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4^{th} Lecture on Differential Equation

(MA-1150)

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What have we learnt?

- Exact First Order Differential Equation

- Methods of solving Exact differential equations
  - ✓ Method I (Best method)
  - ✓ Method 2 (Alternative approach)
  - ✓ Method 3 (Another alternative approach)

- Exact Homogeneous Equations

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What will we learn today?

- Integrating Factors
- Method of solving non-exact first order and first degree ode:
  - ✓ Rule I
  - ✓ Rule II
  - ✓ Rule III
  - ✓ Rule IV

Integrating Factors

If the given first order eqⁿ is not exact then it can be made exact by multiplying it by an integrating factors (I F).

- **Definition:** If the first order and first degree ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact in a rectangle domain D but the differential eqⁿ

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

Integrating Factors

is exact in D, then $\mu(x,y)$ is called an integrating factor of the given ODE.

Ex:

Let us consider a non exact ODE $y\,dx + 2x\,dy = 0$. Multiplying by y, it becomes exact. So IF for this ODE is y.

Question: If an IF exists, is it unique?

It is answered by the following theorem —

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Integrating Factors

Theorem: If the equation $M\,dx + N\,dy = 0$ has one and only one solution, then there exists an infinity of integrating factors.

Proof: Let the general solution be $f(x,y) = c$, where c is the arbitrary constant.

Integrating Factors

Now we have by taking the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Since $f(x,y) = c$ is the general solution of

$$M dx + N dy = 0,$$

Integrating Factors

the relation $\frac{\partial f}{\partial x} / M = \frac{\partial f}{\partial x} / N$ must hold identically.

Hence $\exists$ a function u such that

$$\frac{\partial f}{\partial x} = uM \quad \text{and} \quad \frac{\partial f}{\partial y} = uN$$

Integrating Factors

Now we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \mu(M\,dx + N\,dy).$$

i.e, μ is an integrating factor to make the differential eq^n $M\,dx + N\,dy = 0$ be exact.

Let $\phi(f)$ be a function f.

Integrating Factors

Then $\phi(f)\,df = \mu\,\phi(f)(M\,dx + N\,dy)$.

or $\mu\,\phi(f)(M\,dx + N\,dy) = \phi(f)\,df$.

The R.H.S is exact. So, $\mu\,\phi(f)$ is also an I.F. Since $\phi(f)$ is an arbitrary function of f, it is evident to say that there exists an infinite of integrating factors.

Integrating Factors

Example:

1. For ODE $x\,dy - y\,dx = 0$, the integrating factors are $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}$ and $\frac{1}{x^2+y^2}$.

Solⁿ: (i)

$$x\,dy - y\,dx = 0$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = 0 \quad \Rightarrow d\left(y/x\right) = 0.$$

$$\Rightarrow y/x = c$$

Integrating Factors

(ii)

$$\frac{x\,dy - y\,dx}{y^2} = 0$$

$$\Rightarrow -\frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow -d\left(x/y\right) = 0$$

$$\Rightarrow x/y = \text{constant}$$

$$\Rightarrow y/x = c.$$

Integrating Factors

(iii)

$$\frac{x\,dy - y\,dx}{xy} = 0$$

$$\Rightarrow \frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \ln\left|y/x\right| = \ln c$$

$$\Rightarrow \left|y/x\right| = c.$$

$$\Rightarrow y/x = c.$$

Integrating Factors

(iv)

$$\frac{x\,dy - y\,dx}{x^2+y^2} = 0$$

or $$\frac{d\left(y/x\right)}{1+\left(y/x\right)^2} = 0$$

or $$\tan^{-1}\left(y/x\right) = C_1$$

or $$y/x = C.$$

Integrating Factors

Now we will describe the methods for getting the I.F.

