Lecture 9

Review of Dimensional Analysis & Similitude

Dimensionless Parameters for Turbomachinery

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Dimensional Analysis: Similitude

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Dimensionless parameters provide number of advantages

- *With appropriate choice of dimensionless parameters the* ***performance of machines can be characterized using only a few key variables****.*
- *Given data on one size machine, performance can be predicted for different size machines.*
- *Given data on one set of operating conditions, behavior at different operating condition can be predicted.*
- *Enable designers to pick a particular machine shape of maximum efficiency.*

Application area dictates which dimensionless parameters are important.

Dimensional Analysis: Similitude

Need for dimensional analysis?

- *Considering practical problem, a typical fluid mechanics problem in which experimentation is required*
  - Consider the steady flow of an incompressible Newtonian fluid through a long, smooth walled, horizontal, circular pipe.
    - ***An important characteristic of this system, which would be of interest to an engineer designing a pipeline, is the pressure drop per unit length that develops along the pipe as a result of friction.***
- *The first step in the planning of an experiment to study this problem would be to decide on the factors, or variables, that will have an effect on the pressure drop.*
  - Pressure drop per unit length depends on FOUR variables:
    - $\Delta p_l = f(D,\rho,\mu,V)$ ***, size (D); speed (V); fluid density (ρ); fluid viscosity (μ)***

Dimensional Analysis: Similitude

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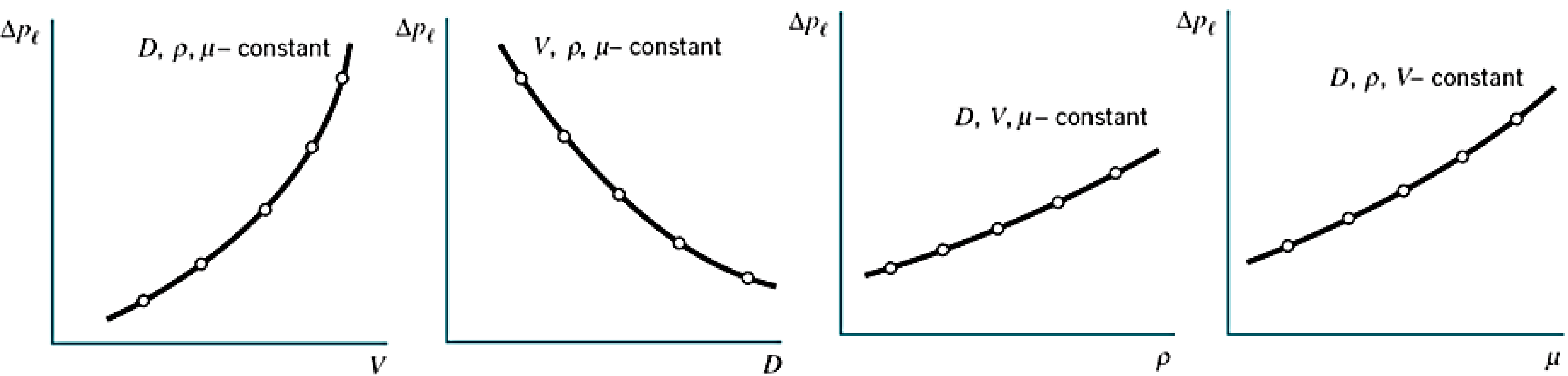
Difficulty is to determine the functional relationship between the pressure drop and the various factors that influence it.

- *Systematic Approach or Parametric Study or Critical Analysis*
  - To perform the experiments in a meaningful and systematic manner, it would be **<u>necessary to change one of the variable, while holding all other constant</u>**, and measure pressure drop.



Dimensional Analysis: Similitude

Fortunately, there is a much simpler approach to the problem that will eliminate the difficulties described earlier.

