भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

2nd Lecture on Transform Techniques

(MA-2120)

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What have we learnt so far?

- Introduction to Integral Transforms
- Introduction and Motivation to Laplace Transforms
- Improper Integral
- Existence of Improper Integral

What will we learn today?

- Function of Exponential Order
- Conditions for existence of Laplace Transform
- Laplace Transform of some elementary functions
- Basic properties of Laplace Transform

Evaluate the Laplace transform of $f(t)=1,\ t\geq 0$.

Solⁿ:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \lim_{R\to\infty}\int_0^R e^{-st} f(t)\,dt$$

$$\mathcal{L}[1] = \lim_{R\to\infty}\int_0^R e^{-st}\,dt = \lim_{R\to\infty} \frac{e^{-sR}-1}{-s} = \frac{1}{s}$$

Since $\lim_{R\to\infty} e^{-sR} = 0$ provided s is real and positive.

Q. What will it be the result if s is complex number?

Solⁿ: Let $s = x+iy$.

$$\mathcal{L}[f(t)] = \mathcal{L}[1] = \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^R$$

$$= \lim_{R\to\infty} -\frac{1}{s}\left[e^{-sR} - 1\right]$$

$$\lim_{R\to\infty} e^{-sR} = \lim_{R\to\infty} e^{-(x+iy)R} = \lim_{R\to\infty} e^{-xR} \cdot e^{-iyR}, \text{ since } \left|e^{-iyR}\right| = 1.$$

$= 0$, for $x > 0$ only

i.e, $Re(s) > 0$.

$$\boxed{\mathcal{L}[1] = \lim_{R\to\infty} -\frac{1}{s}\left[e^{-sR} - 1\right] = \frac{1}{s}}$$, provided $Re(s) > 0$

when s is complex number.

Before going to talk about the Sufficient Condition for existence of Laplace Transform, we need to know another definition of "Functions of Exponential order".

Functions of Exponential order:

Definition: A function $f : [0, \infty) \to \mathbb{R}$ is said to be of exponential order α if $\exists$ constants $\alpha > 0$ and $M > 0$ such that for some $t_0 > 0$,

$$\boxed{|f(t)| \le M e^{\alpha t}, \quad \forall\, t \ge t_0}.$$

This is the Growth Condition for $f(t)$ for large values of t.

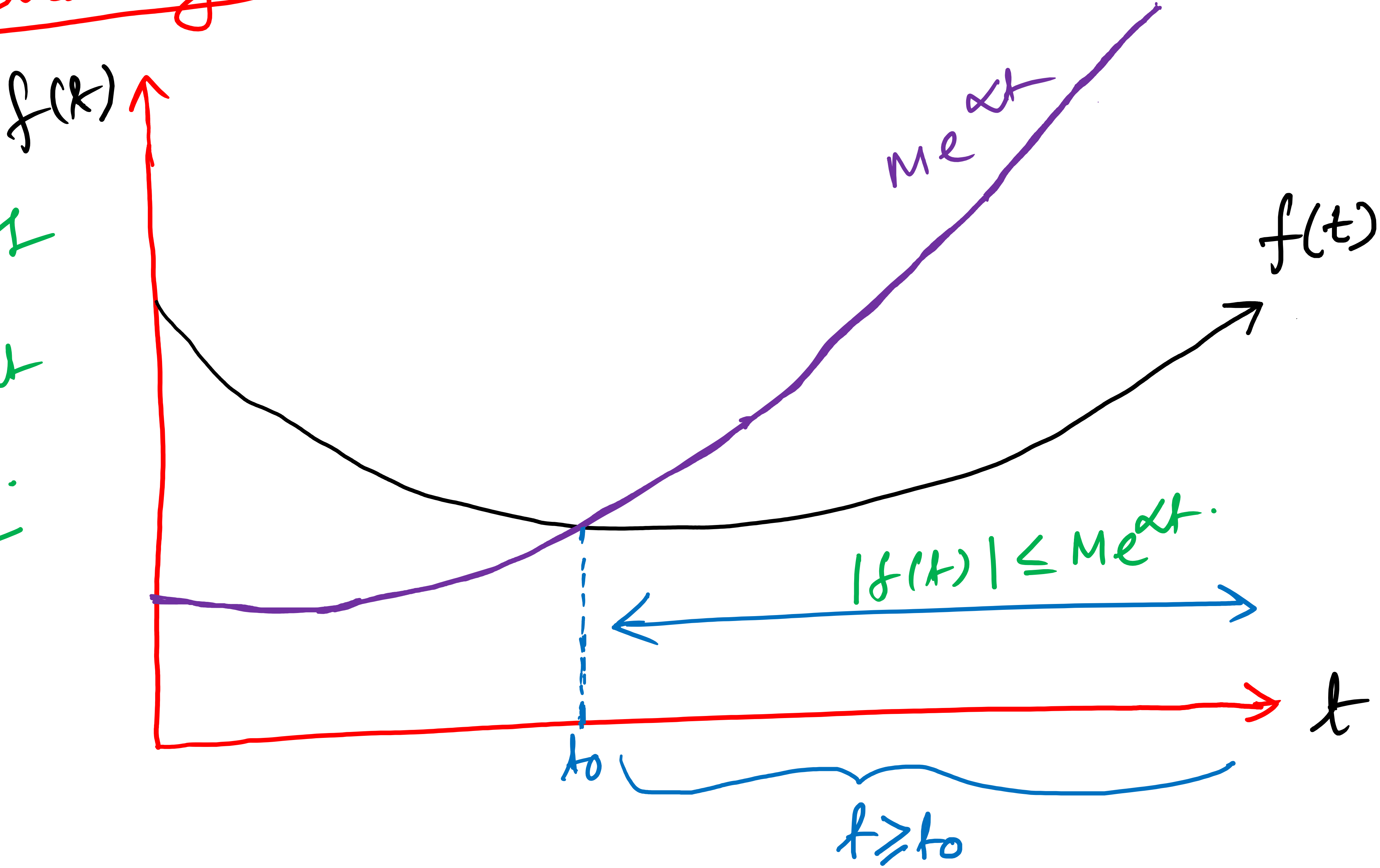
Since $|f(t)| \le M e^{\alpha t} \Rightarrow \boxed{e^{-\alpha t} |f(t)| \le M}$

