भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

8th Lecture on Transform Techniques

(MA-2120)

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What will we learn today?

- Applications of the Laplace Transform

Applications of Laplace Transform :

Ex: Find $f(t)$ as a solution of the integral eqⁿ —

$$f(t) = t + e^{-2t} + \int_0^t f(\tau)\, e^{2(t-\tau)}\, d\tau$$

Solⁿ: $$\int_0^t f(\tau)\, e^{2(t-\tau)}\, d\tau = f * g(t)$$

where $g(t) = e^{2t}$.

Taking the Laplace Transform —

$$F(s) = \frac{1}{s^2} + \frac{1}{s+2} + F(s)\frac{1}{(s-2)}$$

$$\text{or } \left(1 - \frac{1}{s-2}\right)F(s) = \frac{1}{s^2} + \frac{1}{s+2}$$

$$\text{or } F(s) = \frac{s-2}{s-3}\left[\frac{s^2+s+2}{s^2(s+2)}\right]$$

$$F(s) = \frac{1}{45}\left[\frac{14}{s-3} - \frac{5}{s} + \frac{30}{s^2} + \frac{36}{s+2}\right]$$

Taking inverse L.T. –

$$f(t) = \frac{1}{45}\left[14e^{3t} - 5 + 30t + 36e^{-2t}\right]$$

Example:

Solve the integral eqⁿ —

$$x(t) = e^{-t} + \int_0^t \sin(t-\tau)\, x(\tau)\, d\tau$$

Try it.

$$\boxed{x(t) = 2e^{-t} + t - 1}$$

Ex: Using Convolution, solve the initial value problem —

$$y'' + 9y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0.$$

Soln: Taking L. T.,

$$Y(s) = \frac{3}{(s^2+9)^2}$$

Taking inverse L. T.

$$y(t) = \mathcal{L}^{-1}\left[\frac{3}{(s^2+9)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{3} \cdot F(s) \cdot G(s)\right]$$

where $F(s) = G(s) = \frac{3}{(s^2+9)}$

we have $f(t) = \sin 3t$.

$$y(t) = \frac{1}{3} f(t) * g(t)$$

$$= \frac{1}{3} \int_0^t \sin 3\tau \sin(3t - 3\tau)\, d\tau$$

$$= \frac{1}{6} \int_0^t \left[\cos 3(2\tau - t) - \cos 3t\right] d\tau$$

$$= \frac{1}{6} \left[\frac{1}{6} \sin(6\tau - 3t) - \tau \cos 3t\right]\Big|_0^t$$

$$= \frac{1}{18}\left(\sin 3t - 3t \cos 3t\right).$$

Example: **Systems of differential eqⁿs:**

$$\frac{dy}{dt} = -z, \quad \frac{dz}{dt} = y, \quad y(0) = 1,$$

$$z(0) = 0.$$

Solⁿ:

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -\mathcal{L}[z]$$

$$\text{or, } sY(s) - 1 = -Z(s)$$

$$\text{and } \mathcal{L}\left[\frac{dz}{dt}\right] = \mathcal{L}[y]$$

$$\Rightarrow \quad s\,Z(s) = Y(s)$$

$$\therefore \quad s\,Y(s) - 1 = -Z(s) \quad — \; 1$$

$$\text{and} \quad s\,Z(s) = Y(s)$$

$$\text{or} \quad Z(s) = \frac{1}{s}Y(s) \quad — \; 2$$

$\therefore$ From 1 and 2, we have

$$s\,Y(s) - 1 = -\frac{1}{s}Y(s)$$

$$\text{or} \quad s^2 Y(s) - s = -Y(s)$$

or $(s^2+1)\,Y(s) = s$

or $Y(s) = \dfrac{s}{s^2+1}$

Taking Inverse L. T., we have

$y(t) = \cos t$.

$z = -y' = \sin t$.

Ex: Solve

$$y' + z' + y + z = 1$$
$$y' + z = e^t, \quad y(0) = -1, \; z(0) = 2$$

Sol^n: $y(t) = 1 - 2e^t + te^t$

and $z(t) = 2e^t - te^t$.

(Try it)!