(i) Rule-I : Inspection method & For homogeneous

$IF = \frac{1}{Mx+Ny}$, $Mx+Ny \neq 0$,

$IF = \frac{1}{x^2}, \frac{1}{y^2}$ or $\frac{1}{xy}$

if $Mx+Ny=0$

(ii) Rule-II

(iii) Rule-III

(iv) Rule-IV

(v) Rule-V

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(Rule-I) : Examples (Inspection)

We know –
$$d(y/x) = \frac{x\,dy - y\,dx}{x^2}$$

Ex: ① $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$

or, $x\,dx + y\,dy + \dfrac{\dfrac{x\,dy - y\,dx}{x^2}}{\dfrac{x^2+y^2}{x^2}} = 0$

or, $\frac{1}{2}\,d(x^2+y^2) + \dfrac{d(y/x)}{1+(y/x)^2} = 0$

Integrating, $\frac{1}{2}(x^2+y^2) + \tan^{-1} y/x = c_1$

or, $\underline{x^2+y^2 + 2\tan^{-1} y/x = c}$, where $c = 2c_1$ is an arbitrary constant

This is the General solution.

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(Rule - I:) Examples (Inspection)

Ex. 2

$$x\cos(y/x)\,(y\,dx + x\,dy) = y\sin(y/x)\,(x\,dy - y\,dx)$$

Multiplying an integrating factor $\frac{1}{x^2}$, we have

or, $$\frac{x\cos(y/x)\,(y\,dx + x\,dy)}{x^2} = \frac{y\sin(y/x)\,(x\,dy - y\,dx)}{x^2}.$$

or, $$\cos(y/x)\,\frac{d(xy)}{x} = y\sin(y/x)\,d(y/x)$$

or, $$\frac{d(xy)}{xy} = \frac{\sin(y/x)}{\cos(y/x)}\,d(y/x)$$

Integrating, or, $\log|(xy)| = -\log|\cos(y/x)| + \log c$

or, $\boxed{|xy\cos(y/x)| = c}$ — This is the general solution

(Rule I:) Examples (Inspection)

Ex. 3 $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, Not exact.

or, $xy^2\,dx - x^2y\,dy = e^{1/x^3}\,dx$

or, $xy(y\,dx - x\,dy) = e^{1/x^3}\,dx$

Multiplying an IF as $\frac{1}{x^4}$, we have

$$\frac{xy(y\,dx - x\,dy)}{x^4} = \frac{e^{1/x^3}\,dx}{x^4}$$

or, $-\,\frac{y}{x}\,d\left(\frac{y}{x}\right) = \frac{e^{1/x^3}}{x^4}\,dx$

Examples

Integrating, we have

$$-\int \frac{y}{x}\, d\left(\frac{y}{x}\right) = -\frac{1}{3}\int e^{1/x^3}\, d\left(\frac{1}{x^3}\right) + c.$$

$$d\left(\frac{1}{x^3}\right) = -\frac{3}{x^4}.$$

or, $$-\frac{1}{2}\left(\frac{y}{x}\right)^2 = -\frac{1}{3}e^{1/x^3} + c$$

or, $$\boxed{3y^2 - 2x^2 e^{1/x^3} + 6cx^2 = 0}$$

This is the general solution.

Equations which are homogeneous but not exact

Theorem: If $M\,dx + N\,dy = 0$ is not exact but M and N are both homogeneous functions of x and y of same degree n, then the integrating factor is $\dfrac{1}{Mx+Ny}$, provided $Mx+Ny \neq 0$. Here the given ODE is defined on the rectangular domain D.

(Rule I).

For finding IF:

Ex. $x^2y\,dx - (x^3+y^3)\,dy = 0 \quad \text{--- (1)}$

The eqn is homogeneous but not exact

$M = x^2y, \quad N = -(x^3+y^3)$

$\dfrac{\partial M}{\partial y} = x^2, \quad \dfrac{\partial N}{\partial x} = -3x^2$, Not exact.

