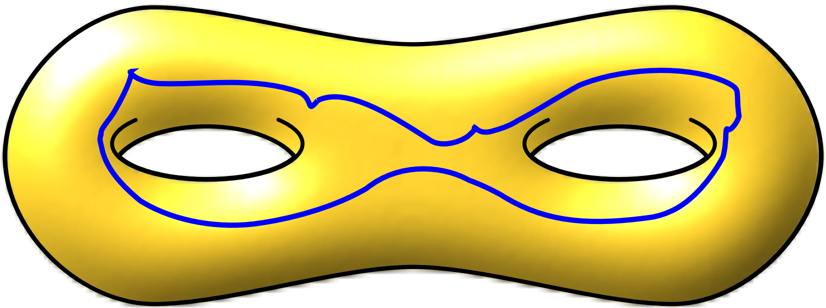

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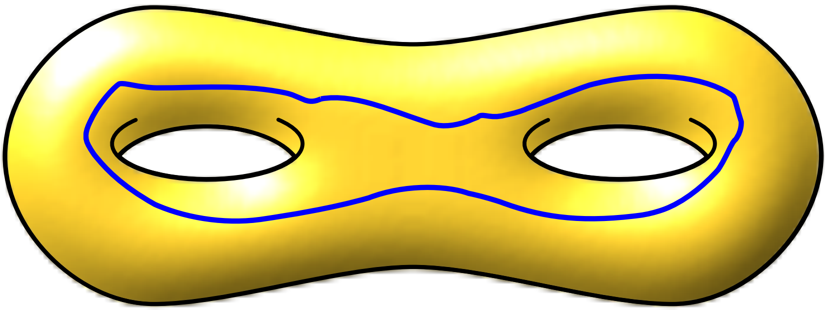

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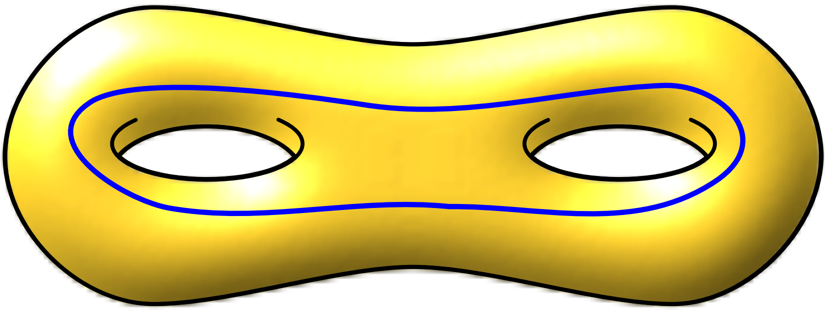

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CINQUIÈME COMPLÉMENT

À

L'ANALYSIS SITUS

Rendiconti del Circolo matematico di Palermo, t. 18, p. 45-110 (1904).

1. Nous avons vu au paragraphe précédent qu'il est relativement aisé de reconnaître si un cycle donné est homologue à un cycle non bouclé, ou si deux cycles donnés sont respectivement homologues à deux cycles qui ne se coupent pas. Nous allons dans le présent paragraphe examiner une question analogue :

Comment reconnaître si un cycle donné est *équivalent* à un cycle non bouclé, ou si deux cycles donnés sont équivalents à deux cycles qui ne se coupent pas ?

Mais avant d'aborder cette question, revenons sur la définition de l'équivalence.

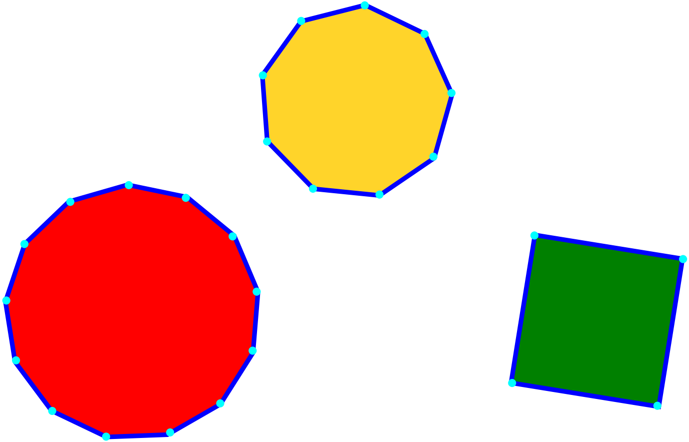
Jusqu'ici nous avons toujours entendu cette équivalence de la façon suivante :

Quand nous écrivons

$$C \equiv C',$$

How can we recognize when a given cycle is equivalent to a simple cycle?

What is a surface?



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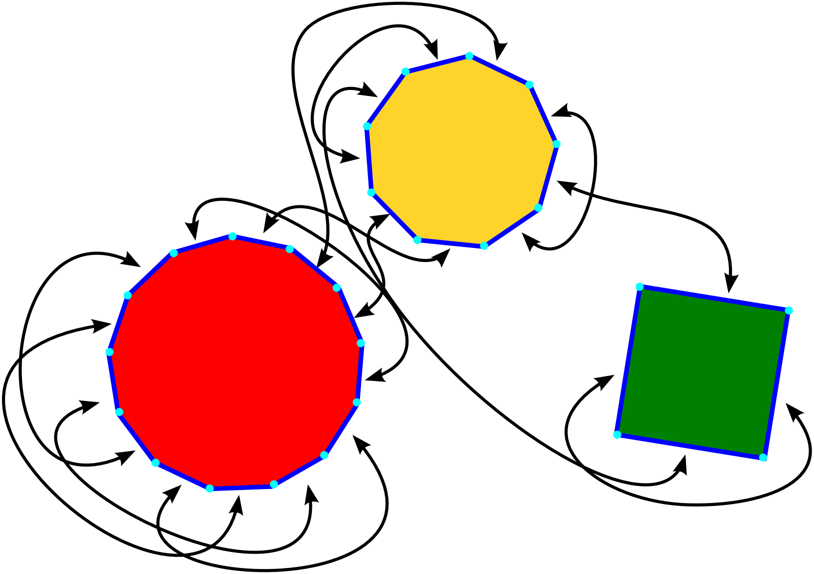

