(1) $f(x) = \sqrt{x+1}$, Domain $= [-1, \infty)$
Range $= [0, \infty)$

$g(x) = 1/x$, Domain $= \mathbb{R} \setminus \{0\}$
$= (-\infty, 0) \cup (0, \infty)$
Range $= (-\infty, 0) \cup (0, \infty)$

The fn. $g \circ f$
defined if Range of $f \subseteq$ Dom. of g

But, 0 is in here   0 not in here

So, restrict f to $(-1, \infty)$
so that range $= (0, \infty)$

Thus, Domain of $g \circ f$ = Domain of restricted f
$= (-1, \infty)$

$$g \circ f(x) = \frac{1}{\sqrt{x+1}}$$

Range of $g \circ f = (0, \infty)$

The fn. $f \circ g$

Need Range of $g \subseteq$ Domain of f

$= (-\infty, 0) \cup (0, \infty)$ $\quad\quad$ $[-1, \infty)$

So, must remove $(-\infty, -1)$ from

Thus, the restricted domain of g is

$[-1, 0) \cup (0, \infty)$

This is the domain of $f \circ g$.

$f \circ g = \sqrt{\frac{1}{x} + 1}$ ← Range consists of all $y \geq 0$ except $y = 1$

Range $= [0, 1) \cup (1, \infty)$.

(2) Let $y = f(x)$ be a fn. Domain $= D$

Image ↗ ↖ Argument

Range $= R$ (not $\mathbb{R}$)

In this problem, we are trying to find out what happens to the fn. if the argument/image is

- Shifted by "h"
- Scaled by a factor "c"

- <u>Shifting the argument</u> <u>$x \to x+h$</u>

Consider the function $y = g(x)$

where $g(x) = f(x+h)$

g is defined at x if $x+h \in D$

i.e. if $x \in D - h = D'$, say

$= \{z - h \mid z \in D\}$

Domain of g.

So, the domain of g = domain of f "shifted" by "h" units to the <u>left</u>.

if h is negative, then the shifting is to the <u>right!</u>

What about the range of $g(x)$?

Stays intact!

if f attains the value y at $x \in D$,

i.e. $f(x) = y$,

then g attains the same value at $x-h \in D'$. Namely,

$$g(x-h) = f(x-h+h) = f(x) = y$$

- <u>Shifting the image by "k"</u>

Now, consider $g(x) = f(x) + k$

Then Domain of g = Domain of f.

If $y \in$ Range of g, then

$$y = g(x) = f(x) + k$$

That is, the range of g

= range of f "shifted" "above" by "k" units.

If $k < 0$, then this means, we have to shift the graph "down" by k-units.

- <u>Scaling the argument by "c" $\neq 0$.</u>

  Let $c \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

  and $g(x) = f(cx)$

  First assume that $c > 0$.

In this case, g is defined at x if $cx \in D$, i.e.

$$x \in D/c = \{z/c : z \in D\}.$$

So the domain "shrinks" or "expands" depending on whether $c > 1$ or $c < 1$.

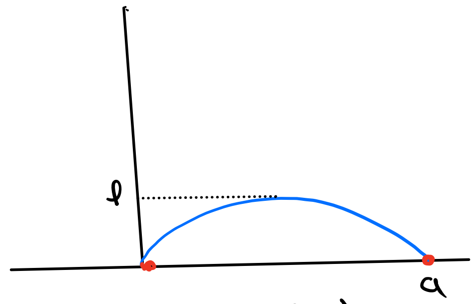
How about the range of g?

Same as that of f.

