WORKSHEET - 2 - KEY

1. Determine the derived sets of the following sets.

   (a) $E = \{x^2 : x \in \mathbb{R}\} = [0, \infty)$

   So, $E' = [0, \infty)$

   (b) $E = \{1/x : x \in \mathbb{R}^*\} = \mathbb{R}^*$

   So, $E' = \mathbb{R}$

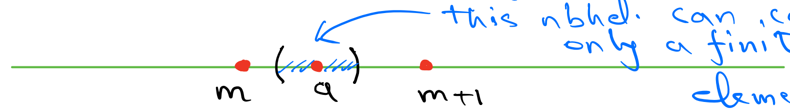
   (c) $E = \{m + 1/n : m, n \in \mathbb{N}\}$ Ans: $\overline{E} = E$

   We show that $E' = \mathbb{N} \subseteq E$

   If $m \in \mathbb{N}$, fixed, then by Arch. Prop., every nbhd. of m contains all but finitely many elements of the form $m + \frac{1}{n}$.
   $\Rightarrow$ every nbhd. of m contains ∞-ly many elements of E $\Rightarrow$ $m \in E'$. Hence, $\mathbb{N} \subseteq E'$.

   Next suppose, $a \in E'$. Since $x > 1$ $\forall$ $x \in E$, deduce that $a \geq 1$. By Arch. Property, $\exists!$ $m \in \mathbb{N}$ s.t. $m \leq a < m+1$. Claim: $a = m$. Else, if $m < a < m+1$, then $\exists$ a nbhd. of a which contains at most finitely many elements of E. (Follow the arguments given to show that 0 is the only limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

   But this implies that $a \notin E'$. Thus $a = m$ $\Rightarrow$ $E' \subseteq \mathbb{N}$, as required.



this nbhd. can contain only a finite no. of elements of E.

2. Let $E = \{x \in \mathbb{Q} : 0 < x < 1\}$. Describe $\overline{E}$. $\overline{E} = [0,1]$

Claim: $E' = [0,1]$!

Let $x \in \mathbb{R}$. If $x \in [0,1]$, then every nbhd. of x contains ∞-ly many elements of E (by the density property of rationals). While if $x \notin [0,1]$, then $\exists$ a deleted nbhd. I of x s.t.

$I \cap E = \phi$.

means $\Longleftrightarrow$

3. Let $E \subset \mathbb{R}$ and $\ell \in \mathbb{R}$. Then prove that ℓ is a limit point of E if and only if for every integer $n \in \mathbb{N}$, the deleted neighbourhood $\left(a - \frac{1}{n}, a + \frac{1}{n}\right)'$ contains an element of E.

"$\Rightarrow$" direction is straightforward

"$\Leftarrow$" By the Arch. Property, every deleted nbhd. of a contains a nbhd. of the form $\left(a-\frac{1}{n}, a+\frac{1}{n}\right)$. Now finish the proof.

4. Show that every nonempty open subset of $\mathbb{R}$ is a union (not necessarily a finite union) of intervals.

If $U \subseteq \mathbb{R}$ is open, & $x \in U$, then $\exists$ a nbhd. I_x of x s.t. $x \in I_x \subseteq U$

Thus,

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} I_x \subseteq U$$

since $I_x \subseteq U \ \forall x \in U$

$\bigcup_{x \in U} I_x \subseteq U$

Hence $U = \bigcup_{x \in U} I_x$ intervals.

5. Let $\{U_\alpha : \alpha \in \mathcal{I}\}$ be a collection of open subsets of $\mathbb{R}$. Prove that $U = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ is also an open subset of $\mathbb{R}$.

$x \in U \Rightarrow x \in U_\alpha$ for some $\alpha \in \mathcal{I}$

U_α - open $\Rightarrow \exists$ a nbhd. I_x of x s.t. $I_x \subseteq U_\alpha$

$\Rightarrow I_x \subseteq U$

$\Rightarrow U$ is open.

6. Now let $\{V_\alpha : \alpha \in \mathcal{I}\}$ be a collection of closed subsets of $\mathbb{R}$. Use the previous exercise to show that $\bigcap_{\alpha \in \mathcal{I}} V_\alpha$ is closed.

Let $V = \bigcap_{\alpha \in \mathcal{I}} V_\alpha$. Then $V^c = \bigcup_{\alpha \in \mathcal{I}} V_\alpha^c$ (De'Morgan's Law)

V_α - closed $\Rightarrow V_\alpha^c$ - open

by 5), V^c is open

$\Rightarrow V$ is closed!