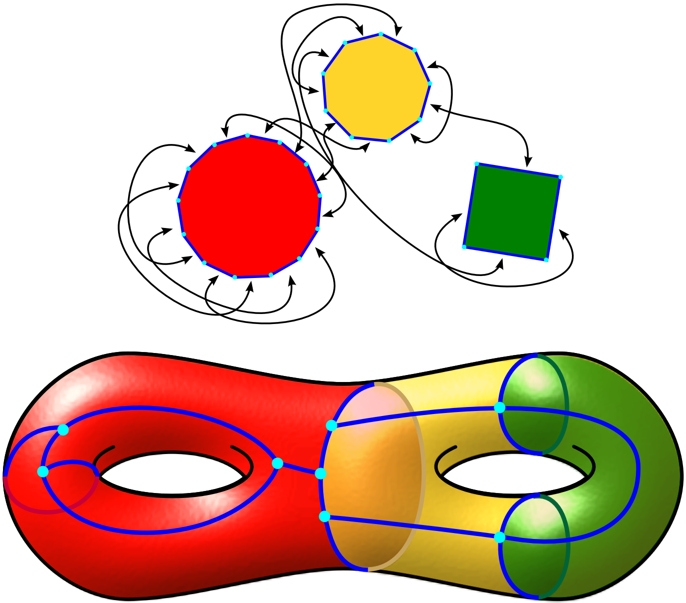

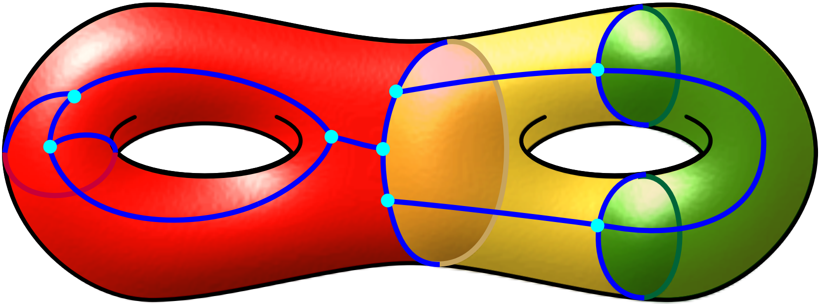

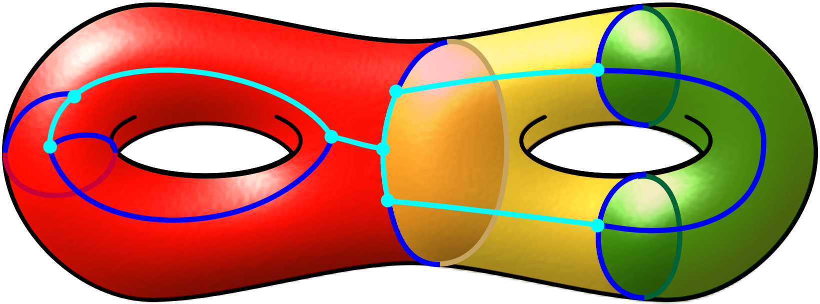

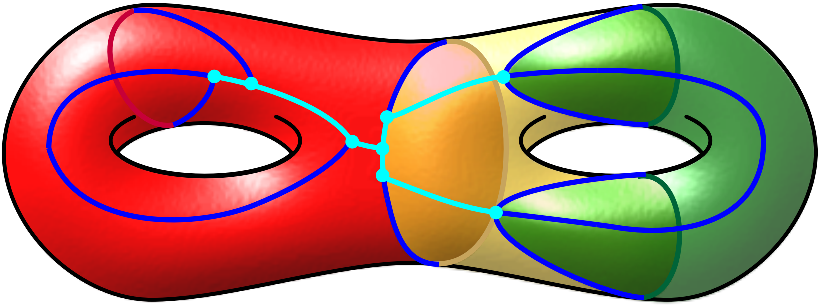

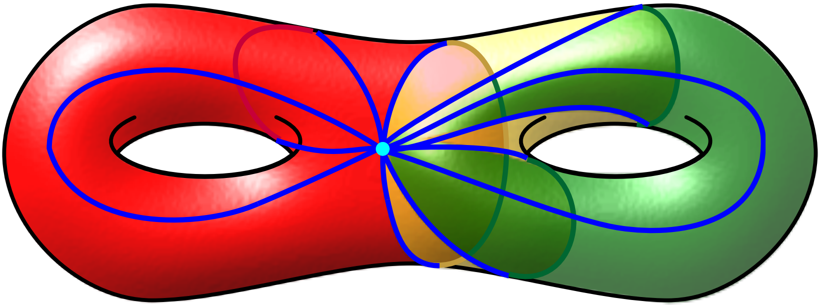

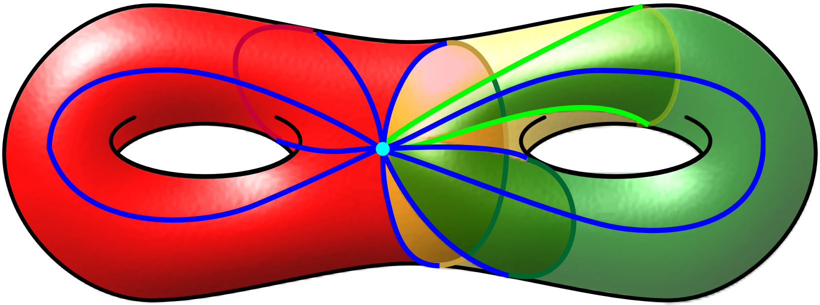

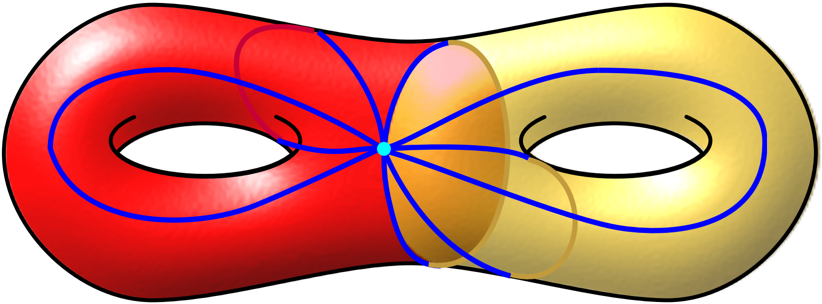

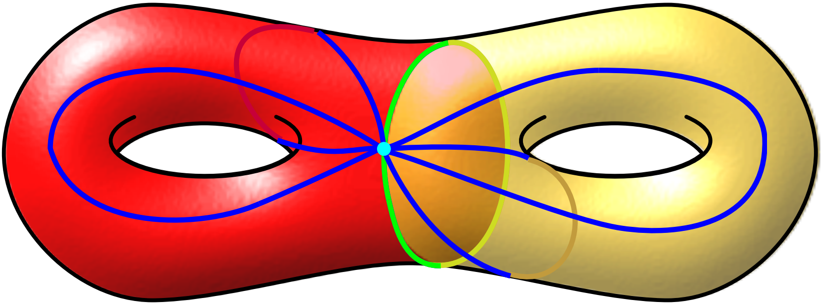

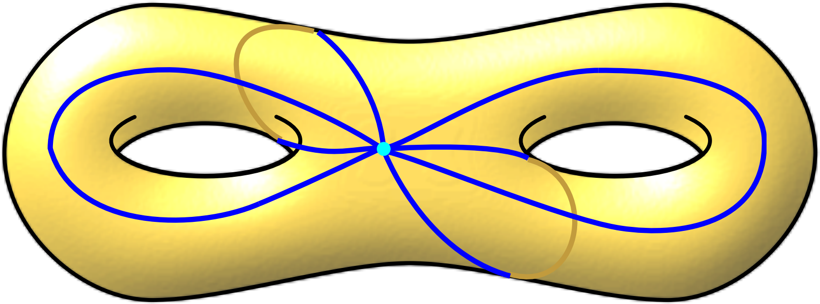

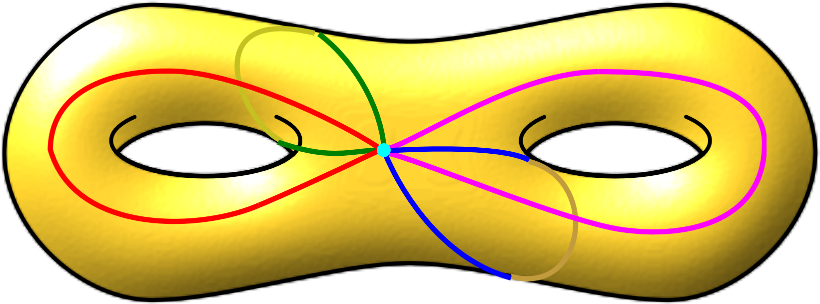

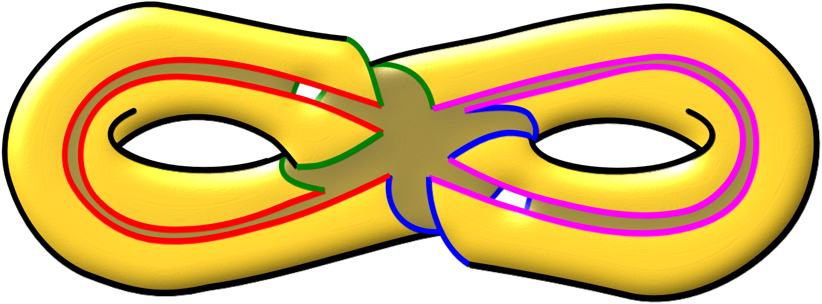

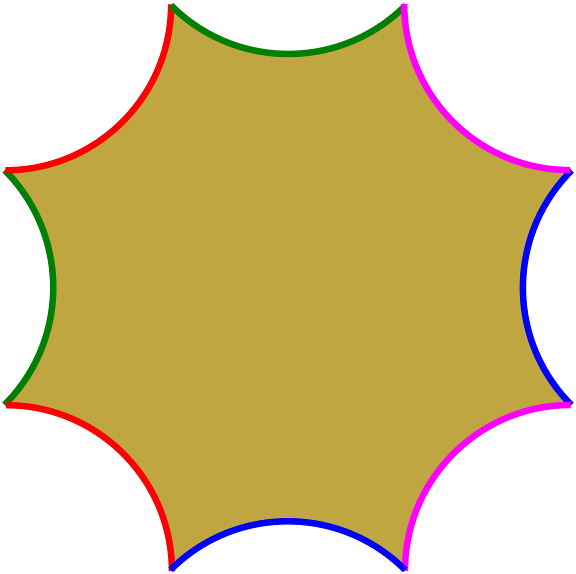

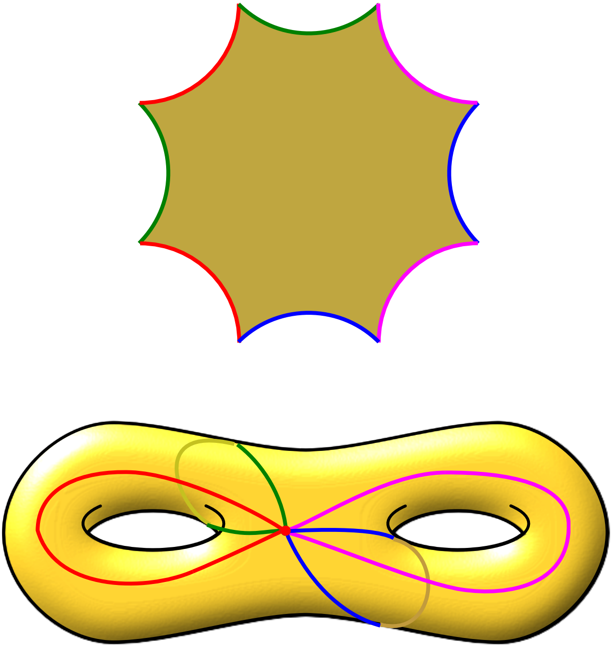

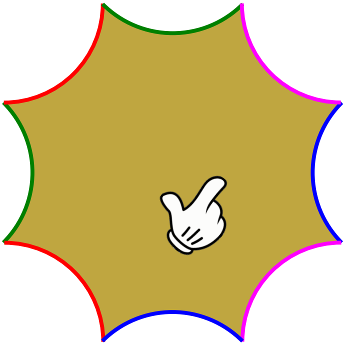

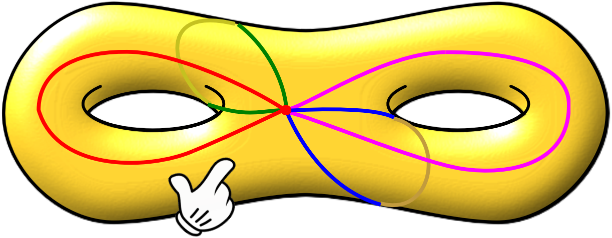

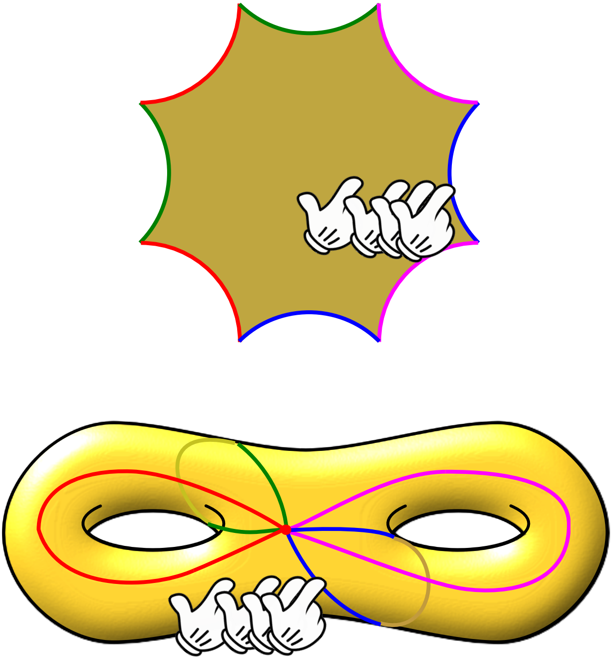

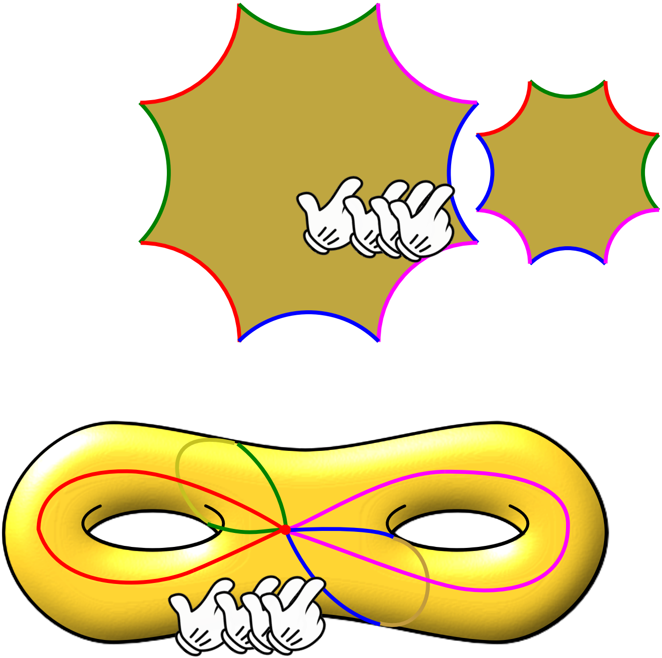

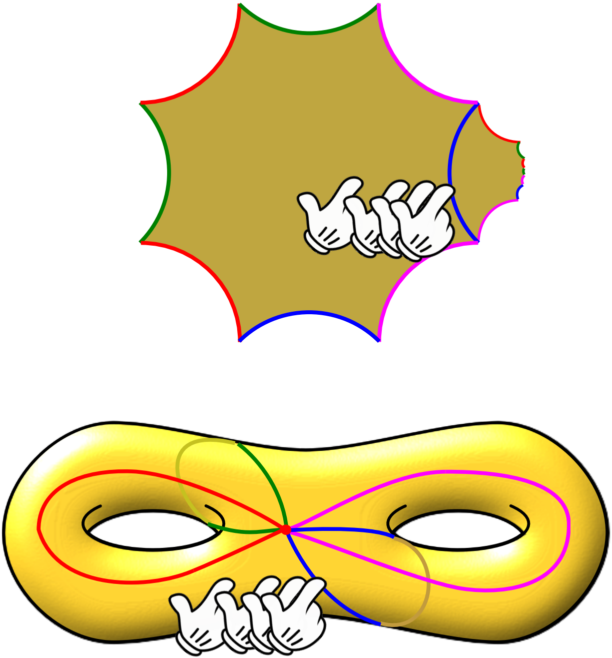

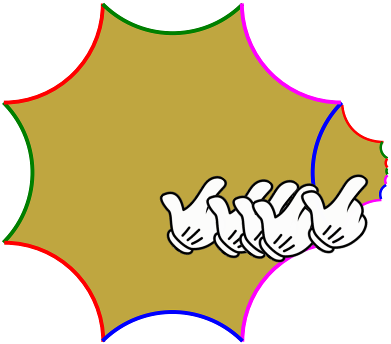

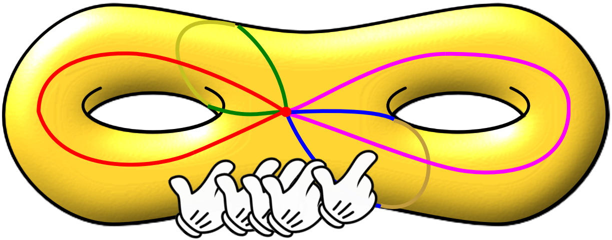

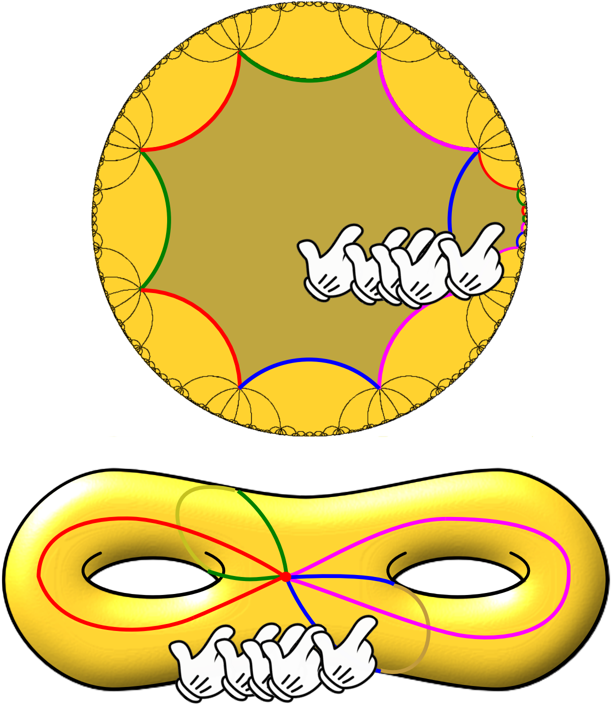

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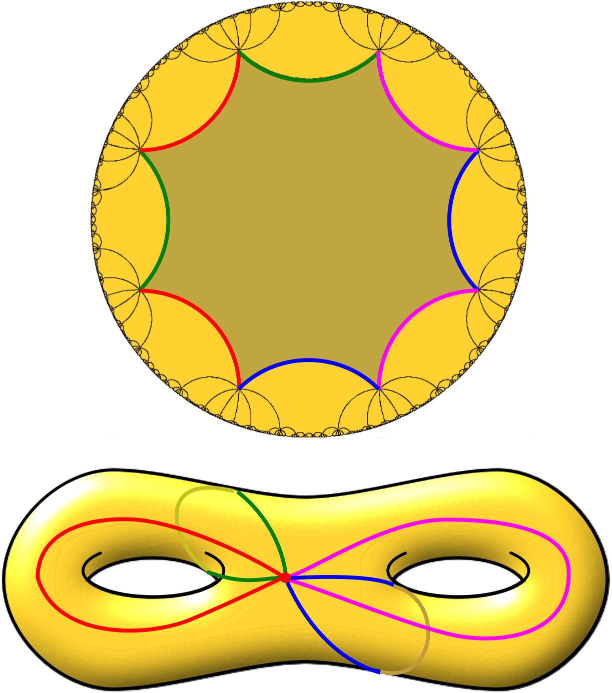

