भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

9th Lecture on Transform Techniques

(MA-2120)

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What will we learn today?

- Fourier Transform
- Fourier Series

Fourier Transform:

Fourier Transform is one of the important integral transform. This transform (i.e, Fourier Transform, Fourier Sine Transform or Fourier Cosine Transform) can be applied to solve many linear initial value and boundary value problems in applied mathematics, mathematical physics and engineering sciences. These transforms are very effective for solving the differential eqns.

Question: How can it be applied to solve differential equations?

Ans: The procedure is same as Laplace Transform

Fourier Transform:

Same Procedure like Laplace Transform

Differential equations/ Integral equations $\xrightarrow{\text{Fourier Transform}}$ Transformed Algebraic equations

Tough to solve Directly

Inverse Fourier Transform

More easy to solve

Solution to the differential equations or Integral equations

Fourier Transform:

Applications:

(i) Solve the ODEs / PDEs in different areas. This transform is more popular to solve Heat equations.

Differential eqⁿs.

(ii) Solve Integral equations.

(iii) Analysis of Stationary signals or real time signal processing; make an effective use of the Fourier Transform in time and frequency domains i.e. in t and ω.

Many more applications like Laplace Transform.

Fourier Transform:

We know the integral transform of a function $f(t)$ defined in $a \le t \le b$ and which is denoted by

$$\boxed{T\{f(t)\} = F(s) = \int_a^b K(s,t)\, f(t)\, dt}$$

For Fourier Transform, the kernel $K(s,t)$ is defined by e^{-ist} and a and b are $-\infty$ and ∞, respectively. See next page.

Fourier Transform:

Let $f(t)$ be piecewise continuous function on $(-\infty, \infty)$. Assume that $f(t)$ is absolutely convergent, i.e., $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges.

Then the Fourier transform of $f(t)$ denoted by $\mathcal{L}[f(t)]$ and is defined as

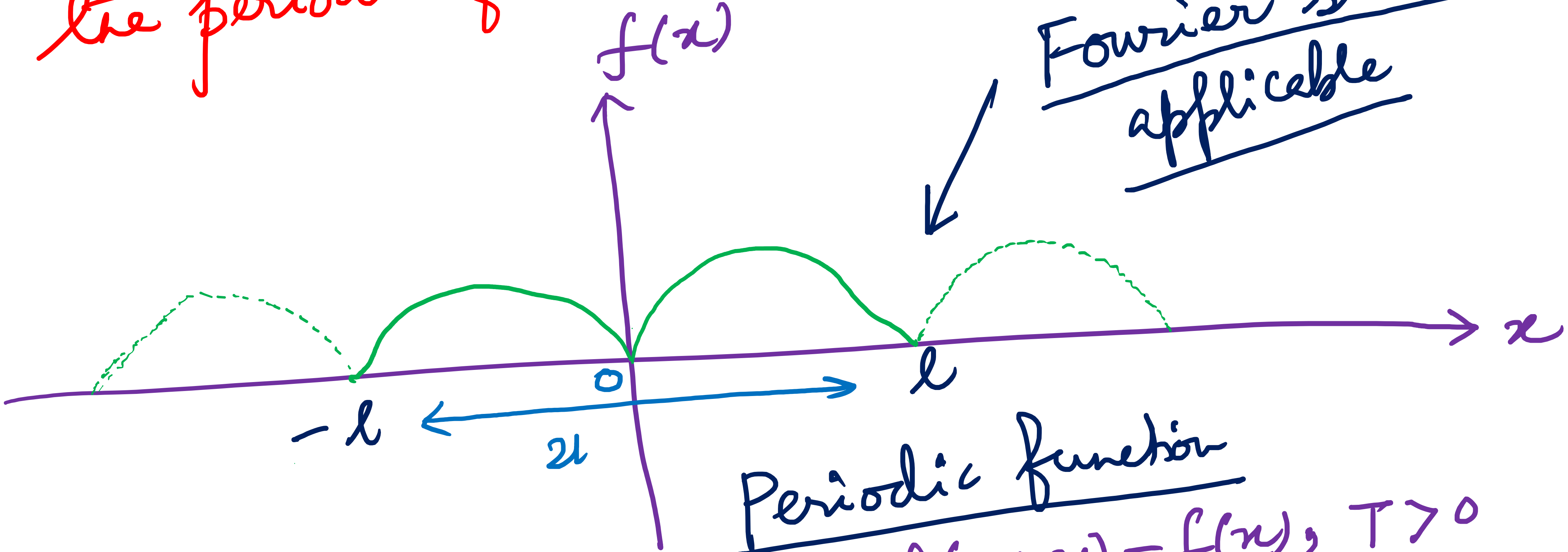
$$\boxed{\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(\omega).}$$