Ex: Solve the matrix differential system —

$$\frac{dX}{dt} = AX, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{and } A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Sol^n: This system is equivalent to

$$\frac{dx_1}{dt} - x_2 = 0$$

and $$\frac{dx_2}{dt} + 2x_1 - 3x_2 = 0$$

with $x_1(0) = 0$ and $x_2(0) = 1$.

Taking L.T. $\Rightarrow$ $sX_1(s) - X_2(s) = 0$

and $2X_1(s) + (s-3)X_2(s) = 1$

This system has the solutions

$$X_1(s) = \frac{1}{s^2-3s+2} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$X_2(s) = \frac{s}{s^2-3s+2} = \frac{2}{s-2} - \frac{1}{s-1}$$

Taking Inverse L.T.,

$x_1(t) = e^{2t} - e^t$ and $x_2(t) = 2e^{2t} - e^t$

$$X(t) = \begin{pmatrix} e^{2t} - e^{t} \\ 2e^{2t} - e^{t} \end{pmatrix}.$$

Ex : Second order coupled differential system :

$$\frac{d^2x_1}{dt^2} - 3x_1 - 4x_2 = 0 \text{ and } \frac{d^2x_2}{dt^2} + x_1 + x_2 = 0, \quad t > 0$$

with initial conditions

$x_1(0) = x_2(0) = 0$, $x_1'(0) = 2$, $x_2'(0) = 0$.

Sol^n:

By using L.T.

$$(s^3-3)X_1(s) - 4X_2(s) = 0$$

and $X_1(s) + (s^2+1)X_2(s) = 0$

Then $X_1(s) = \dfrac{2(s^2+1)}{(s^2-1)^2}$

$$= \frac{(s+1)^2 + (s-1)^2}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}$$

Taking inverse $\mathcal{L}$.T. —

$$\boxed{x_1(t) = t\left(e^{t} + e^{-t}\right)}$$

$$X_2(s) = -\frac{2}{(s^2-1)^2}$$

$$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right]$$

Taking inverse $\mathcal{L}$.T.

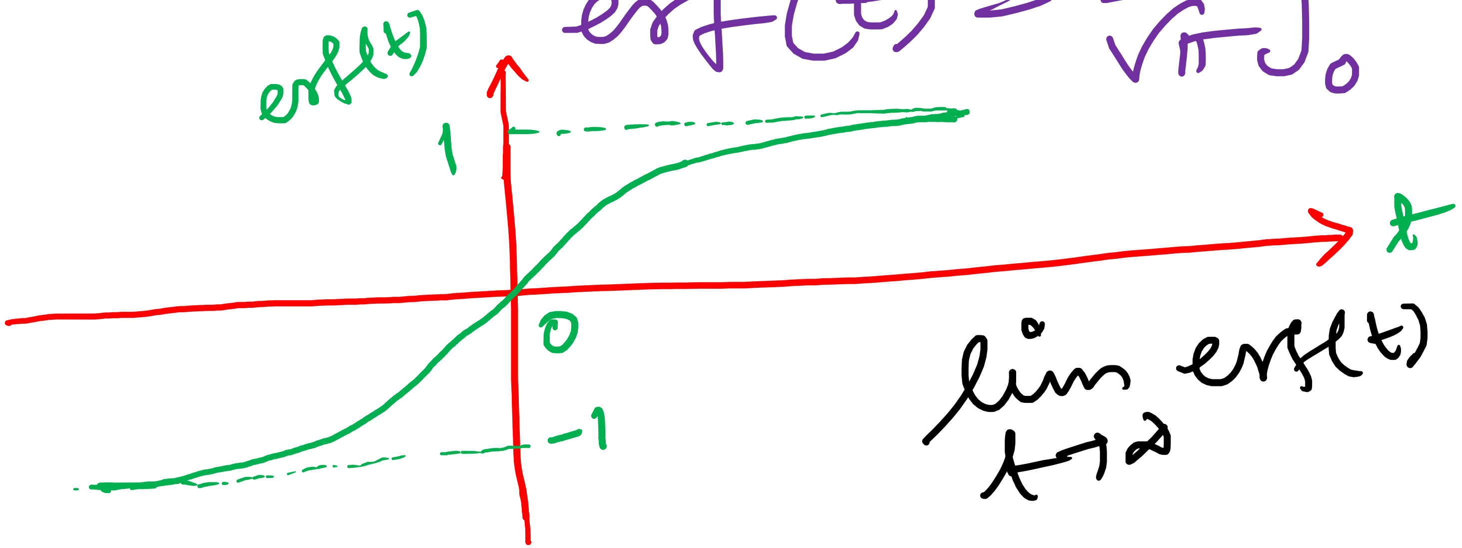
$$\Rightarrow \boxed{x_2(t) = \frac{1}{2}\left(e^{t} - e^{-t} - t e^{t} - t e^{-t}\right)}$$

