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Extra notes related to 9th Lecture on ODE

(MA-1150)

For nth order linear homogeneous ODE:

The nth order linear homogeneous ODE is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0.$$

The auxiliary eq^n is

$$\boxed{a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} \cdot m + a_n = 0}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n number of roots of the above eq^n.

Case-I : Roots are real and distinct

Roots are $\alpha_1, \alpha_2, \dots, \alpha_n$,

now the general solution is

$$\boxed{y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \cdots + c_n e^{\alpha_n x}}$$

Case-II : Roots are real but repeated :

Roots are : $\alpha_1 = \alpha_2 = \alpha, \alpha_3, \alpha_4, \dots, \alpha_n$

Then the general solution is

$$\boxed{y(x) = (c_1 + c_2 x)e^{\alpha x} + c_3 e^{\alpha_3 x} + c_4 e^{\alpha_4 x} + \cdots + c_n e^{\alpha_n x}}$$

<u>Case-III</u> : <u>Roots are complex and nonrepeated</u>

Let roots be $\alpha_1 = \alpha + i\beta$, $\alpha_2 = \alpha - i\beta$, $\alpha_3, \ldots \ldots \alpha_n$. where $\alpha_3 \ldots \alpha_n$ are real and distinct

The general solⁿ is

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{\alpha_3 x} + \cdots + c_n e^{\alpha_n x}$$

Case-IV : Roots are complex and repeated:

Let roots be $\alpha_1 = \alpha + i\beta$, $\alpha_2 = \alpha + i\beta$, $\alpha_3 = \alpha - i\beta$,

$\alpha_4 = \alpha - i\beta$, α_5, α_6, ... α_n.,

Then the general sol^n is

$$y(x) = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\,\cancel{\cos\beta x}\ \sin\beta x\right] + c_5 e^{\alpha_5 x} + \cdots + c_n e^{\alpha_n x}$$

Ex. $$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$$

or $(D^3 - 3D^2 + 4)y = 0.$

The auxiliary eqn is $m^3 - 3m^2 + 4 = 0$

or $(m+1)(m^2 - 4m + 4) = 0$

$m = -1, 2, 2$

$\boxed{m = -1 : \ e^{-x}}$, $\boxed{\begin{array}{l} m = 2 ; \quad e^{2x} \\ \qquad\qquad xe^{2x} \end{array}}$

The general solution is

$$y(x) = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}.$$

Ex $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

or, $(D^4 + 8D^2 + 16)y = 0.$

The A. E. is $m^4 + 8m^2 + 16 = 0$

or, $(m^2+4)^2 = 0 \Rightarrow m = \pm 2i, \pm 2i.$

$$\boxed{\begin{aligned} m = 2i : \quad & e^{2ix} \longrightarrow \cos 2x \\ & xe^{2ix} \longrightarrow x\cos 2x \end{aligned}}$$

linearly independent solutions

$$\boxed{\begin{aligned} m = -2i : \quad & e^{-2ix} \longrightarrow \sin 2x \\ & xe^{-2ix} \longrightarrow x\sin 2x \end{aligned}}$$

linearly independent solutions.

Now the general solution is

$$\begin{aligned} y(x) &= c_1 \cos 2x + c_2 x \cos 2x + c_3 \sin 2x + c_4 x \sin 2x \\ &= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x. \end{aligned}$$

Ex $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

or $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

The auxiliary eqn is $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

or, $m = 1, 1, \pm 2i$

$\boxed{\begin{array}{l} m=1: e^x \\ \quad\quad\; xe^x \end{array}}$, $\begin{array}{l} m = 2i : e^{2ix} \rightarrow \cos 2x \\ m = -2i : e^{-2ix} \rightarrow \sin 2x. \end{array}$

The general solution is -

$$y(x) = (c_1 + c_2 x)e^{x} + (c_3 \cos 2x + c_4 \sin 2x)$$

<u>Ex.</u> $\dfrac{d^4y}{dx^4} + a^4 y = 0$

or $(D^4 + a^4)\,y = 0.$

The auxiliary eqn is $m^4 + a^4 = 0.$

$$\Rightarrow m = \pm \frac{1}{\sqrt{2}}(1+i)a$$

$$= \pm \frac{1}{\sqrt{2}}(1-i)a$$

$$\boxed{m = \frac{1}{\sqrt{2}}(1+i)a} \Rightarrow e^{\frac{1}{\sqrt{2}}(1+i)ax} = e^{\frac{a}{\sqrt{2}}x + i\frac{a}{\sqrt{2}}x} \rightarrow \boxed{e^{\frac{a}{\sqrt{2}}x} \cdot \cos\left(\frac{a}{\sqrt{2}} \cdot x\right)}$$

$$\boxed{m = \frac{1}{\sqrt{2}}(1-i)a} \Rightarrow e^{\frac{1}{\sqrt{2}}(1-i)ax} = e^{\frac{a}{\sqrt{2}}x - i\frac{a}{\sqrt{2}}x} \rightarrow \boxed{e^{\frac{a}{\sqrt{2}}x} \cdot \sin\left(\frac{a}{\sqrt{2}} \cdot x\right)}$$

$$\boxed{m = -\frac{1}{\sqrt{2}}(1-i)a}$$

$$= -\frac{a}{\sqrt{2}} + i\frac{a}{\sqrt{2}} = e^{-\frac{a}{\sqrt{2}}x} \cdot e^{i\frac{a}{\sqrt{2}}x} \rightarrow \boxed{e^{-\frac{a}{\sqrt{2}}x} \cdot \cos\left(\frac{a}{\sqrt{2}}x\right)}$$

$$\boxed{m = -\frac{1}{\sqrt{2}}(1+i)a} = -\frac{a}{\sqrt{2}} - i\frac{a}{\sqrt{2}} = e^{-\frac{a}{\sqrt{2}}x} \cdot e^{-i\frac{a}{\sqrt{2}}x} \rightarrow \boxed{e^{-\frac{a}{\sqrt{2}}x} \sin\left(\frac{a}{\sqrt{2}}x\right)}$$

The general solution is

$$y(x) = e^{\frac{a}{\sqrt{2}}x}\left(c_1 \cos\left(\frac{a}{\sqrt{2}}x\right) + c_2 \sin\left(\frac{a}{\sqrt{2}}x\right)\right)$$

$$+ e^{-\frac{a}{\sqrt{2}}x}\left(c_3 \cos\left(\frac{a}{\sqrt{2}}x\right) + c_4 \sin\left(\frac{a}{\sqrt{2}}x\right)\right)$$

Ex. $(D^4 - a^4)y = 0$

The auxiliary eqn is

$m^4 - a^4 = 0 \Rightarrow m = \pm a, \pm ai$

The general soln is

$y(x) = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$