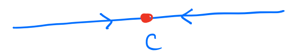
Limits

$$\boxed{\lim_{x\to c} f(x) = \ell}$$



x approaching continuously to c

What does it mean?

A sequential definition

Let us try to understand the meaning of

$$\lim_{x\to 0} f(x) = \ell$$ via a sequential approach

Consider $$\boxed{\lim_{x\to 0} (x^2+1)}$$

$\lim_{n\to 0} f(x_n) = \ell$ for every sequence $x_n \to 0$

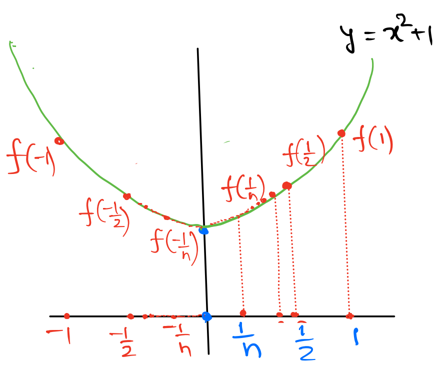
We say that the Right-Hand-limit

$$\lim_{x\to 0^+} (x^2+1) \text{ exists}$$

if $$\lim_{n\to\infty} \left(\frac{1}{n^2}+1\right) \text{ exists.}$$

$f(x) = x$ if $x = \frac{1}{n}$

$f(x) = 1$ o.w.



It is clear that $$\lim_{n\to\infty} \left(\frac{1}{n^2}+1\right) = 1$$

$$\rightarrow \lim_{\frac{1}{n}\to 0} \left(\frac{1}{n^2}+1\right) = 1$$

Thus, given $\varepsilon > 0$, $\exists\ n_0$ s.t.

$$\left|\frac{1}{n^2}+1-1\right| = \frac{1}{n^2} < \varepsilon \quad \forall\ n \geq n_0$$

In fact, can take $n_0 = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$

Thus, $n > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \left|\left(1+\frac{1}{n^2}\right)-1\right| < \varepsilon$

$\frac{1}{n} < \frac{1}{1/\sqrt{\varepsilon}} \Rightarrow \left|\left(1+\frac{1}{n^2}\right)-1\right| < \varepsilon$

$\delta = \sqrt{\varepsilon}$

Rephrasing: for every $\varepsilon > 0$ $\exists$ some $\delta > 0$

s.t. $\frac{1}{n} < \delta \Rightarrow \left|\left(1+\frac{1}{n^2}\right)-1\right| < \varepsilon$

Thus, for every $\varepsilon > 0$ $\exists$ some $\delta > 0$

Domain value $< \delta$

$\Rightarrow$ |Range value $- 1| < \varepsilon$

<u>Definition</u>. $\boxed{\lim_{x \to 0^+} f(x) = \ell_+}$ if for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t.

$0 < x < \delta \Rightarrow |f(x) - \ell_+| < \varepsilon$.

<u>Similarly</u> $\boxed{\lim_{x \to 0^-} f(x) = \ell_-}$ (left-hand-limit)

if for every $\varepsilon > 0$ $\exists$ a $\delta > 0$

s.t

$-\delta < x < 0 \Rightarrow |f(x) - \ell_-| < \varepsilon$

Defn. of a Limit.

$\boxed{\lim_{x\to 0} f(x) = \ell}$ if $\ell_+ = \ell_-$

(then $\ell = \ell_+ = \ell_-$)

Or $\boxed{\lim_{x\to 0} f(x) = \ell}$ if for every $\varepsilon > 0$

$\exists$ a $\delta > 0$ s.t.