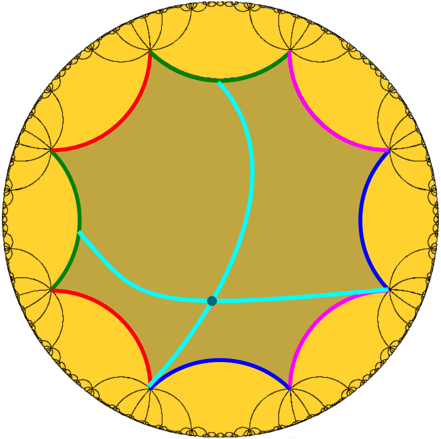

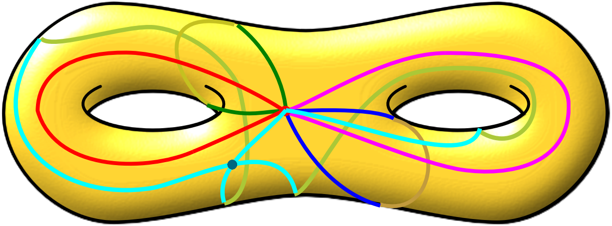

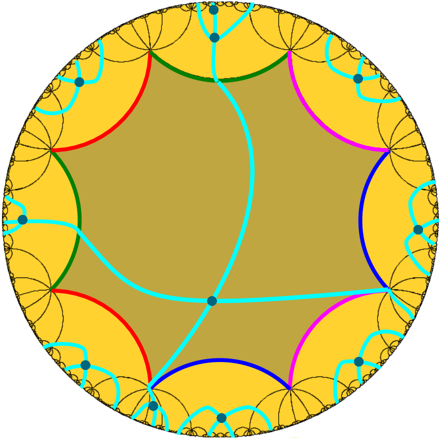

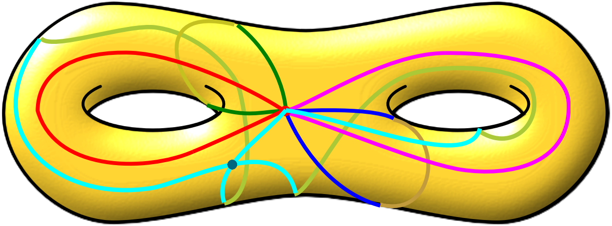

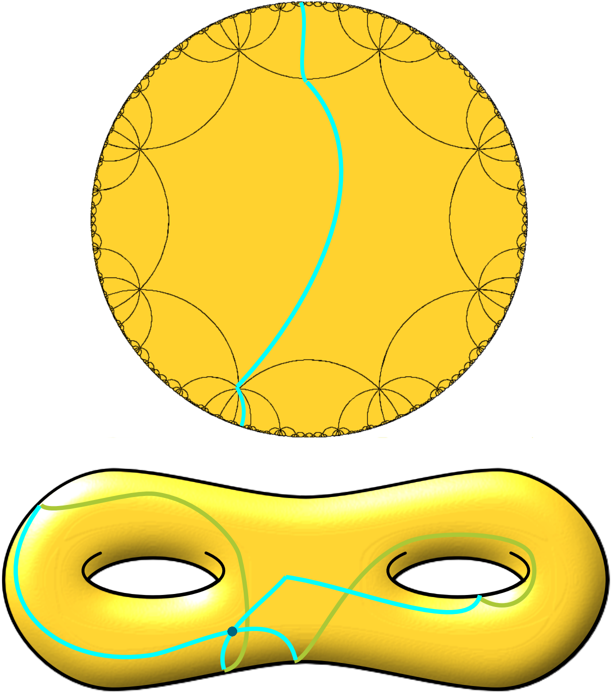

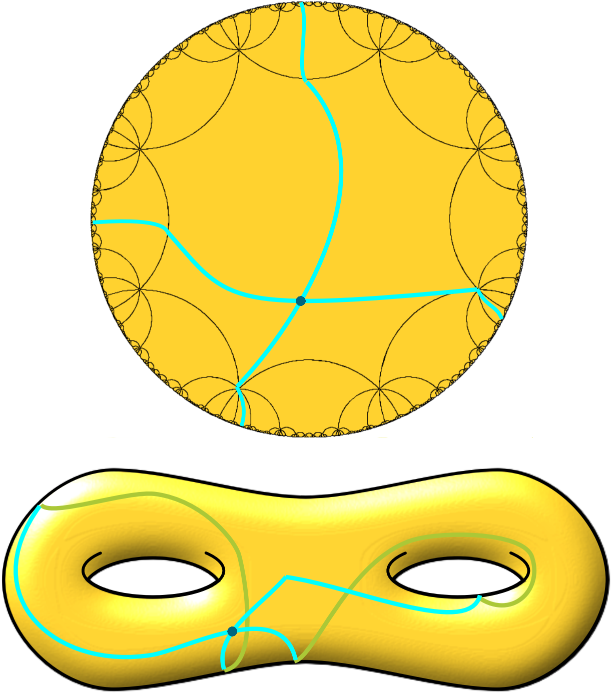

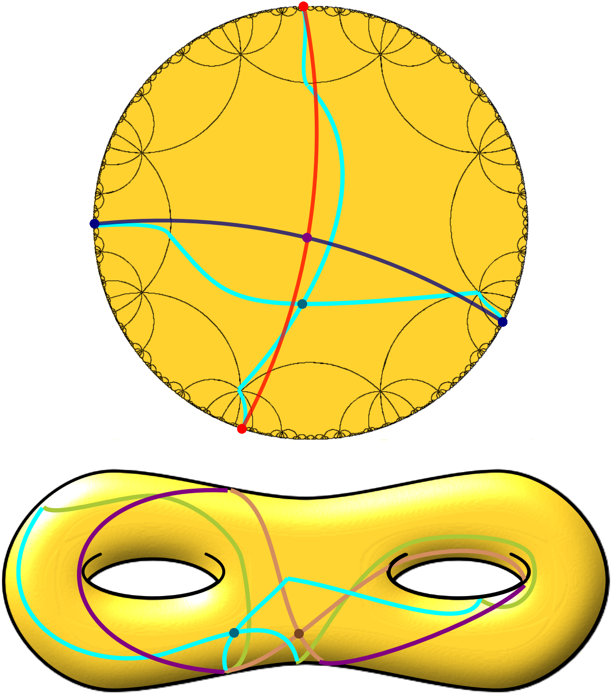

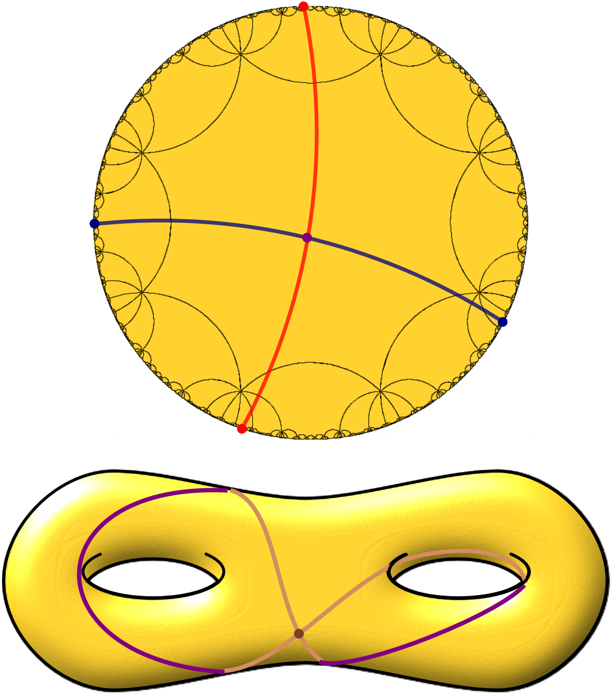

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ALGORITHMS FOR JORDAN CURVES ON COMPACT SURFACES

BY BRUCE L. REINHART*

(Received October 20, 1960)

As is well known, free homotopy classes of mappings of the circle into a space correspond to conjugacy classes in the fundamental group. If the space is two-dimensional, such a free homotopy class may or may not contain a simple closed (that is, Jordan) curve. Our purpose here is to give an algorithm for determining which free homotopy classes admit such a curve in the case of compact surfaces of negative Euler number. Our algorithm is applicable to orientable and non-orientable surfaces, with or without boundary. This problem was first discussed by Dehn [4] and later by Baer [1] and Goeritz [5]. All of these studies are based on cutting the surface into spheres with 3 holes and deriving a (hopefully) canonical form for each free homotopy class by pasting together curves on each sphere with holes. Except for the surface of genus 2 without boundary, no complete answer has been achieved in this way. We shall treat the problem globally by imbedding the fundamental group into the group of motions of the hyperbolic plane, in the spirit of Poincaré [14] and numerous works of Nielsen. As a preliminary, we give in §1 a simple method for assigning to each word in the usual generators of the fundamental group a curve on the surface which has double points only in the neighborhood of the base point, and no other multiple points. We call such a curve an indicating curve for the word. In § 2, we apply the fact that each motion of the hyperbolic plane which arises in our problem leaves fixed a unique geodesic, its axis. The projection of an axis onto the surface is the unique geodesic lying in the free homotopy class containing the motion of which it is axis. For orientable surfaces, a free homotopy class admits a simple closed curve if and only if its geodesic is simple [14, p. 467], while

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	closed surface	**counting**	**free homotopy**	**special feature**
Chillingworth ’69				winding number
Birman & Series ’84			✓	retraction onto a graph
Cohen & Lustig ’87		✓	✓	retraction onto a graph
Lustig ’87	✓	✓	✓	canonical representative
de Graaf & Schrijver ’97	✓	✓	✓	Reidemeister moves
Paterson ’02	✓	✓	✓	Reidemeister moves
Gonçalves et al. ’05	✓	✓	✓	algebraic approach

Input: Given a curve c represented by a closed walk of length ℓ on a combinatorial surface of complexity n:

Counting

We can compute the geometric intersection number $i(c)$ in $O(n+\ell^2)$ time.

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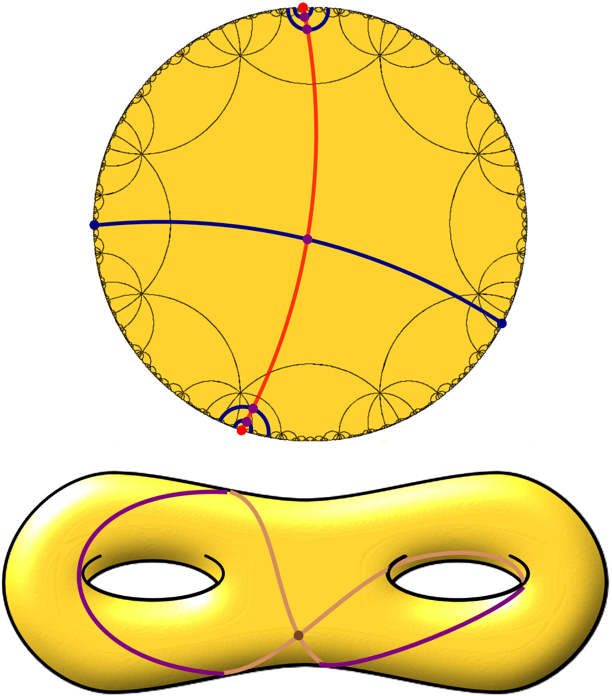
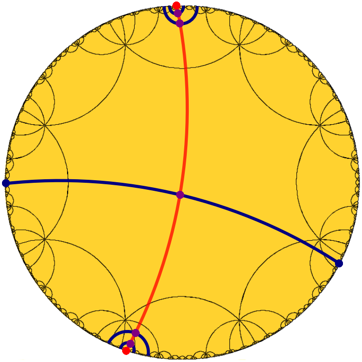
Detecting simple curves

We can decide if c is homotopic to a simple curve in $O(n+\ell\log^2\ell)$ time.





- If a curve c is primitive its lifts are uniquely determined by their limit points.
- If τ is the hyperbolic translation corresponding to a lift $\tilde{c}_0$ of a primitive c then

$$\imath(c) = \frac{1}{2}|\{\text{set of pairs of limit points crossing } \tilde{c}_0\}/\langle\tau\rangle|$$

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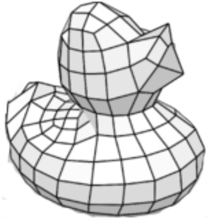
The plan

For a given curve c

1. Determine the primitive root of c.
2. Count the number of classes of crossing pairs of limit points (for the root of c).
3. Use adequate formula if c is not primitive.

Hyp : All faces are quadrilaterals.



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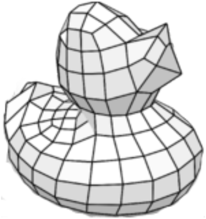

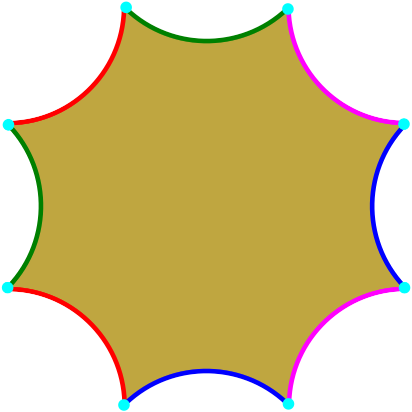

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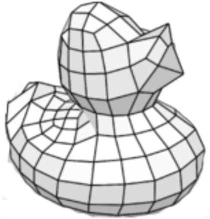

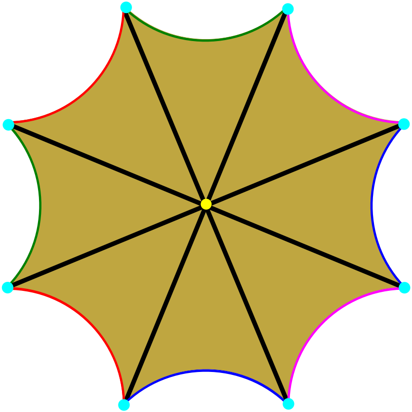

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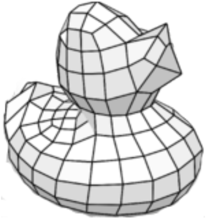

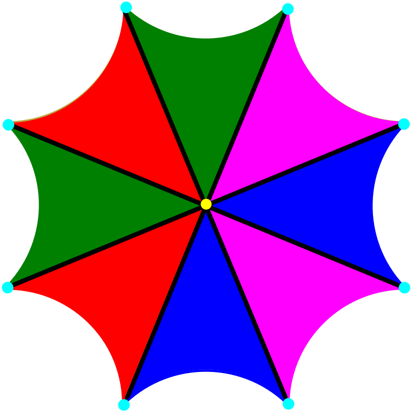

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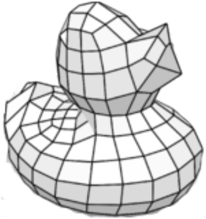

Discrete curvature

$$\kappa_s = 1 - \frac{d_s}{2} + \frac{c_s}{4}$$

$d_s :=$ degree of s and $c_s :=$ number of incident corners.

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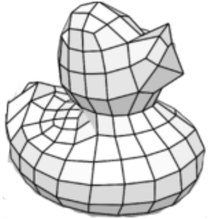

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$$\sum_{s \in S} \kappa_s = \chi$$

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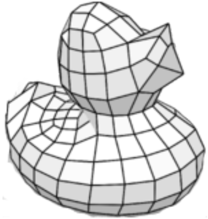

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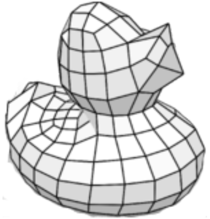

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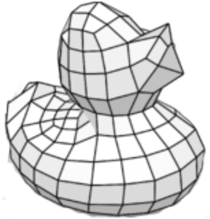

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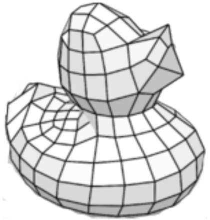

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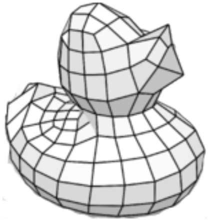

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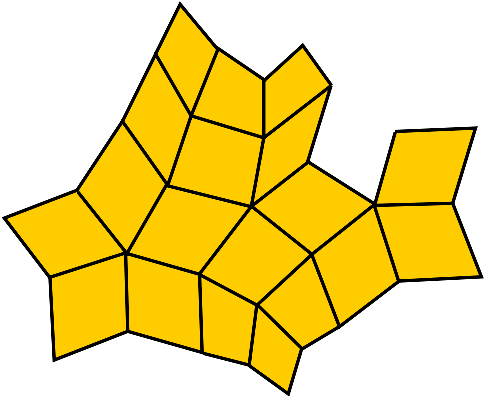
The 4 brackets Theorem

Hyp : All faces are quadrilaterals and all internal vertices have degree ≥ 4.



