ME5053: Soft Robotics

Assignment 1

ME21BTECH11001

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Question 1

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Question 1 :—

$$f(x_1, x_2) = 9x_1^2 - 9x_2^2 + x_1^2 - 1 + (x_2 - 1)^2$$
$$= 10x_1^2 - 9x_2^2 + x_2^2 - 2x_2$$
$$= 10x_1^2 - 8x_2^2 - 2x_2$$

(a) Gradient :

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 20x_1 \\ -16x_2 - 2 \end{bmatrix}$$

at $x_0 = [0 \ -1]^T$ $\quad \nabla f(x_0) = \begin{bmatrix} 0 \\ 14 \end{bmatrix}$

Hessian :

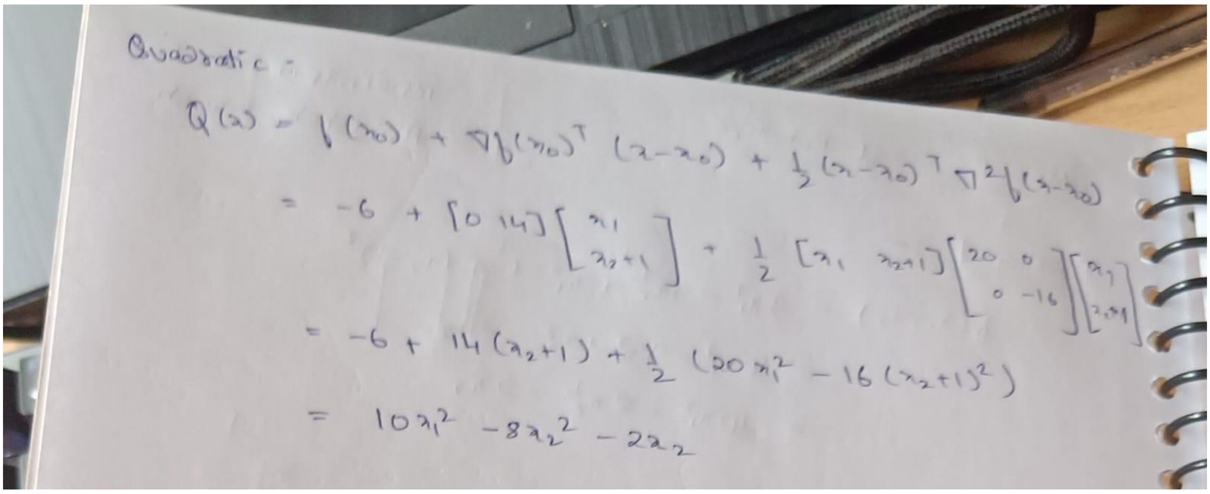
$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\nabla^2 f(x_0) = \begin{bmatrix} 20 & 0 \\ 0 & -16 \end{bmatrix}$$

(b) Linear :

$$L(x) = f(x_0) + \nabla f(x_0)^T (x - x_0)$$
$$= f(0, -1) + [0, 14] \begin{bmatrix} x_1 - 0 \\ x_2 - (-1) \end{bmatrix}$$

$$= 14x_2 + 8$$



Quadratic:

$$Q(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f (x - x_0)$$

$$= -6 + \begin{bmatrix} 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2+1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2+1 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2+1 \end{bmatrix}$$