$Mx+Ny = -y^4 (\neq 0)$. Hence $IF = \dfrac{1}{Mx+Ny} = -\dfrac{1}{y^4}$.

Examples

Now multiplying differential eqn ① by this I F, we get

$$-\frac{x^2}{y^3}dx + \frac{x^3}{y^4}dy + \frac{1}{y}dy = 0$$

or, $-\frac{x^2}{y^2}\left\{\frac{y\,dx - x\,dy}{y^2}\right\} + \frac{1}{y}dy = 0$

or, $-\frac{x^2}{y^2}d\left(\frac{x}{y}\right) + \frac{1}{y}dy = 0$

Integrating, $-\frac{x^3}{3y^3} + \log|y| = \log c$

or, $|y| = c\,e^{x^3/3y^3}$

$\Rightarrow \boxed{y = c\,e^{x^3/3y^3}} \rightarrow$ General Soln

$\underline{\underline{y > 0}}$

Examples

Ex. 2 $\quad (2xy+x^2)\dfrac{dy}{dx} = 3y^2+2xy$

or, $(3y^2+2xy)\,dx - (2xy+x^2)\,dy = 0.$

$M = 3y^2+2xy, \quad N = -(2xy+x^2), \quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, not exact.

$M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 2.

Hence $\dfrac{1}{Mx+Ny} = \dfrac{1}{(3y^2+2xy)x-(2xy+x^2)y} = \dfrac{1}{xy(x+y)}$

$I.F = \dfrac{1}{xy(x+y)}$

Multiplying the given differential eqn by I F, we obtain

$$\frac{y(3y+2x)}{xy(x+y)}\,dx - \frac{x(2y+x)}{xy(x+y)}\,dy = 0.$$

Examples

Now, this equation is exact.

So we have by applying method 2 (you can apply method 1 or method 3)

$$\int \frac{3y+2x}{x(x+y)}\,dx \quad \text{(assuming } y \text{ to be constant)}$$

$$= \int \left(\frac{3}{x} - \frac{1}{x+y}\right) dx = 3\log|x| - \log|(x+y)|$$

$$-\int \frac{2y+x}{y(x+y)}\,dy \quad \text{(term of } y \text{ only)} = -\int \left[\frac{1}{y} + \frac{1}{x+y}\right] dy \quad \text{(term of } y \text{ only)}$$

discard this term, since it is not function y only.

$$= -\int \frac{1}{y}\,dy = -\log|y|$$

Examples

Now the general solution is

$$3\log|x| - \log|(x+y)| - \log|y| = \log c \quad , \quad \underline{x>0, y>0}$$

or, $\log x^3 - \log(y(x+y)) = \log c$

or, $\dfrac{x^3}{y(x+y)} = c$

$\Rightarrow \boxed{x^3 = cy(x+y)} \Rightarrow$ This is the G.S.

where c is an arbitrary constant.

Examples

(iii) **Theorem:** If the functions $M(x,y)$ and $N(x,y)$ in the eqn $M(x,y)\,dx + N(x,y)\,dy = 0$ are homogeneous functions of degree n and $Mx + Ny = 0$, then $\frac{1}{xy}$ or $\frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{x^2+y^2}$ is an integrating factor.

Proof: $\Rightarrow$ If $Mx + Ny = 0 \ \forall (x,y) \in D$, then $M/N = -y/x$.

Then the differential eqn reduces to

$$\frac{M}{N}\,dx + dy = 0 \quad \text{or,} \quad -\frac{y}{x}\,dx + dy = 0$$

$$\text{or,} \quad x\,dy - y\,dx = 0.$$

This is the reduced ODE.

Therefore, $\frac{1}{xy}$ or $\frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{x^2+y^2}$ is an integrating factor of the reduced ODE. (Proved)

Finding Integrating Factor (Rule-II)

Theorem (Rule-II): Let the differential equation be in the form of $M(x,y)\,dx + N(x,y)\,dy = 0, \forall (x,y) \in D$

If $\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$ is a function of x alone, $f(x)$ (say), then $e^{\int f(x)\,dx}$ is an integrating factor.