7. Show that the intersection of finitely many open sets is open. Deduce that a finite union of closed sets is closed.

A, B-closed
$\Rightarrow A^c, B^c$-open
$\Rightarrow A^c \cap B^c$-open
$\Rightarrow A \cup B$-closed

Suffices to prove for two open sets (then extend by induction)

So, let A, B be open, and $x \in A \cap B$. $\exists$ nbhds. $x \in I_x \subseteq A$ & $J_x \subseteq B$. Note that either $I_x \subseteq J_x$ or $J_x \subseteq I_x$.

Let I be the smaller of I_x and J_x, then

$I \subseteq A$ and $I \subseteq B \Rightarrow I \subseteq A \cap B$.

8*. Suppose $E \subset \mathbb{R}$ has no limit points. Show that E is at most countable. (Hint: Since every point of such a set E is an isolated point, so each point x of E has a neighbourhood I_x such that $I_x \cap E = \{x\}$. Fix some rational $q_x \in I_x$. Now show that the map $E \to \mathbb{Q}$ sending $x \to q_x$ is well defined and injective. Now conclude!)

This may not work. The correct hint is as follows: Let ε_x be the radius of I_x. Consider $J_x = (x - \frac{\varepsilon_x}{2}, x + \frac{\varepsilon_x}{2})$. Fix $q_x \in J$, and

$f : E \longrightarrow \mathbb{Q}$

$f(x) = q_x$

Solution: Clearly, $x_1 = x_2 \Rightarrow q_{x_1} = q_{x_2}$, so $f(x)$ is well defined.

Now, $f(x_1) = f(x_2) \Rightarrow q_{x_1} = q_{x_2} \Rightarrow |x_i - q_{x_1}| < \frac{\varepsilon_{x_i}}{2}$ for $i = 1$ & 2.

Thus, $|x_1 - x_2| \leq |x_1 - q_{x_1}| + |x_2 - q_{x_2}|$

$< \frac{\varepsilon_{x_1}}{2} + \frac{\varepsilon_{x_2}}{2} \leq \max\{\varepsilon_{x_1}, \varepsilon_{x_2}\} \Rightarrow x_1 \in J_{x_2}$ or $x_2 \in J_{x_1}$

$\Rightarrow x_1 = x_2$ (by construction of J_x)

Thus $E \hookrightarrow \mathbb{Q}$, hence countable

9. Show that following sequences are convergent using the definition of a limit (that is, use the ε-definition of limit).

(a) $\frac{1}{2^n}$

Given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t $\frac{1}{n} < \varepsilon \quad \forall\, n \geq n_0$ (By AP 2 WOP)

Thus, if $n \geq n_0$, then

$$\left|\frac{1}{2^n}\right| < \frac{1}{n} < \varepsilon \quad \forall\, n \geq n_0 \implies \frac{1}{2^n} \to 0.$$

(b) $\frac{\cos n}{n}$

$$\left|\frac{\cos n}{n}\right| \leq \frac{1}{n} < \varepsilon \quad \forall\, n \geq n_0$$

$$\implies \frac{\cos n}{n} \to 0.$$

10. Evaluate the following limits in whichever way you want. But state your method clearly.

(a) $\frac{\ln n}{n}$

$$n = e^{\ln n} = 1 + \ln n + \frac{(\ln n)^2}{2!} + \cdots > 1 + \ln n + \frac{(\ln n)^2}{2}.$$

Thus

$$0 < \frac{\ln n}{n} < \frac{\ln n}{1 + \ln n + \frac{(\ln n)^2}{2}} \implies \frac{\ln n}{n} \to 0$$

null sequences

an explanation is given at the end.

(b) $n^{1/n}$

$$n^{1/n} = e^{\frac{\ln n}{n}} \to e^0 = 1.$$

(c) $\left(1+\frac{x}{n}\right)^n$ where $x \in \mathbb{R}$ Ans: e^x

$\left(1+\frac{x}{n}\right)^n = e^{\left(n \ln\left(1+\frac{x}{n}\right)\right)}$, so, suff. to show $n \ln\left(1+\frac{x}{n}\right) \to 1$

$x=0$ is easy! So assume $x \neq 0$

By Arch. Prop. $\exists\ n_0$ s.t. $|x| < n\ \forall\, n \geq n_0$.

Take $n \geq n_0$, so that $0 < \frac{|x|}{n} < 1$

Thus, $n \ln\left(1+\frac{x}{n}\right) = n\left(\frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \cdots\right) = x\Big(1 - \underbrace{\frac{x}{2n} + \frac{x^2}{3n^2} - \cdots}_{\text{call } a_n}\Big)$

Let $a_n = \frac{x}{2n} - \frac{x^2}{3n^2} + \frac{x^3}{4n^3} - \cdots = \frac{x}{n}\left(\frac{1}{2} - \frac{x}{3n} + \frac{x^2}{4n^2} - \cdots\right)$

