भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

5^{th} Lecture on Transform Techniques

(MA-2120)

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What did we learn in previous class?

- Scaling Property
- Laplace Transform of Derivatives
- Differentiation of Laplace Transform
- Division by t
- Laplace Transform of Integral

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What will we learn today?

- Laplace Transform of Periodic functions
- Initial Value Theorem
- Final Value Theorem
- Convolution Theorem
- Inverse Laplace Transform

Laplace Transform of Periodic function :

Def^n : A function $f(t)$, $t \geq 0$ is said to be periodic $T > 0$, if $f(t+T) = f(t)$, $\forall\, t \geq 0$.

Examples : $\sin t$, $\cos t$, Triangular wave and square wave.

Theorem: If $\mathcal{L}[f(t)] = F(s)$ and $f(t)$ is periodic function of period T, then

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \underbrace{\int_0^T f(t)\, e^{-st}\, dt}_{\mathcal{L}[f_1(t)]}$$

where $f_1(t) = \begin{cases} f(t), & 0 \le t < T \\ 0, & t > T. \end{cases}$

Proof:

Let $F(s) = \mathcal{L}[f(t)]$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T f(t)\, e^{-st}\, dt + \int_0^\infty e^{-s(u+T)} f(u+T)\, du$$

$$\boxed{\begin{array}{l}\text{Let } u = t - T \\ du = dt\end{array}}$$

$$= \int_0^T f(t)\, e^{-st}\, dt + e^{-sT} \int_0^\infty f(u+T)\, e^{-su}\, du$$

$$= \int_0^T f(t)\, e^{-st}\, dt + e^{-sT} \int_0^\infty f(u)\, e^{-su}\, du$$

[since $f(u+T) = f(u)$]

$$= \int_0^T f(t)\, e^{-st}\, dt + e^{-sT} \mathcal{L}[f(t)]$$

Assuming that $\mathcal{L}[f(t)]$ exists

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt.$$

(Proved)

Some examples are :

(i) **Triangular wave :**

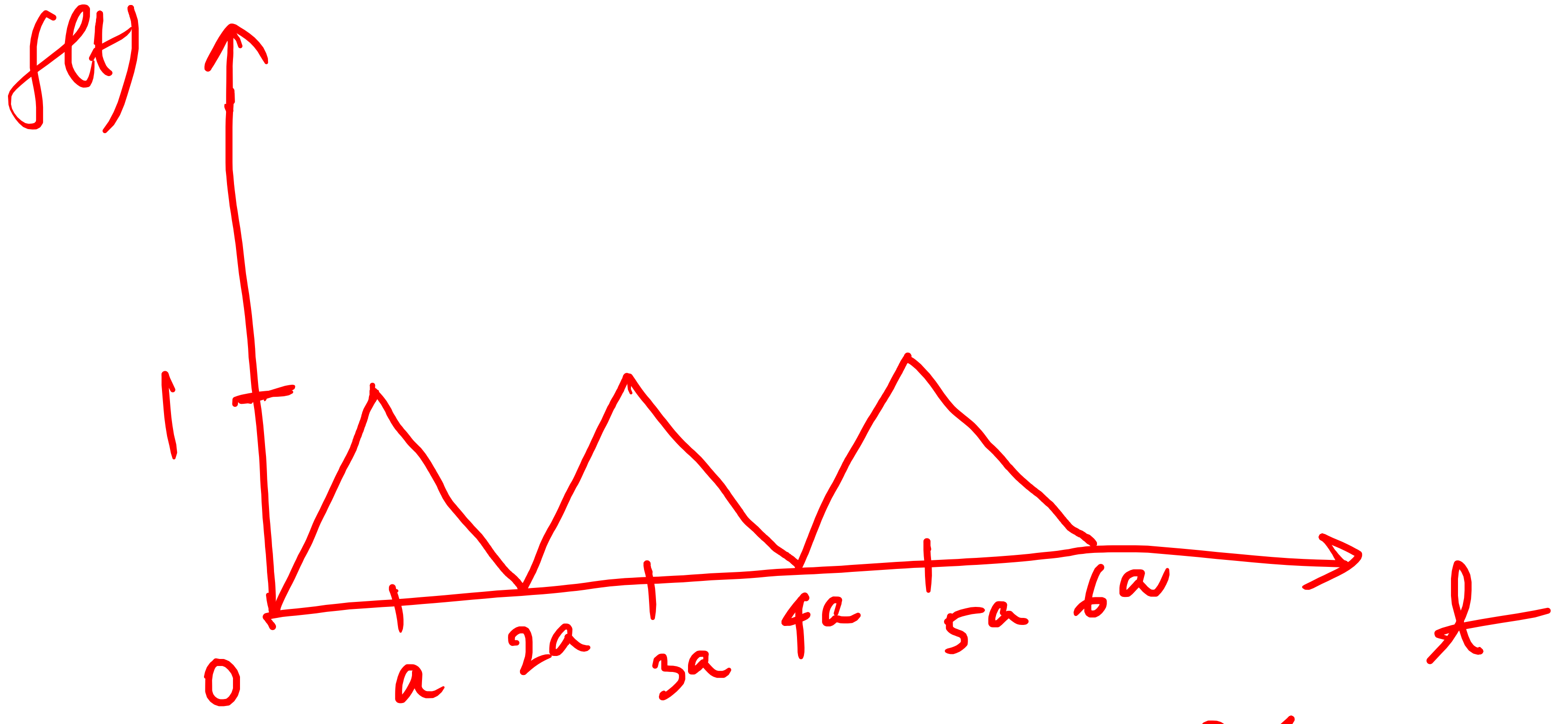
$$f(t) = \begin{cases} t/a, & 0 \le t < a \\ (2a-t)/a, & a \le t \le 2a. \end{cases}$$