Laplace Transform of Some Special functions :

The error function :

The error function is defined by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du.$$



$$\lim_{t\to\infty} \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du$$
$$= \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1.$$

The complement of error function is defined as

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t)$$

$$= \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-u^2}\, du.$$

Now $\mathcal{L}\left[\operatorname{erf}(\sqrt{t})\right]$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\, du.$$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right) du$$

$$= \frac{2}{\sqrt{\pi}} \left[u - u^3/3 + \frac{u^5}{5 \times 2!} - \frac{u^7}{7 \times 3!} + \cdots \right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \times 2!} - \frac{t^{7/2}}{7 \times 3!} + \cdots \right]$$

$$\mathcal{L}\left[\operatorname{erf}(\sqrt{t})\right]$$

$$= \frac{2}{\sqrt{\pi}}\left\{ \frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{\Gamma\frac{5}{2}}{3\, s^{5/2}} + \frac{\Gamma\frac{7}{2}}{5(2!\, s^{7/2})} - \frac{\Gamma\frac{9}{2}}{7\times(3!\, s^{9/2})} + \cdots \right\}$$

$$= \frac{1}{s^{3/2}}\left\{ 1 - \frac{1}{2}\times\frac{1}{s} + \frac{1\times 3}{2\times 4}\frac{1}{s^2} - \frac{1\times 3\times 5}{2\times 4\times 6}\frac{1}{s^3} + \cdots \right\}$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2}$$

$$= \frac{1}{s^{3/2}} \left(\frac{s}{1+s}\right)^{1/2} = \frac{1}{s\sqrt{s+1}}$$

Ex: $\mathcal{L}^{-1}\left[\dfrac{1}{\sqrt{s}\,(s-a)}\right] = e^{at}/\sqrt{a}\;\operatorname{erf}(\sqrt{at})$

Sol^n:

$$\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}\,(s-a)}\right] = \frac{1}{\sqrt{\pi t}} * e^{at}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}}\, e^{a(t-\tau)}\, d\tau$$

$$= \frac{2e^{at}}{\sqrt{\pi a}} \int_0^{\sqrt{at}} e^{-x^2}\, dx \qquad \left[\begin{array}{l}\text{Let}\\ \sqrt{a\tau} = x\end{array}\right]$$

$$= e^{at}/\sqrt{\pi}\;\operatorname{erf}\left(\sqrt{at}\right)$$

Ex: $\mathcal{L}^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$

Sol^n:

$$\mathcal{L}^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = 1 * \frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}$$

Since $\mathcal{L}^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2}e^{-a^2/4t}\Big/\sqrt{\pi t^3}$.

$$\mathcal{L}^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}}\,d\tau$$

Let $x = a/2\sqrt{\tau}$,

$$\mathcal{L}^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{2}{\sqrt{\pi}}\int_{a/2\sqrt{t}}^{\infty} e^{-x^2}\,dx$$

$$= \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

Try this Example :

$$\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}+a}\right] = \frac{1}{\sqrt{\pi t}} - a\,e^{ta^2}\operatorname{erfc}(a\sqrt{t})$$

Hints: $\frac{1}{\sqrt{s}+a} = \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s}(\sqrt{s}+a)}$

