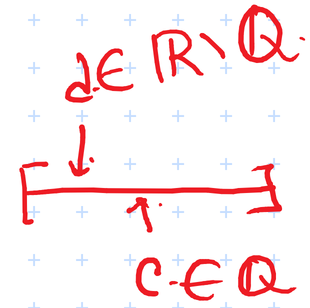
Exercise 1.5.

Let $\boxed{0 < a < b}$ and $f : [a,b] \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} 1 + x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Is x integrable?



Let $[a_1, a_2] \subseteq [a,b]$.

There exists a rational number $c \in [a_1, a_2]$ and an irrational number $d \in [a_1, a_2]$.

Then
$$\boxed{\begin{array}{l} \sup \{ f(x) \mid x \in [a_1, a_2] \} \geq 1 \\ \inf \{ f(x) \mid x \in [a_1, a_2] \} = 0 \end{array}}$$

Now, let $P = \{ a = x_0, x_1, \dots, x_n = b \}$ be a partition of $[a,b]$. Then $m_i(f) = 0$ for all i.
$M_i(f) > 1$

$$\Rightarrow L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = 0 \quad \forall P \in \mathcal{P}[a,b]$$

$$U(P,f) = \sum_{i=1}^{n} \underbrace{M_i(f)}_{\geq 1} (x_i - x_{i-1}) \geq \sum_{i=1}^{n} (x_i - x_{i-1}) = \boxed{b - a > 0}$$
$\forall P \in \mathcal{P}[a,b]$.

$$\Rightarrow L(f) = \sup_{P \in \mathcal{P}([a,b])} L(P,f) = 0$$

$$U(f) = \inf_{P \in \mathcal{P}([a,b])} U(P,f) \geq b - a > 0$$

$\inf \{ x_\alpha \mid x_\alpha > 0 \} = 0$

$\inf \{ \frac{1}{n} \mid n \in \mathbb{N} \} = 0$

Thus f is not integrable.

Exercise 2.3

Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions such that the set $\{x \in [a,b] \mid f(x) \neq g(x)\}$ is finite. If g is integrable, then show that f is integrable and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

$\to$ Let $h : [a,b] \to \mathbb{R}$ be given by $h(x) = f(x) - g(x)$.

Since g is integrable, we see that

h is integrable $\iff$ f is integrable.

($\Rightarrow$) $f(x) - g(x)$ is integrable

$\Rightarrow (f(x) - g(x)) + g(x)$ is integrable.

$\Rightarrow f(x)$ is integrable.

($\Leftarrow$) $f(x)$ is integrable $\Rightarrow f(x) - g(x)$

Thus the problem reduces to solving the following problem:

Suppose $h : [a,b] \to \mathbb{R}$ be a function such that $h(x) \neq 0$ for only finitely many points. Then h is integrable and $\int_a^b h = 0$

Further reduction $\to$ Let $a_1, \ldots, a_n \in [a,b]$.

$$h(x) = \begin{cases} c_i & \text{for } x = a_i,\ i = 1, \ldots, n. \\ 0 & \text{if } x \neq a_i \text{ for all } i = 1, \ldots, n. \end{cases}$$

We may write $\boxed{h(x) = c_1 \boxed{f_1(x)} + \cdots + c_n \boxed{f_n(x)}}$

where $f_i(x) = \begin{cases} 1 & \text{if } x = a_i \\ 0 & \text{if } x \neq a_i. \end{cases}$ for $i = 1, \ldots, n$.

If $f_1, \ldots, f_n$ are integrable, then so is h.

Thus it is enough to show that

Let $f : [a,b] \to \mathbb{R}$ be function given by

$$f(x) = \begin{cases} 1 & \text{for } x = c. \\ 0 & \text{otherwise} \end{cases}$$

for some $c \in [a,b]$. Then f is integrable and $\boxed{\int_a^b f = 0.}$

Riemann Condition:

$L(P, f) = 0$

$\boxed{U(P_\varepsilon, f)}$

$= \frac{\varepsilon}{2} < \varepsilon.$

$f, \boxed{g}$ integrable

$h = f - g$.

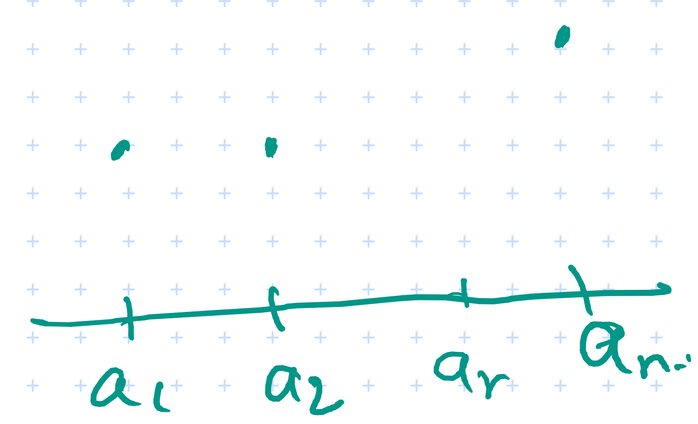
$\boxed{h \text{ is integrable}}$

$\iff \boxed{f \text{ is integrable.}}$

$\Rightarrow h$ int.

$\Rightarrow f - g$ int. g int.

$\Rightarrow f - g + g = f$ int.



