भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

10th Lecture on ODE

(MA-1150)

What have we learnt in previous class?

- Second and Higher Order Homogeneous ODE with constant coefficient

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Today's Class

- Second and Higher Order non-Homogeneous ODE with constant coefficient
- General Method for finding PI
- Short methods for finding PI

III. Solution of nonhomogeneous eqn:

A non homogeneous linear eqn is of the form as

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x), \quad \underline{a \neq 0.}}$$

First step: We need to find the solution of homogeneous eqn - $\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$

Let $y_1(x)$ and $y_2(x)$ be the linearly independent solutions.

Now the complementary function (C.F.) is

$$\boxed{y_c(x) = c_1 y_1(x) + c_2 y_2(x)} \Rightarrow \underline{\text{C.F.}}$$

Second step:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$$

as $(aD^2 + bD + c)y = r(x)$

or $f(D)y = r(x)$, where $f(D) = aD^2 + bD + c$

$$\boxed{f(D)y = r(x)}$$

The particular integral (P.I.) is

$$y_p(x) = \frac{1}{f(D)} \cdot r(x)$$

$$= [f(D)]^{-1} \cdot r(x).$$

Now the complete solution of the linear non-homogeneous eqn is

$y(x) =$ Complementary function (C.F.) + Particular Integral (P.I)

$$\boxed{y(x) = y_c(x) + y_p(x)}$$

$$\Rightarrow \boxed{y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)}$$

Theorem: If $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent solutions of the homogeneous linear equation $a_0(x)y^{(n)}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y = 0$, then $c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ is the general solution of the corresponding linear homogeneous equation and if $y_p(x)$ is any particular solution (a solution not containing any arbitrary constants) of the non-homogeneous equation $a_0(x)y^{(n)}(x) + \cdots + a_n(x)y(x) = r(x)$, then the general solution is given by

$$\boxed{y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{\text{Complementary function (C.F.)}} + \underbrace{y_p(x)}_{\text{particular integral or Solution.}}}$$

Determining Particular Integral:

The linear nonhomogeneous ODE is

$$\boxed{f(D)\,y = X},$$

where $f(D) = aD^2 + bD + c$, this is the D-operator in 2nd order linear nonhomogeneous ODE

or

$f(D) = (a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)$, this is the D-operator in nth order non-homogeneous linear ODE.

and X is function of x and non zero term in ODE.

Now, $PI = \frac{1}{f(D)} \cdot X = [f(D)]^{-1} X$

where $[f(D)]^{-1}$ is called as inverse operator.

Note: $f(D)$ can be expressed as

$(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$, where $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of the auxiliary eqn of the homogeneous linear ODE $(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = 0$.

Therefore, PI for nth order linear non-homogeneous ODE is given by

$$\boxed{PI = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)} \cdot X}$$

For 2nd order ODE,

$$\boxed{PI = \frac{1}{(D-\alpha_1)(D-\alpha_2)} X.}$$

$$PI = \frac{1}{(D-\alpha_1)(D-\alpha_2)} \cdot X$$

$$= \left[\frac{A}{D-\alpha_1} + \frac{B}{D-\alpha_2}\right] X,$$

where A and B are constants

$$= A \cdot \frac{1}{D-\alpha_1} \cdot X + B \cdot \frac{1}{D-\alpha_2} \cdot X$$

Now you need to find out the solutions of $\left(\frac{1}{D-\alpha_1}\right) \cdot X$ and $\left(\frac{1}{D-\alpha_2}\right) \cdot X$ to get P.I.

General Method for P.I:

Theorem: $$\boxed{\frac{1}{D-\alpha} X = e^{\alpha x} \int X e^{-\alpha x} dx}$$

Proof: Let $y = \frac{1}{D-\alpha} X$

or, $Dy - \alpha y = X$

or, $$\boxed{\frac{dy}{dx} - \alpha y = X}$$

which is linear non homogeneous first order ODE.

$$IF = e^{\int -\alpha\,dx} = e^{-\alpha x}$$

Multiplying IF, we have

$$\frac{d}{dx}\left(y e^{-\alpha x}\right) = X e^{-\alpha x}$$

Integrating, $\quad y e^{-\alpha x} = \int X e^{-\alpha x}\,dx$

$$y = e^{\alpha x}\int X e^{-\alpha x}\,dx$$

$$\Rightarrow \boxed{\frac{1}{D-\alpha}X = e^{\alpha x}\int X e^{-\alpha x}\,dx}$$

