भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

# 9th Lecture on ODE

(MA-1150)

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# What have we learnt so far?

- Linear Second and Higher order Ordinary Differential Equation
- Existence and uniqueness theorem
- Linear independence and dependence
- Wronskian
- Some fundamental theorems

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# Today's Class

- Some extra concept on Wronskian
- Homogeneous linear ode with constant coefficients

Now see some important concept on Wronskian of functions $f_1(x), f_2(x), \dots, f_n(x)$ :

The functions $f_1(x), f_2(x), \dots, f_n(x)$ are linearly dependent $\Rightarrow W(f_1, f_2 \dots, f_n; x) = 0, \forall x \in I$

But converse is not true, i.e, $W(x)=0$ does not imply that the functions

$f_1(x), f_2(x), \ldots, f_n(x)$ are linearly dependent.

please see the problems.

**Ex:** Show that $y_1(x) = x^2$ and $y_2(x) = x|x|$ are linearly independent on $-\infty < x < \infty$, however, wronskian vanishes for every real values of $x$.

**Ans:** Consider $c_1 y_1 + c_2 y_2 = 0$, $x \in R$.

we have to show that

$$c_1 = c_2 = 0.$$

For $x \geq 0$, $c_1 x^2 + c_2 x^2 = 0$

$$\Rightarrow \boxed{c_1 + c_2 = 0} \quad \text{--- (1)}$$

For $x < 0$, $c_1 x^2 - c_2 x^2 = 0$

$$\Rightarrow \boxed{c_1 - c_2 = 0} \quad \text{--- (2)}$$

From (1) and (2) we have,

$$c_1 = c_2 = 0.$$

which shows that $y_1(x)$ and $y_2(x)$ are linearly independent functions on $\mathbb{R}$

**Hoewever**, the wronskian for $x \geq 0$

$$W(y_1, y_2; x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0,$$

for $x < 0$, $W(y_1, y_2; x) \underline{= 0.}$

**Ex:** Show that $x^2$ and $x|x|$ are linearly dependent on $[-1,0]$ and $[0,1]$, respectively.

**Sol$^n$:** **Case I:** Let $x \in [-1,0]$. Then $x|x| = -x^2$. Hence we have

$$\boxed{c_1 x^2 + c_2 x|x| = 0}$$ when $c_1 = 1$ and $c_2 = 1$

It is showing that two functions $x^2$ and $x|x|$ are linearly dependent on $[-1,0]$

because there exists constants $c_1$ and $c_2$ not both zero such that

$$c_1 x^2 + c_2 x|x| = 0 \text{ for all } x \text{ in } [-1,0].$$

Case II: Let $x \in [0,1]$. Then $x|x| = x^2$.

Hence we have $c_1' x^2 + c_2' x|x| = 0$ when

$$c_1' = 1 \text{ and } c_2' = -1$$

It is showing that two functions $x^2$ and $x|x|$ are linearly dependent on $[0,1]$ because there exists constants $c_1'$

and $c_2'$ not both zero such that

$$c_1' x^2 + c_2' x|x| = 0 \text{ for all } x \text{ in } [0,1].$$

**Ex:** Show that the functions $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the real line, even though the wronskian cannot be computed.

(Try it) (Home work)

Note: The wronskian of $n$ functions exists if all the functions $y_1, y_2, \ldots, y_n$ are differentiable $(n-1)$ times on the interval $I$. If any one or more functions are not differentiable then the wronskian does not exist.

**Theorem:** Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly independent solution of $L[y]=0$ on an interval $I$. Then every solution of $L[y]=0$ can be expressed uniquely as

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)} \quad \forall x \in I.$$

where $c_1, c_2, \ldots,$ and $c_n$ are constants.

This is known as general solution of linear homogeneous ODE $L[y]=0$.

**Proof:** Let $y(x)$ be any solution of $L[y]=0$ on $I$. At a point $x_0 \in I$, we calculate $y, y', \dots, y^{(n-1)}$ to obtain

$$y(x_0)=b_1,\ y'(x_0)=b_2, \dots\ y^{(n-1)}(x_0)=b_n. \quad \text{—} (1)$$

Let us assume that $y(x)$ can be expressed as a linear combination of $y_1, y_2, \dots, y_n$,

$$y(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \text{ for some constants } c_1, c_2, \dots, c_n.$$

Then from (1) we have

$$c_1 y_1(x_0) + c_2 y_2(x_0) + \cdots + c_n y_n(x_0) = b_1$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) + \cdots + c_n y_n'(x_0) = b_2$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = b_n$$

Since the wronskian $W(x_0) \neq 0$ for linearly independent solutions $y_1, \ldots, y_n$ at $x_0$, $c_1, c_2 \ldots$ and $c_n$ can be determined uniquely.