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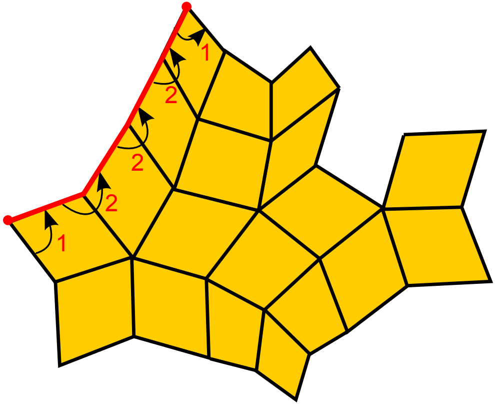

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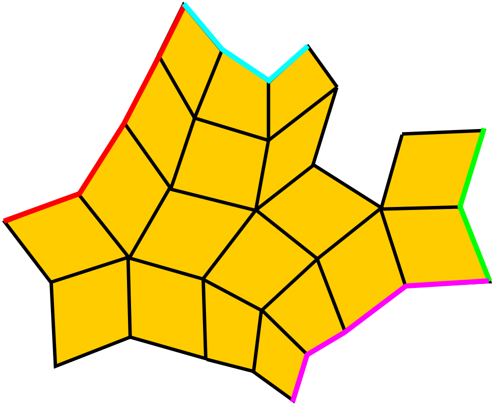

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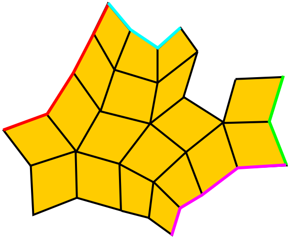

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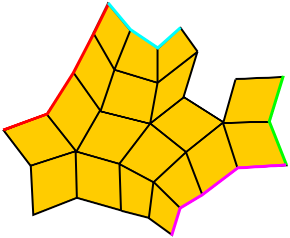

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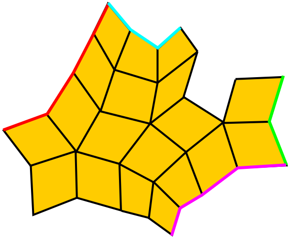

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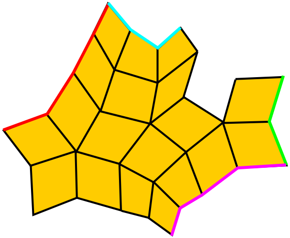

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Hence, on the boundary: $\#\{s \mid c_s = 1\} \geq \#\{s \mid c_s \geq 3\} + 4$.
$\square$

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Corollary

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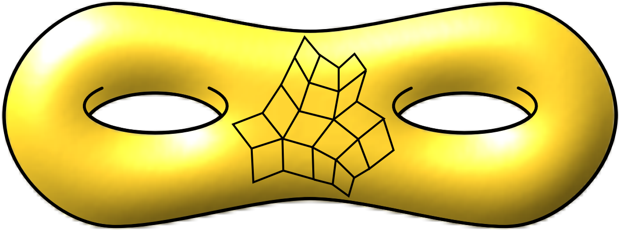

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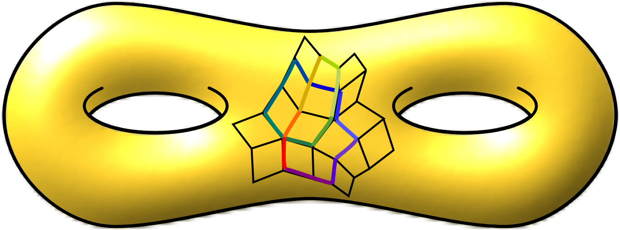

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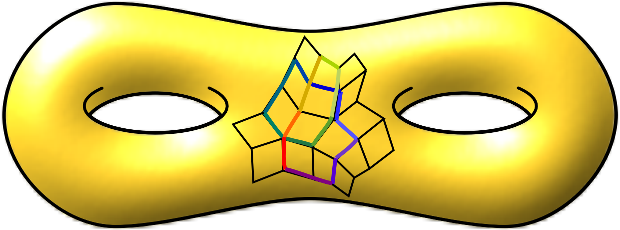

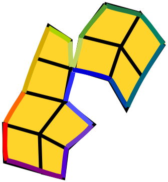

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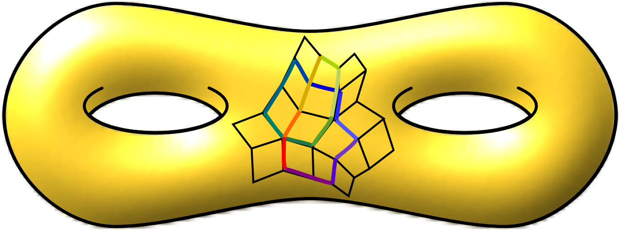

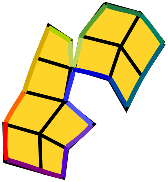

Apply the 4 brackets Theorem to this disk. $\square$

Hyp : All faces are quadrilaterals and all internal vertices have degree > 4.

The 5 brackets Theorem

The boundary of a non-singular disk with at least one interior vertex has at least 5 brackets.

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Canonical representative

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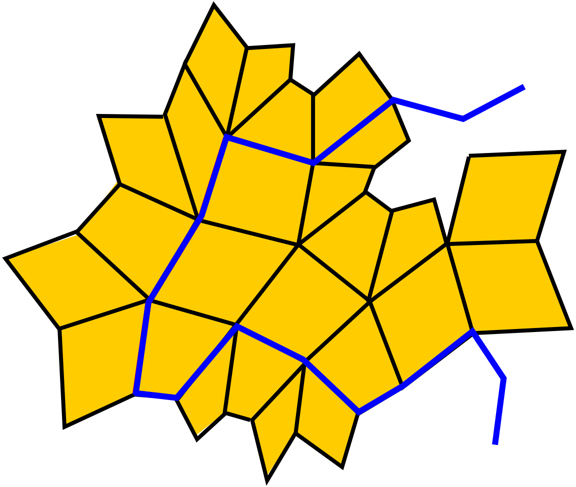

We shorten by removing brackets and spurs

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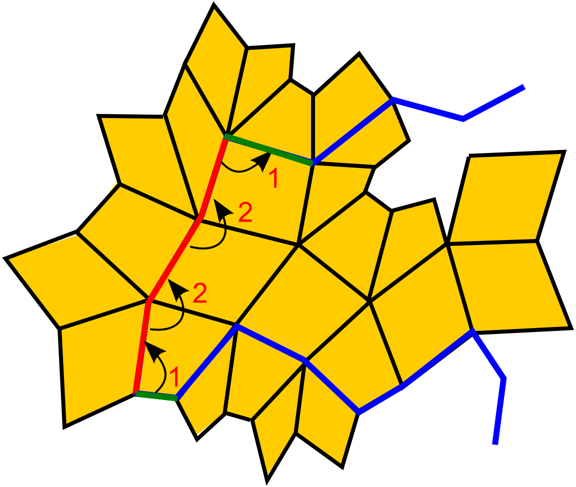

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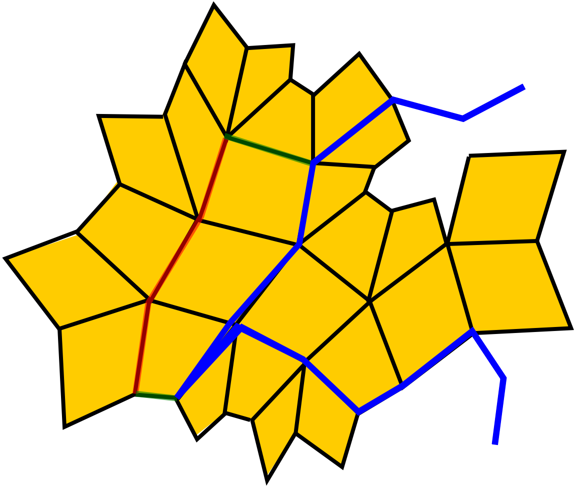

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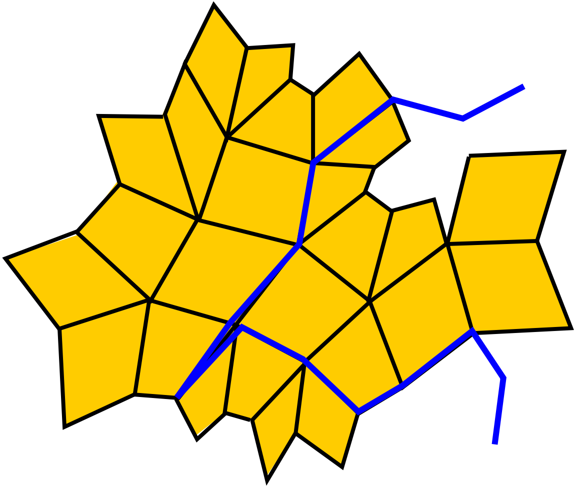

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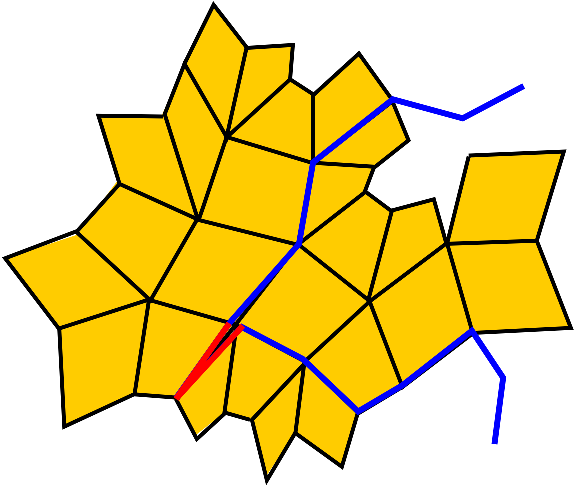

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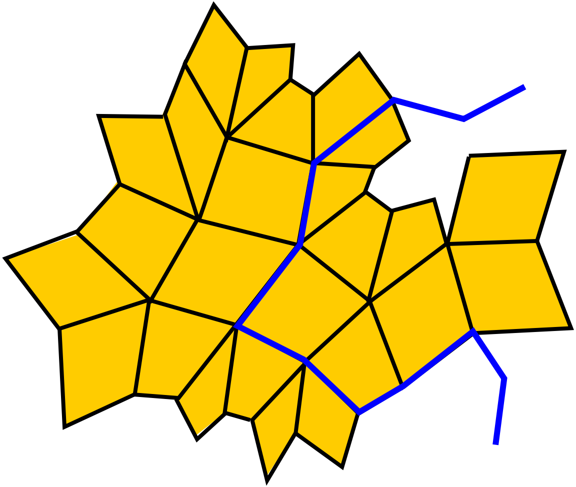

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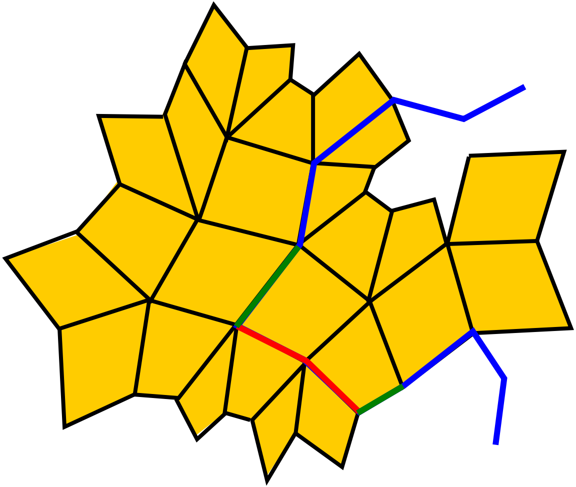

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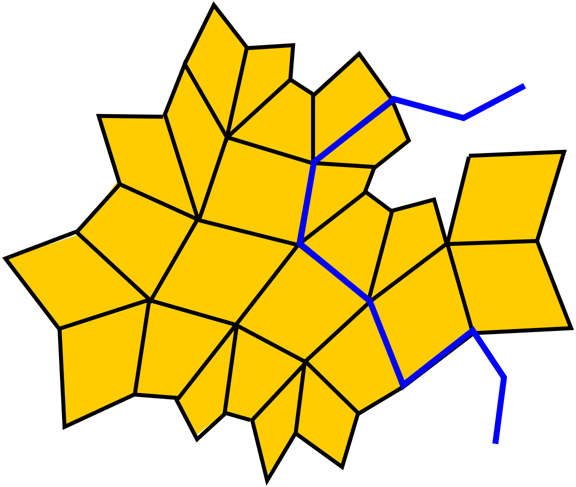

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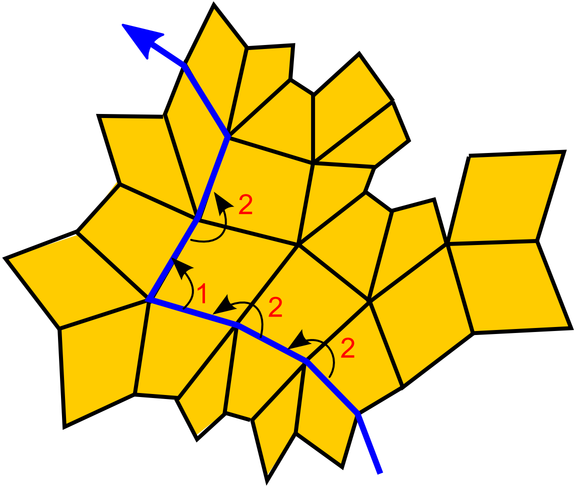

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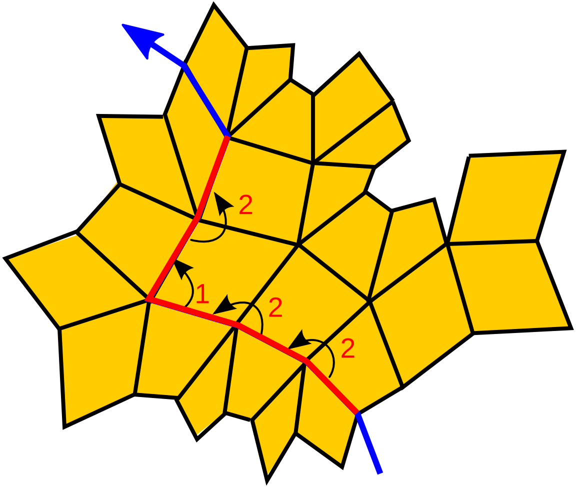

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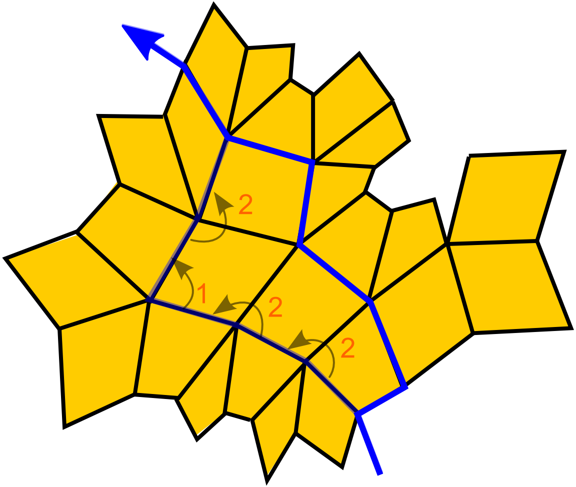

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Canonical representative

L. and Rivaud '12, Erickson and Whittlesey '13

After removing all spurs and brackets and pushing to the right as much as possible, we obtain a canonical representative. It can be computed in linear time.