Proved

Now, $\frac{1}{D-\alpha_1} \cdot X$

$$= e^{\alpha_1 x} \int X e^{-\alpha_1 x} dx$$

$$= X_1(x) \text{ (Say)}$$

and $\frac{1}{D-\alpha_2} \cdot X$ $= e^{\alpha_2 x} \int X e^{-\alpha_2 x} dx$

$$= X_2(x) \text{ (Say)}$$

Now $PI = A \cdot \frac{1}{D-\alpha_1} \cdot X + B \cdot \frac{1}{D-\alpha_2} \cdot X$

$$\boxed{PI = A \cdot X_1(x) + B \cdot X_2(x)}$$

This is the obtained particular Integral by a general method

Example:

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

or, $(D^2+a^2)y = \sec ax$

First Step: To find the complementary function

Now homogeneous equation is

$$(D^2+a^2)y=0 \Rightarrow f(D)y=0, \text{ where } f(D)=D^2+a^2$$

The auxiliary equation is

$$f(m)=0 \text{ or } m^2+a^2=0 \text{ or } m=\pm ia$$

The complementary function (CF) which is the general solution of the homogeneous eqn, is

$$\boxed{y_c(x) = C_1 \cos ax + C_2 \sin ax,}$$

where c_1 and c_2 are arbitrary constants.

Second part: To find the particular Integral (PI):

$$PI = \frac{1}{f(D)} \sec ax = \frac{1}{D^2 + a^2} \cdot \sec ax$$

$$= \frac{1}{(D+ia)(D-ia)} \cdot \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

Now, $\frac{1}{D-ia} \cdot \sec ax$

$$= e^{iax}\int e^{-iax} \sec ax \, dx$$

$$= e^{iax}\int \frac{\cos ax - i \sin ax}{\cos ax} \, dx$$

$$= e^{iax}\left[x + \frac{i}{a}\log \cos ax\right]$$

$$= \left(\cos ax + i \sin ax\right)\left(x + \frac{i}{a}\log \cos ax\right)$$

$$= \left(x \cos ax - \frac{1}{a}\sin ax \log \cos ax\right)$$
$$+ i\left(x \sin ax + \frac{1}{a}\cos ax \log \cos ax\right)$$

Similarly, $\frac{1}{D+ia}\sec ax$

$$= \left(x\cos ax - \frac{1}{a}\sin ax \log\cos ax\right) - i\left(x\sin ax + \frac{1}{a}\cos ax \log\cos ax\right)$$

Hence $P.I. = \frac{1}{2ia}\cdot\frac{1}{D-ia}\sec ax - \frac{1}{2ia}\cdot\frac{1}{D+ia}\sec ax$

$$= \frac{1}{2i a}\left[2i\left(x \sin ax + \frac{1}{a}\cos ax \log(\cos ax)\right)\right]$$

$$= \frac{x \sin ax}{a} + \frac{\cos ax \log \cos ax}{a^2}$$

The Complete Solution is

$= C.F. + P.I.$

$$= c_1 \cos ax + c_2 \sin ax + \frac{x \sin ax}{a} + \frac{\cos ax \log \cos ax}{a^2}.$$

Short Methods of finding Particular Integral (PI) in Special Cases: operator D:

Note: Special forms of X (e^{ax}, $\sin ax$, $\cos ax$)

- when X is e^{ax}:

The given ODE is
$$\boxed{f(D)y = X}, \text{ where } X = e^{ax}$$

Particular Integral (PI)

$$= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

[put $D = a$]

provided $f(a) \neq 0$.

$$\therefore \boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0}$$

Now we can prove that, see next

Proof: If a is constant, then

$$f(D)\,e^{ax} = f(a)\,e^{ax}$$

Note that $D e^{ax} = a e^{ax}$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$D^3 e^{ax} = a^3 e^{ax}$$

$$\cdots \qquad \cdots$$

In general, $D^n e^{ax} = a^n e^{ax}$

$$\boxed{\begin{array}{l}\text{Consider} \\ f(D) = D^n \\ \quad \Downarrow \\ f(a) = a^n\end{array}}$$

$$\Rightarrow f(D)\,e^{ax} = f(a)\,e^{ax}$$

Now we have

$$f(D)e^{ax} = f(a)e^{ax}$$

$$\Rightarrow e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\Rightarrow \frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

Proved.

③ When $f(a) = 0$:

When $f(a) = 0$, then $(D-a)$ is a factor in $f(D)$.