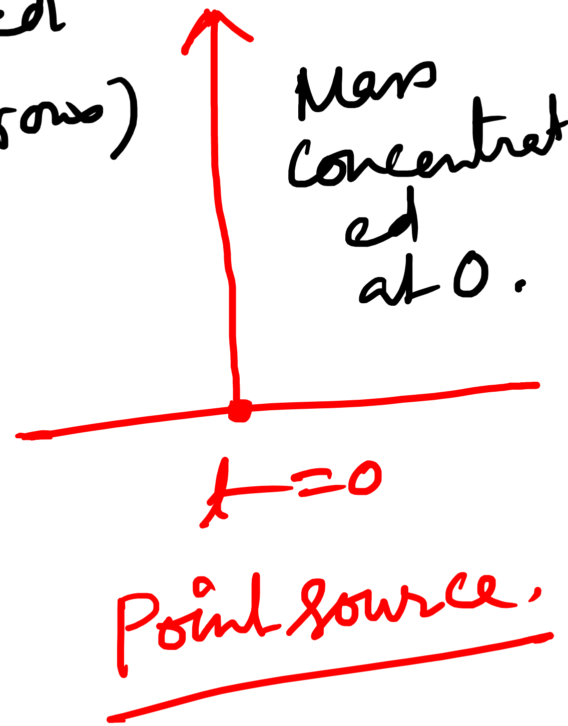
Laplace Transform of Dîrac Delta function :

(it is operator or distribution)

Impulse function

Dirac delta function : (generalized functions)

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0 \end{cases}$$

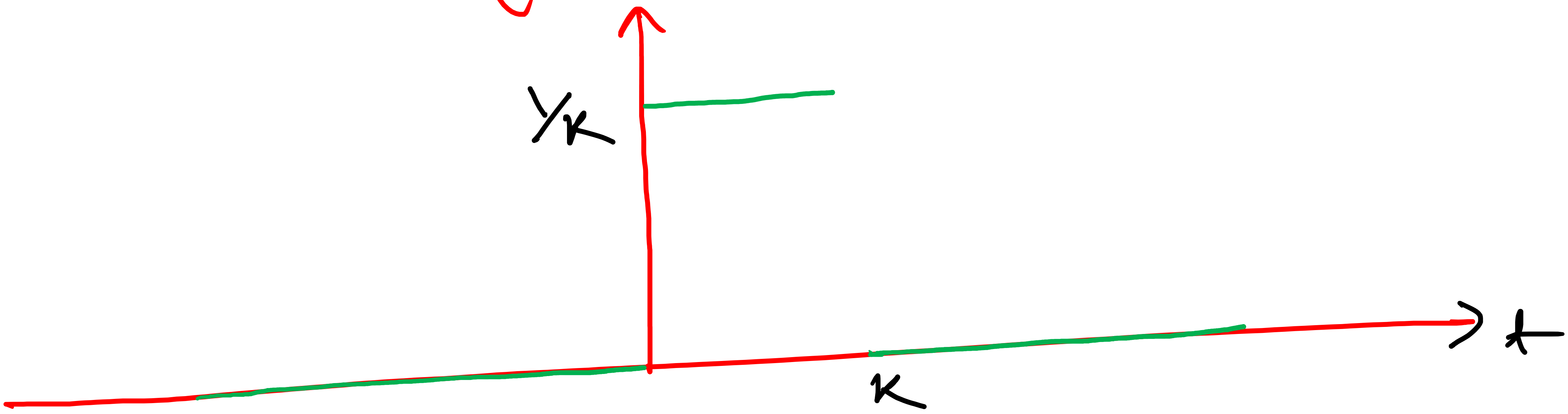


Total point mass concentrated at 0 is unity.

$$\boxed{\int_{-\infty}^{\infty}\delta(t)\,dt = 1}$$

$$\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t\neq a \end{cases}$$

$$\delta_K(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{K}, & 0 \le t < K \\ 0, & t \geq K \end{cases}$$



$$\delta_K(t) = \frac{1}{K}\left[H(t) - H(t-K)\right]$$

where $H(t)$ is the Heaviside function.