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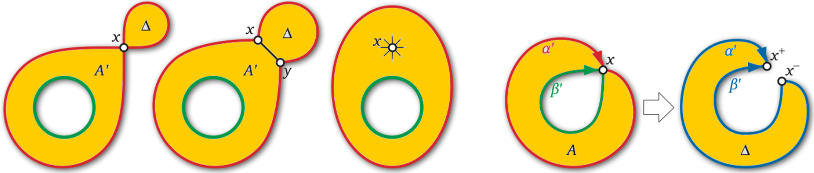

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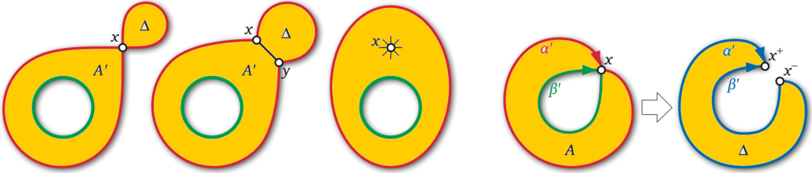

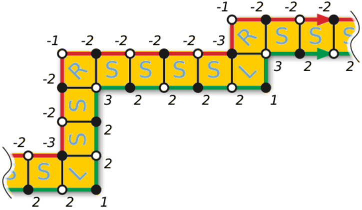

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Corollary I

One can decide if two curves are homotopic in linear time.

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After removing all spurs and brackets and pushing to the right as much as possible, we obtain a canonical representative. It can be computed in linear time.

Corollary I

One can decide if two curves are homotopic in linear time.

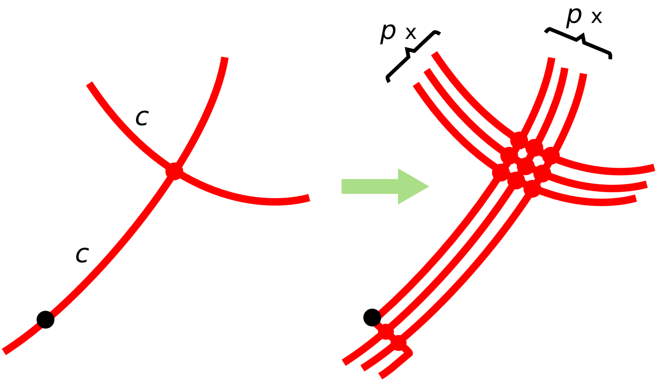
Corollary II

One can compute the primitive root of a curve in linear time.

Non-primitive curves

Lemma

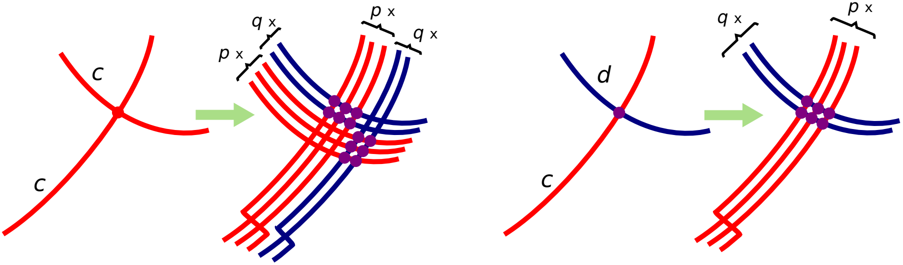
$$\imath(c^p) = p^2 \times \imath(c) + p - 1$$



Non-primitive curves

Lemma

$$\imath(c^p, d^q) = \begin{cases} 2pq \times \imath(c) & \text{if } c \sim d \text{ or } c \sim d^{-1}, \\ pq \times \imath(c,d) & \text{otherwise.} \end{cases}$$



The plan

For a given curve c

1. Determine the primitive root of c.
2. Count the number of classes of crossing pairs of limit points (for the root of c).
3. Use adequate formula if c is not primitive.

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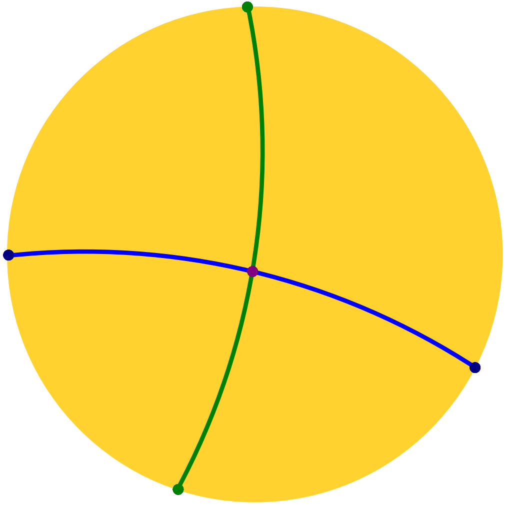
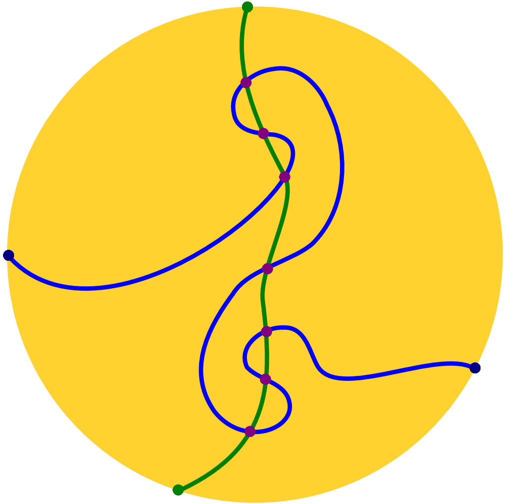
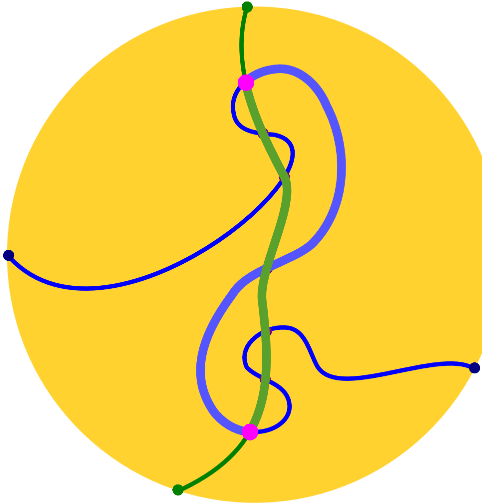




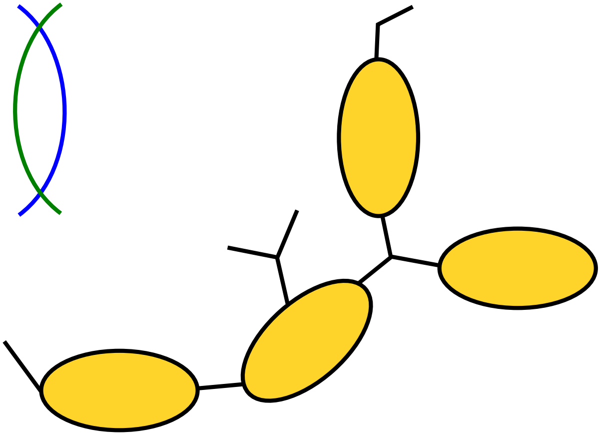


Double paths

A double path is a pair of homotopic paths.

Lemma

Double paths in a canonical (in fact geodesic) representative are quasi-flat.



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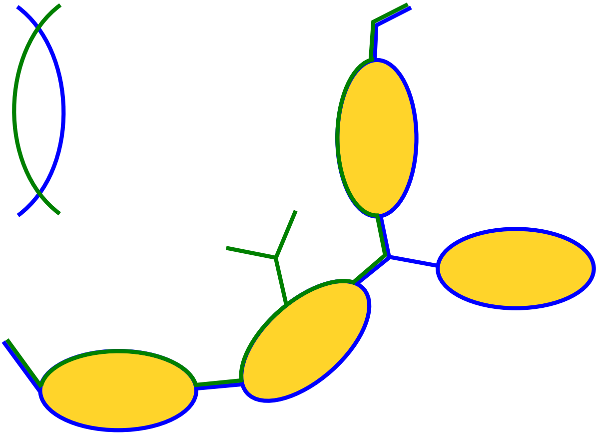

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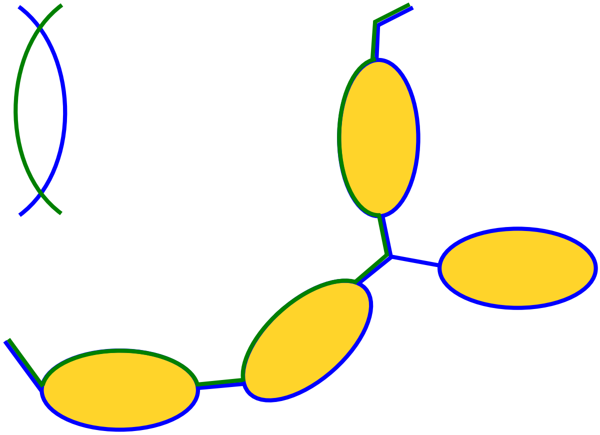

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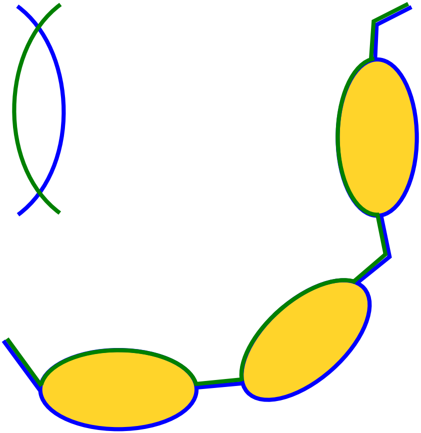

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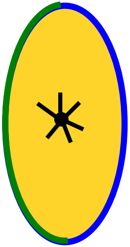

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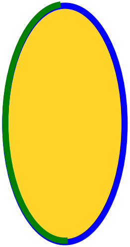

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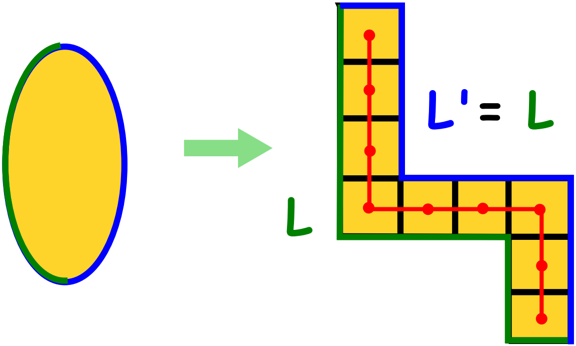

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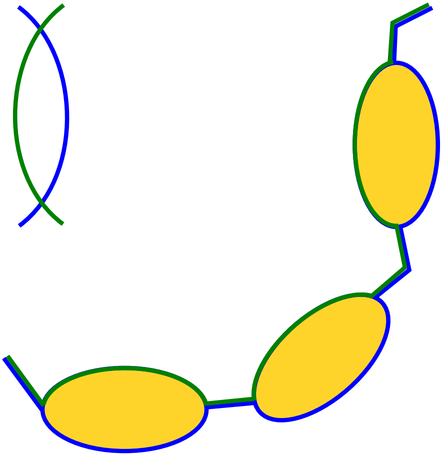

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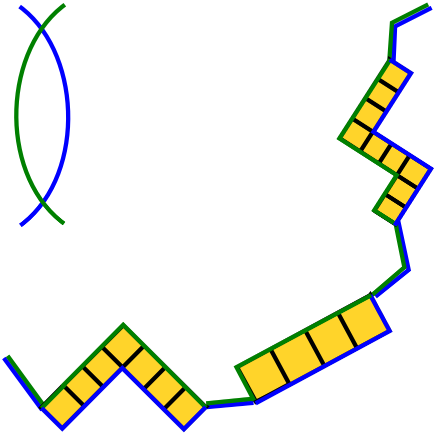

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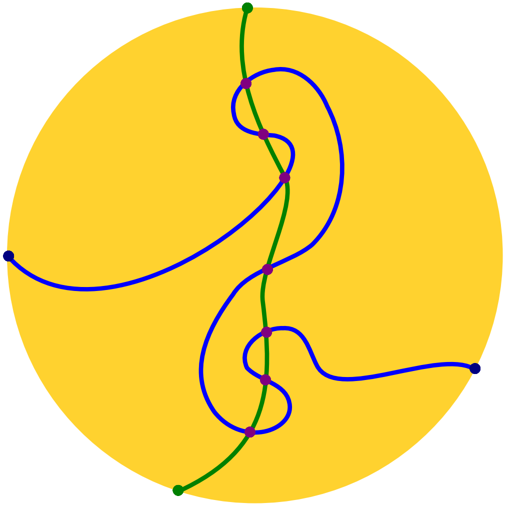

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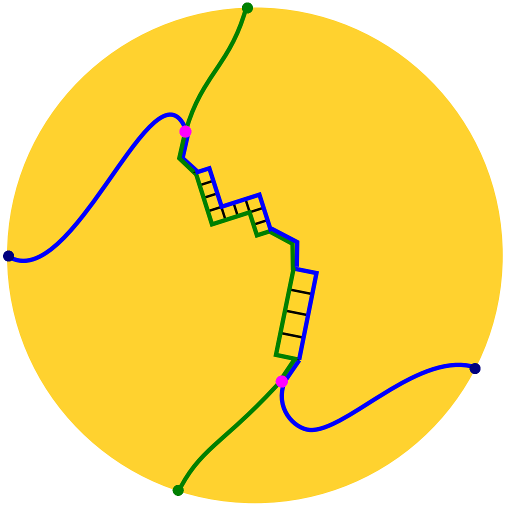

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Corollary

Each class of crossing pairs of limit points are uniquely identified by a maximal crossing double path.