Consider $f(D) = (D-a)\,g(D)$

Now $P.I. = \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{(D-a)\,g(D)} e^{ax}$

$= \dfrac{1}{(D-a)} \cdot \dfrac{1}{g(a)} e^{ax}$

$$= \frac{1}{(D-a)} \cdot \frac{e^{ax}}{g(a)}, \quad \text{Since } g(a) \neq 0.$$

$$= \frac{1}{g(a)} \frac{1}{D-a} e^{ax}$$

$$\boxed{\text{use } \frac{1}{D-a} \cdot X = e^{ax} \int X e^{-ax} \, dx}$$

$$= \frac{e^{ax}}{g(a)} \int e^{ax} \cdot e^{-ax} \, dx, \quad \boxed{X = e^{ax}}$$

$$= \frac{e^{ax}}{g(a)} \int dx = \frac{x e^{ax}}{g(a)}$$

$$PI = \frac{x e^{ax}}{g(a)}$$

Theorem: If $f(a)=0$, then $f(D)$ must have a factor of the type $(D-a)^r$. Then

$$\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}}$$

③ **Summary:** when $X = e^{ax}$,

(i) when X is of the form e^{ax},

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0$$

(ii) when $f(a) = 0$, use the theorem

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}.$$

Example: $y'' - 2y' - 3y = 3e^{2x}$

or $(D^2 - 2D - 3)y = 3e^{2x}$

To find CF: The auxiliary equation of the corresponding homogeneous equation is

$$(m^2 - 2m - 3) = 0$$

or $(m-3)(m+1) = 0$

$m = 3, -1,$

The complementary function (CF) is given by

$$y_c(x) = c_1 e^{-x} + c_2 e^{3x},$$ where c_1 and c_2 are arbitrary constants, and e^{-x} and e^{3x} are two linearly independent solutions.

<u>To find PI</u>: we have $f(D) = D^2 - 2D - 3$

and $X = 3e^{2x}$.

The particular Integral is $= \frac{1}{f(D)} \cdot X$

$$= \frac{1}{D^2 - 2D - 3} 3e^{2x} = 3 \cdot \frac{1}{D^2 - 2D - 3} e^{2x}$$

$$= 3 \cdot \frac{1}{2^2 - 2 \cdot 2 - 3} \cdot e^{2x} \quad \left[\begin{array}{l} \text{Put} \\ D = 2 \end{array} \right]$$

$$= -e^{2x}$$

The Complete Solution is $= CF + PI$

$$= y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{3x} - e^{2x}$$

Ex. $(D^3 - D^2 - D + 1)y = e^x$

To find CF: The auxiliary eqn is

$$m^3 - m^2 - m + 1 = 0$$

$$\text{or } (m-1)(m^2-1) = 0$$

$$\text{or } (m-1)^2(m+1) = 0.$$

$$\Rightarrow m = +1, +1, -1$$

$$m = +1 \longrightarrow e^x, \quad \longrightarrow xe^x \qquad m = -1 \longrightarrow e^{-x}$$

The CF is $(c_1 + c_2 x)e^x + c_3 e^{-x}$

where c_1, c_2 and c_3 are arbitrary constants

$$P.I = \frac{1}{D^3 - D^2 - D + 1} e^x$$

$$= \frac{1}{(D-1)^2 (D+1)} e^x$$

$$= \frac{1}{(D-1)^2} \frac{e^x}{1+1}$$

(Put $D=1$ in operator $(D+1)$)

$$= \frac{1}{(D-1)^2} \frac{e^x}{2}$$

$$= \frac{1}{2} \frac{1}{(D-1)^2} e^x$$

$$\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}}$$

$$= \frac{1}{2} \cdot \frac{x^2}{2!} e^x = \frac{x^2 e^x}{4}$$

Now the complete solution is

$$CF + PI = (c_1 + c_2 x)e^{x} + c_3 e^{-x} + \frac{x^2}{4} e^{x}$$

Ex, $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = e^{2x} + e^{x} + 3e^{-x}$

