भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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First Lecture on Differential Equation

(MA-1150)

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What will we learn in this class?

- Classification on differential equation
- Applications of differential equation
- Order and degree of ODE
- Definition of linear and non-linear ODE
- Notion of solutions to ODE

Differential Equation

➢Definition:

An equation involving **derivatives or differentials** of one or more dependent variables with respect to one or more independent variables is called a **Differential Equation**.

Derivatives: If $y = y(x)$,

$y \to$ dependent variable

$x \to$ independent variable

(i) First order – $\frac{dy}{dx}$,

(ii) Second order – $\frac{d^2y}{dx^2}$

(iii) nth order – $\frac{d^ny}{dx^n}$

$\Rightarrow$ If $u = u(x, y)$,

(i) First order – $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$,

(ii) Second order – $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$,

Partial $\Rightarrow$ derivatives

Ex.

1. $\frac{d^2y}{dx^2} + y = e^x$
2. $\frac{dy}{dx} = x + \sin x$
3. $\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 7x^2$
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Ordinary Differential Equation

➢Definition:

A differential equation involving **derivatives with respect to a single independent variable** is called **Ordinary Differential Equation**.

Ex.

1. $\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^5 + y = x^2$
2. $\frac{d^2y}{dx^2} + x^2\cos y = 0$
3. $x\,dy - y\,dx + 2xy\,dy = 0$

The general form of 'n'th order ordinary Differential equation is

$$\boxed{F(x, y(x), y'(x), \ldots y^{(n)}(x)) = 0}$$

where,

$$y^{(n)} = \frac{d^ny}{dx^n}.$$

Application of ODE

- If we consider that a particle is moving with velocity 10 m/s, then find the distance moved by this particle at 10 sec.

Sol^n: Let $y(t)$ be the distance moved by the particle at time t.

$$\frac{dy}{dt} = 10 \Rightarrow \boxed{y(t) = 10t + c}$$

Application of ODE

Since the particle does not start to move at time $t=0$, $y(0)=0$, which gives $y(0)=0=c \Rightarrow c=0$.

$$\therefore \boxed{y(t)=10t} \Rightarrow \boxed{y(10)=100\text{ m}}$$

Application of ODE

Newton's second law of motion states that the acceleration a of the body with mass m is proportional to the total force F acting on it.

$$\therefore \boxed{F = ma} \Rightarrow ma = F$$

$$\Rightarrow m\frac{d^2y}{dt^2} = mg$$

Application of ODE

a) $\boxed{\frac{d^2y}{dt^2} = g}$ This is 2nd order ODE.

If we consider the air resisting force which is proportional to the velocity, then total force acting on the body is $mg - K\frac{dy}{dt}$, where K is the proportionality constant.

Application of ODE

Then we have

$$\boxed{m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dt}}$$

This is second order linear ode.

Application of ODE

A Free sphere dropping in a Air:

Consider a free sphere dropping in the air from a static state. Let t denote the time, $v(t)$ the velocity of the sphere, m the mass, and g the acceleration of gravity. Assume that the air resistance on the sphere is $a\,v^2(t)$, where a is constant. Now by Newton's second law of motion, we have first order and first degree ODE —

$$m\frac{dv(t)}{dt} = mg - a v^2(t)$$

with initial condition —

$$v(0) = 0.$$

IVP
Initial value problem
we will discuss on it later

This is non-linear ODE.

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Also there are many applications like Study of harmonic oscillation, the one dimensional fluid flow in the tube or channel, the population model, epedemic model (System of first order ODE), etc.

Partial Differential Equation

➢Definition:

A differential equation involving **partial derivatives with respect to more than one independent variables** is called **Partial Differential Equation**.

Ex.

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (Laplace equation) ← Elliptic PDE

2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (wave equation) → Hyperbolic PDE

3. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (heat equation) — Parabolic PDE.

The general form of PDE is

$$F\left(x, y, u(x,y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots\right) = 0$$

Applications of PDE

- In the real life situation, most of the problems consist of PDE. There are many applications in different areas in science and engineering like wave motion, heat transfer, drugs and oxygen transport in the human body, i.e., convective-diffusion models, etc..