$$f(t+T) = f(t+2a) = f(t)$$

Ex, Find the Laplace Transform of the periodic function defined by triangular wave

$$f(t) = \begin{cases} t/a, & 0 \le t \le a \\ (2a-t)/a & a \le t \le 2a. \end{cases}$$

and $f(t+2a) = f(t)$



Triangular wave.

Sol^n: we have $T = 2a$. Therefore

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a \frac{t}{a}\, e^{-st}\,dt + \int_a^{2a} \frac{(2a-t)}{a}\, e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{as^2}\left(1-2e^{-as}+e^{-2as}\right)\right]$$

$$= \frac{\left(1-e^{-as}\right)^2}{as^2\left(1-e^{-2as}\right)}$$

$$= \frac{e^{as/2} - e^{-as/2}}{as^2\left(e^{as/2} + e^{-as/2}\right)} = \frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$$

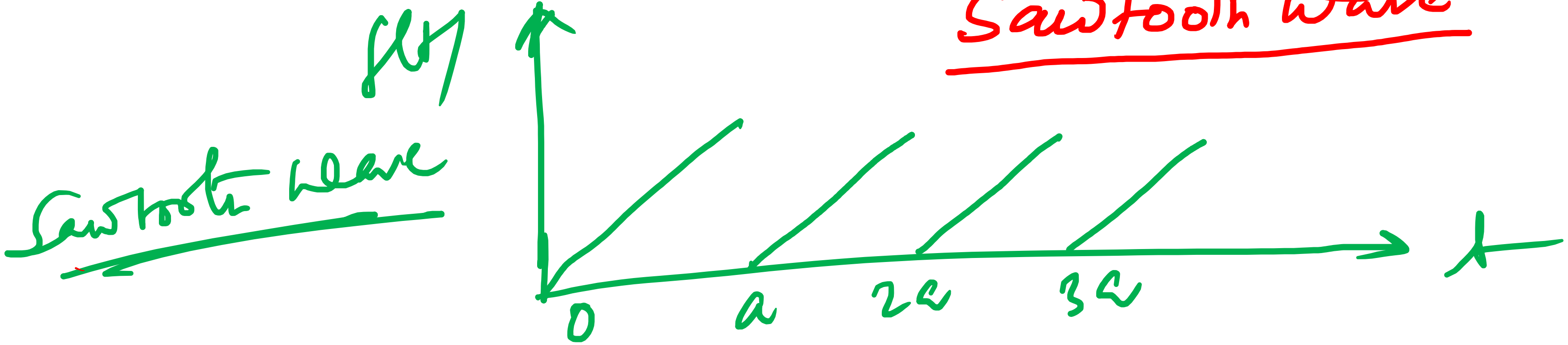
$s > 0$

Ex: $\mathcal{L}[f(t)]$, where $f(t)$ is defined by

Sawtooth wave $f(t) = t$, $0 \le t \le a$,

$f(t+a) = f(t)$.