Solⁿ: $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + e^{x} + 3e^{-x}$

The linear operator

$$f(D) = D^3 - 5D^2 + 8D - 4$$
$$= (D-1)(D-2)^2$$

The auxiliary equation is

$$(m-1)(m-2)^2=0$$

$$\Rightarrow m=1,\ 2,\ 2$$

The complementary function (CF) is

$$c_1e^{x}+(c_2+c_3x)e^{2x}$$

The particular Integral is

$$\frac{1}{(D-1)(D-2)^2}\left\{e^{2x}+e^{x}+3e^{-x}\right\}$$

$$= \frac{1}{(D-1)(D-2)^2} e^{2x} + \frac{1}{(D-1)(D-2)^2} e^{x} + \frac{3}{(D-1)(D-2)^2} e^{-x}$$

Now $\frac{1}{(D-1)(D-2)^2} e^{2x}$

$$= \frac{1}{(D-2)^2 (D-1)} e^{2x}$$

$$= \frac{1}{(D-2)^2} \cdot \frac{e^{2x}}{(2-1)}$$

[Put $D=2$ in the operator $(D-1)$]

$$= \frac{1}{(D-2)^2} e^{2x}$$

$$= \frac{1}{(D-2)^2} e^{2x}$$

[use the theorem $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$]

$$= \frac{x^2}{2!} e^{2x}$$

$$= \frac{x^2}{2} e^{2x}$$

$$\frac{1}{(D-1)(D-2)^2} e^{x} = \frac{1}{(D-1)} \cdot \frac{e^x}{(1-2)^2}$$

[Put $D=1$ in the operator $(D-2)^2$]

$$= \frac{1}{(D-1)} \cdot e^{x}$$

$$= x e^{x}$$

[use the theorem $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$]

$$\frac{3}{(D-1)(D-2)^2}e^{-x}$$

$$= \frac{3}{(-1-1)(-1-2)^2}e^{-x} \quad \left[\text{put } D=-1\right]$$

$$= -\frac{1}{6}e^{-x}$$

Now the PI is $\frac{x^2}{2}e^{2x} + xe^{x} - \frac{1}{6}e^{-x}$

The general solution is $(CF + PI)$

$$= c_1e^{x} + (c_2 + c_3x)e^{2x} + \frac{x^2}{2}e^{2x} + xe^{x} - \frac{1}{6}e^{-x}.$$

Ex $(D^3-2D^2-5D+6)y=(e^{2x}+3)^2+e^{3x}\cosh x$

This is Cosine hyperbolic function

Sol^n: The linear operator is

$$f(D)=D^3-2D^2-5D+6$$

The auxiliary equation is

$$m^3-2m^2-5m+6=0$$

$$\text{or, } (m-1)(m-3)(m+2)=0$$

$$\Rightarrow m=1,3,-2$$

The CF is $c_1e^{x}+c_2e^{3x}+c_3e^{-2x}$, where c_1,c_2 and c_3 are arbitrary constants

Note that

$$(e^{2x}+3)^2+e^{3x}\cosh x$$

$$=e^{4x}+6e^{2x}+9e^{0\cdot x}+e^{3x}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\left[\text{Since } \cosh x = \frac{e^{x}+e^{-x}}{2}\right]$$

$$=e^{4x}+6e^{2x}+9e^{0.x}+\frac{1}{2}e^{4x}+\frac{1}{2}e^{2x}$$

$$=\frac{3}{2}e^{4x}+\frac{13}{2}e^{2x}+9e^{0.x}$$

PI is $\dfrac{1}{(D-1)(D-3)(D+2)}\left\{(e^{2x}+3)^2+e^{3x}\cosh x\right\}$

$$= \frac{3}{2} \frac{1}{(D-1)(D-3)(D+2)} e^{4x} + \frac{13}{2} \frac{1}{(D-1)(D-3)(D+2)} e^{2x}$$

$$+ 9 \cdot \frac{1}{(D-1)(D-3)(D+2)} e^{0.x}$$

$$= \frac{3}{2} \frac{1}{(4-1)(4-3)(4+2)} e^{4x} + \frac{13}{2} \frac{1}{(2-1)(2-3)(2+2)} e^{2x}$$

$$+ 9 \cdot \frac{1}{(0-1)(0-3)(0+2)} e^{0.x}$$

$$= \frac{e^{4x}}{12} - \frac{13}{8}e^{2x} + 3/2$$

The general solution or Complete Solution

is C.F. + P.I

$$= c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x} + \frac{e^{4x}}{12} - \frac{13}{8}e^{2x} + 3/2$$

(iii) When $X = \sin ax$ or $\cos ax$:

Consider $X = \sin ax$ and ODE is $\boxed{f(D)y = \sin ax}$, where $f(D)$ is linear operator in D.