- *Collecting these variables into two non-dimensional combinations of the variables (called dimensionless product or dimensionless groups)*
  - Only one dependent and one independent variable
  - Easy to set up experiments to determine dependency
  - Easy to present results (one graph)

$$\frac{D\Delta p_{\ell}}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Dependent variable

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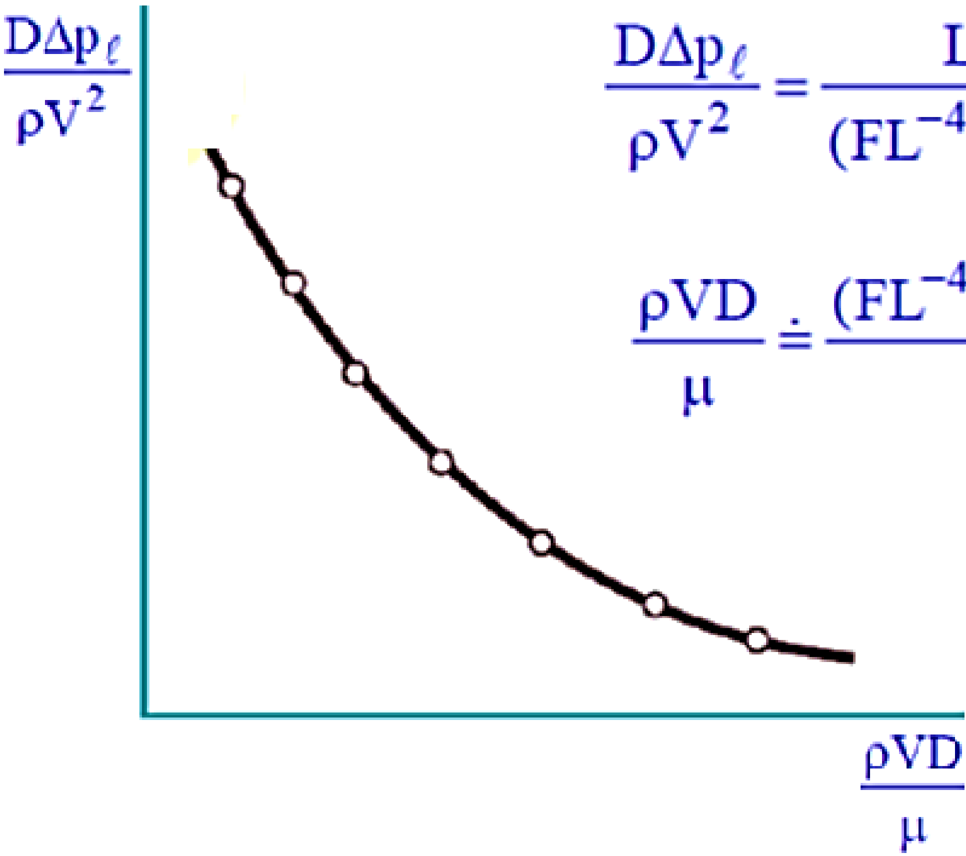
Dimensional Analysis: Similitude

Pressure drop plot in dimensionless parameters.



$$\frac{D\Delta p_\ell}{\rho V^2} = \frac{L(F/L^3)}{(FL^{-4}T^2)(FT^{-1})^2} \doteq F^0L^0T^0$$

$$\frac{\rho V D}{\mu} \doteq \frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)} = F^0L^0T^0$$

dimensionless product or dimensionless groups

Dimensional Analysis: Similitude

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Which dimensionless parameters to form? How many?

- ***Buckingham Pi theorem***
  - If an equation involving <u>k variables</u> is dimensionally homogeneous, it can be reduced to a relationship among <u>**k-r** independent dimensionless products (Pi terms)</u>, where <u>**r** is the minimum number of reference dimensions</u> required to describe the variables.
    - ***Given a physical problem in which the dependent variable is a function of k-1 independent variables.***
    - ***Mathematically, we can express the functional relationship in the equivalent form***

$$u1 = f\,(u2\,,\,u3\,,\ldots\ldots,\,u\,k\,)$$

$$g(u1,\,u2\,,\,u3\,,\ldots\ldots,\,u\,k\,) = 0$$

Where 'g' is an unspecified function, different from 'f'.