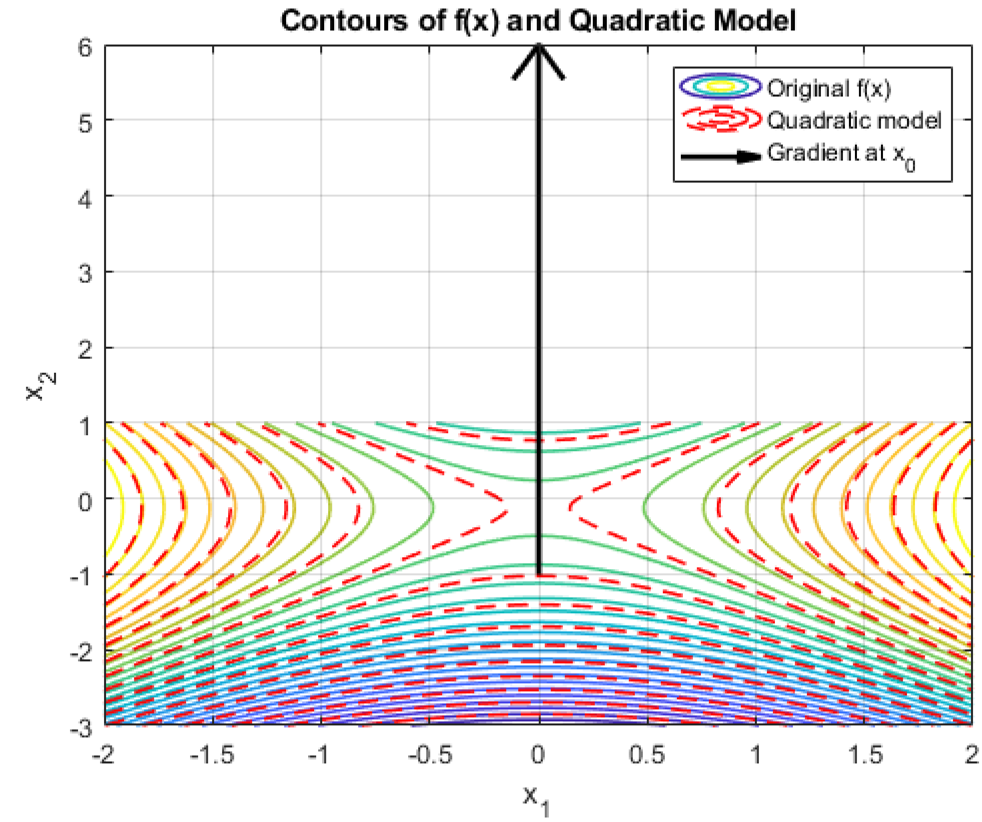
$$= -6 + 14(x_2+1) + \frac{1}{2}(20x_1^2 - 16(x_2+1)^2)$$