Order of Differential Equation

➢ Definition:

The order of the **highest order derivative** involved in a differential equation is called the **Order of Differential Equation**.

Ex. 1 $y = x\frac{dy}{dx} - c\frac{dx}{dy}$

$\Rightarrow y = x\frac{dy}{dx} + \frac{c}{\frac{dy}{dx}} \Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$

This is first order and second degree ODE.

Degree of Differential Equation

➢Definition:

The power of the **highest order derivative** involved in a differential equation is called the **Degree of Differential Equation**.

Ex. (2) $\frac{d^2y}{dx^2} + x^2\cos y = 0 \Rightarrow$ 2nd order and 1st degree.

(3) $y = x\frac{dy}{dx} + a\left\{1 + \frac{d^2y}{dx^2}\right\}^{1/2}$

$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = a\left\{1 + \frac{d^2y}{dx^2}\right\}$

$\Rightarrow$ 2nd order and 1st degree

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Examples

④ $\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^5 + y = x^2 \longrightarrow$ Degree - 3 & order - 2

⑤ $\left\{y + x\left(\frac{dy}{dx}\right)^2\right\}^{4/3} = x\left(\frac{d^2y}{dx^2}\right) \Rightarrow$ order - 2 & degree - 3

⑥ $\left(\frac{d^2y}{dx^2}\right)^{1/3} = \left(y + \frac{dy}{dx}\right)^{1/2} \Rightarrow$ order - 2 & degree - 2

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^{2/3} = \left(y + \frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(y + \frac{dy}{dx}\right)^3$$

Linear Differential Equation

➢ Definition:

A differential equation is called **linear** if

(i) every dependent variable and every derivative occurs in the **first degree** only

and

(ii) **no products** of dependent variables and/or derivatives occur.

For linearity: only y, $\frac{dy}{dx}$ occurs but not in product form

no y^2, $\left(\frac{dy}{dx}\right)^2$, $y\frac{dy}{dx}$, $y^2\frac{dy}{dx}$, $y^2\left(\frac{dy}{dx}\right)^2$ occurs

Nonlinear: If the differential equation is not linear, then it is called nonlinear differential equation.

Examples

Ex. 1. $y = \sqrt{x}\,\frac{dy}{dx} + k\,\frac{dx}{dy}$

$\Rightarrow y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}}$

$\Rightarrow y\,\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + K$

$\Rightarrow$ order - 1

degree - 2

It is nonlinear eqn

Since the first order derivative has second degree.

(2) $y = x\,\frac{dy}{dx} + a\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}$ $\Rightarrow$ This is nonlinear

(3) $\left(\frac{d^2y}{dx^2}\right)^{1/3} = \left(y + \frac{dy}{dx}\right)^{1/2}$ $\Rightarrow$ Nonlinear

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Examples

(4) $\frac{dy}{dx} = 1 + y^2 \Rightarrow$ Nonlinear

(5) $\frac{\partial^2 u}{\partial t^2} = K\left(\frac{\partial^3 u}{\partial x^3}\right)^2 \Rightarrow$ nonlinear, order-3, degree-2

Note: Every linear equation is of first degree but every first degree equation may not be linear.

$$\boxed{\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0}$$ $\Rightarrow$ It is first degree but nonlinear.

Solution of Differential Equation

➢ Definition:

Any relation between the dependent and independent variables (no derivative terms) which **satisfies the differential equation** is called a **Solution or Integral of Differential Equation**.

Ex. (i) $y = ce^{2x}$ is the solution of $\frac{dy}{dx} = 2y$ — (1)

$\Rightarrow y = ce^{2x}$

$\Rightarrow \frac{dy}{dx} = 2ce^{2x}$,

$\frac{dy}{dx} - 2y = 2ce^{2x} - 2ce^{2x} = 0$

So, the solution of (1) satisfies the differential equation (1)

(ii) $y = A/x + B$ is a solution of $\frac{d^2y}{dx^2} + (2/x)\frac{dy}{dx} = 0$.

Family of Curves

➢Definition:

An n **parameter family of curves** is a set of relations of the form

$$\{(x, y): f(x, y, c_1, c_2, \dots, c_n) = 0\}$$

Ex: ① Set of Concentric circles defined by —

$x^2+y^2=c$ — one parameter family of circles if c takes all non-negative real values.