Dimensional Analysis: Similitude

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The Buckingham Pi theorem states that: Given a relation among k variables of the form

- $g(u_1, u_2, u_3, \ldots\ldots, u_k) = 0$

The k variables may be grouped into k-r independent dimensionless products, or **Π terms**, expressible in functional form by

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

$$\text{or} \quad \bar{\phi}(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{k-r}) = 0$$

k??

r??

Π??

Dimensional Analysis: Similitude

The number "**r**" is usually, but not always, equal to the minimum number of independent dimensions required to specify the dimensions of all the parameters.

- *Usually the reference dimensions required to describe the variables will be the* ***basic dimensions M, L, and T or F, L, and T****.*

The theorem does not predict the **functional form of φ.**

- *The functional relation among the* ***independent dimensionless products Π*** *must be determined experimentally.*

Note that, **k-r dimensionless products 'Π terms'** obtained from the procedure are **independent**.

Dimensional Analysis: Similitude

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A **Π term** is not independent if it can be obtained from a product or quotient of the other dimensionless products of the problem.

E.g.

$$\Pi_5 = \frac{2\Pi_1}{\Pi_2\Pi_3} \quad \text{or} \quad \Pi_6 = \frac{\Pi_1^{3/4}}{\Pi_3^{2}}$$

then neither Π5 nor Π6 is independent of the other dimensionless products or dimensionless groups

Buckingham Pi Theorem

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Determination of Pi Terms

- *Several methods can be used to form the **dimensionless products, or pi term**, that arise in a dimensional analysis.*
  - METHOD of repeating variables is one such method.
- *Regardless of the method to be used to determine the dimensionless products, one begins by **listing all dimensional variables** that are known (or believed) to **affect** the given **flow phenomenon**.*

Eight steps listed hereafter outline a recommended procedure for determining the Π terms.

Buckingham Pi Theorem

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Step 1 List all the variables.

- *List all the dimensional variables involved.*
  - Keep the number of variables to a minimum, so that we can minimize the amount of laboratory work.
- *All variables must be independent.*
  - E.g. If the cross-sectional area of a pipe is an important variable, **either the *area* or the pipe *diameter*** could be used, but not both, since they are obviously not independent.
    - ***$\gamma = \rho \times g$, that is, γ, ρ, and g are not independent.***

Viz. : For pressure drop per unit length, k = 5. (All variables are Δp_l, D, μ, ρ, and V)

$$\Delta p = f\,(D, \rho, \mu, V)$$

Buckingham Pi Theorem

Step 2 Express each of the variables in terms of basic dimensions.

- *Find the number of reference dimensions.*
  - Select a set of fundamental (primary) dimensions.
    - ***For example: MLT, or FLT.***

Viz.: For pressure drop per unit length , we choose FLT.

$$\Delta p_\ell \doteq FL^{-3} \qquad D \doteq L \qquad \rho \doteq FL^{-4}T^{2}$$

$$\mu \doteq FL^{-2}T \qquad V \doteq LT^{-1}$$

r=3

Buckingham Pi Theorem

Step 3 Determine the required number of pi terms.

- *Let k be the number of variables in the problem.*
- *Let r be the number of reference dimensions (primary dimensions) required to describe these variables.*
- *Then, the number of pi terms is k-r*

Example: For pressure drop per unit length k=5, r = 3, the number of **pi terms** is $k - r = 5 - 3 = 2$.

Buckingham Pi Theorem

Step 4 Select a number of repeating variables, where the number required is equal to the number of reference dimensions.

- *Select a set of* ***r dimensional variables*** *that includes all the primary dimensions (****repeating variables****).*
- *These repeating variables will all be combined with each of the remaining parameters to form* ***Pi terms****.*
  - No repeating variables should have dimensions that are power of the dimensions of another repeating variable.