Finding Integrating Factor (Rule-II)

of the given differential equation.

Proof: The given differential eqⁿ is

$$M\,dx + N\,dy = 0 \quad \text{—} ① ,\ \forall (x,y) \in D.$$

Now we have to prove that $e^{\int f(x)\,dx}$ is an integrating factor of eqⁿ ①, where $f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$.

Finding Integrating Factor (Rule-II)

That means we have to show that after multiplying eqn ① by $e^{\int f(x)\,dx}$, the reduced eqn is exact in D.

Now we have

$$e^{\int f(x)\,dx} M\,dx + e^{\int f(x)\,dx} N\,dy = 0$$

or, $M'\,dx + N'\,dy = 0$ — ②

Finding Integrating Factor (Rule-II)

where $M' = M e^{\int f(x)\,dx}$

and $N' = N e^{\int f(x)\,dx}$.

Aim is to prove $\boxed{\dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}}$

Here $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = f(x)$

Finding Integrating Factor (Rule-II)

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + N f(x)$$

Multiplying by $e^{\int f(x)\,dx}$, we have

$$\Rightarrow \frac{\partial M}{\partial y} e^{\int f(x)\,dx} = \frac{\partial N}{\partial x} e^{\int f(x)\,dx} + N f(x) e^{\int f(x)\,dx}$$

$$\Rightarrow \frac{\partial}{\partial y}\left\{M e^{\int f(x)dx}\right\} = \frac{\partial}{\partial x}\left\{N e^{\int f(x)\,dx}\right\}$$

Finding Integrating Factor (Rule-II)

a, $\boxed{\dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}}$, $\forall (x,y) \in D$.

which is the condition for exactness of the differential equation –

$$M'dx + N'dy = 0, \quad \forall (x,y) \in D.$$

$\Rightarrow$ This implies that $e^{\int f(x)dx}$ is the integrating factor for eq^n (1).

Example

Example: Solve $(x^3+xy^4)\,dx + 2y^3\,dy = 0$

Sol^n:

Here $M(x,y) = x^3 + xy^4$.

$N(x,y) = 2y^3$.

$$\frac{\partial M}{\partial y} = 4xy^3, \quad \frac{\partial N}{\partial x} = 0.$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$ The eq^n is not exact.

Example

Here $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

$$= \frac{1}{2y^3}\left(4xy^3 - 0\right) = 2x = f(x).$$

Integrating factor (IF)

$$= e^{\int f(x)\,dx} = e^{\int 2x\,dx} = \underline{\underline{e^{x^2}}}.$$

Examples

Multiplying by IF, we have

$$\boxed{e^{x^2}(x^3+xy^4)\,dx + e^{x^2}\,2y^3\,dy = 0}$$

⇒ Here you can apply any of three methods for exact ode.

You try it as your Home work.

[Multiplying by 2]

$$\text{or}\quad 2e^{x^2}x^3\,dx + \left\{y^4 e^{x^2}\cdot 2x\,dx + e^{x^2}\cdot 4y^3\,dy\right\} = 0$$

[The grouping method is applied here]

Integrating,

$$\int 2x^3 e^{x^2}\,dx + \int d\left(y^4\cdot e^{x^2}\right) = c\,,$$ where c is an arbitrary constant

$$\text{or}\quad \int e^{x^2}\cdot x^2\cdot d(x^2) + \int d\left(y^4\cdot e^{x^2}\right) = c$$

$$\text{or}\quad e^{x^2}(x^2-1) + e^{x^2}\cdot y^4 = c \Rightarrow \boxed{e^{x^2}\left(x^2-1+y^4\right) = c}$$

This is General Solⁿ.