② Set of circles defined by —

$(x-c_1)^2+(y-c_2)^2=c_3$, is a three parameter family of circles if c_1 and c_2 takes all real values, and c_3 takes all non-negative real values.

Examples

Note: Solution of a differential equation is a family of curves.

Ex,

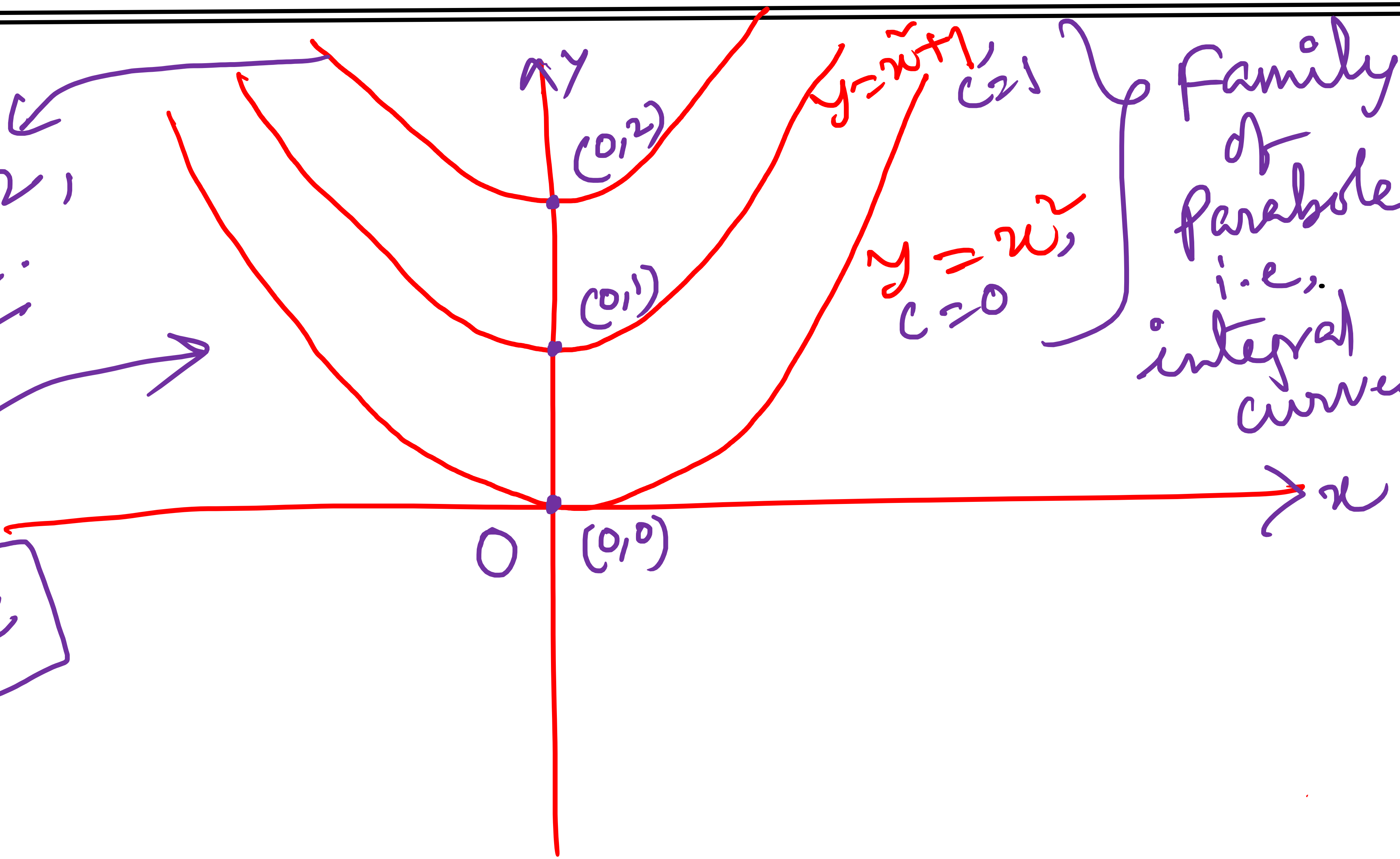
$$\frac{dy}{dx} = 2x \quad a, \quad y = x^2 + c$$

This Solution represents the family of parabola.