Example: For pressure drop per unit length (r = 3) select ρ, V, D.

Buckingham Pi Theorem

Step 5 Form a pi term

- *by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.*
- *Set up dimensional equations, combining the variables selected in Step 4 with each of the other variables (nonrepeating variables) in turn, to form dimensionless groups or dimensionless product.*

There will be k – r equations.

Buckingham Pi Theorem

Example: Forming Pi term using pressure drop per unit length

$$\Pi_1 = \Delta p_l \underbrace{D^a \, V^b \, \rho^c}_{}$$

Repeating Variables

Non-dimensionalization process

$$(FL^{-3})(L)^a (LT^{-1})^b (FL^{-4}T^2)^c \doteq F^0 L^0 T^0$$

$$F: 1 + c = 0$$

$$L: -3 + a + b - 4c = 0$$

$$T: -b + 2c = 0$$

$$\Rightarrow a = 1, b = -2, c = -1$$

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

Buckingham Pi Theorem

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Step 6 Repeat Step 5 for each of the remaining nonrepeating variables.

$\Pi_2 = \mu\, D^a\, V^b\, \rho^c$

$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$$F: 1 + c = 0$$

$$L: -2 + a + b - 4c = 0$$

$$T: 1 - b + 2c = 0$$

$$\Rightarrow a = -1, b = -1, c = -1$$

$$\Rightarrow \Pi_2 = \frac{\mu}{DV\rho}$$

Buckingham Pi Theorem

Step 7 Check all the resulting pi terms to make sure they are dimensionless.

- *Check to see that each group obtained is dimensionless.*

Example: For pressure drop per unit length .

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq F^0 L^0 T^0 \doteq M^0 L^0 T^0$$

$$\Pi_2 = \frac{\mu}{D V \rho} \doteq F^0 L^0 T^0 \doteq M^0 L^0 T^0$$

Buckingham Pi Theorem

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Step 8 Express the final form as a relationship among the pi terms, and think about what is means.

Express the result of the dimensional analysis.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, ,,, \Pi_{k-r})$$

Example: For pressure drop per unit length

$$\frac{\Delta p_\ell D}{\rho V^2} = \tilde{\phi}\left(\frac{\mu}{DV\rho}\right)$$

- *Dimensional analysis will not provide the form of the function.* ***The function can only be obtained from a suitable set of experiments***

Buckingham Pi Theorem

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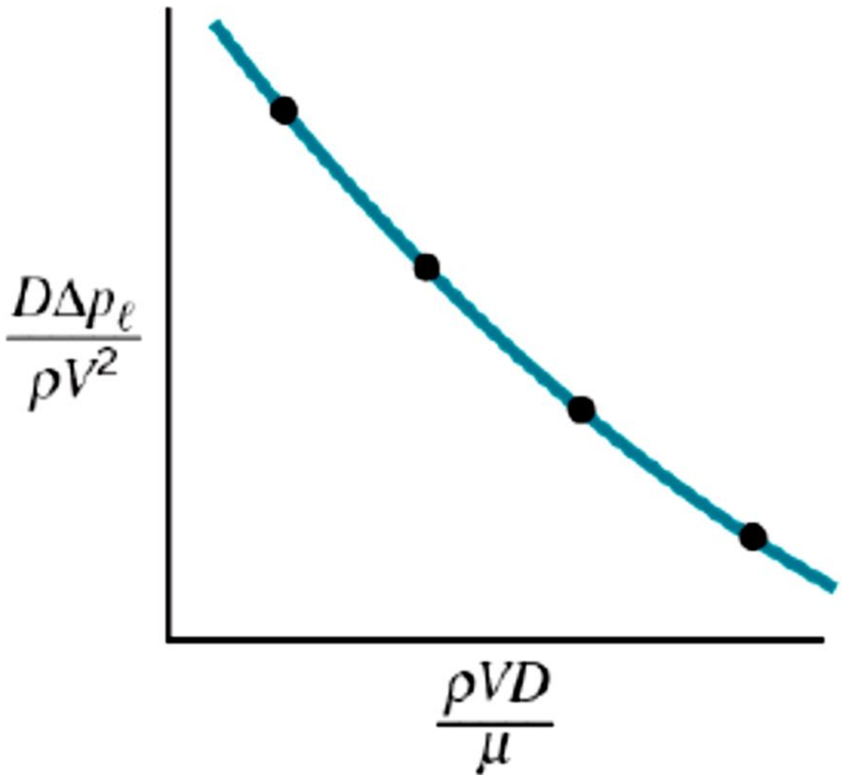
The pi terms can be rearranged. For example, Π2, could be expressed as