* Before we will go in details of the Fourier Transform, we need to know the root of this. How is it developed? What is the actual meaning of this transform?

To know all the answers, we have to study the <u>Fourier Series</u> and therefore <u>Fourier Integral</u>.

Fourier Series :

* Fourier Series can be applied to approximate the periodic function.



Fourier series is applicable

Periodic function

$f(x+2l) = f(x), \; T > 0$

Let $f(x)$ be a periodic function of period $2l$ defined on $[-l, l]$, i.e, $f(x+2l) = f(x)$.

Suppose $f(x)$ has the expansion —

$$f(x) = \frac{a_0}{2} + \left[a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + \cdots\right] + \left[b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \cdots\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \quad \text{— (1)}$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are the coefficients and need to be determined.

⊛ Orthogonal functions :

Two functions $\phi(x)$ and $\psi(x)$ are orthogonal on the interval $[a, b]$ if

$$\boxed{\int_a^b \phi(x)\,\psi(x)\,dx = 0}$$

Consider the set of orthogonal functions

$$\left\{1, \cos\left(\frac{\pi x}{l}\right), \cos\left(\frac{2\pi x}{l}\right), \ldots, \sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right), \ldots\right\}$$

which are orthogonal on the interval $[-l, l]$. These functions have the following properties —

(i) $$\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right) dx$$
$$= 0$$

(ii) $$\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \cos\left(\frac{n\pi x}{l}\right) dx$$
$$= \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx$$
$$= 0, \quad m \neq n.$$

(iii) $$\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = 0 \quad \forall\ m \text{ and } n.$$

(iv) $\int_{-l}^{l} \cos^2\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} \sin^2\left(\frac{m\pi x}{l}\right) dx$

$= l.$

where m and n are integers.

Now we want to determine the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ by using the orthogonal properties of the trigonometric functions.

Assumption: Term by term integration can be performed on eq ①.

Now integrating eq ① term by term on the interval $[-l, l]$, we have —

$$\int_{-l}^{l} f(x)\,dx = \frac{a_0}{2}\int_{-l}^{l} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)dx + b_n \int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$a_1 \quad \boxed{a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx}.$$

Now multiply both sides of (1) by $\cos\left(\frac{m\pi x}{l}\right)$ and integrate term by term on the interval $[-l, l]$, we obtain

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{a_0}{2}\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)dx$$

$$+\sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx + b_n\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

Now we have by using the orthogonal properties,

$$\int_{-l}^{l} f(x) \cos\left(\frac{m\pi x}{l}\right) dx = l\, a_m$$

$$\text{or } \boxed{a_m = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{m\pi x}{l}\right) dx.}$$

Multiplying both sides of (1) by $\sin\left(\frac{m\pi x}{l}\right)$ and integrating term by term on the interval $[-l, l]$, we get

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \frac{a_0}{2}\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)dx$$

$$+\sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx\right.$$

$$\left.+\,b_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

By using orthogonal properties, we have

$$\int_{-l}^{l} f(x) \sin\left(\frac{m\pi x}{l}\right) dx = l\, b_m$$

or

$$\boxed{b_m = \frac{1}{l}\int_{-l}^{l} f(x) \sin\left(\frac{m\pi x}{l}\right) dx}$$

Now we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \quad \text{--- (1)}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$

This infinite Series is known as Fourier Series. ← orthogonal series

If the period of the function $f(x)$ is 2π, and the function is defined on $[-\pi, \pi]$, then we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] \quad \text{--- (2)}$$

where

$$(3) \quad \begin{cases} a_0 = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, dx \\[2ex] a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\[2ex] \text{and } b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx \, dx. \end{cases}$$

Note: From the definition of the definite integrals, if $f(x)$ is continuous or piecewise continuous on $[-\pi, \pi]$, then the integrals (3) exist and $f(x)$ can be expanded as Fourier Series.