$\Rightarrow \lim_{t \to \infty} e^{-\alpha t} |f(t)|$ exists and has the finite value.

So equivalently, one can say that a function $f:[0,\infty) \to \mathbb{R}$ is said to be exponential order α if

$$\boxed{\lim_{t\to\infty} e^{-\alpha t}|f(t)| = \text{finite value}}$$

Geometrically,



$f(t)$ does not grow faster than $Me^{\alpha t}$ for $t \geq t_0$.

Examples:

① Show that the function $f(t) = t^n$ has exponential order α for any $\alpha > 0$ and any $n \in \mathbb{N}$.

Sol^n: To show that $f(t) = t^n$ is of exponential order α, we need $\lim\limits_{t\to\infty} e^{-\alpha t}|f(t)| =$ finite quantity.

Now for $n=1$: $\lim\limits_{t\to\infty} e^{-\alpha t}|t| = \lim\limits_{t\to\infty} e^{-\alpha t}\, t$

$$= \lim_{t\to\infty} \frac{t}{e^{\alpha t}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t\to\infty} \frac{1}{\alpha e^{\alpha t}} = 0, \quad \left(\text{By L'Hospital's Rule}\right)$$

$$\lim_{t\to\infty} e^{-\alpha t}|t| = 0, \text{ for any } \alpha > 0$$

<u>For $n=2$:</u> $$\lim_{t\to\infty} e^{-\alpha t}|t^2| = \lim_{t\to\infty} \frac{t^2}{e^{\alpha t}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t\to\infty} \frac{2}{\alpha^2 e^{\alpha t}} \quad \left(\text{use L'Hospital Rule two times}\right)$$

$$= 0, \text{ for any } \alpha > 0$$

Similarly we have for any $n \in \mathbb{N}$

$$\lim_{t\to\infty} e^{-\alpha t}|t^n| = \lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t\to\infty} \frac{n!}{\alpha^n e^{\alpha t}} \quad \left(\text{By L'Hospital's Rule repeatedly } n \text{ times}\right)$$

$$= 0 \quad \text{for any } \alpha > 0.$$

Ex ②: Show that e^{t^2} is not of exponential order α.

Sol^n: $\lim_{t\to\infty} e^{-\alpha t}|e^{t^2}| = \lim_{t\to\infty} e^{-\alpha t} \cdot e^{t^2}$

$= \infty$ for all values of α.

Example ③:

(i) If $f(t)$ is a bounded function then $f(t)$ is of exponential order. [Ex. $\sin t$ and $\cos t$]

(ii) Any polynomial in t is of exponential order.

Note: If two functions are of exponential order, then their sum is also an exponential order.

(iii) $\ln(t)$ is of exponential order.

Since $\ln(t) \leq t \;\; \forall\, t > 0$ and t is of exponential order,

$\ln(t)$ is also an exponential order.

note: If $g(t)$ is the function of exponential order and $f(t) \le g(t)$, $\forall t > 0$, then $f(t)$ is also a function of exponential order.