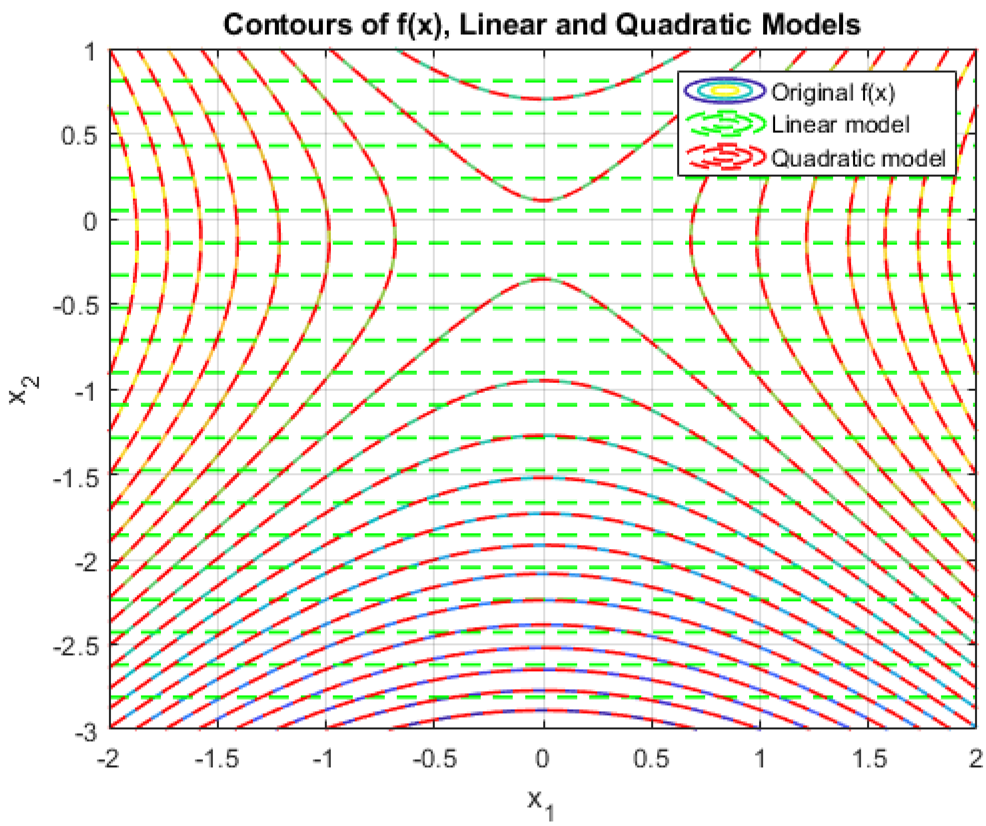
$$= 10x_1^2 - 8x_2^2 - 2x_2$$

Code:

```
% Define grid
[x1, x2] = meshgrid(linspace(-2,2,100), linspace(-3,1,100));

% Original function
f = 10*x1.^2 - 8*x2.^2 - 2*x2;

% Linear model (s = x - x0 = [x1, x2 + 1])
s1 = x1;
s2 = x2 + 1;
m1 = -6 + 14*s2;

% Quadratic model
m2 = -6 + 14*s2 + 10*s1.^2 - 8*s2.^2;

% Plotting
figure;
contour(x1, x2, f, 30, 'LineWidth', 1.2); hold on;
contour(x1, x2, m2, 15, '--r', 'LineWidth', 1.2);
quiver(0, -1, 0, 14, 0.5, 'k', 'LineWidth', 2);
legend('Original f(x)', 'Quadratic model', 'Gradient at x_0');
title('Contours of f(x) and Quadratic Model');
xlabel('x_1'); ylabel('x_2');
grid on;

% Plot linear model as separate figure
figure;
contour(x1, x2, f, 20, 'LineWidth', 1.2); hold on;
contour(x1, x2, m1, 20, '--g', 'LineWidth', 1.2);
contour(x1, x2, m2, 20, '--r', 'LineWidth', 1.2);
legend('Original f(x)', 'Linear model', 'Quadratic model');
title('Contours of f(x), Linear and Quadratic Models');
xlabel('x_1'); ylabel('x_2');
grid on;
```

Plots:


Contours of f(x) and Quadratic Model
Original f(x)
Quadratic model
Gradient at x_0
x_1
x_2


Contours of f(x), Linear and Quadratic Models
Original f(x)
Linear model
Quadratic model
x_1
x_2

Question 2

Question 2 :—

$\min f(x) = x_1^2 + x_2^2$

sub to $g_1(x) = 25 - x_1 x_2 \le 0 \qquad g_2(x) = 2 - x_1 \le 0$

Let Lagrangian

$L = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$

① KKT conditions are

stationarity : $\nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 g_2(x) = 0$

Primal feasibility : $g_1(x) \le 0 \ , \ g_2(x) \le 0$

Dual feasibility : $\lambda_1 \geq 0 \ , \ \lambda_2 \geq 0$

Complementary Slackness : $\lambda_1 g_1(x) = 0 \ , \ \lambda_2 g_2(x) = 0$

$\Rightarrow \quad 2x_1 - \lambda x_2 - \lambda_2 = 0 \ , \ 2x_2 - \lambda_1 x_1 = 0$

$25 - x_1 x_2 \le 0 \qquad 2 - x_1 \le 0$

$\lambda_1 \geq 0 \qquad \lambda_2 \geq 0$

$\lambda_1 (25 - x_1 x_2) = 0 \ , \quad \lambda_2 (2 - x_1) = 0$

② Applying KKT conditions

$$\begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0$$

Assume both constraints active

$\rightarrow \quad x_1 x_2 = 25 \qquad x_1 = 2 \quad \Rightarrow \quad x_1 = 2 \ , \ x_2 = 12.5$