Finding Integrating Factor (Rule-III)

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Theorem (Rule-III): Consider the differential equation $M(x,y)\,dx + N(x,y)\,dy = 0, \forall (x,y) \in D$. If
$$\frac{1}{M(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$$
is a function of y alone, say $\phi(y)$, then $e^{-\int \phi(y)\,dy}$ is an integrating factor of $M\,dx + N\,dy = 0, \forall (x,y) \in D$.

Finding Integrating Factor (Rule-III)

Proof: It is exactly similar to the previous theorem (Rule-II).

You try it. (Home work)

Example

Example: Solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

Soln: The given eqn is not exact since

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

and

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{8xy^3e^y + 8xy^2 + 4}{2xy^4e^y + 2xy^3 + y} = \frac{4}{y} = \phi(y)$$

$$IF = e^{-\int \phi(y)\,dy} = \frac{1}{y^4}.$$

Example

Multiplying the given differential eq^n by I F, we have -

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - 3\frac{x}{y^4}\right)dy = 0$$

This is exact.

So, we have by applying method 2,

$$\int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx \quad \text{(assuming } y \text{ as constant)}$$

$$= x^2e^y + x^2/y + x/y^3$$

Examples

and

$$\int \left(x^2 e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right) dy \quad \text{(terms of function } y \text{ only)}$$

$$= \int 0 \, dy = 0.$$

Now the required general solⁿ is –

$$\boxed{x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$$

, where c is an arbitrary constants.

For Finding Integrating Factor (Rule-IV)

Theorem (Rule-IV): Consider the differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$. If $Mx - Ny \neq 0$ then $\frac{1}{Mx - Ny}$ is an integrating factor of the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ where $M(x,y)$ and $N(x,y)$ are written as $M(x,y) = y f_1(xy)$ and $N(x,y) = x f_2(xy)$ $\forall (x,y) \in D$, where D is rectangular domain.

For Finding Integrating Factor (Rule-IV)

Proof: The given eq^n is $M\,dx + N\,dy = 0$

where $M = y f_1(xy)$ and $N = x f_2(xy)$. —— (1)

Now $Mx - Ny = xy\{f_1(xy) - f_2(xy)\} \neq 0$

We have to prove that $\frac{1}{Mx - Ny}$ is an integrating factor of the eq^n

$M\,dx + N\,dy = 0, \ \forall (x,y) \in D.$

For Finding Integrating Factor (Rule-IV)

To check that multiply the eqⁿ ① by $\frac{1}{Mx-Ny}$, we have -

$$\frac{M}{Mx-Ny}dx + \frac{N}{Mx-Ny}dy = 0.$$

or $M'dx + N'dy = 0$ —— ②

where $M' = \frac{M}{Mx-Ny}$ and $N' = \frac{N}{Mx-Ny}$.

For Finding Integrating Factor (Rule-IV)

Aim is to show that

$$\boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}, \quad \forall (x,y) \in D$$

$$\frac{\partial M'}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{f_1}{x(f_1-f_2)}\right\} = \frac{-f_2 \frac{\partial f_1}{\partial y} + f_1 \frac{\partial f_2}{\partial y}}{x(f_1-f_2)^2}$$

For Finding Integrating Factor (Rule-IV)

Similarly $\frac{\partial N'}{\partial x}$

$$= \frac{\partial}{\partial x}\left\{\frac{f_2}{y(f_1-f_2)}\right\} = \frac{f_1\frac{\partial f_2}{\partial x} - f_2\frac{\partial f_1}{\partial x}}{y(f_1-f_2)^2}.$$

Now $\frac{\partial M'}{\partial y} - \frac{\partial N'}{\partial x}$

$$= \frac{f_2\left(-y\frac{\partial f_1}{\partial y} + x\frac{\partial f_1}{\partial x}\right) + f_1\left(y\frac{\partial f_2}{\partial y} - x\frac{\partial f_2}{\partial x}\right)}{xy(f_1-f_2)^2}$$

For Finding Integrating Factor (Rule-IV)

we have $M = y f_1(xy)$, let $X = xy$.