(iv) If $f(t)$ is not an exponential order then $e^{f(t)}$ is also not of exponential order.

Example: Whether the following functions are of exponential order or not.

(i) e^{-2t} (ii) $\cos t^2$ (iii) $t^2 \sin t$ (iv) e^{t^3}

Try it! (Homework).

Now we will Study the Sufficient Conditions for existence of Laplace Transform.

Theorem: (Sufficient Conditions for existence of Laplace Transform)

If a function $f:[0,\infty)\to\mathbb{R}$ is continuous or piecewise continuous in every finite interval $(0,R)$ and of exponential order α, then the Laplace Transform of $f(t)$ exists for all s provided $Re(s) > \alpha$.

Moreover, under these condition, Laplace integral $\int_0^\infty e^{-st} f(t)\,dt$ converges absolutely.

Proof: Since $f(t)$ is of exponential order α, then

$$|f(t)| \leq M_1 e^{\alpha t}, \; \forall\, t \geq t_0 \quad \text{--- (1)}$$

Also $f(t)$ is piecewise continuous on $[0,\infty)$, then

$$|f(t)| \le M_2 \quad , \quad 0 < t < t_0 \quad — (2)$$

From eqn (1) and (2) we have

$$|f(t)| \le M e^{\alpha t} \quad , \quad t > 0 \, ,$$

where $M = \max\{M_1, M_2\}$.

Then, $\displaystyle\int_0^R |e^{-st} f(t)| \, dt \le \int_0^R |e^{-(x+iy)t}| M e^{\alpha t} dt$

Let $s = x + iy$.

$$\int_0^R \left| e^{-st} f(t) \right| dt \le M \int_0^R e^{-(x-\alpha)t}\, dt$$

$$= \frac{M}{x-\alpha} - \frac{M}{(x-\alpha)} e^{-(x-\alpha)R}$$

Letting $R \to \infty$ and noting $Re(s) = x > \alpha$, we have

$$\boxed{\int_0^\infty \left| e^{-st} f(t) \right| dt \le \frac{M}{x-\alpha}}$$

This imply that the Laplace integral

converges absolutely and hence it converges for $Re(s) > \alpha$. (Proved)

Note: we have

$$\left|\int_0^\infty e^{-st} f(t)\, dt\right| \leq \int_0^\infty \left|e^{-st} f(t)\right| dt$$

$$\leq \frac{M}{Re(s) - \alpha}$$

for $Re(s) > \alpha$.

(i) If $Re(s) \to \infty$,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s) \to 0.$$

(ii) If $\mathcal{L}[f(t)] \not\to 0$ as $s \to \infty$ (or $Re(s) \to \infty$), then i.e. if $F(s) \not\to 0$ as $s \to \infty$ (or $Re(s) \to \infty$), then $F(s)$ can not be the Laplace transform of any continuous function $f(t)$.

Example: Consider functions $F_1(s) = 1$ and $F_2(s) = \frac{s}{s+1}$

These are not Laplace transform of any continuous function since $F_1(s) \not\to 0$ as $s \to \infty$ and $F_2(s) \not\to 0$ as $s \to \infty$.

Similarly for $F(s) = s$ or s^2

Laplace transform does not exist

Since $\lim_{s \to \infty} F(s) \neq 0$.

- Consider a function $f(t) = e^{at^2}$, $a > 0$ does not have the existence of Laplace transform through it is continuous but is not of the exponential order because $\lim_{t \to \infty} e^{at^2 - st} = \infty$.

Note: The conditions stated in the existence theorem are sufficient and not necessary conditions. If these conditions are satisfied then the Laplace transform must exist. If these conditions are not satisfied, then the Laplace transform may or may not exist.

Note: The conditions for existence of Laplace transform are not satisfied. However the Laplace transform may exist.

Example: Consider the function $f(t) = t^{-1/2}$.

This function is not continuous on any interval $[0, R]$ because it has limit ∞ at $t \to 0$ from the right. However $\int_0^R e^{-st}\, t^{-1/2}\, dt$ exists for all $R > 0$.

$$\mathcal{L}\left[t^{-1/2}\right] = \int_0^\infty e^{-st}\, t^{-1/2}\, dt$$

Let $u = st$

$du = s\,dt$

$\Rightarrow dt = \dfrac{du}{s}$.

$t = u/s$.

$$= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^{-1/2} \frac{du}{s}$$

$$= \frac{1}{s^{1/2}} \int_0^\infty e^{-u}\, u^{-1/2}\, du$$

$$\Gamma m = \int_0^\infty e^{-u}\, u^{m-1}\, du$$

$$= \frac{1}{s^{1/2}} \int_0^\infty e^{-u}\, u^{\frac{1}{2}-1}\, du$$

$$= \frac{1}{s^{1/2}} \cdot \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{s^{1/2}} = \sqrt{\pi/s}.$$

$s > 0$.