$2x_1 - \lambda_1 x_2 - \lambda_2 = 0$ ———— ①

$2x_2 - \lambda_1 x_1 = 0$ ———— ②

from ② $\quad 2x_2 = \lambda_1 x_1 \Rightarrow \lambda_1 = 12.5$

from ① $\quad 2x_1 = \lambda_1 x_2 + \lambda_2 \Rightarrow \lambda_2 = -152.25$

$\lambda_2 < 0 \longrightarrow$ violates dual feasibility

$\longrightarrow$ constraint g_2 is inactive

So only $g_1(x)$ is active $(\lambda_1 > 0, \lambda_2 = 0)$

from stationarity

$$\Rightarrow \left.\begin{aligned} 2x_1 - \lambda_1 x_2 &= 0 \\ 2x_2 - \lambda_1 x_1 &= 0 \end{aligned}\right\} \quad \begin{aligned} \lambda_1 &= 2x_1/x_2 \\ \lambda_1 &= 2x_2/x_1 \end{aligned}$$

$\Rightarrow x_1 = x_2$

from constraint $25 = x_1 x_2 \Rightarrow x_1 = x_2 = 5$

Optimal soluth $\quad x^* = [5 \; 5]^T$

$$f(x^*) = 5^2 + 5^2 = 50$$

③ $\quad L(x, \lambda_1) = x_1^2 + x_2^2 + \lambda_1(25 - x_1 x_2)$

Hessian of Lagrangian

$$\nabla^2 L = \begin{bmatrix} 2 & -\lambda_1 \\ -\lambda_1 & 2 \end{bmatrix} \quad @ \; [5 \; 5]^T, \; \lambda_1 = 2 \quad \nabla^2 L = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Let $d \in C$ satisfying linearized constraints

$\nabla g_1^T d = 0 \Rightarrow -x_2 d_1 - x_1 d_2 = 0 \Rightarrow d_1 = -d_2$

take $\quad d = [1 \; -1]^T$

Second order sufficient conditions $\longrightarrow$

$$d^T \nabla^2 L \, d = [1 \; -1] \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 8 > 0$$

(4)

$\min f(x) = x_1^2 + x_2^2$

s/t $\quad g_1(x) = 25.1 - x_1 x_2 \leq 0 \qquad g_2(x)\ 1.9 - x_1 \leq 0$

assume both constraint active

$x_1 x_2 = 25.1 \qquad x_1 = 1.9 \implies x_1 = 1.9,\ x_2 \approx 13.21$

from KKT stationarity :

$2x_1 - \lambda_1 x_2 - \lambda_2 = 0$

$2x_2 - \lambda_1 x_1 = 0$

Solving we get $\lambda_1 \approx 13.93$

& $\lambda_2 = -180.19 < 0$

$\lambda_2 < 0 \longrightarrow$ dual infeasibility $(\lambda_1 \geq 0,\ \lambda_2 = 0)$

try only $g_1(x)$ constraint

from stationarity,

$$\left.\begin{aligned} 2x_1 - \lambda_1 x_2 &= 0 \\ 2x_2 - \lambda_1 x_1 &= 0 \end{aligned}\right\} \quad x_1 = x_2$$