Now $\boxed{PI = \frac{1}{f(D)} \sin ax}$

$\Rightarrow$ Consider $f(D) = \phi(D^2)$, where $\phi(D^2)$ is a polynomial in D. So $f(D)$ is associated with operator D^2.

$$PI = \frac{1}{f(D)} \sin ax$$

$$= \frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax$$

$$\therefore \boxed{\frac{1}{f(D)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax}$$

Now we will prove this, see next.

$$D^2 \sin ax = -a^2 \sin ax$$

$$D^4 \sin ax = a^4 \sin ax$$

$$= (-a^2)^2 \sin ax$$

or, $$(D^2)^2 \sin ax = (-a^2)^2 \sin ax$$

$$(D^2)^3 \sin ax = (-a^2)^3 \sin ax$$

$$(D^2)^n \sin ax = (-a^2)^n \sin ax$$

Since $\phi(D^2)$ is a polynomial in D^2

$$\boxed{\phi(D^2) \sin ax = \phi(-a^2) \sin ax}$$

$$f(D)\sin ax = \phi(-a^2)\sin ax$$

$$\sin ax = \frac{1}{f(D)}\phi(-a^2)\sin ax$$

$$= \phi(-a^2)\frac{1}{f(D)}\sin ax$$

$$\text{or } \frac{1}{\phi(-a^2)}\sin ax = \frac{1}{f(D)}\sin ax$$

$$\Rightarrow \boxed{\frac{1}{f(D)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax}$$

Hence Proved.

(ii) Similarly

$$\frac{1}{f(D)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax.$$

Ex $(D^2-4)y = \sin 2x$

The linear operator is $f(D) = D^2-4$.

$= (D-2)(D+2)$

The auxiliary eq^n is $m^2-4=0 \Rightarrow m=2,-2$

the C.F. is $\underline{c_1 e^{-2x} + c_2 e^{2x}}$, where c_1 and c_2 are arbitrary constants

$$PI = \frac{1}{D^2-4}\sin 2x$$

$$= \frac{1}{-2^2-4}\sin 2x \qquad \left[\text{Put } D^2 = -2^2\right]$$

$$= -\frac{1}{8}\sin 2x$$

The general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}\sin 2x$$

Ex $(3D^2+2D-8)y = 5\cos x.$

The linear operator is

$$f(D) = 3D^2+2D-8$$
$$= (3D-4)(D+2)$$

The auxiliary eq^n is

$$(3m-4)(m+2) = 0$$
$$m = 4/3, -2$$

The C.F. is $c_1 e^{4x/3} + c_2 e^{-2x}$, where c_1 and c_2 are arbitrary constants

PI is $\frac{1}{3D^2+2D-8} 5\cos x$

$= 5 \cdot \frac{1}{3\cdot(-1^2)+2D-8} \cos x$ [Put $D^2 = -(1)^2$
$= -1$]

$= 5 \cdot \frac{1}{2D-11} \cos x$

$= 5 \cdot \frac{2D+11}{(2D-11)(2D+11)} \cos x$

$= 5 \cdot \frac{2D+11}{4D^2-121} \cos x$

$= 5 \cdot \frac{2D+11}{4(-1)-121} \cos x$

$$= \frac{5}{-125} \cdot (2D+11)\cos x$$

$$= -\frac{1}{25}\left(-2\sin x + 11\cos x\right)$$

$$= \frac{2\sin x - 11\cos x}{25}$$

$$=$$

The general solution is

$$y = c_1 e^{4/3 x} + c_2 e^{-2x} + \frac{2\sin x - 11\cos x}{25}$$

(III) Consider $X = e^{ax} \cdot V(x)$, where V is function of x.

Then the particular integral

is $\frac{1}{f(D)} e^{ax} V$

$= e^{ax} \frac{1}{f(D+a)} \cdot V$

[put $D = D + a$]

This is very useful for complicated functions X

$\Rightarrow \boxed{\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} \cdot V}$

we will use this for finding P.I.

Ex $(D^2+4)y = \sin 2x$, A.E is $m^2+4=0$

$m = \pm 2i$

The CF is $c_1 \cos 2x + c_2 \sin 2x$, where c_1 and c_2 are arbitrary constants

To find PI :

$$\frac{1}{(D^2+4)} \sin 2x,$$

now if you put $D^2 = -2^2 = -4$ in (D^2+4), you will get $\frac{1}{0}$ form.