The pulse has the height $\frac{1}{K}$ and is of duration of K. As $K \to 0$, the amplitude of the pulse $\longrightarrow \infty$.

$$\boxed{\delta(t) = \lim_{K \to 0^+} \delta_K(t)}$$

Filtering Property of Dirac delta function:

Theorem: Let $f(t)$ be continuous and integrable in $[0,\infty)$. Then

$$\boxed{\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)}$$

Note:

$$\delta_k(t) = \frac{1}{k}\left[H(t) - H(t-k)\right]$$

$$\delta(t) = \lim_{k\to 0} \delta_k(t) = \lim_{k\to 0} \frac{H(t) - H(t-k)}{k} = H'(t)$$

$$\therefore \boxed{\delta(t) = H'(t)}$$

Laplace Transform of Dirac Delta function:

The delta function does not satisfy the conditions of the existence theorem. However we can obtain the Laplace Transform of the delta function by its definition.

$$\delta_K(t-a) = \frac{1}{K}\left[H(t-a) - H(t-a-K)\right]$$

$$\mathcal{L}\left[\delta_K(t-a)\right] = \frac{1}{K}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+K)s}}{s}\right]$$

$$= \frac{e^{-as}}{Ks}\left(1 - e^{-Ks}\right).$$

Taking the limit —

$$\lim_{k \to 0^+} \frac{1 - e^{-ks}}{k} = s.$$

$$\therefore \lim_{k \to 0^+} \mathcal{L}\left[\delta_k(t-a)\right] = \mathcal{L}\left[\delta(t-a)\right] = \frac{e^{-as} \cdot s}{s}$$

$$= e^{-as}.$$

when $a = 0$, $\mathcal{L}[\delta(t)] = 1$

and $\mathcal{L}^{-1}[\delta(t)] = 1$.

Here the basic property of the Laplace Transform $F(s) \to 0$, as $s \to \infty$ is voilated but of course δ is not a function but a linear operator.

This result also can be obtained by filtering property for the function $f(t) = e^{-st}$.

we have $\boxed{\int_0^\infty e^{-st}\,\delta(t-a)\,dt = f(a) = e^{-as}.}$

<u>Remark:</u> $\mathcal{L}[H'(t)] = \mathcal{L}[\delta(t)] = 1$

Example : Find the solⁿ of the initial value theorem —

$$y'' + 2y' + 5y = \delta(t-2), \quad y(0) = 0, \; y'(0) = 0.$$

Solⁿ:

$$\mathcal{L}[y'' + 2y' + 5y] = \mathcal{L}[\delta(t-2)]$$

$$\Rightarrow \boxed{s^2 Y(s) + 2s Y(s) + 5Y(s) = e^{-2s}}$$

$$Y(s) = \frac{e^{-2s}}{(s+1)^2 + 4}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+4}\right] = \frac{1}{2} e^{-t} \sin 2t.$$

$$\mathcal{L}^{-1}\left[Y(s)\right] = y(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s+1)^2+4}\right]$$

$$= \frac{1}{2} e^{-(t-2)} \sin 2(t-2)\, H(t-2)$$

$$y(t) = \begin{cases} 0, & 0 \leq t < 2 \\ \frac{1}{2} e^{-(t-2)} \sin 2(t-2), & t \geq 2. \end{cases}$$

Bessel function:

This important function is the solution of the Bessel equation of order ν

$$\boxed{t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-\nu^2)y = 0}$$

and is given by

$$\boxed{J_\nu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{2n+\nu}}{2^{2n+\nu}\, n!\,(n+\nu)!}}$$

$$J_\nu(at) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (at)^{2n+\nu}}{2^{2n+\nu} \cdot n!\,(n+\nu)!}$$