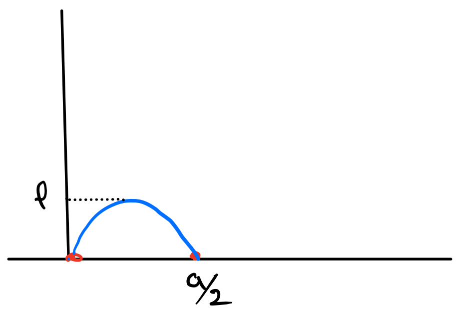
If $f(\overset{\in D}{x}) = y$, then

$$g(\underset{\in D/c}{x/c}) = f(c \cdot x/c) = f(x) = y$$

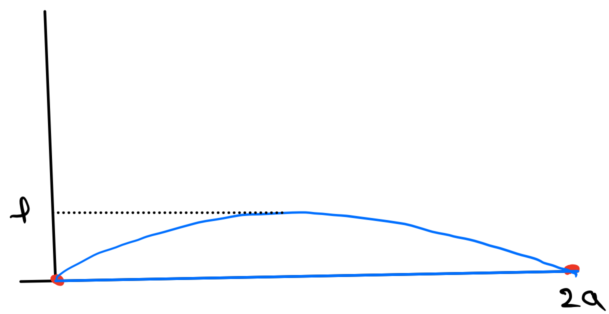
Accordingly, the graph "contracts" or "expands" dep. on c.



$y = f(x)$



$y = f(2x)$



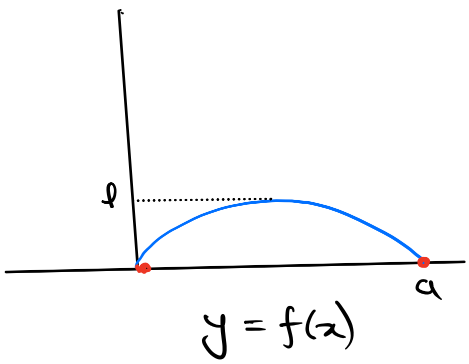
$y = f(x/2)$

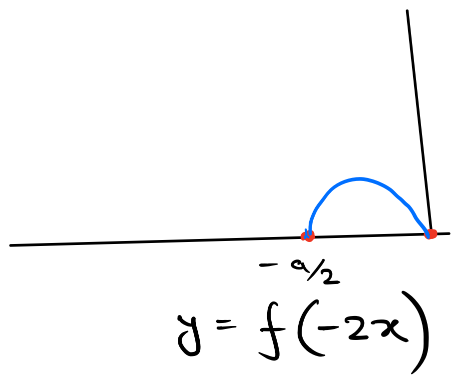
Now consider the case that $c = -1$.
Then g is defined at x if
f is defined at $-x$, since

$$g(x) = f(-x)$$

Thus, domain of g is $-D = \{-x : x \in D\}$
This effects a "reflection"
about the <u>y-axis.</u>

Generally, if $c < 0$, then we have a combination of both reflection (about y-axis) + a contraction/expansion.





- <u>Scaling the image by "c" ≠ 0</u>

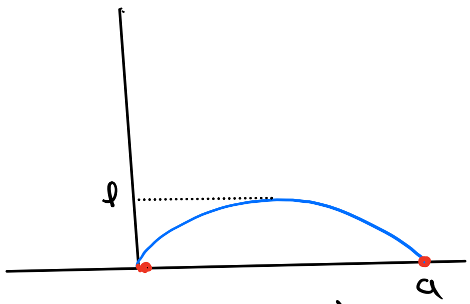
Let $g(x) = c f(x)$

Then Domain of g = domain of f

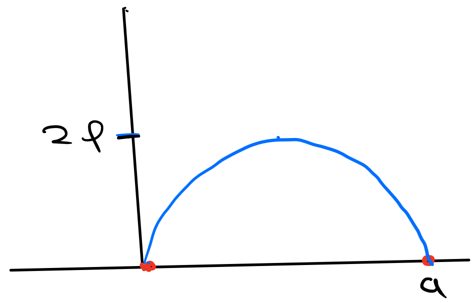
First assume that $c > 0$.

Range of $g = cR = \{c \cdot y : y \in R\}$.