From constraint, $25.1 = x_1 x_2 \implies x_1 = x_2 = \sqrt{25.1}$

check $g_2(x) \longrightarrow 1.9 - 5.01 < 0$

Optimal soln $\quad x^* = [5.01 \quad 5.01]^T$

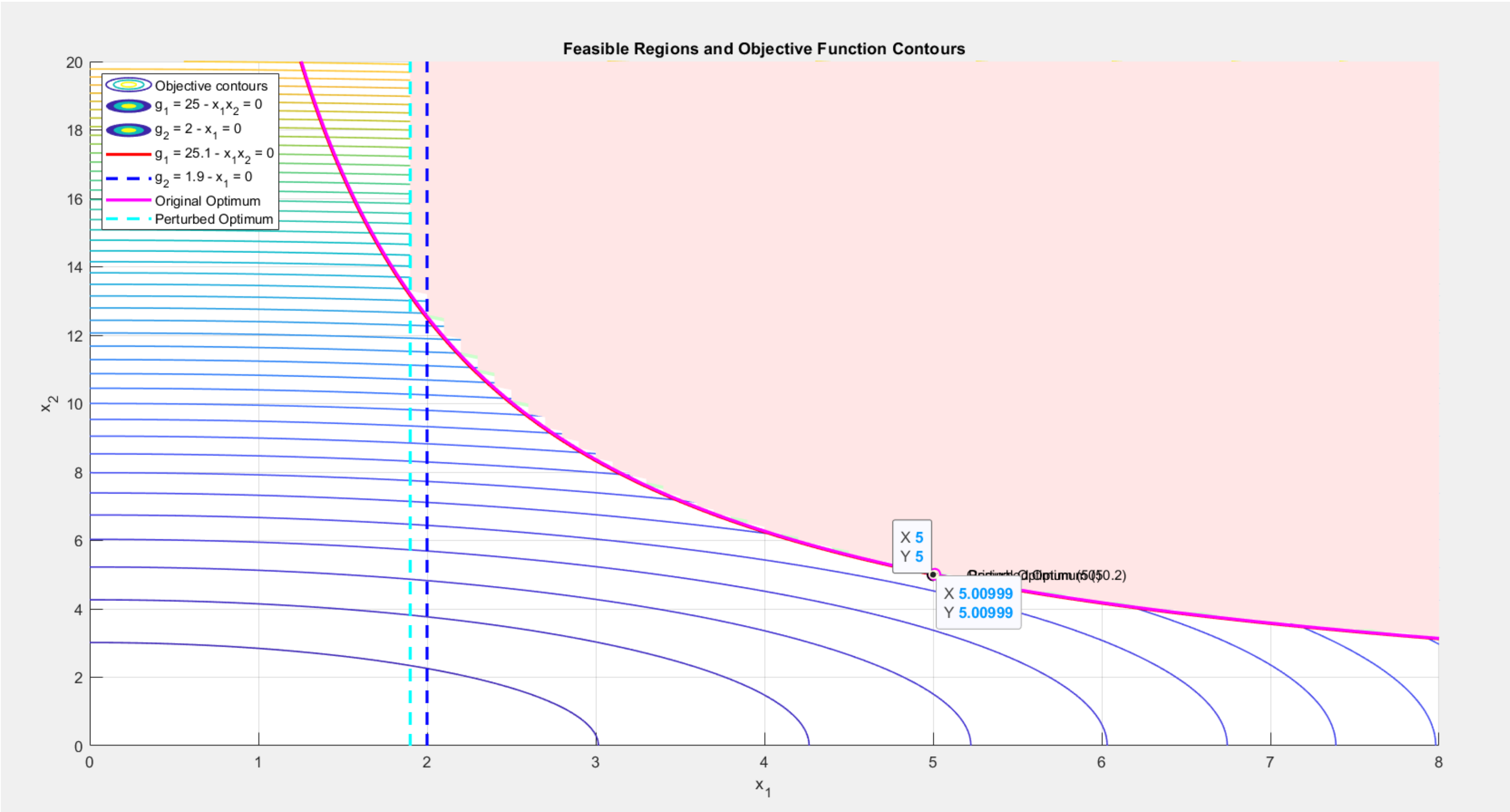
$f(x^*) = 50.2$

Code:

```
clc; clear;



% Grid for contour plots
[x1, x2] = meshgrid(0:0.1:8, 0:0.1:20);
f = x1.^2 + x2.^2;

% Constraints (original and perturbed)
g1 = 25 - x1 .* x2;
```