$\Rightarrow M = y f_1(X)$

$\Rightarrow \boxed{f_1(X) = M/y}$

$\Rightarrow \dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_1}{\partial X} \dfrac{\partial X}{\partial y}$

$= x \dfrac{\partial f_1}{\partial X}$

$\Rightarrow \boxed{y \dfrac{\partial f_1}{\partial x} = X \dfrac{\partial f_1}{\partial X}}$.

For Finding Integrating Factor (Rule-IV)

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial X}\frac{\partial X}{\partial x}$$

$$= y\frac{\partial f_1}{\partial X}$$

$$\text{or, } \boxed{x\frac{\partial f_1}{\partial x} = xy\frac{\partial f_1}{\partial X} = X\frac{\partial f_1}{\partial X}}$$

$$\therefore \boxed{y\frac{\partial f_1}{\partial y} = x\frac{\partial f_1}{\partial x}}.$$

For Finding Integrating Factor (Rule-IV)

Similarly we have

$$\boxed{y\frac{\partial f_2(xy)}{\partial y} = x\frac{\partial f_2(xy)}{\partial x}}$$

So we have $\frac{\partial M'}{\partial y} - \frac{\partial N'}{\partial x} = 0$

as $\boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$ (Proved)

For Finding Integrating Factor (Rule-IV)

Note: If $Mx - Ny = 0$, then $\frac{M}{N} = y/x$ and the eqⁿ becomes to $\boxed{x\,dy + y\,dx = 0}$ $\Rightarrow xy = c$. general solution

Since, $M\,dx + N\,dy = 0$

or $\frac{M}{N}\,dx + dy = 0$

or, $y\,dx + x\,dy = 0$

Example

Example: Solve

$$(x^2y^2+xy+1)\,y\,dx-(x^2y^2-xy+1)\,x\,dy=0$$

Sol^n: The given differential eq^n is not exact.

$M(x,y)=(x^2y^2+xy+1)y$, and $N(x,y)=-(x^2y^2-xy+1)x$

$$\boxed{\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}}$$

Example

Now $Mx - Ny = (x^2y^2 + xy + 1)xy + (x^2y^2 - xy + 1)xy$

$$= 2xy(x^2y^2+1)$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy(x^2y^2+1)}$$

multiplying by I.F, we have -

$$\frac{x^2y^2+xy+1}{2x(x^2y^2+1)}\,dx - \frac{x^2y^2-xy+1}{2y(x^2y^2+1)}\,dy = 0$$

or, $\frac{x^2y^2+xy+1}{x(x^2y^2+1)}\,dx - \frac{x^2y^2-xy+1}{y(x^2y^2+1)}\,dy = 0$

This is exact

Examples

So, we have by applying method 2,

$$\int \frac{x^2y^2+xy+1}{x(x^2y^2+1)}\,dx \quad (\text{assuming } y \text{ as constant})$$

$$= \int \frac{x^2y^2+1}{x(x^2y^2+1)}\,dx + \int \frac{y\,dx}{x^2y^2+1}$$

$$= \int \frac{dx}{x} + \int \frac{d(xy)}{1+x^2y^2} = \log|x| + \tan^{-1}(xy)$$

Examples

and

$$-\int \frac{x^2y^2 - xy + 1}{y(x^2y^2+1)}\, dy$$ (terms of function y only)

$$= -\int \frac{1}{y}\, dy = -\log|y|$$

Then the general sol^n is

$$\tan^{-1}(xy) + \log|x| - \log|y| = c$$

$$\Rightarrow \boxed{\tan^{-1}(xy) + \log\left|\left(\frac{x}{y}\right)\right| = c}$$

where c is an arbitrary constant.

$$\Rightarrow \tan^{-1}(xy) + \log\left(\frac{x}{y}\right) = c.$$