Sawtooth wave



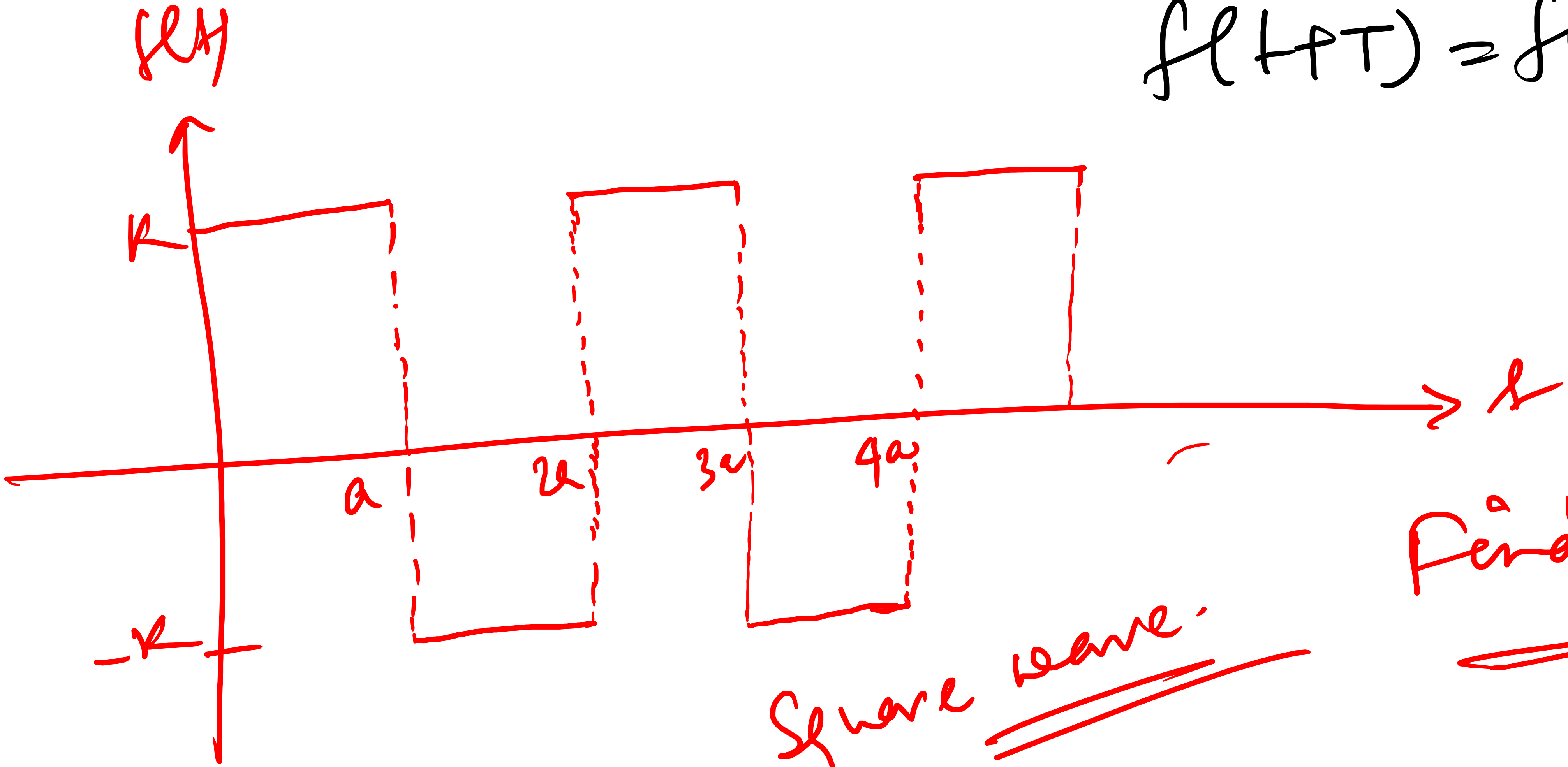
Sol^n: $\frac{1}{s^2} - \frac{a e^{-as}}{s(1-e^{-as})}, \; s>0.$

(Try).

Square wave:

$$f(t) = \begin{cases} K, & 0 \le t < a \\ -K & a \le t < 2a. \end{cases}$$

$$f(t+T) = f(t+2a) = f(t)$$



Square wave.