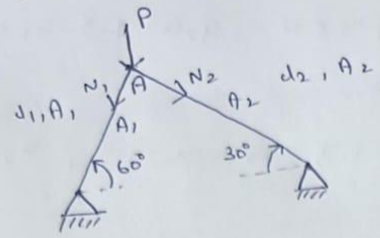
```
g2 = 2 - x1;

g1_pert = 25.1 - x1 .* x2;
g2_pert = 1.9 - x1;

% Feasible region masks
feasible_orig = double((g1 <= 0) & (g2 <= 0));
feasible_pert = double((g1_pert <= 0) & (g2_pert <= 0));

% Plot
figure;
hold on;

% Contours of the objective function
contour(x1, x2, f, 50, 'LineWidth', 1);

% Shade feasible regions
contourf(x1, x2, feasible_orig, [1 1], 'FaceColor', [0.8 1 0.8], 'LineColor',
'none');
contourf(x1, x2, feasible_pert, [1 1], 'FaceColor', [1 0.9 0.9], 'LineColor',
'none');

% Constraints (boundaries)
contour(x1, x2, g1, [0 0], 'r', 'LineWidth', 2);         % g1 = 0
contour(x1, x2, g2, [0 0], 'b--', 'LineWidth', 2);       % g2 = 0
contour(x1, x2, g1_pert, [0 0], 'm', 'LineWidth', 2);    % g1 pert = 0
contour(x1, x2, g2_pert, [0 0], 'c--', 'LineWidth', 2);  % g2 pert = 0

% Optimal points
plot(5, 5, 'ko', 'MarkerSize', 8, 'MarkerFaceColor', 'k');
text(5.2, 5, 'Original Optimum (50)', 'FontSize', 9);

x_opt = sqrt(25.1);
plot(x_opt, x_opt, 'mo', 'MarkerSize', 8, 'MarkerFaceColor', 'm');
text(x_opt + 0.2, x_opt, sprintf('Perturbed Optimum (%.1f)', 2*x_opt^2),
'FontSize', 9);

% Labels and legend
xlabel('x_1'); ylabel('x_2');
title('Feasible Regions and Objective Function Contours');

legend({'Objective contours', ...
        'g_1 = 25 - x_1x_2 = 0', ...
        'g_2 = 2 - x_1 = 0', ...
        'g_1 = 25.1 - x_1x_2 = 0', ...
        'g_2 = 1.9 - x_1 = 0', ...
        'Original Optimum', ...
        'Perturbed Optimum'}, ...
        'Location', 'northwest');

grid on;
axis([0 8 0 20]);
```

Plots:


Feasible Regions and Objective Function Contours
Objective contours
g1 = 25 - x1x2 = 0
g2 = 2 - x1 = 0
g1 = 25.1 - x1x2 = 0
g2 = 1.9 - x1 = 0
Original Optimum
Perturbed Optimum
X 5
Y 5
X 5.00999
Y 5.00999
x1
x2
0
1
2
3
4
5
6
7
8
0
2
4
6
8
10
12
14
16
18
20

Question 3

Question 3 :–



① Equilibrium eqⁿˢ

$$\left.\begin{aligned} N_1 \sin 60^\circ + N_2 \sin 30^\circ &= P \\ N_1 \cos 60^\circ &= N_2 \cos 30^\circ \end{aligned}\right\} \Rightarrow \begin{aligned} N_1 &= \frac{\sqrt{3}}{2} P \\ N_2 &= \frac{1}{2} P \end{aligned}$$

where N_1 & N_2 are internal axial forces

(i) Total strain energy

$$U = \sum_{i=1}^{2} \frac{N_i^2 l_i}{2 E_i A_i}$$

$$\Rightarrow \quad U = \frac{3}{8} \frac{P^2 l_1}{E_1 A_1} + \frac{P^2 l_2}{8 E_2 A_2}$$

using $\quad P \delta_A = 2U$

obj functⁿ $\Rightarrow \quad \delta_A = \frac{2U}{P} = \frac{3}{4} \frac{P l_1}{E_1 A_1} + \frac{P l_2}{4 E_2 A_2}$

(ii) Constraints :

ⓐ upper limit of stress

$$\sigma_1 = \frac{N_1}{A_1} \le S_y \qquad \sigma_2 = \frac{N_2}{A_2} \le S_y$$

$$\Rightarrow \quad A_1 \geq \frac{N_1}{S_y} \quad \& \quad A_2 \geq \frac{N_2}{S_y}$$