* The family of curves which are the solutions of the ODE are called as integral curves of ODE.

Examples

$y = x^2 + 2$, $c = 2$.



$\boxed{y = x^2 + c}$

Family of Parabola i.e., integral curves.

Formation of Differential Equation

- From a given family of curves containing n arbitrary constants, we can obtain n th order differential equation.

Ex. (1) Obtain the differential eqn satisfied by $xy = ae^{x} + be^{-x} + x^2$, where a & b are arbitrary constants.

$\Rightarrow$ $xy = ae^{x} + be^{-x} + x^2 \longrightarrow \underline{ae^{x} + be^{-x} = xy - x^2}$

Diff. w.r.t x, we have-

$$x\frac{dy}{dx} + y = ae^{x} - be^{-x} + 2x$$

or, $x\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = ae^{x} + be^{-x} + 2$ [diff. again w.r.t x]

or, $x\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} - xy + x^2 - 2 = 0$

This is the required differential eqn.

Examples

(2) Show that $Ax^2+By^2=1$ is the solution of

$$x\left[y\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2\right]=y\left(\frac{dy}{dx}\right)$$

(3) Find the differential eqn of the family of curves $y=e^x(A\cos x+B\sin x)$, where A and B are arbitrary constants.

Ans: $\boxed{y''-2y'+2y=0}$

General and Particular Solution

nth order differential equation ⇒ $F(x, y, y', y'', y''', \dots, y^n) = 0$ → This is general form. (1)

$y^{(n)} = \frac{d^n y}{dx^n}$, nth order derivative

➢Definition of General Solution:

A solution of (1) containing ***n* independent arbitrary constants** is called a **General Solution**.

➢Definition of Particular Solution:

A solution of (1) obtained from a **general solution** of (1) by giving particular values to one or more of the ***n* independent arbitrary constants** is called a **Particular Solution**.

Remark: General Solution of an nth order differential equation will contain n arbitrary Constants.

Examples

Example:

$$y' + xy = 0$$

$$\Rightarrow \boxed{y(x) = c\,e^{-x^2/2}},$$

where c is an arbitrary constant.

General Sol^n

The graphs of this eq^{n}s are called as solution curves, form a family of bell-shaped curves.

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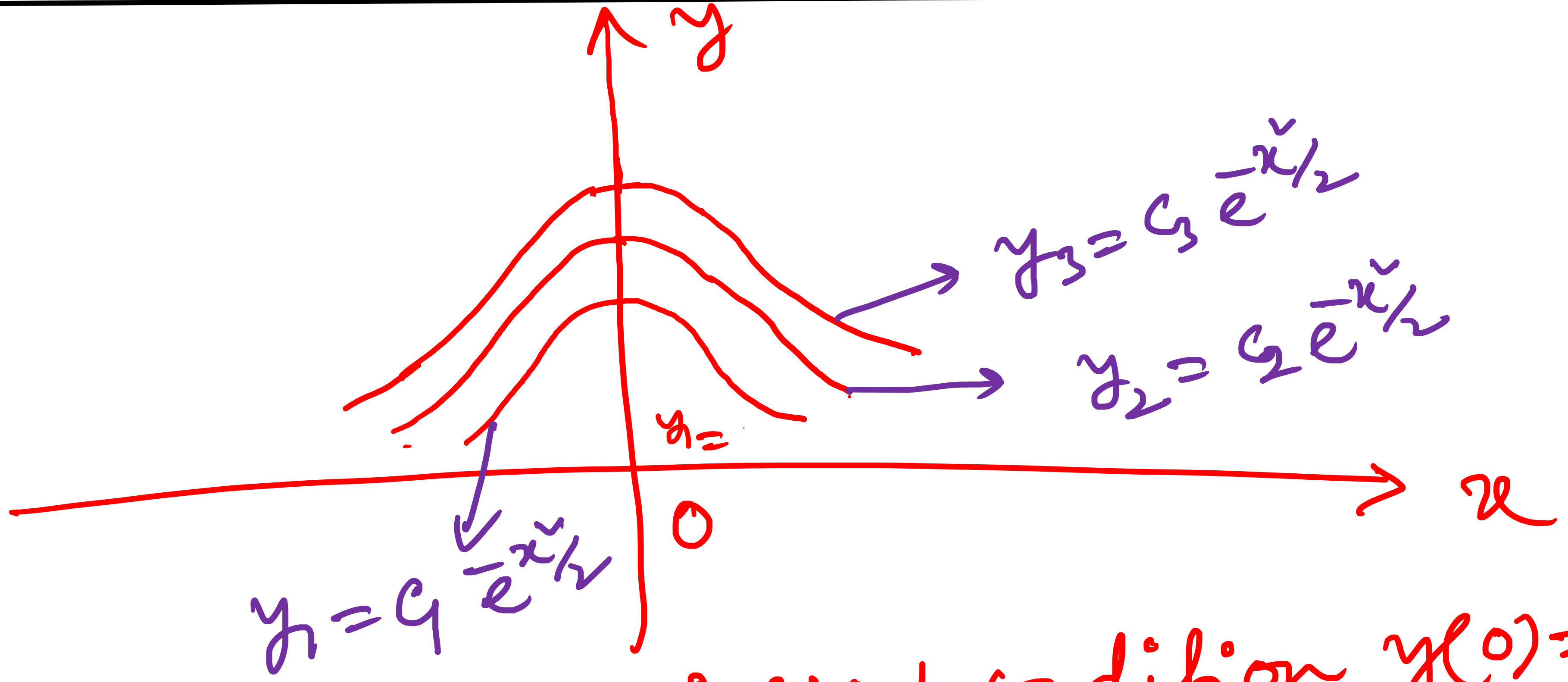
Examples