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Each class of crossing pairs of limit points are uniquely identified by a maximal crossing double path.

Lemma

The set of maximal double paths can be computed in quadratic time.

PROOF. A pair of indices (i, j) may occur at most twice in the set of maximal double paths. $\square$

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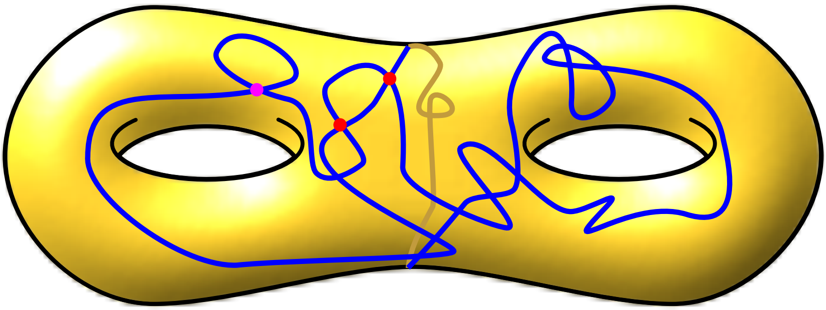
Despré and L. '16

The geometric intersection number of one (two) curve(s) can be computed in quadratic time.

Computing an actual minimal configuration



Hass and Scott '85

A curve with excess intersection has either a monogon or a singular bigon.

Computing an actual minimal configuration

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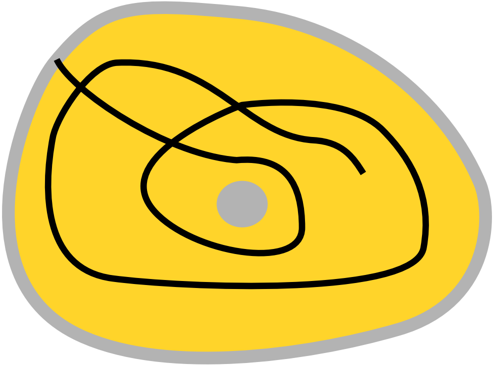

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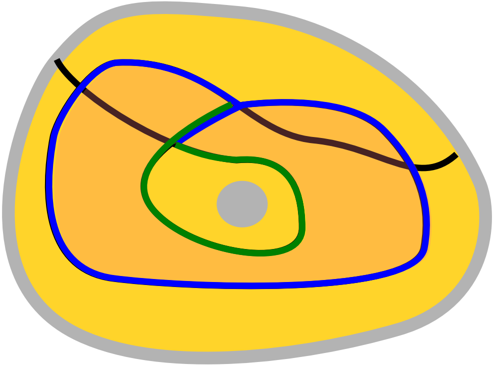

Bigon swapping

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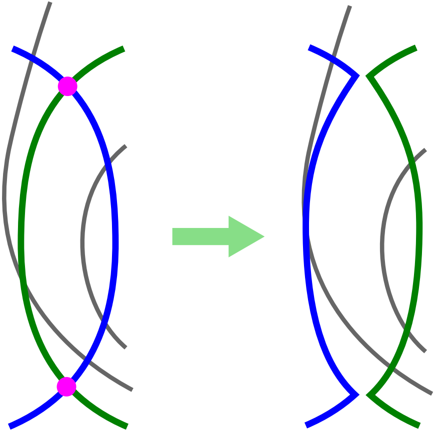

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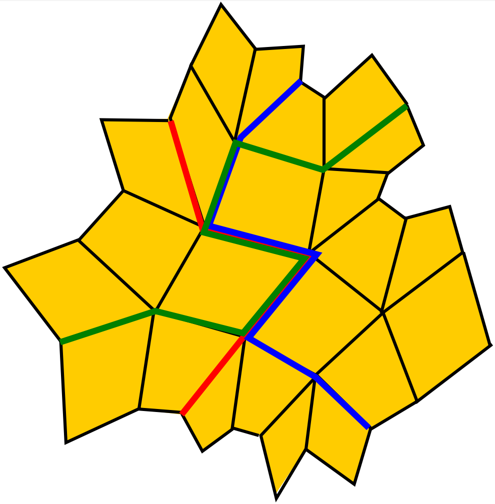
Given c, an homotopic *perturbation* with a minimal number of intersections can be computed in $O(|c|^4)$ time.



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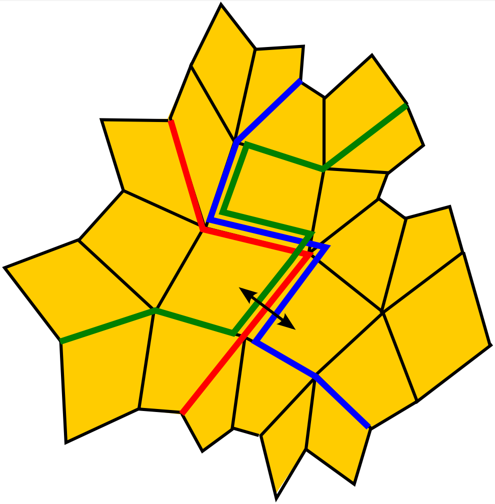
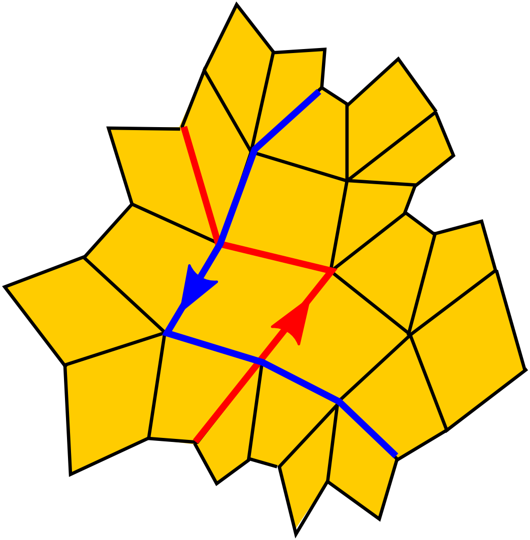
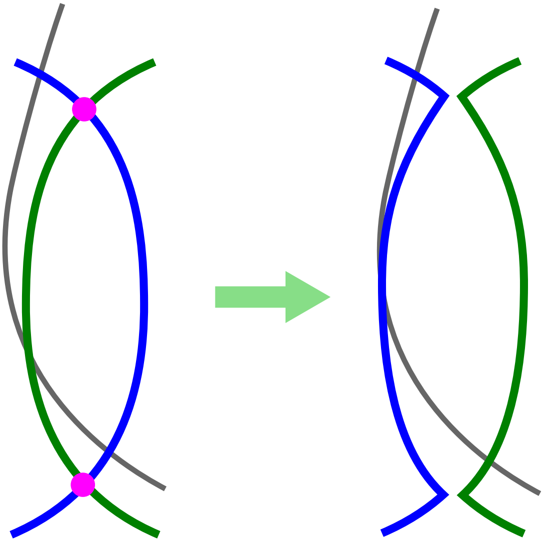


Detecting simple curves



Canonical $\neq$ minimal

Detecting simple curves



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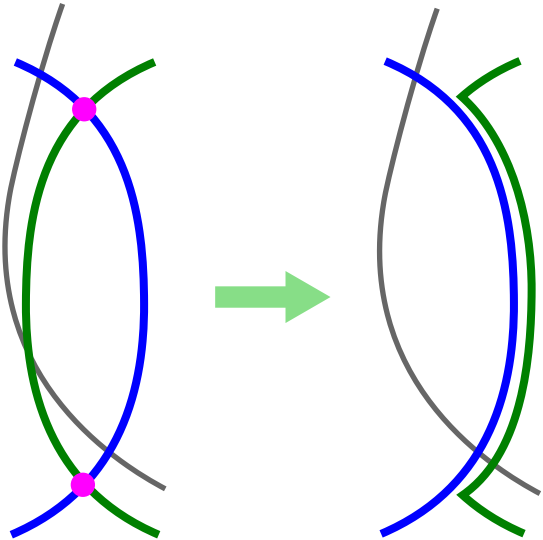

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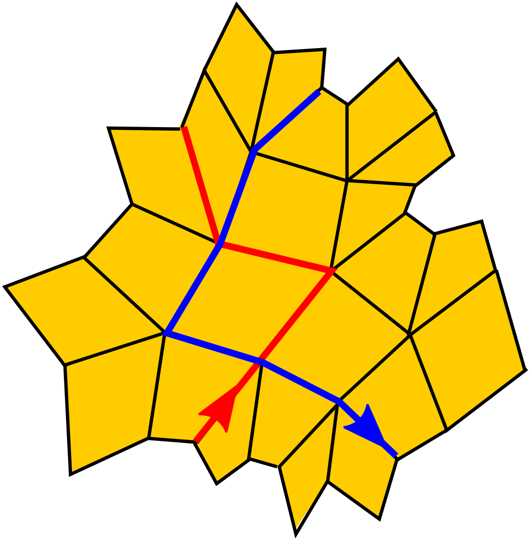

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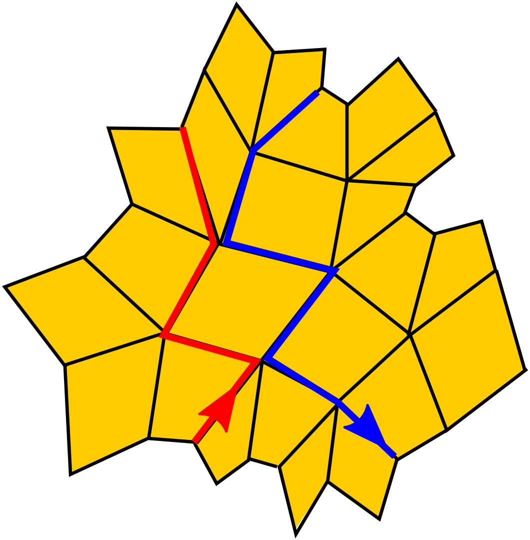

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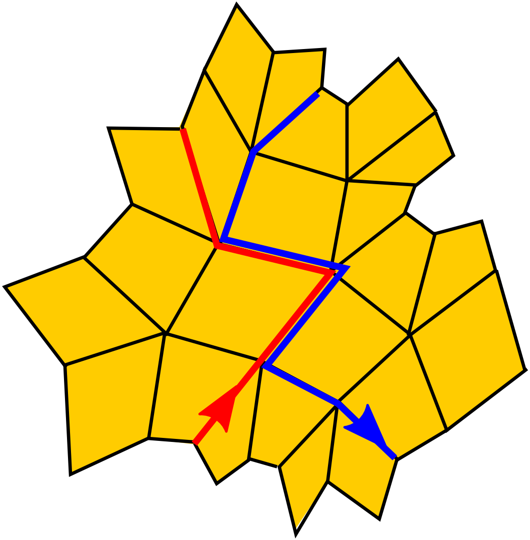
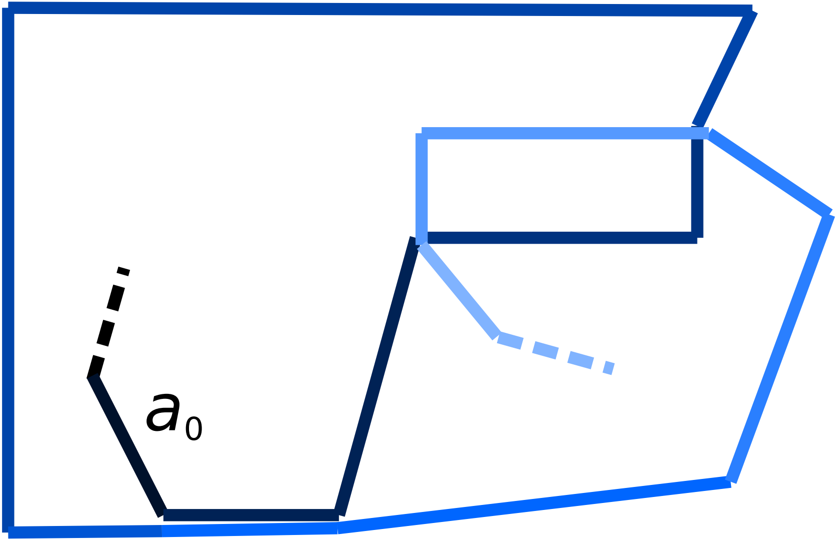


The unzip algorithm



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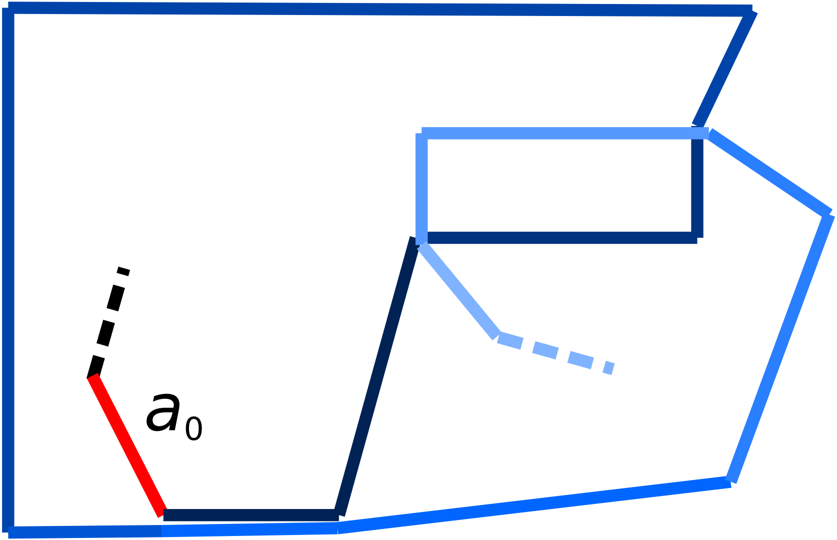