So, it is not the right concept.
we will go for another technique

$$= \frac{1}{D^2+4} \sin 2x.$$

Now $e^{2ix} = (\cos 2x + i \sin 2x)$

$\therefore$ $\sin 2x$ is the imaginary part of e^{2ix}.

So we can write the PI as

Imaginary part of $\frac{1}{D^2+4} e^{2ix}$

Now $\frac{1}{D^2+4} e^{2ix}$

$= e^{2ix} \frac{1}{(D+2i)^2+4} \cdot 1$

use

$\left[\frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(D+a)} \right]$

$= e^{2ix} \frac{1}{D^2+4iD-4+4} \cdot 1$

$$= e^{2ix} \frac{1}{(D^2+4iD)} \cdot 1$$

$$= e^{2ix} \frac{1}{4iD\left(1+\frac{D}{4i}\right)} \cdot 1$$

$$= e^{2ix} \cdot \frac{1}{4iD} \cdot \left(1+D/4i\right)^{-1} \cdot 1$$

since $\left[\left(1+\frac{D}{4i}\right)^{-1} = \left(1-\frac{D}{4i}+\cdots\right) \cdot x^0 = 1\right]$

$$= e^{2ix} \cdot \frac{1}{4i} \cdot \frac{1}{D} \cdot 1 = \frac{e^{2ix}}{4i} \cdot x$$

$$= \frac{x(\cos 2x + i \sin 2x) \cdot i}{4i \cdot i}$$

$$= -\frac{x}{4}\left(-\sin 2x + i\cos 2x\right)$$

Another Technique

$$\frac{1}{D^2+4} e^{2ix}$$

$$= \frac{1}{(D-2i)(D+2i)} e^{2ix} = \frac{1}{(D-2i)} \cdot \frac{e^{2ix}}{(2i+2i)}$$

$\boxed{\text{put } D = 2i}$

$$= \frac{1}{(D-2i)} \frac{e^{2ix}}{4i}$$

$$= \frac{1}{4i} \frac{1}{D-2i} e^{2ix}$$

[use theorem $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$]

$$= \frac{1}{4i} \cdot x e^{2ix} = -\frac{i}{4} x e^{2ix}$$

$$= -\frac{x}{4} \{ -\sin 2x + i \cos 2x \}$$

So, $\frac{1}{D^2+4} e^{2ix} = -\frac{x}{4} \{ -\sin 2x + i \cos 2x \}$

$$\frac{1}{D^2+4}\sin 2x$$

$$= \text{Imaginary part of } \frac{1}{D^2+4} e^{2ix}$$

$$= \text{Imaginary part of } \left\{ \frac{x}{4}\sin 2x - i\frac{x}{4}\cos 2x \right\}$$

$$= -\frac{x}{4}\cos 2x,$$ This is the particular integral.

The general solution is $\boxed{c_1\cos 2x + c_2\sin 2x - \frac{x}{4}\cos 2x}$

Ex find the PI of $(D^2+4)y = x\sin^2 x$

$$PI = \frac{1}{(D^2+4)} x\sin^2 x$$

$$x\sin^2 x = \frac{x}{2}(1-\cos 2x) = x/2 - x/2 \cos 2x$$

Now $$\frac{1}{D^2+4}\left[x\sin^2 x\right] = \frac{1}{D^2+4}\left[x/2 - x/2 \cos 2x\right]$$

$$= \frac{1}{2}\frac{1}{D^2+4}\cdot x - \frac{1}{2}\frac{1}{D^2+4}\, x\cos 2x$$

Now $\dfrac{1}{D^2+4}\cdot x$

$$= \frac{1}{4\left(1+D^2/4\right)}\cdot x$$

$$= \frac{1}{4}\cdot\left(1+D^2/4\right)^{-1} x$$

$$= \frac{1}{4}\left(1 - D^2/4 + \cdots\right) x$$

$$= \frac{1}{4}\left(1 - \frac{D^2}{4} + \cdots\right)x = \frac{x}{4}.$$

and $\frac{1}{D^2+4} x \cos 2x$

Now $\cos 2x$ is the real part of e^{2ix}.