$c_1 \quad c_2 \quad c_n$

$\uparrow \quad \uparrow \quad \uparrow$

$a_1, a_2, \ldots a_n$

$f_1(x) = \begin{cases} 1 & \text{if } x = a_1 \\ 0 & \text{otherwise.} \end{cases}$

$f_2 = \begin{cases} 1 & \text{if } x = a_2. \\ 0 & \text{otherwise.} \end{cases}$

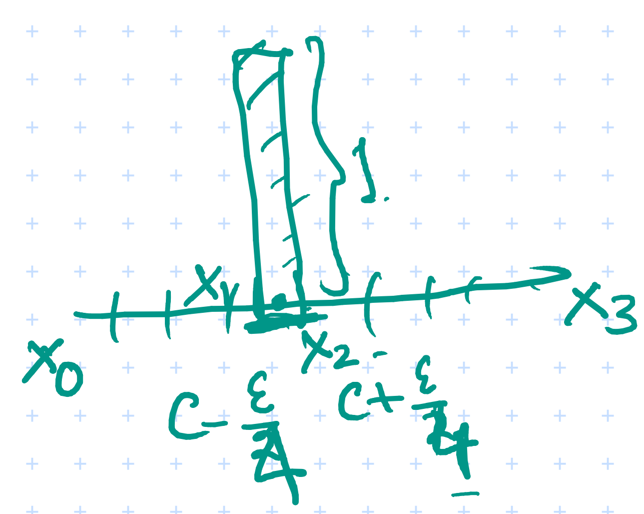
$\vdots$

$f_n = \begin{cases} 1 & \text{if } x = a_n \\ 0 & \text{otherwise.} \end{cases}$

$h(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x).$

$c_1 f_1 + c_2 f_2$

$\int_a^b f_i = 0 \Rightarrow \int_a^b h = 0.$



Exercise 3.6 (To be evaluated).

Suppose $f:[a,b]\to\mathbb{R}$ is an integrable function such that $f(x)\geqslant 0$ for all $x\in[a,b]$. Show that

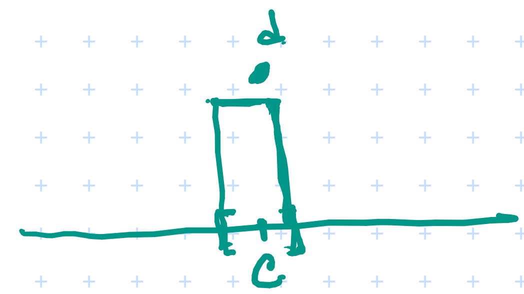
ⓐ $\int_a^b f(x)\,dx \geqslant 0$

Follows from ~~algebraic~~ order property:

$g(x)=0$ on $[a,b]$

$$\int_a^b f(x) \geqslant \int_a^b g(x) = 0.$$

ⓑ If f is continuous and $\int_a^b f(x)\,dx=0$, then $f(x)=0$ for all $x\in[a,b]$.



Suppose if possible $f(x)\neq 0$

$\Rightarrow \exists\, c\in[a,b]$ and $\boxed{d>0}$ s.t. $\boxed{f(c)=d}$.

$\Rightarrow$ Since f is continuous at c, there exists $\delta>0$ s.t. $f(x)>\frac{d}{2}$ for all $\boxed{x\in(c-\delta,c+\delta)}$.

$\boxed{[c-\frac{\delta}{2},c+\frac{\delta}{2}]}\subseteq(c-\delta,c+\delta)$

$\Rightarrow$ for all $x\in[c-\frac{\delta}{2},c+\frac{\delta}{2}]$, $\boxed{f(x)>\frac{d}{2}}$

$$\Rightarrow \int_a^b f(x)\,dx \overset{\text{domain}}{\underset{\text{addivity}}{=}} \boxed{\int_a^{c-\frac{\delta}{2}} f(x)\,dx} + \boxed{\int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f(x)\,dx} + \boxed{\int_{c+\frac{\delta}{2}}^{b} f(x)\,dx}$$

$f(x)\geqslant 0$ $\forall x\in[a,b]$

$$\geqslant 0 + \delta\frac{d}{2} + 0 > 0.$$

a contradiction.

$f(x)>\frac{d}{2}$..

$$\int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f > \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} \frac{d}{2} = \delta\cdot\frac{d}{2}>0$$

Therefore $f(x)=0$ $\forall x\in[a,b]$.

ⓒ Show that ⓑ is false if f is not continuous.

Take $f(x)=\begin{cases}1 & \text{if } x=b\\ 0 & \text{if } x\in[a,b).\end{cases}$

$f(x)\geqslant 0$.

$\int f = 0$

Exc. 4.5

Let $f:[a,b]\to\mathbb{R}$ be an integrable function. Define $G:[a,b]\to\mathbb{R}$ by

$$G(x) := \int_x^b f(t)\,dt.$$

Then G is continuous on $[a,b]$. If f is continuous at $c\in[a,b]$, then G is differentiable at c with $G'(c) = -f(c)$.

Ans. Let $F(x) := \int_a^x f(t)\,dt.$

By domain additivity, $\int_a^b f(t)\,dt = \int_a^x f(t)\,dt + \int_x^b f(t)\,dt$

$$\Rightarrow \boxed{G(x)} = \underbrace{\int_a^b f(t)\,dt}_{\text{constant}} - \boxed{F(x)}$$

Since, f is intergable, by FTC (1), F is continuous, Since $\int_a^b f(t)\,dt$ is a constant, we see that $G(x)$ is continuous.

If f is continuous at c, then by FTC (1), F is differentiable at c and $F'(c) = f(c)$. Again, since $\int_a^b f(t)\,dt$ is constant. G is differentiable at c and.

$$G'(c) = -F'(c) = -f(c).$$