(b) Volume constraint

$$A_1 l_1 + A_2 l_2 \le V^* = V_{max}$$

(2) Given, $E_1 = E_2 = 210$ GPa

$l_1 = 1m \quad l_2 = 1.73$ m

$P = 10$ kN $\quad S_y = 250$ mPa

Putting values we get

$$\delta_A = \frac{3}{4} \frac{10^4 \times 1}{210 \times 10^9 A_1} + \frac{10^4 \times 1.73}{4 \times 210 \times 10^9 A_2}$$

$$= \frac{1}{840 \times 10^5} \left(\frac{3}{A_1} + \frac{1.73}{A_2} \right)$$

subject to $A_1 \geq 3.464 \times 10^{-5}$

$A_2 \geq 2 \times 10^{-5}$

& $A_1 \cdot 1 + A_2 \cdot 1.73 \le V_{max}$

at critical conditions $V_{max} \approx 10^{-4} m^2 \left[\frac{P}{S_y} \cdot 1 + \frac{P}{S_y} 1.73 \right]$

assuming both constraint active

$A_1 = 3.464 \times 10^{-5} \quad A_2 = 2 \times 10^{-5}$

$V = A_1 \cdot 1 + A_2 \cdot 1.73 = 6.924 \times 10^{-5} < 10^{-4}$

$\rightarrow \lambda_3 < 0 \rightarrow$ inactive

Let $K = \frac{10^4}{210 \times 10^9}$

Lagrangian,

$$L = K \left(\frac{0.75}{A_1} + \frac{0.4325}{A_2} \right) + \lambda_1 \left(\frac{8660}{A_1} - 250 \times 10^6 \right)$$

$$+ \lambda_2 \left(\frac{5000}{A_2} - 250 \times 10^6 \right) + \lambda_3 \left(A_1 + 1.73 A_2 - 10^{-4} \right)$$

Applying KKT,

from stationarity

$$-K\frac{0.75}{A_1^2} - \lambda_1\frac{8660}{A_1^2} + \lambda_3 = 0$$

$$-K\frac{0.4325}{A_2^2} - \lambda_2\frac{5000}{A_2^2} + 1.73\lambda_3 = 0$$

Vol constraint inactive $\lambda_3 = 0$

we get $\lambda_1 = -K\frac{0.75}{8660}$ & $\lambda_2 = -K\frac{0.4325}{5000}$

$\Rightarrow \lambda_1, \lambda_2 < 0 \longrightarrow$ violates dual feasibility

Taking Vol constraint & one stress constraint active

Trying $A_1 = 3.464 \times 10^{-5}$

Vol constraint $A_2 = \frac{V_{max} - A_1}{1.73} = 3.776 \times 10^{-5}$

$$\sigma_2 = \frac{5000}{A_2} \approx 132.4 \times 10^6 \text{ Pa} < 250 \text{ MPa}$$

$\Rightarrow \quad A_1 = 3.464 \times 10^{-5} \text{ m}^2$

$\quad\quad A_2 = 3.776 \times 10^{-5} \text{ m}^2$

$\Rightarrow \delta_A = 1.5764$ mm

Question 4

Question 4 :—

(a) Assume $y_i = 1/A_i$

$\Rightarrow$ obj function

$$\delta_A = \frac{3}{4}\frac{Pd_1}{E_1}y_1 + \frac{1}{4}\frac{Pd_2}{E_2}y_2$$

with constraints

$$\frac{N_1}{A_1} \le S_y \Rightarrow y_1 \le \frac{S_y}{N_1}$$

$$y_2 \le \frac{S_y}{N_2}$$

& vol constraint $\frac{d_1}{y_1} + \frac{d_2}{y_2} \le V^*$

let $K = \frac{10^4}{210 \times 10^9}$

$y_1 \le 2.887 \times 10^4$, $y_2 \le 5 \times 10^4$, $\frac{1}{y_1} + \frac{1.73}{y_2} \le 10^{-4}$

on $K(0.75y_1 + 0.4325\, y_2)$

Lagrangian

$$L = K(0.75y_1 + 0.4325y_2) + \lambda_1(8660y_1 - 250\times10^6) + \lambda_2(5000y_2 - 250\times10^6) + \lambda_3\left(\frac{1}{y_1} + \frac{1.73}{y_2} - 10^{-4}\right)$$