$$|x| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

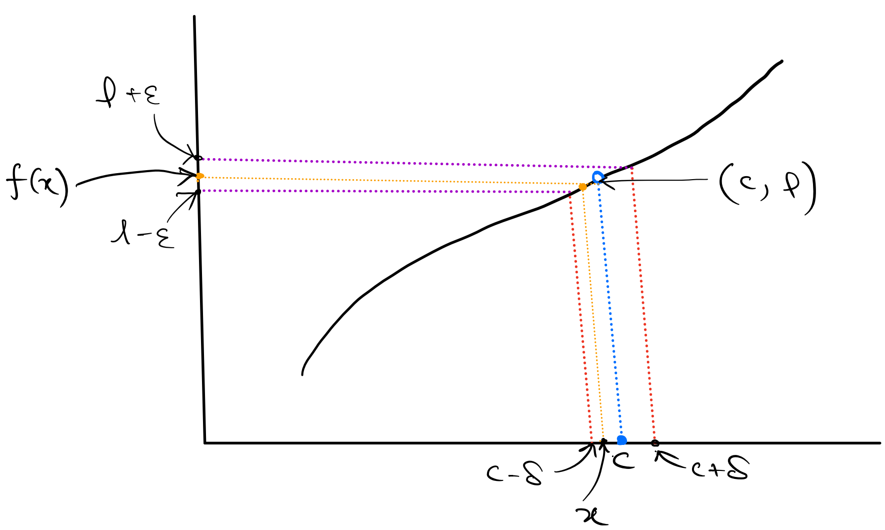
If 0 is replaced by c, Then

$\boxed{\lim_{x\to c} f(x) = \ell}$ if for every $\varepsilon > 0$

$\exists$ a $\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

$\frac{\sin x}{x}$



Examples.

- $\lim_{x \to 2} x^2 = 4$

To show: given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|x-2| < \delta \Rightarrow |x^2-4| < \varepsilon$$

When is $|x^2-4| < \varepsilon$?

$$|x^2-4| = |x-2|\,|x+2|$$

Assume $0 < \varepsilon < 1$

Claim: If $|x^2-4| < \varepsilon$, then

$$|x+2| \leq 5$$

Else, $|x+2| > 5 \Rightarrow |x| > 3$

$\Rightarrow x^2 - 4 > 5 > \varepsilon$, a contradiction

Thus, $|x+2| \leq 5$.

Next, $|x^2-4| = |x-2|\,|x+2|$

$\leq 5|x-2| < \varepsilon$

whenever $|x-2| < \varepsilon/5$.

Thus, $|x-2| < \varepsilon/5 \Rightarrow |x^2-4| < \varepsilon$

$$\delta = \varepsilon/5.$$

- $f(x) = \frac{|x|}{x}$, $x \neq 0$.

$\lim_{x \to 0^+} f(x) = 1$, $\lim_{x \to 0^-} f(x) = -1$

and $\lim_{x \to 0} f(x)$ does not exists.

Showing $\lim_{x \to 0^+} f(x) = 1$

To show: given $\varepsilon > 0$, $\exists$ δ s.t.

$0 < x < \delta \Rightarrow \left| \frac{|x|}{x} - 1 \right| < \varepsilon$

But if $x > 0$, then

$\left| \frac{|x|}{x} - 1 \right| = 0 < \varepsilon$

So, <u>any</u> $\delta > 0$ would work!

Similarly, for $\lim_{x \to 0^-} f(x) = -1$,

need to show that given $\varepsilon > 0$

$\exists$ a $\delta > 0$ s.t.

$-\delta < x < 0 \Rightarrow \left| \frac{|x|}{x} + 1 \right| < \varepsilon$

But, $x < 0 \Rightarrow \left| \frac{|x|}{x} + 1 \right| = 0$

So, any $\delta > 0$ would work.

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

To show: given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.
$0 < |x| < \delta \Rightarrow \left| \frac{\sin x}{x} - 1 \right| < \varepsilon$.

Once again, we seek to know
What range $|x| < \delta$ guarantees that
$\left| \frac{\sin x}{x} - 1 \right| < \varepsilon$?