Find $\mathcal{L}[f(t)]$

(9) Initial value Theorem:

Let $f(t)$ be continuous on $[0,\infty)$ and of exponential order. If $f'(t)$ is piecewise continuous on $[0,\infty)$, then

$$\boxed{f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)}, \quad (s \text{ real})$$

If $f(t)$ is continuous on $(0,\infty)$, then

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

Proof: $\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)$

Taking limit

$$\lim_{s\to\infty} \mathcal{L}[f'(t)] = \lim_{s\to\infty} s\,\mathcal{L}[f(t)] - f(0)$$

$$\text{or} \quad 0 = \lim_{s\to\infty} s\,\mathcal{L}[f(t)] - f(0)$$

$$\text{or} \quad f(0) = \lim_{s\to\infty} s\,\mathcal{L}[f(t)] = \lim_{s\to\infty} s\,F(s).$$

Example : If $\mathcal{L}[f(t)] = \dfrac{s+1}{(s-1)(s+2)}$

then $f(0) = \lim_{s\to\infty} s\left(\dfrac{s+1}{(s-1)(s+2)}\right)$

$= 1.$

(10) Final value Theorem:

Let $f(t)$ be continuous on $[0,\infty)$ and be of exponential order and if $f'(t)$ is piecewise continuous on $[0,\infty)$ and $\lim_{t\to\infty} f(t)$ exists,

then

$$\boxed{\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{s\to 0} s\,\mathcal{L}[f(t)] \\ &= \lim_{s\to 0} s\,F(s) \quad [s \text{ real}]\end{aligned}}$$

Ex: Let $f(t) = \sin t$.

Then $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s^2+1} = 0$

but $\lim_{t\to\infty} \sin t$ does not exist.

Note: we may deduce that if $\lim_{s\to 0} sF(s) = L$, then either $\lim_{t\to\infty} f(t) = L$ or this limit does not exist.

(12) Convolution Theorem:

(a) Convolution: Let $f(t)$ and $g(t)$ be defined in $[0,\infty)$. Then, the convolution of $f(t)$ and $g(t)$, denoted by $(f*g)(t)$ is defined by

$$\boxed{(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \quad t \geq 0.}$$

Example:

If $f(t)=e^t$, $g(t)=t$, then

$$(f*g)(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$=\int_0^t e^{\tau}(t-\tau)\,d\tau$$

$$=e^t-t-1$$

The following properties of convolution are —

(i) $f * g = g * f$ (commutative)

(ii) $f * (g_1 + g_2) = f * g_1 + f * g_2$ (distributive)

(iii) $(f * g) * h = f * (g * h)$ (associative)

(iv) $(f * 0) = (0 * f) = 0$.

(v) $c(f * g) = cf * g = f * cg$, where c is a constant.

But $f*1 = 1*f \neq f$.

For example, $f(t) = t$, we have

$$(1*f)(t) = \int_0^t 1\cdot(t-x)\,dx$$

$$= t^2/2 \neq t = f(t).$$

Theorem: Let $f(t)$ and $g(t)$ be piecewise continuous function on $[0,\infty)$ and be of exponential order α. Then

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$$

$$= F(s)\cdot G(s);\quad Re(s) > \alpha.$$

Proof: You try it.

Example: $\mathcal{L}[e^{at} * e^{at}] = \mathcal{L}[e^{at}]\,\mathcal{L}[e^{at}]$

$$= \frac{1}{(s-a)^2}$$

Ex: $\mathcal{L}[\sin\omega t * \sin\omega t]$

$$= \frac{\omega^2}{(s^2+\omega^2)^2}$$

Inverse Laplace Transform

$$f(t) \longrightarrow \boxed{\mathcal{L}[f(t)]} \longrightarrow F(s) \longrightarrow \boxed{\mathcal{L}^{-1}[F(s)]} \longrightarrow f(t)$$

Laplace Transform

Inverse Laplace Transform

Here t-space $\xrightarrow{\mathcal{L}}$ s-space $\xrightarrow{\mathcal{L}^{-1}}$ t-space

original space.

$$\boxed{\mathcal{L}^{-1}\left(\mathcal{L}[f(t)]\right) = f(t)}$$

If $\mathcal{L}[f(t)] = F(s)$, $Re(s) > \alpha$, then

$$\mathcal{L}^{-1}[F(s)] = f(t), \quad t \geq 0.$$

② Inverse Laplace Transform :

If $F(s)$ be the Laplace transform of a function $f(t)$, i.e., if $\mathcal{L}[f(t)] = F(s)$, then $f(t)$ is called the Inverse Laplace Transform of the function $F(s)$ and it is written as $\boxed{f(t) = \mathcal{L}^{-1}[F(s)]}$,

where $\mathcal{L}^{-1}$ is called the Inverse Laplace Transformation operator.