Applying KKT

from stationarity

$$0.75K + \lambda_1 8660 - \frac{\lambda_3}{y_1^2} = 0$$

$$0.4325K + \lambda_2 5000 - \frac{\lambda_3 1.73}{y_2^2} = 0$$

Vol constraint, $\lambda_3 = 0$ would yield λ_1 & $\lambda_2 < 0$
$\rightarrow$ violating dual feasibility

Taking vol constraint & one stress constraint active

Trying, $y_1 = 2.887 \times 10^4 \ m^{-2}$ 0.433

$$\frac{1.73}{y_2} = 10^{-4} - \frac{1}{y_1}$$

$$\Rightarrow y_2 = \underline{2.64 \times 10^4} < 5 \times 10^4 \ m^{-2}$$

$$\Rightarrow A_1 = \frac{1}{y_1} = 3.463 \times 10^{-5} \ m^2$$

$$A_2 = \frac{1}{y_2} = 3.787 \times 10^{-5} \ m^2$$

$$\delta_A = \frac{10^4}{210 \times 10^9}\left(0.75 \times 2.887 \times 10^4 + 0.4325 \times 2.64 \times 10^4\right)$$

$$= 1.5747 \text{ mm}$$

(b) Adding Buckling constraint

for Euler buckling
$$P_{cr} = \frac{\pi^2 E I_i}{l_i^2}, \quad I_i = \alpha A_i^2$$

$$\Rightarrow \text{for compression bars, } f_i \le \frac{\pi^2 E \alpha A_i^2}{l_i^2}$$

Bar 1 in compression and Bar 2 in tension
$\Rightarrow$ adding constraint only for bar 1

$$8660 \le \frac{\pi^2 E_1 \alpha A_1^2}{l_1^2} \Rightarrow A_1 \geq \sqrt{\frac{8660 \cdot l_1^2}{\pi^2 E_1 \alpha}}$$

Obj funct^n

$$\min \ \delta_A = \frac{P}{E}\left(\frac{0.75}{A_1} + \frac{0.4325}{A_2}\right)$$

with sub/to

Stress constraint : $A_1 \geq 3.464 \times 10^{-5}$

$A_2 \geq 2 \times 10^{-5}$

Buckling constraint : $A_1 \geq \sqrt{\frac{8660}{\pi^2 E \alpha}}$

Volume constraint : $A_1 + 1.73 A_2 \leq V_{max} = 10^{-4}$

Lagrangian :

$$L = \frac{P}{E}\left(\frac{0.75}{A_1} + \frac{0.4325}{A_2}\right) + \lambda_1\left(\frac{8660}{A_1} - 250 \times 10^6\right)$$
$$+ \lambda_2\left(\frac{5000}{A_2} - 250 \times 10^6\right) + \lambda_3\left(A_1 + 1.73 A_2 - 10^{-4}\right)$$
$$+ \lambda_4\left(8660 - \frac{\pi^2 E_1 \alpha A_1^2}{l_1^2}\right)$$

Applying KKT

from stationarity,

$$-\frac{P}{E}\left(\frac{0.75}{A_1^2}\right) - \frac{8660 \lambda_1}{A_1^2} + \lambda_3 - \lambda_4 \pi^2 E_1 \alpha \, 2 A_1 = 0$$

$$-\frac{P}{E}\left(\frac{0.4325}{A_2^2}\right) - \frac{5000 \lambda_2}{A_2^2} + 1.73 \lambda_3 = 0$$

$$A_1^* = \max\left(3.464 \times 10^{-5}, \sqrt{\frac{8660}{\pi^2 \, 210 \times 10^9 \, \alpha}}\right)$$

$$A_2^* = \max\left(2 \times 10^{-5}, \frac{10^{-4} - A_1^*}{1.73}\right)$$