Laplace transformation exist. but the function is not piecewise continuous.

Example: Consider $f(t) = 2t\,e^{t^2}\cos(e^{t^2})$.

Here $f(t)$ is continuous on $[0,\infty)$ but not of exponential order.

However the $\mathcal{L}[f(t)]$ exists.

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}\, 2t\, e^{t^2}\cos(e^{t^2})\,dt$$

$$= e^{-st}\sin(e^{t^2})\Big|_0^\infty + s\int_0^\infty e^{-st}\sin(e^{t^2})\,dt$$

$$= -\sin(1) + s\,\underbrace{\mathcal{L}[\sin(e^{t^2})]}_{\text{exists.}}$$

Laplace transform of some elementary functions:

1. Find $\mathcal{L}[e^{at}]$.

Sol^n:

$$\mathcal{L}[e^{at}] = \lim_{R\to\infty} \int_0^R e^{at} e^{-st}\, dt$$

$$= \lim_{R\to\infty} \int_0^R e^{-(s-a)t}\, dt$$

$$= \lim_{R\to\infty} \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^R$$

$$= \frac{1}{s-a}, \text{ provided } Re(s)>a \text{ or } s>a.$$

$$\boxed{\lim_{R\to\infty} e^{-(s-a)R} = 0, \text{ provided } s>a \text{ or } Re(s)>a}$$

Example: (i) $\mathcal{L}\left[e^{iat}\right] = \dfrac{1}{s-ia}$, Provided $Re(s) > 0$

Try it! (**Home work**).

(ii) $\mathcal{L}\left[e^{-iat}\right] = \dfrac{1}{s+ia}$, $Re(s) > 0$

② Find $\mathcal{L}[t]$.

$$\mathcal{L}[t] = \int_0^\infty e^{-st} \cdot t \, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot t \, dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{-st} \cdot t}{-s} - \frac{e^{-st}}{s^2} \right]_0^R$$

$$\lim_{R\to\infty} e^{-sR} = 0.$$

$$= \frac{1}{s^2}, \; \underline{s>0}.$$

Similarly $\mathcal{L}[t^2] = \frac{2!}{s^3}$, $\mathcal{L}[t^3] = \frac{3!}{s^4}$

when n is positive integer, $\boxed{\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}}$, $\underline{\underline{s>0}}$

③ $\mathcal{L}[t^n]$ when n is an positive integer.

$n = 1, 2, 3, \ldots$

Solⁿ:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0$$

④ Find $\mathcal{L}[t^\alpha]$ when α is non integer.

Solⁿ:

$$\mathcal{L}[t^\alpha] = \int_0^\infty e^{-st} t^\alpha \, dt$$

$$= \int_0^\infty e^{-u} \cdot \left(\frac{u}{s}\right)^\alpha \frac{du}{s}$$

$$= \frac{1}{s^{\alpha+1}} \cdot \int_0^\infty e^{-u} u^\alpha \, du$$

Let $st = u$

$s\,dt = du$

$dt = \frac{du}{s}$.

and $t = u/s$

$$= \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-u} \, u^{(\alpha+1)-1} \, du \qquad \left[\Gamma m = \int_0^{\infty} u^{m-1} \cdot e^{-u} du \ (m>0) \right]$$

$$= \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1), \quad \text{provided } (\alpha+1)>0 \text{ and } s>0$$

$$\boxed{\mathcal{L}[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha>-1 \text{ and } s>0}$$

where α is non-integer

and when α is an integer only

$$\boxed{\mathcal{L}[t^{\alpha}] = \frac{\alpha!}{s^{\alpha+1}}}$$

Properties of Laplace Transforms:

① **Linearity:** If $f(t)$ and $g(t)$ are two functions whose Laplace transform exist, then for any two constants α and β, we have

$$\boxed{\begin{aligned}\mathcal{L}[\alpha f(t)+\beta g(t)] &= \alpha\,\mathcal{L}[f(t)]+\beta\,\mathcal{L}[g(t)] \\ &= \underbrace{\alpha F(s)+\beta G(s)}_{\text{region of convergence?}}\end{aligned}}$$

Consider $F(s)$ has region of convergence $s > c_1$ $(c_1 > 0)$

and $G(s)$ has region of convergence $s > c_2$ $(c_2 > 0)$

Then the region of convergence for

$\alpha F(s) + \beta G(s)$ is $s > \max\{c_1, c_2\}$