So, on the same domain, the images are scaled by c



$y = f(x)$



$y = 2f(x)$

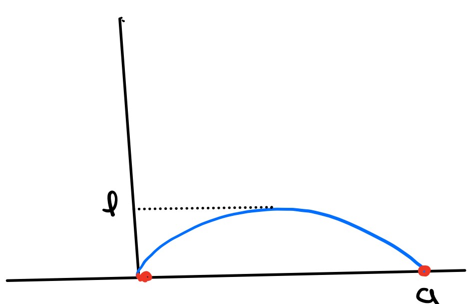
If $c = -1$, then $f(x) \longrightarrow -f(x)$

meaning, the curve is reflected about the <u>x-axis.</u>

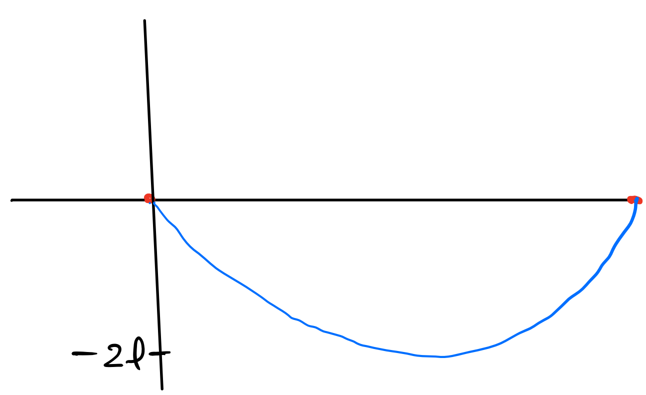
In general if $c < 0$, then graphing

$$g(x) = cf(x)$$

involves both scaling and reflecting about x-axis.

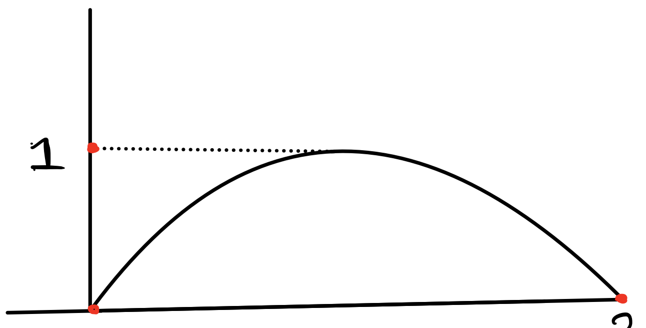


$y = f(x)$



$y = -2f(x)$

For instance, the answer to 2. (h) is as follows.