$$\Pi_2 = \frac{\rho VD}{\mu}$$

$$\Rightarrow \frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right)$$



Selection of Variables

One of the most important, and difficult, steps in applying dimensional analysis to any given problem is the selection of the variables that are involved.

There is no simple procedure whereby the variable can be easily identified.

- *Generally, one must rely on a good understanding of the phenomenon involved and the governing physical laws.*

If extraneous variables are included, then too many "Pi terms" appear in the final solution, and it may be difficult, time consuming, and expensive to eliminate these experimentally.

Selection of Variables

If important variables are omitted, then an incorrect result will be obtained; and again, this may prove to be costly and difficult to ascertain.

Most engineering problems involve certain simplifying assumptions that have an influence on the variables to be considered.

- *Usually solution approaches are kept as simple as possible, even if some accuracy is sacrificed.*

Selection of Variables

A suitable balance between simplicity and accuracy is a desirable goal.

Variables can be classified into three general group:

- *Geometry: lengths and angles.*
- *External Effects: produce, or tend to produce, a change in the system.*
  - Such as force, pressure, velocity, or gravity.
- *Material properties: relate the external effects and the responses.*

Selection of Variables

Choice of repeating variables

- *Repeating variables are those which we think will appear in all or most of the* ***pi terms****, and are a influence in the problem.*

Some rules which should be followed are

- *When combined, these repeating variables variable must contain all of dimensions (M, L, T) (That is not to say that each must contain M,L and T).*
- *A combination of the repeating variables must* <u>*not*</u> *form a dimensionless group.*
- *The repeating variables* <u>*do not*</u> *have to appear in all* ***pi terms****.*

Selection of Variables

- *The repeating variables should be chosen to be **measurable in an experimental investigation**.*
- *They should be of major interest to the designer.*
  - For example, pipe diameter (dimension L) is more useful and measurable than roughness height (also dimension L).

In fluids it is usually possible to take **ρ, u** and **d** as the three repeating variables.

- *This freedom of choice results in there being many different **pi groups** which can be formed - and all are valid. There is not really a wrong choice.*

Uniqueness of Pi Terms

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The Pi terms obtained depend on the somewhat arbitrary selection of repeating variables. For example, in the problem of studying the pressure drop in a pipe.

$\Delta p_l = f\,(D, \rho, \mu, V)$

Selecting D,V, and ρ as repeating variables:

$$\frac{\Delta p_\ell D}{\rho V^2} = \Phi_1\left(\frac{\rho V D}{\mu}\right)$$

Selecting D,V, and μ as repeating variables:

$$\frac{\Delta p_\ell D^2}{V\mu} = \Phi_2\left(\frac{\rho V D}{\mu}\right)$$

Uniqueness of Pi Terms

Both are correct, and both would lead to the same final equation for the pressure drop. There is not a unique set of pi terms which arises from a dimensional analysis.