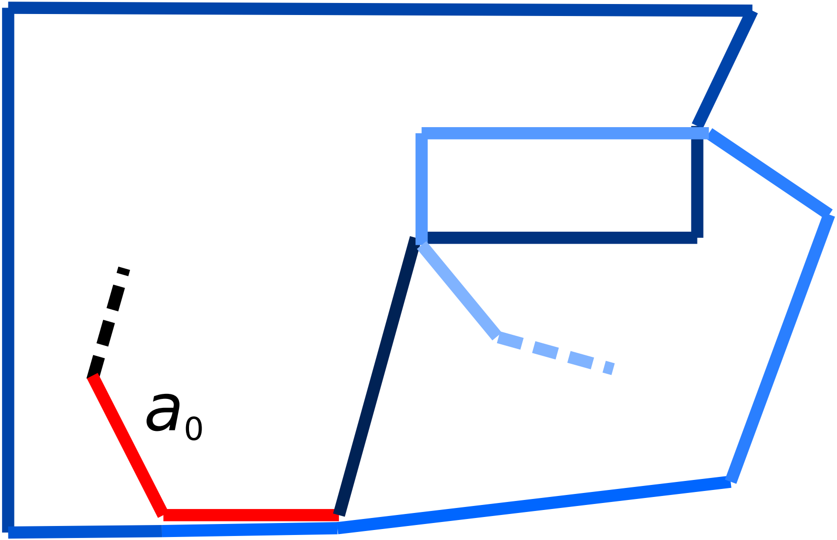
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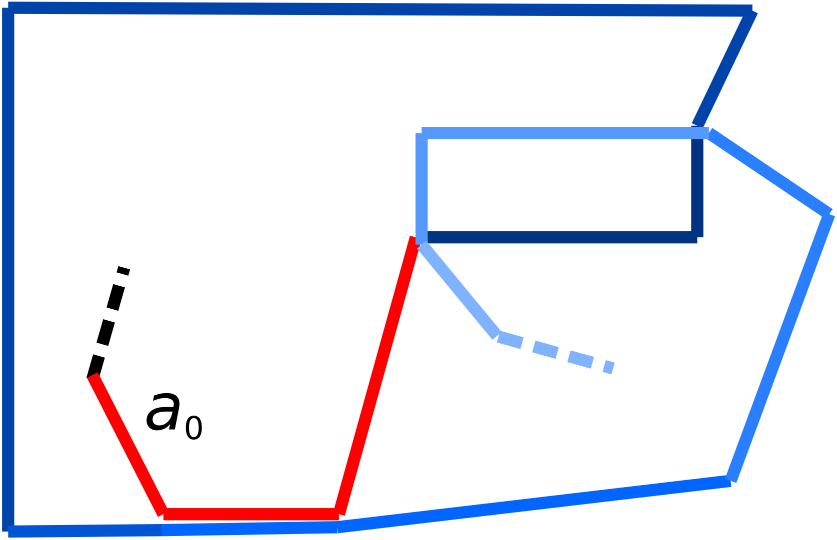

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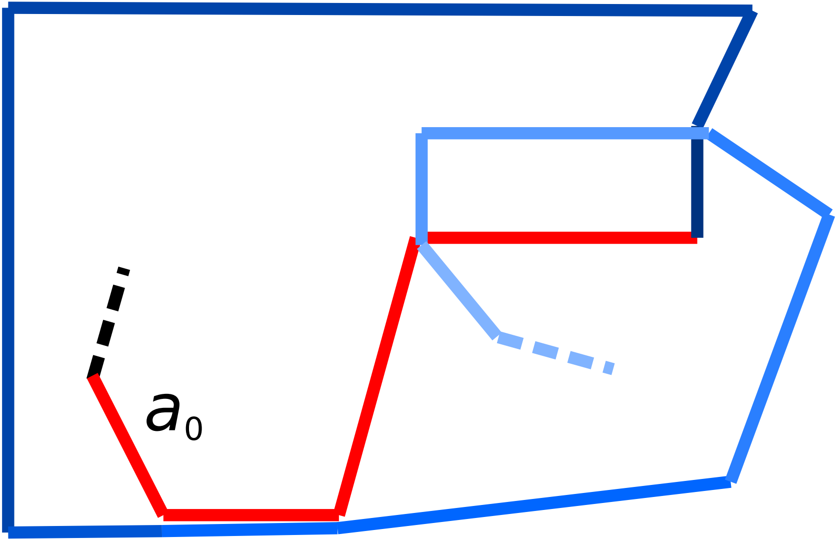

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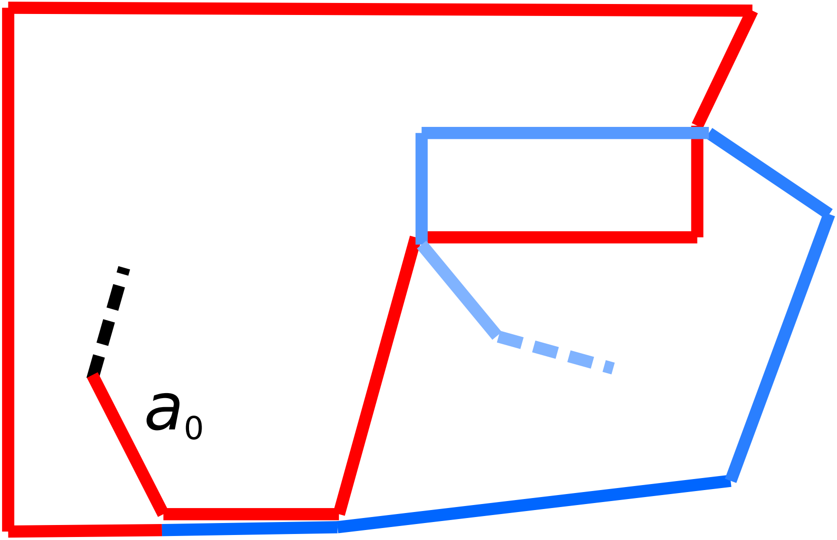

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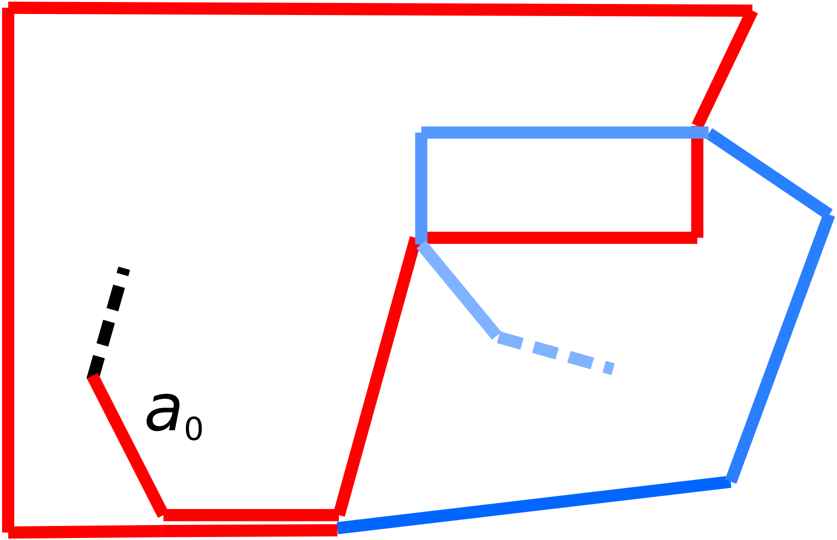

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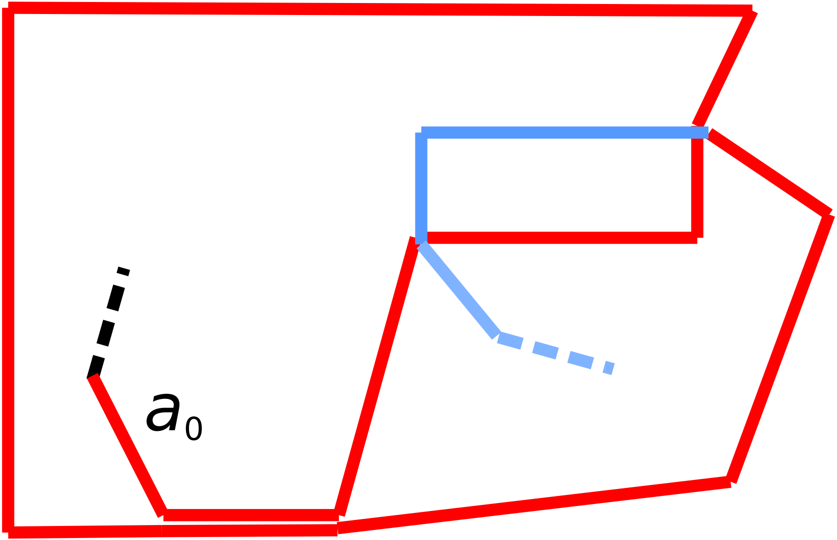

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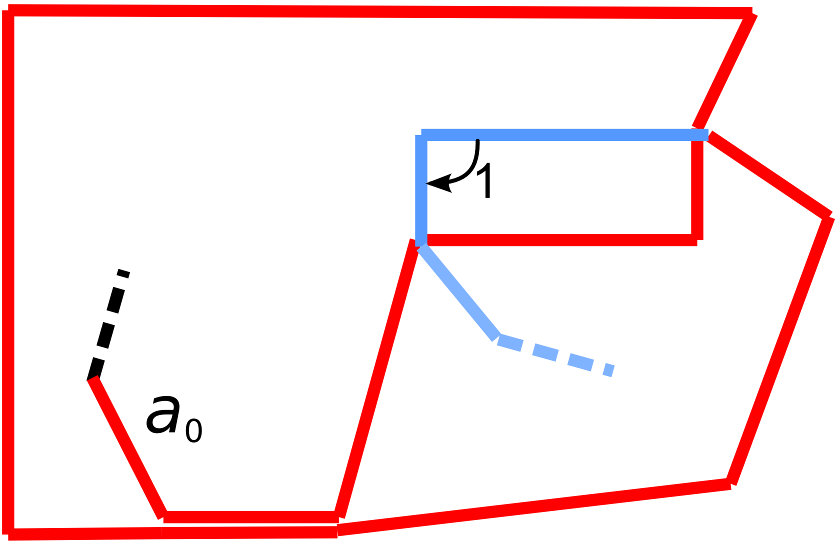

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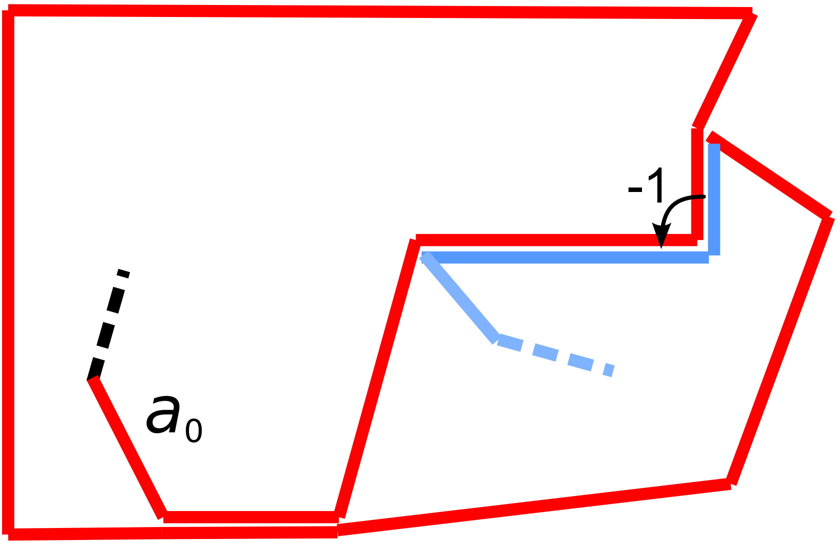

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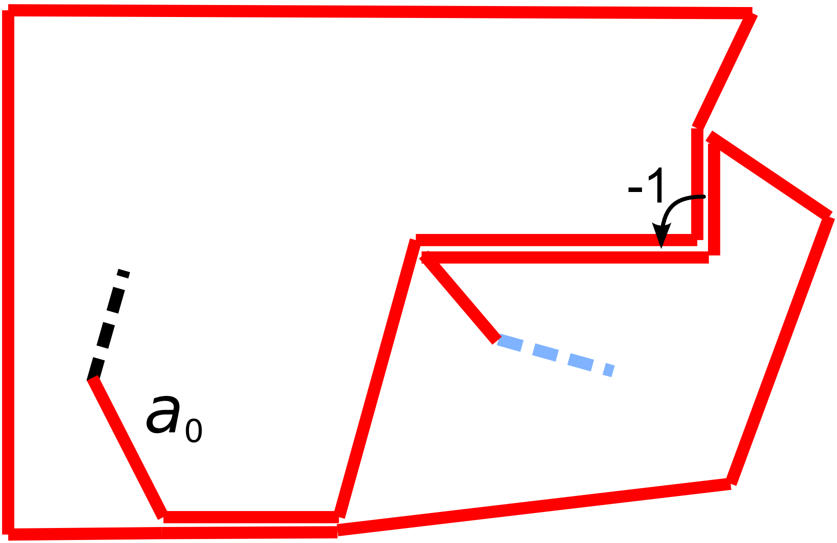
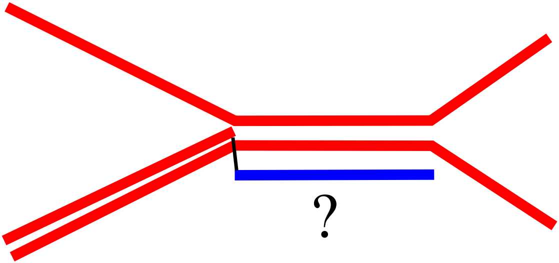


The unzip algorithm: Insertion



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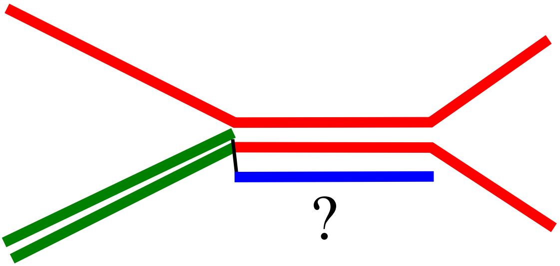

Use ordering of previous arcs

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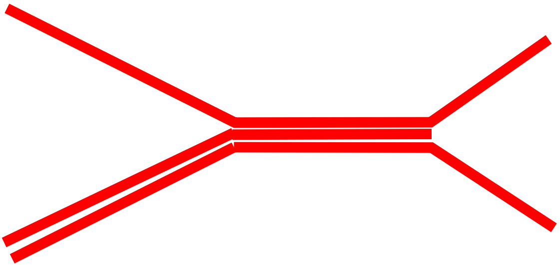