Inverse Laplace Transform of Some elementary functions :

① $\mathcal{L}[1] = 1/s. \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

② $\mathcal{L}[t] = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t.$

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \Rightarrow \mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n.$

where n is positive integer.

or, $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = t^n/n!$

(3) $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}.$$

(4) $\mathcal{L}[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1, \; s > 0.$

$$\Rightarrow \; t^{\alpha} = \mathcal{L}^{-1}\left[\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right]$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)}}$$

Q. Let $\mathcal{L}^{-1}[F(s)] = f(t)$. Now can we find any function $g(t) \neq f(t)$ such that $\mathcal{L}^{-1}[F(s)] = g(t)$? (uniqueness problem of the inverse Laplace transform)

Example: $\mathcal{L}[\sin at] = \dfrac{a}{s^2+a^2}, \; s>0$

So, $\mathcal{L}^{-1}\left[\dfrac{a}{s^2+a^2}\right] = \sin at, \; t \geq 0$

Let $f(t) = \sin at, \; t \geq 0$.

Let $g(t) = \begin{cases} \sin at, & t > 0 \\ 1, & t = 0. \end{cases}$ (discontinuous at $t=0$)

Then $$\boxed{\mathcal{L}\left[g(t)\right] = \frac{a}{s^2+a^2}}$$

Since alternating a function at a single point (or even at a finite no. of points) does not alter the value of the Laplace integral by the properties of Riemann integral or Improper Riemann integral.

This example gives that $\mathcal{L}^{-1}[F(s)]$ can be more than one function, infact infinitely many at least when we are considering functions with discontinuities

Note: we have for any $t > 0$

$$\boxed{\int_0^t \underbrace{\left(g(u) - f(u)\right)}_{\text{null function}} du = 0}$$

Def^n of Null function: A function $n(t)$ such that $\int_0^t n(u)\,du = 0$ for any $t > 0$ is called a null function.

Theorem: If $x(t)$ is a null function, then $\mathcal{L}[x(t)] = 0$ for any $s>0$.

Proof:

$$\mathcal{L}[x(t)] = \int_0^\infty e^{-st} x(t)\, dt$$

$$= \left[e^{-st} \underbrace{\int_0^t x(u)\, du}_{0} \right]_0^\infty + s \int_0^\infty e^{-st} \left(\underbrace{\int_0^t x(u)\, du}_{0} \right) dt$$

$$= 0 + s \cdot 0 = 0 \, . \quad s>0$$

Corollary: If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[f(t)+n(t)] = F(s)$, for any null function $n(t)$.

$$\Rightarrow \boxed{\mathcal{L}^{-1}[F(s)] = f(t)+n(t), \text{ where } n(t) \text{ is the null function.}}$$

Theorem: (Lerch's theorem):

If $F(s) = \mathcal{L}[f(t)]$ and $F(s) = \mathcal{L}[g(t)]$, $f(t) \neq g(t)$, for $t \geq 0$, then $f(t) - g(t)$ is a null function.

In particular, if $f(t)$ and $g(t)$ are continuous function with the same Laplace transform $F(s)$, then $f(t) \equiv g(t)$.

So if we restrict our attention to functions that are continuous on $[0,\infty)$, then the inverse Laplace transform is uniquely defined.

Also one can say the uniqueness of the inverse Laplace transform in this way. See next page.

Remark:

If we restrict ourselves to functions $f(t)$ which are piecewise continuous in every finite interval $0 \le t \le t_0$ and of exponential order for $t > t_0$, then the inverse of Laplace transform of $F(s)$ i.e. $\mathcal{L}^{-1}[F(s)] = f(t)$ is unique.

Note: While evaluating inverse Laplace transform of any function, we shall always assume such uniqueness unless otherwise stated.