$$e^{2ix} = \cos 2x + i \sin 2x.$$

So, $\frac{1}{D^2+4} x \cos 2x$

$$= \text{Real part of } \frac{1}{D^2+4} x e^{2ix}$$

Now $\dfrac{1}{D^2+4}\, x e^{2ix}$

$$= e^{2ix} \frac{1}{(D+2i)^2+4}\, x$$

$$= e^{2ix} \cdot \frac{1}{D^2+4iD-4+4} \cdot x$$

$$= e^{2ix} \frac{1}{D^2+4iD} \cdot x$$

$$= e^{2ix} \frac{1}{4iD\left(1+\frac{D}{4i}\right)} \cdot x$$

$$= e^{2ix}\frac{1}{4iD}\cdot\left(1+D/4i\right)^{-1}x \qquad \left[(1+x)^{-1} = 1-x+x^2-\dots\right]$$

$$= e^{2ix}\cdot\frac{1}{4iD}\left(1-D/4i+\dots\right)x \qquad \left[\begin{array}{l} Dx=1 \\ D^2x=0 \end{array}\right]$$

$$= e^{2ix}\cdot\frac{1}{4iD}\cdot\left(x-\frac{1}{4i}\right)$$

$$= -\frac{i}{4}e^{2ix}\left[\frac{x^2}{2}-\frac{1}{4i}x\right]$$

$$= -i/4\left(\cos 2x+i\sin 2x\right)\left(x^2/2 - x/4i\right)$$

$$
\begin{aligned}
&= -\frac{i}{8}\left(\cos 2x + i\sin 2x\right)\left(x^2 + \frac{ix}{2}\right) \\
&= -\frac{1}{8}\left(-\sin 2x + i\cos 2x\right)\left(x^2 + \frac{ix}{2}\right) \\
&= -\frac{1}{8}\left\{\left(-x^2\sin 2x - \frac{x}{2}\cos 2x\right) + i\left(x^2\cos 2x \right.\right. \\
&\qquad\qquad \left.\left. - \frac{x}{2}\sin 2x\right)\right\}
\end{aligned}
$$

Now $\frac{1}{D^2+4} xe^{2ix}$

$$= -\frac{1}{8}\left\{\left(-x^2\sin 2x - \frac{x}{2}\cos 2x\right) + i\left(x^2\cos 2x - \frac{x}{2}\sin 2x\right)\right\}$$

$$\frac{1}{D^2+4} x\cos 2x$$

[Consider only the Real part]

$$= -\frac{1}{8}\left(-x^2\sin 2x - \frac{x}{2}\cos 2x\right)$$

$$= \frac{x^2\sin 2x}{8} + \frac{x}{16}\cos 2x$$

Now $\dfrac{1}{D^2+4}\, x\sin^2 x$

$$= \frac{x}{8} - \frac{x^2\sin 2x}{16} - \frac{x\cos 2x}{32}$$

This is required particular integral.

Ex $(D^2-1)y = x^2 \cos x$

The auxiliary eqⁿ is $m^2-1=0 \Rightarrow m=\pm 1$

The CF is $c_1 e^{x} + c_2 e^{-x}$.

To find P.I.:

$$\frac{1}{(D^2-1)} x^2 \cos x$$

$$= \text{Real part of } \frac{1}{(D^2-1)} x^2 e^{ix}$$

Now $\frac{1}{D^2-1} x^2 e^{ix}$

$$= e^{ix} \frac{1}{(D+i)^2-1} x^2$$

$$= e^{ix} \frac{1}{(D^2+2iD-1-1)} x^2$$

$$= e^{ix} \frac{1}{D^2+2iD-2} x^2$$

$$= e^{ix} \frac{1}{-2\left(1-\frac{D^2+2iD}{2}\right)} \cdot x^2$$

$$= e^{ix}/-2 \cdot \left(1-\frac{D^2+2iD}{2}\right)^{-1} \cdot x^2$$

$$\left[(1-x)^{-1} = 1+x+x^2+x^3+\dots\right]$$

$$= \frac{e^{ix}}{-2} \cdot \left\{1+\left(\frac{D^2+2iD}{2}\right)+\left(\frac{D^2+2iD}{2}\right)^2+\dots\right\}x^2$$

delete this terms.