Choosing such $c_1, c_2, \ldots$ and $c_n$, we define the function $\psi(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$.

Then we have —

$$\left.\begin{aligned} &L(\psi(x)) = 0 \\ \text{with } &\psi(x_0) = b_1 = y(x_0), \\ &\psi'(x_0) = b_2 = y'(x_0), \\ &\cdots \qquad \cdots \qquad \cdots \\ &\psi^{(n-1)}(x_0) = b_n = y^{(n-1)}(x_0) \end{aligned}\right\} \text{ — } (2)$$

Thus $\psi(x)$ and $y(x)$ are both solutions of $L[y]=0$ on $I$ and they satisfy the same initial conditions at $x_0$. By uniqueness theorem, $y(x)$ must be identically equal to $\psi(x)$ on $I$.

Hence $\boxed{y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)}$

(Proved)

**Note:** The $n$ linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ are called the fundamental solutions of $L[y] = 0$ on I. This set of fundamental solutions forms a basis of the $n$th order linear homogeneous eqn.

This family of solutions $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ which contains all linearly independent solutions, is called the general solution of $L[y] = 0$.

## Differential operator D:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

$$\boxed{\text{operator } D \equiv \frac{d}{dx}}$$

$$D^2 \equiv \frac{d^2}{dx^2}.$$

or, $a_0(x)D^2y + a_1(x)Dy + a_2(x)y = 0$ ,

or, $(a_0D^2 + a_1D + a_2)y = 0$

$\Rightarrow$ $\boxed{(D-m_1)(D-m_2)y = 0.}$ where $m_1$ and $m_2$ are the roots of the given eqn.

For nth order linear ODE,

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

operator, $D^n \equiv \frac{d^n}{dx^n}$

or, $(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0$

or, $\boxed{(D-m_1)(D-m_2)\cdots(D-m_n)y = 0}$ , $m_1, m_2, \ldots m_n$ are $n$ roots of the given eqn

## Solution of second order linear homogeneous eqⁿs with constant coefficients:

Consider the linear homogeneous second order eqⁿ

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$$

, where a, b, c are constants

To solve this eqⁿ, let us consider $y = Ae^{mx}$ be a trial solution, where $A \neq 0$.

Then we have $\frac{dy}{dx} = Ame^{mx}$, and $\frac{d^2y}{dx^2} = Am^2e^{mx}$

Now from the given eqⁿ, we have

$$(am^2 + bm + c)Ae^{mx} = 0$$

$$\boxed{am^2 + bm + c = 0}, \text{ since } Ae^{mx} \neq 0$$

This is an algebraic eqn in m. It is called the characteristic eqn or the auxiliary eqn of the linear homogeneous eqn.

The roots of this equation are called the characteristic roots. The quadratic eqn has the roots

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Now we have the following three cases -

(i) The roots are real and distinct, $m = m_1, m_2$; $m_1 \neq m_2$ if $\underline{b^2 - 4ac > 0}$

(ii) The roots are real and equal say $m = m_1, m_1$ if $b^2-4ac=0$

(iii) The roots are complex if $b^2-4ac<0$.

Case-I: Real and distinct: (Non-repeated roots)

$$m = \frac{-b+\sqrt{b^2-4ac}}{2a} = m_1$$

$$m = \frac{-b-\sqrt{b^2-4ac}}{2a} = m_2$$

So, we can obtain two solutions $e^{m_1 x}$ and $e^{m_2 x}$

The two solutions are linearly independent on any interval I, since the wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$$

Hence the general solution is the linear combination of two linearly independent solutions

i.e, $\boxed{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

(*) See the procedure for getting the above solution

we have 2nd order linear ODE –

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

or, $(aD^2+bD+c)y=0$

or, $(D-m_1)(D-m_2)\,y=0$, Since $m_1$ and $m_2$ are two real and distinct roots.

Now let $(D-m_2)y=z$.

Then $(D-m_1)z=0$

$\Rightarrow Dz-m_1z=0$

or $\dfrac{dz}{dx}-m_1z=0$

or $\dfrac{dz}{z}=m_1\,dx$

or $\log|z|=m_1x+\log|c|$

or, $\log|z/c|=m_1x$ $\Rightarrow$ $\boxed{z=ce^{m_1x}}$

$c$ is an arbitrary constant.

Now $z = c\,e^{m_1 x}$

and we have $(D - m_2)y = z = ce^{m_1 x}$

or $\dfrac{dy}{dx} - m_2 y = ce^{m_1 x}$

This is linear eqⁿ in y.