$y = f(x)$

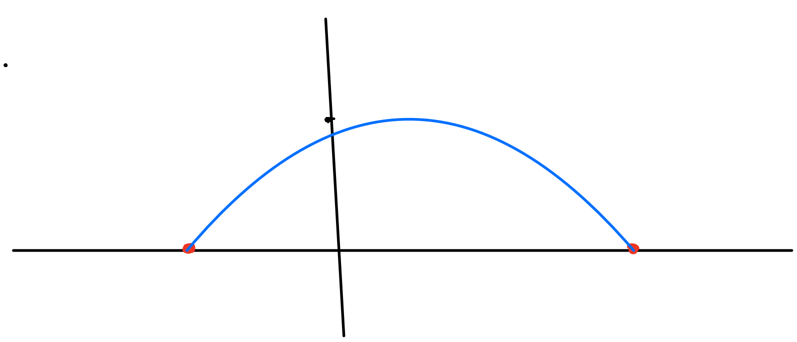
To graph. $y = -f(x+1) + 1$

Step 3. Range of $-f(x+1)$ to be shifted by 1 above

Step 2. Reflect the Image of $f(x+1)$ about x-axis

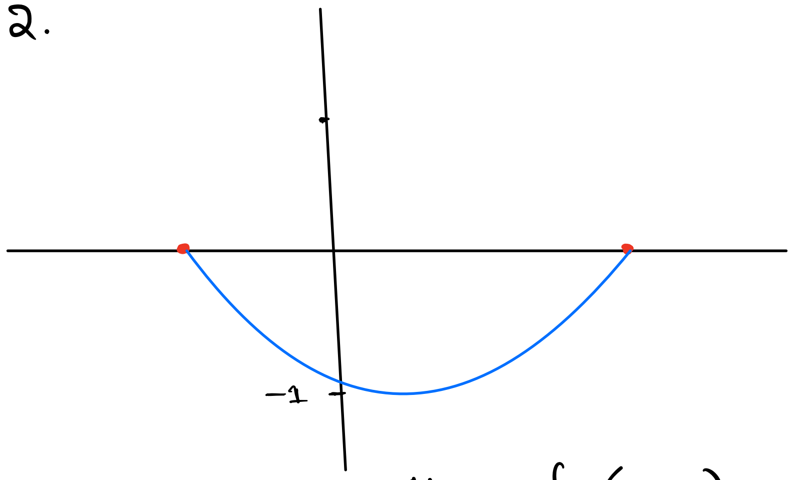
Step 4. domain of $f(x)$ to be shifted by 1 to the left.

Step 1.



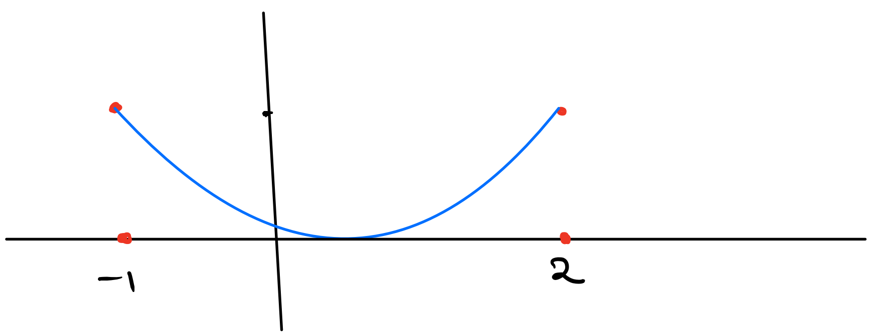
$y = f(x+1)$

Step 2.



$y = -f(x+1)$

Step 3.



$y = -f(x+1) + 1$

(2) (Again!) Yes, it is possible.

e.g. if $f : X \to Y$ is bijective with $X, Y \subseteq \mathbb{R}$,

then $f^{-1} \circ f(x) = x \quad \forall x \in X$.

the graph of $f^{-1} \circ f$ is that of $y = x$ on X.

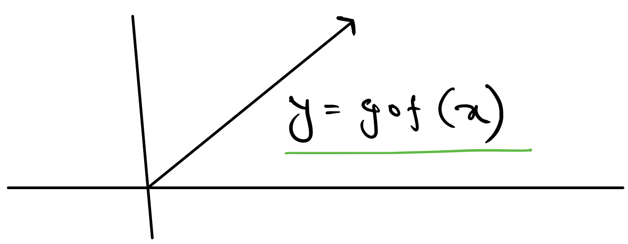
Example. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$f(x) = x^2$

and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$g(x) = \sqrt{x}$ (basically $g = f^{-1}$)

then $g \circ f(x) = x \quad \forall x \in \mathbb{R}_{\geq 0}$



In fact, modifying the idea, any straight line can be obtained this way. Namely, if

$f(x) = x^2$ and $g(x) = a\sqrt{x} + b$,

then $g \circ f(x) = ax + b$ on $\mathbb{R}_{\geq 0}$.

(3) Define
$$\left.\begin{aligned} E_f(x) &= \frac{f(x)+f(-x)}{2} \\ \text{and} \quad O_f(x) &= \frac{f(x)-f(-x)}{2} \end{aligned}\right\} x \in D$$

Then $f(x) = E_f(x) + O_f(x) \quad \forall x \in D$

$\Rightarrow f = E_f + O_f$ on D

it is easy to check that

E_f is even and O_f is odd!

(4) To show:

(i) f is injective

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$$

and, (ii) f is surjective.

If $y \in Y$, then $\exists$ a $x \in X$ s.t. $f(x) = y$.