$$= -\frac{e^{ix}}{2}\left\{1+\frac{D^2}{2}+iD+\frac{1}{4}\left(D^4+4iD^3-4D^2\right)+\dots\right\}x^2$$

$$\left[\begin{aligned} Dx^2 &= 2x \\ D^2x^2 &= 2 \\ D^3x^2 &= 0 \\ D^4x^2 &= 0. \end{aligned}\right]$$

$$= -e^{ix}/2\left\{1+iD+\frac{D^2}{2}-D^2\right\}x^2$$

$$= -\frac{e^{ix}}{2}\left\{x^2 + 2ix - 1\right\}$$

$$= -\frac{1}{2}(x^2 - 1 + 2ix)(\cos x + i\sin x)$$

$$= -\left(\frac{x^2-1}{2} + ix\right)(\cos x + i\sin x)$$

$$= -\left\{\left(\frac{x^2-1}{2}\right)\cos x - x\sin x\right) + i\left(x\cos x + \left(\frac{x^2-1}{2}\right)\sin x\right)\right\}$$

$$\frac{1}{(D^2-1)} x^2 \cos x$$

$$= -\left(\frac{x^2-1}{2}\right)\cos x + x\sin x$$

[Consider only the real part]

$$= -\frac{x^2}{2}\cos x + x\sin x + \frac{1}{2}\cos x.$$

So, the general solution is

$$c_1 e^{x} + c_2 e^{-x} - \frac{x^2}{2}\cos x + x\sin x + \frac{1}{2}\cos x.$$

Ex Find the PI of

$$(D^3+1)y = e^{x/2} \sin \frac{x\sqrt{3}}{2}$$

$$PI = \frac{1}{D^3+1} e^{x/2} \sin \frac{x\sqrt{3}}{2}$$

$$= \text{Imaginary part of } \frac{1}{D^3+1} e^{x/2} \cdot e^{i\frac{x\sqrt{3}}{2}}$$

$$= \text{Im. part of } \frac{1}{D^3+1} e^{\frac{x}{2}(1+i\sqrt{3})}$$

$$= \text{Im. part of } e^{x\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} \frac{1}{\left(0+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3+1} \cdot 1$$

$$= \text{Im. part of } e^{x/2} \cdot e^{ix\sqrt{3}/2} \frac{1}{(D+\omega)^3+1} \cdot 1$$

$$\left[\text{where } \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ and } \omega^2 = +\frac{1}{2} - i\frac{\sqrt{3}}{2} \text{ and } \omega^3 = -1\right]$$

$$= \text{Im. part of } e^{x/2} e^{ix\frac{\sqrt{3}}{2}} \frac{1}{D^3+3D^2\omega+3D\omega^2+\omega^3+1} \cdot 1$$

$$= \text{Im. part of } e^{x/2} e^{ix\frac{\sqrt{3}}{2}} \frac{1}{D^3+3D^2\omega+3D\omega^2} \cdot 1 \quad \left[\text{since } \omega^3 = -1\right]$$

$$= \text{Im. part of } e^{x/2} \cdot e^{ix\frac{\sqrt{3}}{2}} \frac{1}{3D\omega^2\left(1+\frac{D^2+3\omega D}{3\omega^2}\right)} \cdot 1$$

$$= \text{Im. part of } e^{x/2} \cdot e^{ix\sqrt{3}/2} \cdot \frac{1}{3D\omega^2}\left(1 + \frac{D^2+3\omega D}{3\omega^2}\right)^{-1} \cdot x^0$$

$$= \text{Im. part of } e^{x/2} \cdot e^{ix\sqrt{3}/2} \cdot \frac{1}{3D\omega^2} \cdot 1$$

$$= \text{Im. part. of } e^{x/2}\, e^{ix\sqrt{3}/2} \cdot \frac{1}{3\omega^2} \cdot x$$

$$= \text{Im. part of } e^{x/2} \cdot \frac{\left(\cos\frac{x\sqrt{3}}{2} + i\sin\frac{x\sqrt{3}}{2}\right) \cdot \omega \cdot x}{3\omega^3}$$

$$= \text{Im. part of } e^{x/2} \frac{\left(\cos\frac{x\sqrt{3}}{2} + i\sin\frac{x\sqrt{3}}{2}\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right).x}{3.-1}$$

$\omega^3 = -1$

$\omega = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$$= \text{Im. part of } -\frac{xe^{x/2}}{3}\left(\cos\frac{x\sqrt{3}}{2} + i\sin\frac{x\sqrt{3}}{2}\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{xe^{x/2}}{3}\left\{\frac{\sqrt{3}}{2}\cos\frac{x\sqrt{3}}{2} + \frac{1}{2}\sin\frac{x\sqrt{3}}{2}\right\}$$

$$= -\frac{xe^{x/2}}{6}\left\{\sqrt{3}\cos\frac{x\sqrt{3}}{2} + \sin\frac{x\sqrt{3}}{2}\right\}$$

This is the required solution.