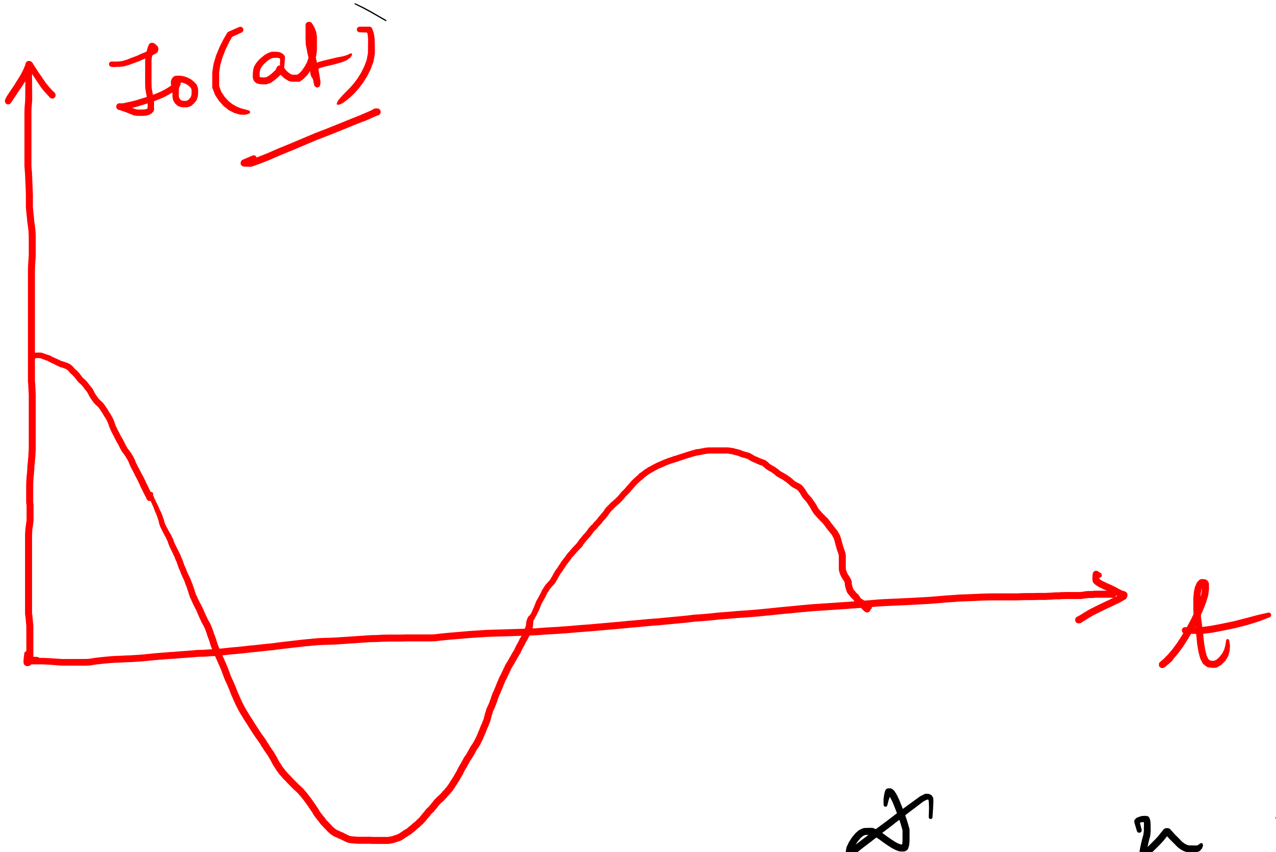
where $(n+\nu)! = \sqrt{n+\nu+1}$

For $\nu = 0$,

$$J_0(at) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} t^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} a_{2n} t^{2n}.$$

where $a_{2n} = \dfrac{(-1)^n a^{2n}}{2^{2n}(n!)^2}$

Here $J_0(at)$ is a bounded function and $|a_{2n}| = \dfrac{|a|^{2n}}{2^{2n}(n!)^2} \leq \dfrac{|a|^{2n}}{(2n)!}$



$$\mathcal{L}\left[J_0(at)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{2^{2n}(n!)^2} \cdot \mathcal{L}\left[t^{2n}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{2^{2n}(n!)^2 s^{2n+1}}$$

$$= \frac{1}{s} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \cdot \left(\frac{a^2}{s^2}\right)^n$$

$$= \frac{1}{s} \cdot \frac{1}{\sqrt{1 + a^2/s^2}} \cdot \qquad Re(s) > |a|$$

$$= \frac{1}{\sqrt{s^2 + a^2}} \cdot$$

[Here we have used the Taylor Series expansion

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} \cdot (n!)^2} x^{2n}, \quad |x|<1$$]

with $x = a/s$.