$2^2 \leq 2^5 \leq 5!$
$2^3 \leq 2^7 \leq 7!$

$$\left| \frac{\sin x}{x} - 1 \right| = \left| -\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right|$$
$$\leq \frac{|x|^2}{2} \left(1 + \frac{|x|^2}{2} + \frac{|x|^4}{2^2} + \cdots \right)$$
$$= \frac{|x|^2}{2 - |x^2|} \quad \left(\text{can take } |x| < \sqrt{2} \right)$$
$$< \varepsilon$$

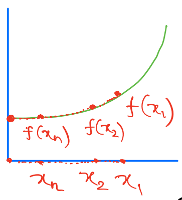
provided, $|x| < \sqrt{\frac{2\varepsilon}{1+\varepsilon}} := \delta$

Thus, $|x| < \sqrt{\frac{2\varepsilon}{1+\varepsilon}} \Rightarrow \left| \frac{\sin x}{x} - 1 \right| < \varepsilon$.

Thm. A Suppose $\lim_{x\to c} f(x) = \ell$.

Then for <u>every</u> sequence $x_n \to c$,

one has $f(x_n) \to \ell$.



Proof. To show: given $\varepsilon > 0$ $\exists$ n_0 s.t.

$$|f(x_n) - \ell| < \varepsilon \quad \forall \ n \geq n_0$$

Now

$$\lim_{x\to c} f(x) = \ell$$

$\Rightarrow$ given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon. \quad - (*)$$

Since $x_n \to c$, $\exists$ n_0 s.t.

$$|x_n - c| < \delta \quad \forall \ n \geq n_0$$

But then

$$(*) \Rightarrow |f(x_n) - \ell| < \varepsilon \quad \forall \ n \geq n_0,$$

as required.

Thm. B Conversely, if $f(x_n) \to \ell$ for every $x_n \to c$, then

$$\lim_{x \to c} f(x) = \ell.$$

Proof. To show: given $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

Prove by contradiction

Assume on the contrary that

$$\lim_{x \to c} f(x) \neq \ell.$$

There is some $\varepsilon > 0$ for which there is no $\delta > 0$

s.t. $|x - c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$

$\Rightarrow$ There is some $\varepsilon > 0$ s.t. for every $\delta > 0$ there is some x satisfying

$$|x - c| < \delta$$

but $|f(x) - \ell| \geq \varepsilon$

Taking $\delta = 1, \frac{1}{2}, \frac{1}{3}, \dots$, we get

$\exists\ x_n$ satisfying

$$|x_n - c| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - \ell| \geq \varepsilon$$

Next observe that $x_n \to c$

but $f(x_n) \not\to \ell$

because the ε-nbhd. $(\ell - \varepsilon, \ell + \varepsilon)$ of ℓ does not contain any x_n.

This contradicts the hypothesis of the Thm.

Therefore, $\lim_{x \to c} f(x) = \ell$.

Corollary. The limit of a function is unique.

(i.e. cannot have $\lim_{x\to c} f(x) = \begin{cases} \ell \\ \ell' \end{cases}$)

Proof. Set $x_n = c + \frac{1}{n}$

Then $x_n \to c$

By the Thm.

$$f(x_n) \to \lim_{x\to c} f(x)$$

Since the limit of a sequence is unique, the result follows.

Sandwich Thm for limits

Let f, g, and h be functions all defined on a deleted nbhd. I of c such that

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in I.$$

If $\lim_{x\to c} f(x) = \ell = \lim_{x\to c} h(x)$,

then $\lim_{x\to c} g(x) = \ell$.

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

but $\frac{\sin x}{x}$ is not defined at $x=0$

Proof. Follows from the Sandwich Thm. of sequences.

Application.

- Consider the Dirichlet's Function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Does $\lim_{x \to 0} f(x)$ exists?

Suppose $\lim_{x \to 0} f(x) = \ell$

$\{\frac{1}{n}\}_n$ is a rational sequence with $\frac{1}{n} \to 0$

$\{\frac{1}{n\sqrt{2}}\}$ is an irrational sequence with $\frac{1}{n\sqrt{2}} \to 0$

By the <u>Thm. A</u>,

- $f(\frac{1}{n}) \to \ell$ & $f(\frac{1}{n\sqrt{2}}) \to \ell$

But $f(\frac{1}{n}) = 0 \ \forall n \Rightarrow f(\frac{1}{n}) \to 0$

and $f(\frac{1}{n\sqrt{2}}) = 1 \ \forall n \Rightarrow f(\frac{1}{n\sqrt{2}}) \to 1$

By uniqueness of limit, deduce that the limit does not exists.

