Quiz-1

1. Question

2 points

Suppose f and g are injective functions $[0,1] \to [0,1]$ with $g(x) \neq 0$ for all $x \in [0,1]$. Which of the following function(s) is injective on $[0,1]$?

Check all that apply.

e.g. if $f(x) = 1 - g(x)$, then $(f+g)(x) = 1 \ \forall x$

$f + g$

in which case $f+g$ is not injective

Option 1

e.g. $f(x) = 1 - x/2$

$g(x) = \frac{1+x}{2}$

$f \cdot g$

If $h(x) = f(x)g(x)$, then $h(0) = \frac{1}{2} = h(1)$, hence h is not injective

Option 2

e.g. if $f(x) = g(x) \ \forall x$, then $f(x)/g(x) = 1 \ \forall x$

f/g

whence f/g is not injective.

Option 3

$f \circ g$

Option 4

2. Question 2 points

Let X be a *finite* set, and $f : X \to X$ be a function. Pick the correct alternative(s).
Check all that apply.

Always true!

$f -$ bijective $\implies f -$ injective

- [x] Option 1

f-inj & X finite
=> |f(X)| = |X|
=> f(X) = X

$f -$ injective $\implies f -$ bijective

=> f-surjective, and hence f-bijective

- [x] Option 2

f-surjective
=> f(X) = X
=> |f(X)| = |X|

$f -$ surjective $\implies f -$ injective

=> f-injective, since X is finite.

- [x] Option 3

None of the above

- [ ] Option 4

3. Question

2 points

Let X be an *infinite* set, and $f : X \to X$ be a function. Pick the correct alternative(s).

Check all that apply.

Always true!

$f -$ bijective $\implies$ $f -$ injective

☑ Option 1

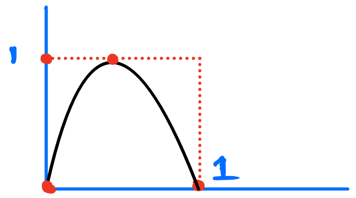
False, e.g. take $f : [0,1] \to [0,1]$
$f(x) = \frac{x+1}{2}$

$f -$ injective $\implies$ $f -$ bijective

Then f - injective but not surjective

☒ Option 2

simpler solution



$f -$ surjective $\implies$ $f -$ injective

solution with a specific example

$f(x) = 4x(1-x)$

By A.P $\geq$ GP, $0 \leq f(x) \leq 1$

If $y \in [0,1]$, the $y = 4x(1-x)$ has a solution

since gives $4x^2 - 4x + y = 0$

discriminant $= 16(1-y) \geq 0$,

hence has a solution.

☒ Option 3

None of the above

☒ Option 4

4. Question

2 points

There is a bijective function $f : (0,1) \to (0,\infty)$

Mark only one oval.

- ☑ True
- ◯ False

$$\begin{tikzcd}
(0,1) \arrow[r, "1/x"', "\text{bijective}"] \arrow[rr, bend right, "1/x - 1"'] & (1,\infty) \arrow[r, "x-1"', "\text{bijective}"] & (0,\infty)
\end{tikzcd}$$

hence, composition bijective

5. Question 2 points

Suppose $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(m+n) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}$, and $f(3) \neq 0$. Then f is injective.

Mark only one oval.

- [x] True
- [ ] False

• $f(0) = f(0+0) = f(0) + f(0)$
$\Rightarrow f(0) = 0$
• $0 = f(0) = f(1-1) = f(1) + f(-1)$
$\Rightarrow f(-1) = -f(1)$.
• Thus, if $n \in \mathbb{N}$, then $f(n) = f(\underbrace{1+1+\cdots}_{n\text{-times}}) = n f(1)$
and $f(-n) = f(\underbrace{-1-1-\cdots-1}_{n\text{-times}}) = n f(-1) = -n f(1)$.
• Now, $f(m) = f(n) \Rightarrow f(m-n) = 0$
$\Rightarrow (m-n) f(1) = 0 \Rightarrow m = n$ since $0 \neq f(3) = 3f(1) \Rightarrow f(1) \neq 0$.

6. Question 2 points

Suppose $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(m+n) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}$, and $f(3) \neq 0$. Then f is surjective.

Mark only one oval.

- [ ] True
- [x] False

$f : \mathbb{Z} \to \mathbb{Z}$, given by $f(m) = 2m$, then
• $f(m+n) = 2(m+n) = 2m + 2n = f(m) + f(n)$
• $f(3) = 6 \neq 0$
But f is not surjective.

7. Question 2 points

Suppose I and J are nonempty intervals. Then possible value(s) of $|I \cap J|$ are

Check all that apply.

- [x] 0 ← $I = (0,1)$, $J = (2,3)$
- [x] 1 ← $I = (0,1]$, $J = [1,2)$
- [ ] 2 ✗
- [x] Infinity ← $I = (0,1)$, $J = (\frac{1}{2}, 2)$
Then $I \cap J = (\frac{1}{2}, 1)$.

If $I \cap J$ is finite then $|I \cap J| = 0$ or 1.
since if $a, b \in I \cap J$ with $a < b$,
then $(a,b) \subseteq I \cap J$.
$\Rightarrow |I \cap J| = \infty$.

8. Question

2 points

Suppose $f : X \to X$ is injective and $g : X \to X$ is bijective. Pick the correct alternative(s).

Check all that apply.

$g \circ f$ is injective

since both injective

- [x] Option 1

$g \circ f$ is surjective

e.g. if g is identity and f is not surj., then $g \circ f = f$, hence not surj.

- [ ] Option 2

$g \circ f$ is bijective

follows from

- [ ] Option 3

$f \circ g$ is injective

since both injective

- [x] Option 4

9. Question

2 points

Let x and y be arbitrary rationals, and z and w be arbitrary irrationals. Pick the correct alternative(s).

Check all that apply.

$z + w$ is irrational

$z = 1+\sqrt{2}$, $w = 1-\sqrt{2}$, then $z+w = 2 \in \mathbb{Q}$

- [x] Option 1

yz is irrational

$y=0$, $z=\sqrt{2}$, then $yz = 0 \in \mathbb{Q}$.

- [x] Option 2

zw is rational

$z=\sqrt{2}$, $w=\sqrt{3}$, then $zw = \sqrt{6} \leftarrow$ irrational

- [x] Option 3

$x + w$ is irrational

if $x+w = q \in \mathbb{Q}$, then $w = q - x \in \mathbb{Q}$, which is impossible

- [x] Option 4

10. Question

2 points

Let $n_1 < n_2 < \cdots$ be an arbitrary sequence of natural numbers. Which of the following statements may *not* be correct?

Check all that apply.

By AP, $\exists\ n \in \mathbb{N}$ s.t. $nx > 1$.
Since $n_k \to \infty$ with k,

$\exists$ a k s.t. $n_k x > 1$

$\exists\ n_k \geq n$ so that
$n_k x \geq nx > 1$

- [x] Option 1

e.g. consider the sequence
$1 < 2 < 3 < \cdots$

$\exists$ a k s.t. $n_k \leq x < n_{k+1}$

and $x = \frac{1}{2}$.
no element of the sequence is $\leq \frac{1}{2}$.

- [x] Option 2

By AP, $\exists\ n \in \mathbb{N}$
s.t. $n \cdot 1 > x$

$\exists$ a k s.t. $x < n_k$

Now, can find an appropriate n_k

- [x] Option 3

By ①, $\exists\ n_k$ s.t.
$n_k x > 1$
$\Rightarrow x > 1/n_k$

$\exists$ a k s.t. $x > 1/n_k$

- [x] Option 4