Example: Obtain the solution of the Bessel eqn:

$$t\frac{d^2x}{dt^2}+\frac{dx}{dt}+a^2t\,x(t)=0,\quad x(0)=1$$

Soln:

$$\mathcal{L}\left[t\frac{d^2x}{dt^2}\right]+\mathcal{L}\left[\frac{dx}{dt}\right]+a^2\mathcal{L}\left[t\,x(t)\right]=0$$

or $$-\frac{d}{ds}\left[\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\}\right]+sX(s)-x(0)-a^2\frac{dX}{ds}=0$$

or $$-\frac{d}{ds}\left[s^2X(s)-sx(0)-x'(0)\right]+sX(s)-x(0)-a^2\frac{dX}{ds}=0$$

or $(s^2+a^2)\frac{dx}{ds} + sX = 0$

$$X(s) = \frac{c}{\sqrt{s^2+a^2}}, \quad c \text{ is arbitrary constant}$$

$$\mathcal{L}^{-1}\left[X(s)\right] = \mathcal{L}^{-1}\left[\frac{c}{\sqrt{s^2+a^2}}\right]$$

$$= c J_0(at).$$

<u>Ex:</u>

$$\mathcal{L}\left[t J_0(at)\right] = s/(s^2+a^2)^{3/2}$$

$$\mathcal{L}\left[e^{-at} J_0(at)\right] = \frac{1}{\sqrt{s^2+2as+2a^2}}$$

$$\mathcal{L}\left[J_1(at)\right] = \frac{1}{a}\left[\frac{s}{\sqrt{s^2+a^2}} - 1\right]$$

by using $J_0'(t) = \underline{-J_1(t)}$.

Ex. $\mathcal{L}\left[e^{-a^2t^2}\right] = \frac{\sqrt{\pi}}{2a} e^{s^2/4a^2} \operatorname{erfc}\left(\frac{s}{2a}\right).$

Try!

Ex $\mathcal{L}\left[I_0(at)\right] = \frac{1}{\sqrt{s^2-a^2}}, \quad Re(s) > |a|$

Modified Bessel function of first kind with order ν.

Ex: $\mathcal{L}^{-1}\left[e^{-\sqrt{s}}\right] = \left(2\sqrt{\pi}\, t^{3/2}\right)^{-1} e^{-1/4t}$.

Solⁿ:

$$\mathcal{L}^{-1}\left[e^{-\sqrt{s}}\right] = \mathcal{L}^{-1}\left[1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \cdots\right]$$

$$\mathcal{L}^{-1}\left[s^{n+1/2}\right] = \frac{t^{-n-3/2}}{\Gamma\left(-n-\tfrac{1}{2}\right)} \quad \text{for } n = 0, 1, 2, \ldots$$

$$\mathcal{L}^{-1}\left[s^{n+1/2}\right] = \frac{(-1)^{n+1}}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n+1}{2} \cdot t^{-n-3/2}$$

$\mathcal{L}^{-1}\left[s^n\right] = 0, \; n = 0, 1, 2, \ldots$

$\Gamma x = \frac{1}{x}\Gamma(x+1), \; x < 0$

$\Gamma\left(-\tfrac{1}{2}\right) = -2\,\Gamma\left(\tfrac{1}{2}\right)$

$$\mathcal{L}^{-1}\left[1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \frac{s^{5/2}}{5!} + \cdots\right]$$

$$= 0 + \frac{t^{-3/2}}{2\sqrt{\pi}} - \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{t^{-5/2}}{3!\sqrt{\pi}} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{t^{-7/2}}{5!\sqrt{\pi}} + \cdots$$

$$= \frac{1}{2\sqrt{\pi}\,t^{3/2}}\left\{1 - \frac{1/4t}{1!} + \frac{(1/4t)^2}{2!} - \frac{(1/4t)^3}{3!} + \cdots\right\}$$

$$= \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{t^{3/2}}\cdot e^{-1/4t} = \left(2\sqrt{\pi}\,t^{3/2}\right)^{-1} e^{-1/4t}.$$