Ex Find the Fourier Series expansion of the periodic function

$$f(x) = x, \quad -\pi \le x \le \pi, \quad f(x+2\pi) = f(x)$$

Sol^n:

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0$$ (x is an odd function on $[-\pi, \pi]$)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx$$

$$= 0$$

($x \cos nx$ is an odd function on $[-\pi, \pi]$)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

[Since $x \sin nx$ is an even f^n on $[-\pi, \pi]$]

$$= 2\pi\left[-x\left(\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$\frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right] = \frac{2}{n}(-1)^{n+1}$$

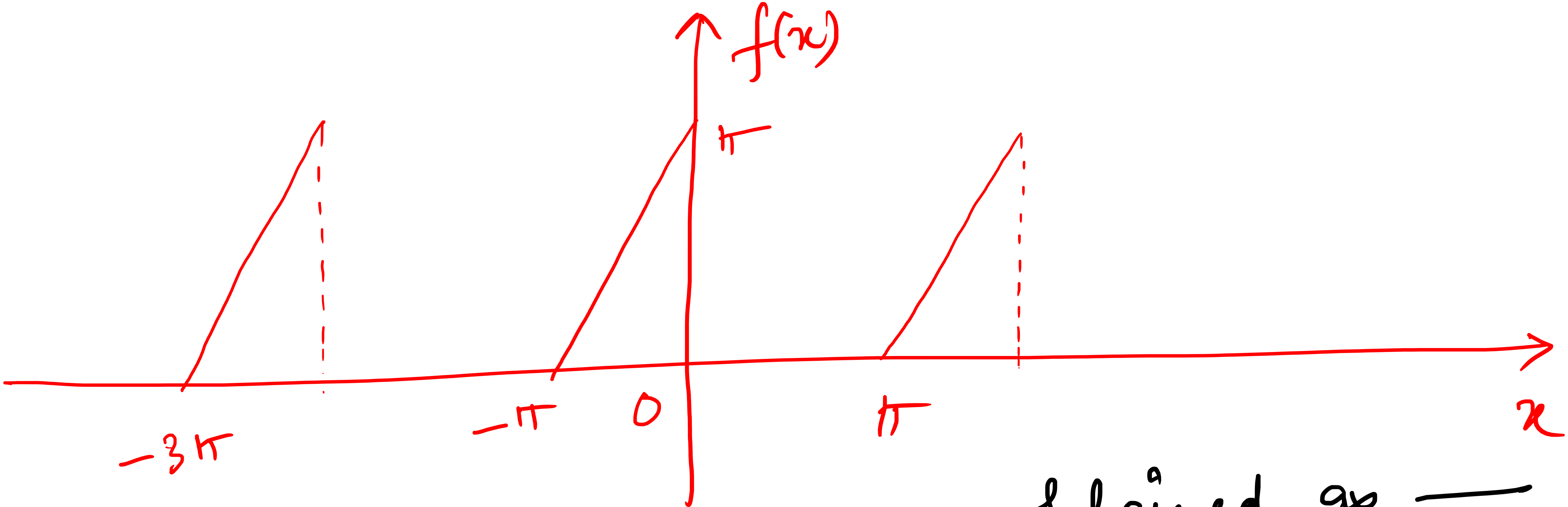
Therefore the Fourier expansion of $f(x) = x$, on $[-\pi, \pi]$ is given by

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Ex. Find the Fourier Series expansion of the periodic function with period 2π,