Now IF $= e^{-\int m_2 dx} = e^{-m_2 x}$

Multiplying by IF, we have

$$e^{-m_2 x}\left(\frac{dy}{dx} - m_2 y\right) = ce^{m_1 x - m_2 x} = ce^{(m_1 - m_2)x}$$

or $d\left(y \cdot e^{-m_2 x}\right) = c\,e^{(m_1 - m_2)x} dx$

Integrating, $\Rightarrow y e^{-m_2 x} = \dfrac{c}{(m_1 - m_2)} e^{(m_1 - m_2)x} + c_2$

as $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, where

$$c_1 = \frac{c}{m_1 - m_2}.$$

So, the general solution of the 2nd order linear homogeneous eq$^n$ is

$$\boxed{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}},$$ when

$m_1$ and $m_2$ are two real and distinct roots.

This is the procedure for getting the general solution.

Example:

$y'' - y' - 6y = 0$

or $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

or $(D^2 - D - 6)y = 0$.

i.e. $f(D)y = 0$ where $f(D) = (D^2 - D - 6)$

Now the auxiliary or characteristic eq$^n$ is obtained

as $f(m) = 0$

or $m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0$

$m = -2, 3$

Two solution are obtained as $e^{-2x}$ and $e^{3x}$

and these solutions are linearly independent

since $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix}$

$= 3e^{-2x}e^{3x} + 2e^{-2x}e^{3x}$

$= 3e^{x} + 2e^{x} = 5e^{x} \; (\neq 0)$

Now the general solution is

$$\boxed{y(x) = Ae^{-2x} + Be^{3x}}$$

Ex. $4y'' - 8y' + 3y = 0$, $y(0) = 1$, $y'(0) = 3$

$4y'' - 8y' + 3y = 0$

or $4\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 3y = 0$

or $(4D^2 - 8D + 3)y = 0$

or $f(D)y = 0$, where $f(D) = 4D^2 - 8D + 3$.

The auxiliary eqn is $f(m) = 0$

or $4m^2 - 8m + 3 = 0$

or $m = \frac{1}{2}, \frac{3}{2}$

Two Solutions are obtain as $e^{x/2}$ and $e^{3x/2}$ and these Solutions are linearly independent since

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x/2} & e^{3x/2} \\ \frac{1}{2}e^{x/2} & 3/2\, e^{3x/2} \end{vmatrix}$$

$$= \frac{3}{2}e^{2x} - \frac{1}{2}e^{2x} = e^{2x} (\neq 0)$$

Now the general solution is

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{x/2} + c_2 e^{3x/2}$$

where $c_1$ and $c_2$ are arbitrary constants.

we have the initial conditions —

$y(0)=1$ and $y'(0)=3.$

$y(0)=1 \Rightarrow y(0)=1=c_1+c_2 \Rightarrow c_1+c_2=1$ or $c_1=1-c_2$

$$y'(x)=\frac{1}{2}c_1e^{x/2}+\frac{3}{2}c_2e^{3x/2}$$

$y'(0)=3$ gives, $y'(0)=3=\frac{1}{2}c_1+\frac{3}{2}c_2$

or $c_1+3c_2=6$

or $1+2c_2=6$ or $c_2=\frac{5}{2}$

$c_1=-\frac{3}{2}$

Now the solution of the initial value problem is

$$y(x) = -\frac{3}{2}e^{x/2} + \frac{5}{2}e^{3x/2}$$

## Case-II : Real and equal roots : (Repeated roots)

The Second order linear homogeneous ODE is given by

$$a y'' + b y' + c y = 0$$

$$\Rightarrow (aD^2 + bD + c) y = 0$$

The auxiliary eqn is $am^2 + bm + c = 0$

$$\Rightarrow m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let roots be equal, so, $b^2 - 4ac = 0$.

Now $m = -b/2a, -b/2a$

Say $\alpha = -b/2a.$

$m = \alpha$. is the repeated root.

Now we have $y_1(x) = e^{\alpha x}$, and $y_2(x) = xe^{\alpha x}$

as two linearly independent Solutions.

So, the general Solution is

$$\boxed{y(x) = (c_1 + c_2 x) e^{\alpha x}}$$

$(aD^2+bD+c)y=0$ when $\alpha$ is the repeated root.

or $(D-\alpha)(D-\alpha)y=0$

This is the procedure for getting the general solution

Let $Z=(D-\alpha)y$

So, we have $(D-\alpha)Z=0$

or $DZ-\alpha Z=0$

or $\frac{dz}{dx}-\alpha z=0$

$\Rightarrow z=c_1 e^{\alpha x}$.