The functions Φ1 and Φ2 will be different because the dependent pi terms are different for the two relationships.

$$\frac{\Delta p_\ell D}{\rho V^2} = \Phi_1\left(\frac{\rho V D}{\mu}\right) \qquad \frac{\Delta p_\ell D^2}{V\mu} = \Phi_2\left(\frac{\rho V D}{\mu}\right)$$

Uniqueness of Pi Terms

EXAMPLE $\Pi_1 = \Phi(\Pi_2, \Pi_3)$

Form a new pi term $\Pi_2' = \Pi_2^a \Pi_3^b$

$$\Rightarrow \Pi_1 = \underbrace{\Phi_1(\Pi_2', \Pi_3) = \Phi_2(\Pi_2, \Pi_2')}_{\text{All are correct}}$$

Note that π_2' may be formed out of different choice of repeating variables. Let's take a look at different Pi's

$$\frac{\Delta p_\ell D}{\rho V^2} = \Phi_1\left(\frac{\rho V D}{\mu}\right) \qquad \frac{\Delta p_\ell D^2}{V\mu} = \Phi_2\left(\frac{\rho V D}{\mu}\right)$$

Uniqueness of Pi Terms

$$\frac{\Delta p_\ell D}{\rho V^2} = \Phi_1\left(\frac{\rho V D}{\mu}\right)$$

Selecting D,V, and ρ as repeating variables:

$$\Rightarrow \frac{\Delta p_\ell D}{\rho V^2} \times \frac{\rho V D}{\mu} = \frac{\Delta p_\ell D^2}{V\mu}$$

$$\Rightarrow \frac{\Delta p_\ell D^2}{V\mu} = \Phi_2\left(\frac{\rho V D}{\mu}\right)$$

Modeling and Similitude

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Objective of Modeling and similitude is <u>to develop the procedures for designing models so that the model and prototype will behave in a similar fashion</u>.

Model Vs Prototype

- *Model ?*
  - A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect.
    - ***Mathematical or computer models may also confirm to this definition, but our interest will be in physical model.***
- *Prototype? The physical system for which the prediction are to be made.*

Modeling and Similitude

Model Vs Prototype continued

- *Models that resemble the prototype but are generally of a different size, may involve different fluid, and often operate under different conditions.*
- *Usually a model is smaller than the prototype.*
- *Occasionally, if the prototype is very small, it may be advantageous to have a model that is larger than the prototype so that it can be more easily studied.*
  - **E.g. large models have been used to study the motion of red blood cells.**

Modeling and Similitude

With the successful development of a valid model, it is possible to predict the behavior of the prototype under a certain set of conditions.

- *There is an inherent danger in the use of models in that predictions can be made that are in error and the error not detected until the prototype is found not to perform as predicted.*

It is crucial that the model be properly designed and tested and that the results be interpreted correctly.

Similarity of Model and Prototype

What conditions must be met to ensure the similarity of model and prototype?

Geometric Similarity

- *Model and prototype have same shape.*
- *Linear dimensions on model and prototype correspond within constant scale factor.*

Kinematic Similarity

- *Velocities at corresponding points on model and prototype differ only by a constant scale factor.*

Dynamic Similarity

- *Forces on model and prototype differ only by a constant scale factor.*

Theory of Models

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The prototype and the model must have the same phenomenon.

For prototype $\Pi_1 = \phi(\Pi_2 ,\Pi_3,,,\Pi_n)$

For model $\Pi_{1m} = \phi(\Pi_{2m} ,\Pi_{3m} ,,,,\Pi_{nm})$

- *The model is designed and operated under the following conditions (called design conditions, also called **similarity requirements or modeling laws**)*
  - $\mathbf{\Pi_2 = \Pi_{2m}}$ $\mathbf{\Pi_3 = \Pi_{3m} \ldots\ldots \Pi_n = \Pi_{nm}}$
- *The measured of Π_{1m} obtained with the model will be equal to the corresponding Π_1 for the prototype.*
  - $\mathbf{\Pi_1 = \Pi_{1m}}$ **Called prediction equation**