(i) If $f(x_1) = f(x_2)$,

then applying 'g' on both sides,

$$g \circ f(x_1) = g \circ f(x_2)$$

$\Rightarrow \quad x_1 = x_2$ (since $g \circ f$ is identity on X)

2nd part

If one of $f \circ g$ or $g \circ f$ is not identity, then f need not be bijective.

For consider

$f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $f(x) = x^2$

$g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $g(x) = \sqrt{x}$

Then $f \circ g(x) = x \ \forall x \in \mathbb{R}_{\geq 0}$

but f is not bijective.

Fails because, $g \circ f(x) = |x| \leftarrow$ not the identity.

(ii) Let $y \in Y$

Now $f \circ g(y) = y$ (since $f \circ g$ is identity on Y)

$\Rightarrow f(g(y)) = y$

$g(y) \in X$, say $g(y) = x$,

then $f(x) = y$.

(5) Apply AP on the +ve real no. x/y.

By AP, $\exists\ n \in \mathbb{N}$ s.t. $n \cdot \frac{x}{y} > 1$

since $y > 0$, multiplying by y,

$$nx > y.$$

(6) By AP $\exists\, n \in \mathbb{N}$ s.t. $nx > 1$

or $\frac{1}{n} < x$

Now, $2^n = (1+1)^n = 1 + n + \cdots > n$

Thus, $\frac{1}{2^n} < \frac{1}{n} < x$.

(7) Let I be any interval.

$\exists\, a, b \in I$ s.t. $a < b$

So that $(a, b) \subseteq I$.

By the Thm., $\exists$ a rational q_1 s.t.

$a < q_1 < b$, so $q_1 \in I$

Similarly $(q_1, b) \subseteq I$

Apply Thm. again to get a q_2 s.t.

$q_1 < q_2 < b$, so $q_2 \in I$.

Keep applying the Thm. over and over

to get $q_1 < q_2 < q_3 < \cdots$ ← all $\in I$.

with $q_n < q_{n+1} < b \;\; \forall n$

and $q_{n+1} \in I$.

So, I contains ∞-ly many rationals.

(8) Let $x < y$ be real nos.

Then $x\sqrt{2} < y\sqrt{2}$

by the result on the existence of a rational between <u>any</u> two real nos., $\exists$ a rational no. m/n satisfying

$$x\sqrt{2} < \frac{m}{n} < y\sqrt{2}$$

Dividing by $\sqrt{2}$,

$$x < \frac{m}{n\sqrt{2}} < y$$

an irrational no.

as required.

(9) Let $S_x = \{ n \in \mathbb{N} : nx > 1 \}$

since $x < 1$, it follows that

$1 \notin S_x$.

Thus, $n \in S_x \Rightarrow n \geq 2$.

Next, AP $\Rightarrow S_x \neq \phi$.

We make use of the Well Ordering Principle (WOP) stating that "every nonempty subset of $\mathbb{N}$ contains a smallest element"

This makes sense because every subset of $\mathbb{N}$ is "bounded from the left"

Thus, S_x has a smallest member, say m,

Then $m-1 \notin S_x$ (since 'm' is already the smallest member of S_x)

$\Rightarrow (m-1)x \leq 1$

Note that $m \geq 2$, so that $m-1 > 0$

It follows that $x \leq \frac{1}{m-1}$

$m \in S_x \Rightarrow mx > 1 \Rightarrow x > \frac{1}{m}$

Thus, $\frac{1}{m} < x \leq \frac{1}{m-1}$.

(10) Let $S =$ Range of $f \subseteq X$.

X - finite $\Rightarrow$ S - finite

Also, $S \neq \emptyset$

Now, every nonempty finite subset of $\mathbb{R}$ has a smallest element.

Thus, S has a smallest element.

Let x be the smallest element of S.

Claim! $f(x) = x$

By the given condition on f,

$$f(x) \leq x$$

If $f(x) < x$, then we have an absurd situation since

$$f(x) \in S$$

and x is the smallest member of S.

It follows that $f(x) = x$, as desired.