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ 0, & 0 \le x < \pi \end{cases}$$

$$f(x+2\pi) = f(x)$$



The Fourier coefficients are obtained as —

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\,dx$$

$$= \pi/2.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi\cos nx\,dx + \int_{-\pi}^{0}x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\sin nx}{n}\right) + \left\{x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(1-\cos n\pi)\right] = \frac{1}{\pi n^2}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, & \text{for } n \text{ even} \\ 2/\pi n^2, & \text{for } n \text{ odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi+x)\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[-\pi/n\right] = \underline{-\frac{1}{n}}.$$

Therefore the Fourier Series expansion is given by —

$$f(x) = \pi/4 + \sum_{n=1}^{\infty} \left[\frac{1}{\pi n^2} \left\{ 1 - (-1)^n \right\} \cos nx - \frac{1}{n} \sin nx \right]$$

$$= \pi/4 + \frac{2}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots \right] - \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \dots \right]$$

Example: Find the Fourier Series expansion of the following periodic function of period 4

$$f(x) = \begin{cases} 2+x, & -2 \le x \le 0 \\ 2-x, & 0 < x \le 2, \end{cases}$$

$$f(x+4) = f(x).$$

Ans: $f(x) = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left[(2n-1) \cdot \frac{\pi x}{2}\right]$

Fourier Series Expansions of Even and odd functions :

Let $f(x)$ be a function defined on $[-l, l]$. Then $f(x)$ is an even function on $[-l, l]$ if $f(-x) = f(x), \ -l \leq x \leq l$

The function is odd if
$$f(-x) = -f(x), \quad -l \leq x \leq l.$$

Examples: $|x|$, x^2, cosine function are even functions.

x, x^3, and sine function are odd function.

For even function $f(x)$: If $f(x)$ is even function on $[-l, l]$, then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx.$$

For odd function :

If $f(x)$ is an odd function on $[-l, l]$, then we have

$$\int_{-l}^{l} f(x)\,dx = 0.$$

If $f(x)$ is an even function on $[-l, l]$, then we have the following Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l},$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

This Series is called as Fourier Cosine Series.

The Fourier Series for an odd function on $[-l, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

This Series is called as Fourier Sine Series.

Example: Find the Fourier Series expansion of the function

$$f(x) = x^2, \quad -2 \le x \le 2$$

Sol^n: Here $f(x) = x^2$ is an even function. We have the following Fourier Cosine Series coefficients —

$$a_0 = \frac{2}{l} \int_0^l f(x)\,dx = \int_0^2 x^2\,dx$$

$$= 8/3$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^2 x^2\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{x^2\sin\left(n\pi x/2\right)}{\left(n\pi/2\right)}\right]_0^2 - 2\int_0^2 x\,\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\,dx$$

$$= -\frac{4}{n\pi}\left[-x\frac{\cos\left(n\pi x/2\right)}{n\pi/2} + \frac{\sin\left(n\pi x/2\right)}{\left(n\pi/2\right)^2}\right]_0^2$$

$$= \frac{16}{n^2\pi^2} \cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$$

Therefore the Fourier Series is given by —

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{2}\right)$$

Question: ① Does the Fourier Series converges at a point $x \in [-l, l]$

Ans: Not always.

② **Pointwise Convergent:** Consider an infinite Series $f_1(x) + f_2(x) + \cdots = \sum_{n=1}^{\infty} f_n(x)$.

This Series is said to be convergent for a given value of x if its Sequence of partial sum —

$S_n(x) = \sum_{k=1}^{n} f_k(x)$ have a finite limit

$$\boxed{\lim_{n \to \infty} S_n(x) = S(x)}$$

Then it is said to be pointwise convergent.

Uniform Convergence: This is a stronger notion of convergence.

A sequence $f_n(x)$ of functions converges uniformly to a function $f(x)$ if the speed of convergence of $f_n(x)$ to $f(x)$ does not depend on x.

Ex: (i) $f_n(x) = x^n$, $x \in [0,1]$

$f_n(x)$ converges to $f(x)$ pointwise

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

Speed of convergence depends on x.

(ii) $f_n(x) = \frac{1}{n} \sin nx, \quad x \in (-\infty, \infty)$

$\downarrow$ uniform Convergence

$f(x) = 0$, Here the speed of Convergence does not depends on x.

Note: Many functions including some discontinuous periodic functions can be expanded in a Fourier Series.