Model Scales

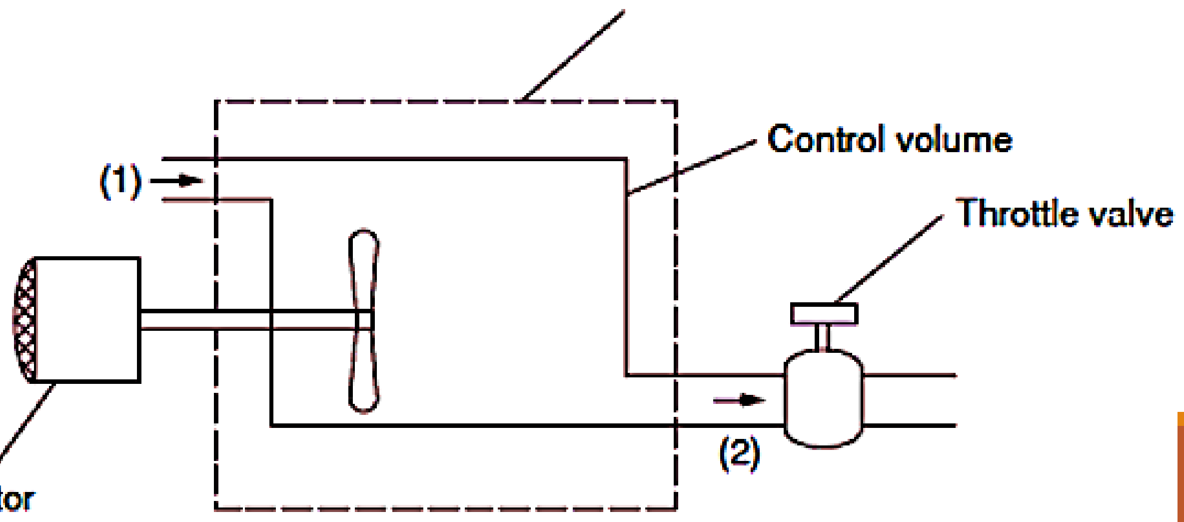
The ratio of a model variable to the corresponding prototype variable is called the scale for that variable.

- *Length Scale* $$\frac{\ell_1}{\ell_2} = \frac{\ell_{1m}}{\ell_{2m}} \Rightarrow \lambda_\ell = \frac{\ell_{1m}}{\ell_1} = \frac{\ell_{2m}}{\ell_2}$$
- *Velocity Scale* $$\lambda_V = \frac{V_m}{V}$$
- *Density Scale* $$\lambda_\rho = \frac{\rho m}{\rho}$$
- *Viscosity Scale* $$\lambda_\mu = \frac{\mu m}{\mu}$$

Control Volume Analysis Approach

Using thermodynamic analysis concept of control volume and control surfaces to analyze turbomachine (a pump here).

- *Listing the parameters that influence the performance and operation of a turbomachine.*
  - Use first law of thermodynamics for energy conversion process in turbomachine (Pump here) & list various variable accounting for loss



Dimensional Analysis of Turbomachines Primary Variables - Q's

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Q_1	Rotational Speed	N	T^{-1}
Q_2	Diameter	D	L
Q_3	Density	ρ	$M \cdot L^{-3}$
Q_4	Volumetric Flow Rate	Q	$L^3 \cdot T^{-1}$
Q_5	Head	H	$L^2 \cdot T^{-2}$
Q_6	Power	P	$M \cdot L^2 \cdot T^{-3}$
Q_7	Viscosity	μ	$M \cdot L^{-1} \cdot T^{-1}$
Q_8	Acoustic Speed	a	$L \cdot T^{-1}$
Q_9	Tip Clearance	ε	L
Q_{10}	Roughness	k	L