If we consider the initial condition $y(0)=3$, we have $c=3$. Therefore we will have the particular sol^n as $\boxed{y(x) = 3e^{-x^2/2}}$

Examples

Ex. 1 Consider $\left(\frac{dy}{dx}\right)^2 - 4y = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y$$

or, $\frac{dy}{dx} = \pm 2\sqrt{y} \Rightarrow \frac{dy}{2y^{1/2}} = \pm\, dx$

$$\Rightarrow y^{1/2} = \pm(x \pm c) \Rightarrow y = (x \pm c)^2$$

This is the general solution and it represents the one parameter family of Parabola.

Particular Solution: Let $c = 0$, then general solⁿ becomes —

$$y = x^2, \longrightarrow \text{Parabola}$$

This is the particular Solⁿ.

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Singular Solution

- **Definition:**

A solution of (1) which can not be obtained from **general solution of (1)** by any choice of the $\boldsymbol{n}$ **independent arbitrary constants** is called a **Singular Solution**.

Examples

Ex. Consider $yy' - xy'^2 = 1$

or, $y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 1$

Let $p = \frac{dy}{dx}$, or, $yp - xp^2 = 1$

or, $y = xp + \frac{1}{p}$ —— (1)

This is the Clairaut's equation and its general form is $\boxed{y = xp + f(p)}$

Diff. w.r.t x, we have

or, $\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$

or, $p = p + x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$

or, $\frac{dp}{dx}\left(x - \frac{1}{p^2}\right) = 0 \Rightarrow \frac{dp}{dx} = 0$ or, $x - \frac{1}{p^2} = 0$

$p = \text{Constant} = c$,

$p^2 = \frac{1}{x}$

or, $p = \pm\frac{1}{\sqrt{x}}$

Examples

Put $p = c$ in equation (1), then we have

$y = cx + \frac{1}{c}$, which is the general solution, It is one parameter family of straight lines

This general sol^n gives the particular sol^n by putting $c = 1$,

Particular Sol^n $\longrightarrow$ $y = x + 1$,

Singular Sol^n: $p^2 = \frac{1}{x}$ and $p = \pm\frac{1}{\sqrt{x}}$, put in (1)

$yp = xp^2 + 1 \Rightarrow yp = x \cdot \frac{1}{x} + 1 \Rightarrow yp = 2$

$\Rightarrow y = 2/p = \pm 2\sqrt{x} \Rightarrow y^2 = 4x$, Parabola

This is the Singular sol^n.