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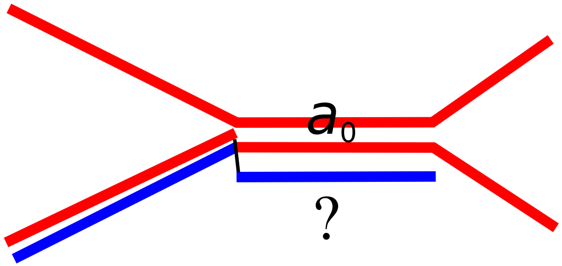

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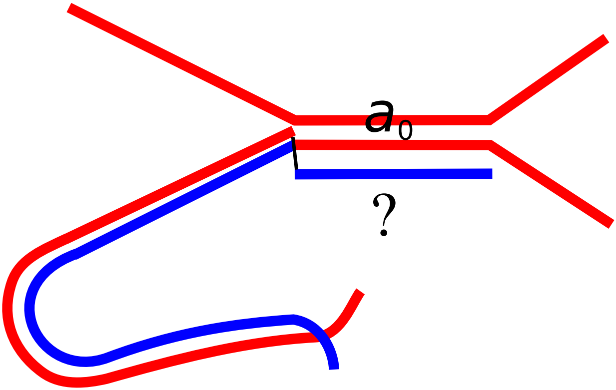

Use the fact that oriented bigons are flat + KMP

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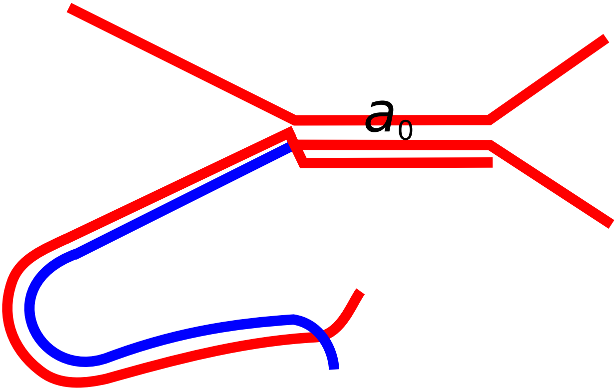

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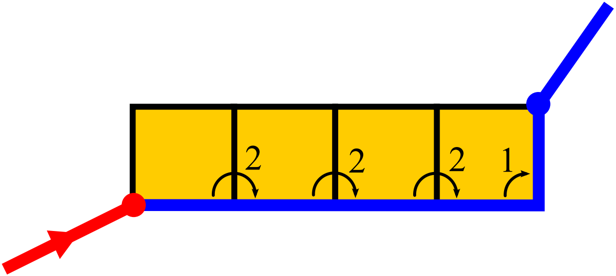
The unzip algorithm: Switchable arcs



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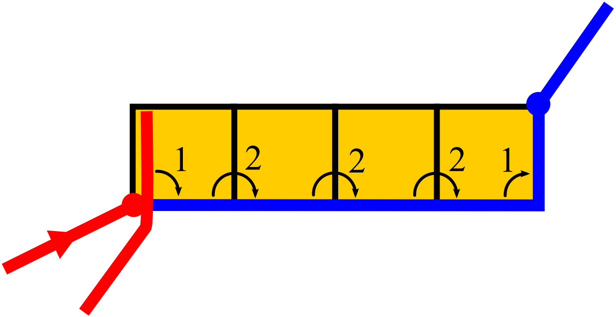

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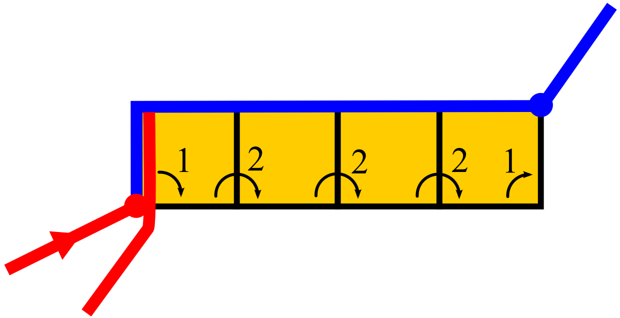

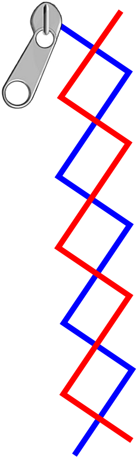

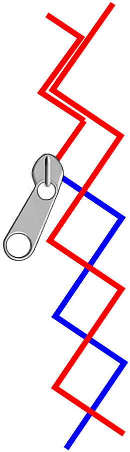

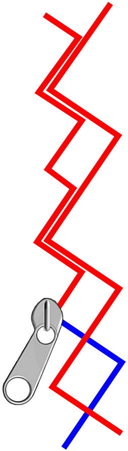

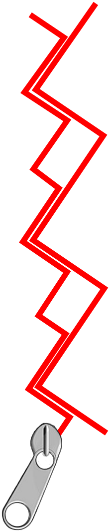

The unzip algorithm

Despré and L. '16

Given a combinatorial curve c, the **unzip algorithm** runs in $O(|c| \log^2 |c|)$ time and returns an embedding of c iff its geometric intersection number is null.

Open problems

- Extend the algorithms to non-orientable surfaces.

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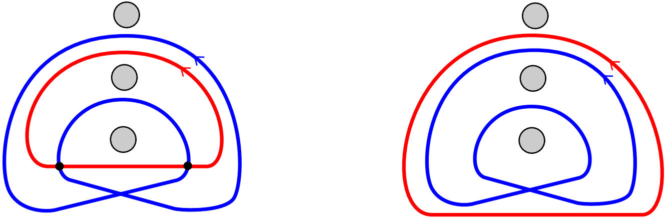
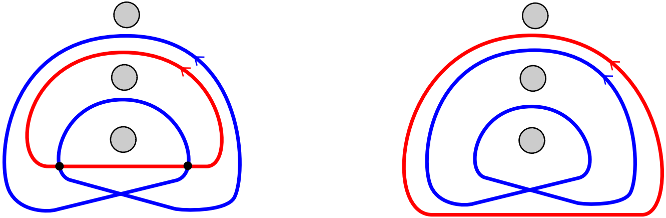
- Extend the algorithms to non-orientable surfaces.
- Propose a polynomial time algorithm to compute a minimal configuration of two curves.



Open problems

- Extend the algorithms to non-orientable surfaces.
- Propose a polynomial time algorithm to compute a minimal configuration of two curves.



- Is quadratic time optimal to just compute the geometric intersection number?

Open problems

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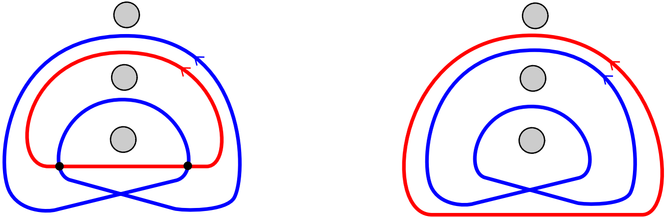
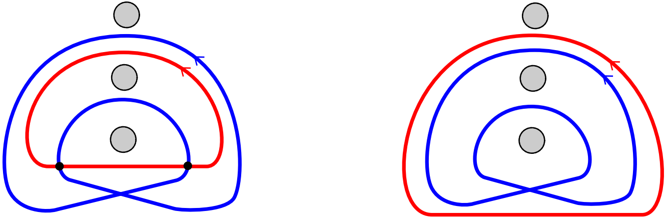


- Is quadratic time optimal to just compute the geometric intersection number?
- Find a better algorithm (less than quartic) to compute a minimal configuration of single curve.

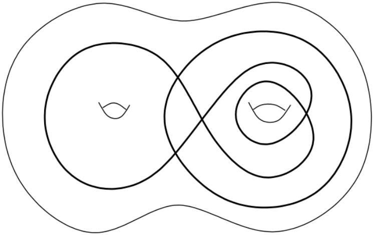
Open problems

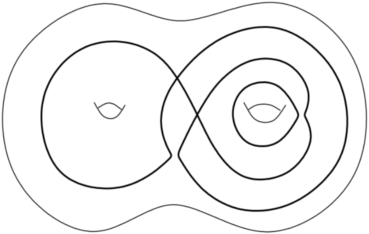
- Extend the algorithms to non-orientable surfaces.
- Propose a polynomial time algorithm to compute a minimal configuration of two curves.



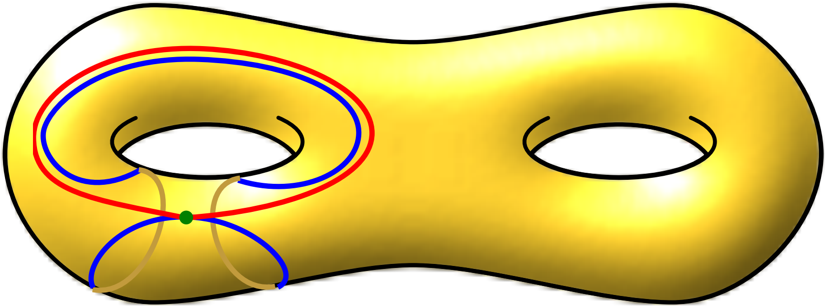
- Is quadratic time optimal to just compute the geometric intersection number?
- Find a better algorithm (less than quartic) to compute a minimal configuration of single curve.
- Does the unzip algorithm extends to nonsimple curves or multiple curves to compute minimal configurations?

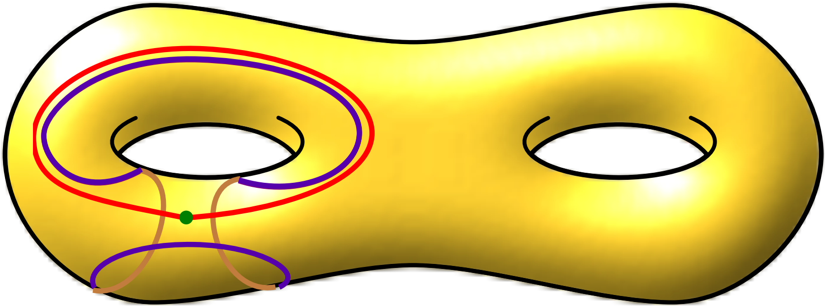
Thank you for your attention

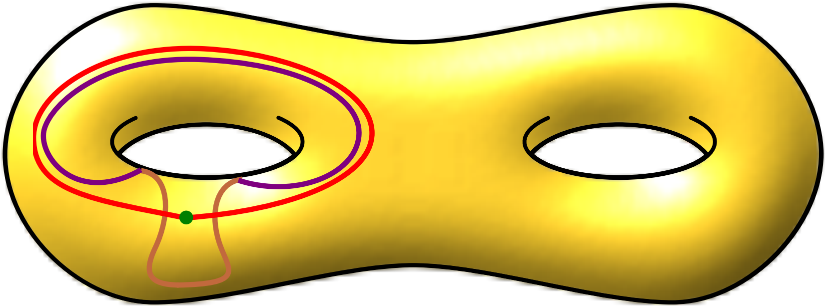


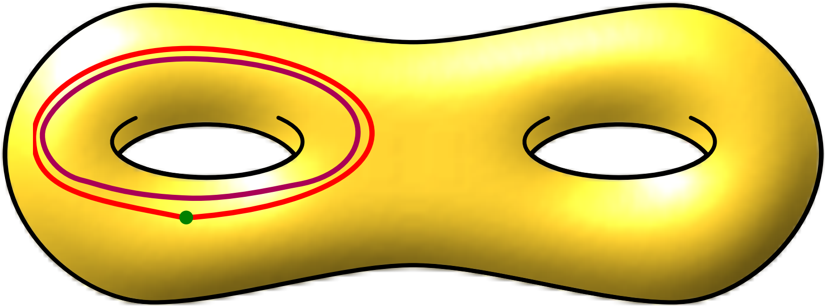


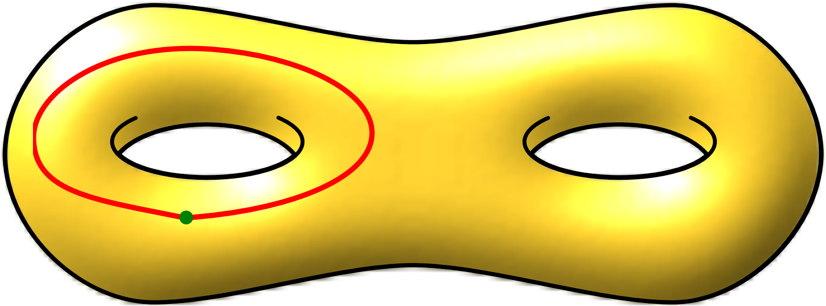
Neumann-Coto '01

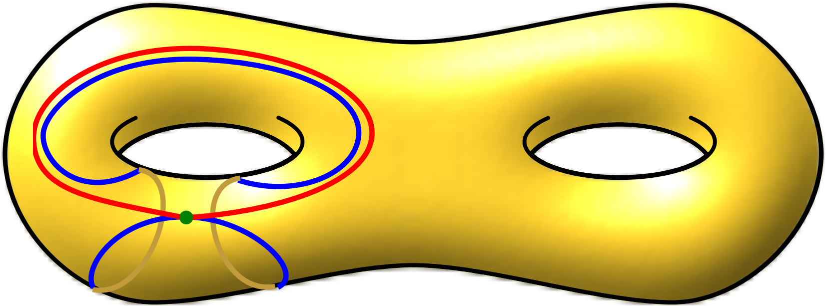


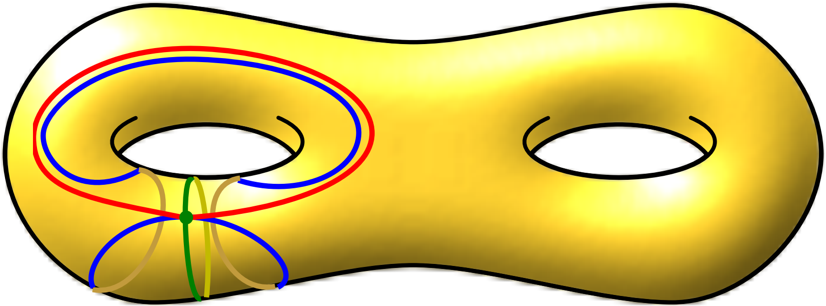












$$a \sim c^{-1} \cdot b \cdot c$$