10 Q's - 3 Dimensions = 7 Π's

Dimensional Analysis of Turbomachines

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$$\Pi_1 = Q_1^{\,a1} \cdot Q_2^{\,b1} \cdot Q_3^{\,c1} \cdot Q_4$$

$$\Pi_2 = Q_1^{\,a2} \cdot Q_2^{\,b2} \cdot Q_3^{\,c2} \cdot Q_5$$

$$\Pi_3 = Q_1^{\,a3} \cdot Q_2^{\,b3} \cdot Q_3^{\,c3} \cdot Q_6$$

$$\Pi_4 = Q_1^{\,a4} \cdot Q_2^{\,b4} \cdot Q_3^{\,c4} \cdot Q_7$$

$$\Pi_5 = Q_1^{\,a5} \cdot Q_2^{\,b5} \cdot Q_3^{\,c6} \cdot Q_8$$

$$\Pi_6 = Q_1^{\,a6} \cdot Q_2^{\,b6} \cdot Q_3^{\,c6} \cdot Q_9$$

$$\Pi_7 = Q_1^{\,a7} \cdot Q_2^{\,b7} \cdot Q_3^{\,c7} \cdot Q_{10}$$

Various Pi terms can be estimated.

$$\boxed{\begin{array}{l} Q_1 = N \\ Q_2 = D \\ Q_3 = \rho \end{array}}$$

Repeating

Variables

Dimensional Analysis of Turbomachines - Pi terms

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$$\Pi_1 = N^a \cdot D^b \cdot \rho^c \cdot Q$$

$$\left(T^{-1}\right)^a \cdot (L)^b \cdot \left(M \cdot L^{-3}\right)^c \left(L^3 \cdot T^{-1}\right) = M^0 \cdot T^0 \cdot L^0$$

$$\Pi_1 = \frac{Q}{ND^3} = \phi = Flow coefficient$$

$$\Pi_2 = N^a \cdot D^b \cdot \rho^c \cdot H$$

$$\left(T^{-1}\right)^a \cdot (L)^b \cdot \left(M \cdot L^{-3}\right)^c \left(L^2 \cdot T^{-2}\right) = M^0 \cdot T^0 \cdot L^0$$

$$\Pi_2 = \frac{gH}{N^2 D^2} = \psi = Energy\ transfer\ or\ head\ coefficient$$

$$\Pi_3 = N^a \cdot D^b \cdot \rho^c \cdot P$$

$$\left(T^{-1}\right)^a \cdot (L)^b \cdot \left(M \cdot L^{-3}\right)^c \left(M \cdot L^2 \cdot T^{-3}\right) = M^0 \cdot T^0 \cdot L^0$$

$$\Pi_3 = \frac{P}{\rho N^3 D^5} = \xi = Power\ coefficient$$

Dimensional Analysis of Turbomachines - Pi terms

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$$\Pi_4 = N^a \cdot D^b \cdot \rho^c \cdot \mu$$

$$\left(T^{-1}\right)^a \cdot (L)^b \cdot \left(M \cdot L^{-3}\right)^c \left(M \cdot L^{-1} \cdot T^{-1}\right) = M^0 \cdot T^0 \cdot L^0$$

$$\Pi_4 = \frac{\mu}{\rho N D^2} = \frac{\mu}{\rho U D} = \frac{1}{\mathrm{Re}}$$

$$\Pi_5 = N^a \cdot D^b \cdot \rho^c \cdot a$$

$$\left(T^{-1}\right)^a \cdot (L)^b \cdot \left(M \cdot L^{-3}\right)^c \left(L \cdot T^{-1}\right) = M^0 \cdot T^0 \cdot L^0$$

$$\Pi_5 = \frac{a}{ND}$$ or more conventionally $$\Pi_5 = \frac{ND}{a}$$

$$\Pi_6 = N^a \cdot D^b \cdot \rho^c \cdot \varepsilon$$

$$\left(T^{-1}\right)^a \cdot (L)^b \cdot \left(M \cdot L^{-3}\right)^c \cdot (L) = M^0 \cdot T^0 \cdot L^0$$

$$\Pi_6 = \frac{\varepsilon}{D}$$ Similarly $$\Pi_7 = \frac{k}{D}$$