Now, $(D-\alpha)y=c_1 e^{\alpha x}$

or $\frac{dy}{dx} - \alpha y = c_1 e^{\alpha x}$

which is linear nonhomogeneous first order ODE.

$$IF = e^{-\int \alpha dx} = e^{-\alpha x}$$

Multiplying IF, we have

$$\frac{d}{dx}\left(e^{-\alpha x} \cdot y\right) = c_1 e^{\alpha x} \cdot e^{-\alpha x} = c_1$$

Integrating, $y e^{-\alpha x} = \int c_1 dx + c_2$

or $y e^{-\alpha x} = c_1 x + c_2$

$y = c_1 x e^{\alpha x} + c_2 e^{\alpha x}$

$\Rightarrow$ or $\boxed{y = (c_2 + c_1 x) e^{\alpha x}.}$ This is the general solution of the second order linear homogeneous ODE $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

where $\alpha$ is the repeated root of the auxiliary eqn $am^2 + bm + c = 0.$

When we have $r$ number of equal or repeated roots say $\alpha$, then the general solution is obtained as

$$\boxed{y(x) = \left(c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \cdots + c_r x^{r-1}\right) e^{\alpha x}}$$

for $r$ number of equal roots $\alpha$.

**Example:** $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$, $y(0) = 2$, $y'(0) = 3$.

$\Rightarrow (D^2 + 6D + 9)y = 0 \longrightarrow f(D)y = 0$, where $f(D) = D^2 + 6D + 9$

Now the auxiliary eq$^n$ is

$$f(m)=0 \Rightarrow m^2+6m+9=0$$

$$\Rightarrow m=-3,-3.$$

The general sol$^n$ is

$$y(x)=(c_1+c_2x)e^{-3x}$$

Here $e^{-3x}$ and $xe^{-3x}$ are two linearly independent solutions, $W(e^{-3x}, xe^{-3x})\neq 0$

$y(0)=2$ gives

$c_1=2$,

$$y(x)=(c_1+c_2x)e^{-3x} \Rightarrow \frac{dy}{dx}=c_2e^{-3x}-3(c_1+c_2x)e^{-3x}$$

or, $y'=c_2e^{-3x}-3(c_1+c_2x)e^{-3x}$

$y'(0) = 3$ gives

$c_2 - 3c_1 = 3$ or $c_2 = 3 + 3c_1 = 3 + 3.2$

$= 9$

$\therefore c_1 = 2, \; c_2 = 9.$

Now the solution to the given IVP is

$$\boxed{y(x) = (2 + 9x)e^{-3x}}$$

## Case-III : Complex Roots :

The 2nd order linear homogeneous ODE is

$$(aD^2+bD+c)y=0.$$

The auxiliary eqn is $am^2+bm+c=0$

$$\Rightarrow m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

When $b^2-4ac<0$, $m = -b/2a \pm i\,\frac{\sqrt{4ac-b^2}}{2a}$

$= \alpha \pm i\beta$. , where $\alpha = -b/2a$

$$\beta = \frac{\sqrt{4ac-b^2}}{2a}$$

Therefore the general solution is

$$= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$, where $c_1$ and $c_2$ are arbitrary constants.

$$= c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$$

$$= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right]$$

We have

$$e^{i\beta x} = \cos\beta x + i\sin\beta x$$

$$e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right],$$ where $A = c_1+c_2$, $B = i(c_1-c_2)$

This is the general solⁿ, when the roots of the A.E. are complex $\alpha+i\beta$, and $\alpha-i\beta$.

Here, we can easily see,
the two linearly independent solutions are
$y_1 = e^{\alpha x}\cos\beta x$ and $y_2 = e^{\alpha x}\sin\beta x$

The wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$

$= \beta e^{2\alpha x} (\neq 0)$.

Thats why the general solⁿ is
$y(x) = c_1 y_1 + c_2 y_2 = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$.

(iii) To find this solution you follow the same procedure as Case - I.

$$\Rightarrow (aD^2+bD+c)y=0$$

$$\text{or } (D-m_1)(D-m_2)y=0$$

since
$m_1 = \alpha + i\beta$
$m_2 = \alpha - i\beta$.

Follow the case I ; Let $z = (D-m_2)y$.

we can obtain $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right].$$

**Example :**

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$$

or $(D^2 - 6D + 13)y = 0$

or $f(D)\,y = 0$, where

$f(D) = D^2 - 6D + 13$

Now the auxiliary eqn (A.E.) is

$f(m) = 0$

or $m^2 - 6m + 13 = 0$

$m = 3 \pm 2i$.

Then the general solution is

$$= c_1 e^{(3+2i)x} + c_2 e^{(3-2i)x}$$

$$= e^{3x}\left(c_1 e^{2ix} + c_2 e^{-2ix}\right)$$

$$= e^{3x}\left[A\cos 2x + B\sin 2x\right]$$

$$\boxed{\begin{array}{l} e^{2ix} \rightarrow \cos 2x \\ e^{-2ix} \rightarrow \sin 2x. \end{array}}$$