Algebra of Limits.

Suppose f and g are defined on a deleted nbhd. of c. Further suppose that $\lim_{x\to c} f(x) = \ell$ & $\lim_{x\to c} g(x) = s$.

Then

(a) $\lim_{x\to c}\left(f(x) \pm g(x)\right) = \ell \pm s$

(b) $\lim_{x\to c}\left(k\cdot f(x)\right) = k\ell \quad \forall\ k \in \mathbb{R}$

(c) $\lim_{x\to c}\left(f(x)\cdot g(x)\right) = \ell s$

(d) If $g(x) \neq 0$ on a nbhd. of c, and $s \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\ell}{s}.$$

Proof. Follows from algebra of sequences.

e.g. Let us prove (a)

$\boxed{\text{Let } x_n \to c}$

$$\lim_{x\to c} f(x) = l \Rightarrow \lim_{n\to\infty} f(x_n) = l$$

Similarly, $\lim_{n\to\infty} g(x_n) = s$

$$\Rightarrow \lim_{n\to\infty}\big(f(x_n) + g(x_n)\big) = l+s$$

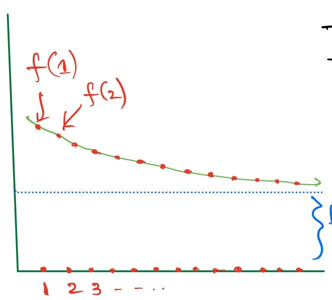
$$\Rightarrow \lim_{x\to c}\big(f(x) + g(x)\big) = l+s$$

(By Thm. B)

Infinite limits

What is the meaning of

$$\lim_{x\to\infty} f(x) = l \ ?$$



It has a similar meaning as that of

$$\lim_{n\to\infty} f(n) = l$$

i.e., given $\varepsilon > 0$ $\exists$ n_0 s.t

$$|f(n) - l| < \varepsilon \quad \forall \ n \geq n_0$$

More generally, we say

$$\lim_{x\to\infty} f(x) = l$$

if for every $\varepsilon > 0$, $\exists$ $M > 0$ s.t.

$$|f(x) - l| < \varepsilon \quad \forall\ x > M.$$

Example.

- $$\lim_{x\to\infty} \frac{1+2x}{1+x} = 2.$$

$$\left|\frac{1+2x}{1+x} - 2\right| = \left|\frac{-1}{1+x}\right| = \frac{1}{1+x} < \varepsilon$$

(As $x \to \infty$, so taking $x > 0$)

$$\Rightarrow x > \frac{1}{\varepsilon} - 1$$

This is M.

<u>When the limit is ∞.</u>

What is the meaning of

$$\lim_{x\to c} f(x) = \pm\infty$$

In order to understand the above limit, we try to understand

$$\lim_{n\to\infty} f\left(c+\frac{1}{n}\right) = \pm\infty$$

Given every $M>0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$\left|f\left(c+\frac{1}{n}\right)\right| > M \quad \forall \; n \geq n_0.$$

Note that $\quad n \geq n_0 \Rightarrow \frac{1}{n} \leq \frac{1}{n_0} := \delta$

Thus, $\left|c+\frac{1}{n}-c\right| = \frac{1}{n} < \delta$

$$\Rightarrow \left|f\left(c+\frac{1}{n}\right)\right| > M$$

Thus, $\lim_{x\to c} f(x) = \pm\infty$ if

given every $M>0$, $\exists$ $\delta>0$ s.t.

$$|x-c|<\delta \Rightarrow |f(x)| > M$$

<u>Example.</u> $\quad \lim_{x\to 1} \frac{1}{|x-1|} = \infty$

$$\frac{1}{|x-1|} > M \Leftrightarrow |x-1| < \frac{1}{M} := \delta.$$