Suff. to show that $a_n \to 0$

Now, by Δ-inequality, we have for $n \geq n_0$

$$\left|\frac{1}{2} - \frac{x}{3n} + \frac{x^2}{4n^2} - \cdots\right| \leq \frac{1}{2} + \frac{|x|}{3n} + \frac{|x|^2}{4n^2} + \cdots \leq 1 + \frac{|x|}{n} + \frac{|x|^2}{n^2} + \cdots = \frac{1}{1-\frac{|x|}{n}}$$

Next, $\frac{1}{n} \to 0 \Rightarrow \frac{|x|}{n} \to 0 \Rightarrow 1 - \frac{|x|}{n} \to 1$

$\Rightarrow \frac{1}{1-\frac{|x|}{n}} \to 1$ since $1 - \frac{|x|}{n} \neq 0\ \forall\, n \geq n_0$

Thus, $0 \leq |a_n| \leq \frac{|x|}{n} \frac{1}{1-\frac{|x|}{n}} \to 0 \Rightarrow a_n \to 0$, as required.

(d) $\left(\frac{n+1}{n-1}\right)^n$ (starts from $n=2$)

$$\left(\frac{n+1}{n-1}\right)^n = \left(1+\frac{2}{n-1}\right)^n = \underbrace{\left(1+\frac{2}{n-1}\right)^{n-1}}_{\to e^2 \text{ by (c)}} \cdot \underbrace{\left(1+\frac{2}{n-1}\right)}_{\to 1}$$

By product rule,

$$\left(\frac{n+1}{n-1}\right)^n \to e^2$$

(e) $n \ln\left(1 - \frac{1}{n}\right)$ $= n\left(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right) = 1 + \frac{1}{2n} + \frac{1}{3n^2} + \cdots \quad \forall\, n>1$

$= 1 + a_n$

$a_n = \frac{1}{2n} + \frac{1}{3n^2} + \cdots$

$$a_n < \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots = \frac{1}{n-1}$$

Thus

$$0 < a_n < \frac{1}{n-1} \quad \forall\, n>1$$

$\Rightarrow \quad a_n \to 0$ (by Generalized Sandwich Thm.)

(f) $\dfrac{n^2}{(1+x)^n}$ where $x > 0$

Observe that for $n \geq 3$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \cdots$$

$$\geq \frac{n(n-1)(n-2)}{6}x^3$$

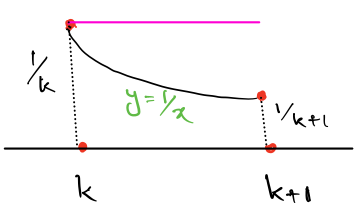
Thus

$$0 < \frac{n^2}{(1+x)^n} \leq \frac{6n^2}{n(n-1)(n-2)x^3} = \underbrace{\underbrace{\frac{1}{1-\frac{1}{n}}}_{\to 1} \cdot \underbrace{\frac{1}{n-2}}_{\to 0} \cdot \underbrace{\frac{6}{x^3}}_{\text{constant}}}_{\to 0} \quad \forall\, n \geq 3$$

$$\Rightarrow \frac{n^2}{(1+x)^n} \to 0$$

(g) $\dfrac{L_n}{n}$ where $L_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

since $\frac{1}{k+1} \leq \frac{1}{x} \quad \forall\, x \in [k, k+1]$, hence



$$\frac{1}{k+1}\int_k^{k+1} dx \leq \int_k^{k+1} \frac{dx}{x} = \ln(k+1) - \ln k$$

$$\frac{1}{k+1} \leq \frac{1}{x} \leq \frac{1}{k} \quad \forall\, x \in [k, k+1]$$

$$\Rightarrow \frac{1}{k+1} \leq \ln(k+1) - \ln k$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln n - \ln(n-1))$$

$$= 1 + \ln n$$

Thus, $\dfrac{L_n}{n} \leq \dfrac{1 + \ln n}{n} \to 0.$

11. Suppose $a_n \to \ell$ and $b_n \to s$, and that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then show that $\ell \leq s$.

Recall, if $c_n \geq 0 \ \forall\, n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} c_n \geq 0$ if it exists! (Prove this)

Now, $b_n - a_n \geq 0 \ \forall\, n$ (given)

hence, $\lim\limits_{n\to\infty} (b_n - a_n) \geq 0$

$$\Rightarrow \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \geq 0$$

$$\Rightarrow s - \ell \geq 0.$$

12. According to Stirling's estimate, one has

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}. \quad (*)$$

Making use of this information, evaluate $\lim\limits_{n\to\infty} (n!)^{1/n}$. $\leftarrow$ $(n!)^{1/n^2}$

Taking $1/n^2$-th power

$$(2\pi n)^{1/n^2} \cdot \left(\frac{n}{e}\right)^{1/n} \cdot e^{\frac{1}{n^2(12n+1)}} < (n!)^{1/n^2} < (2\pi n)^{1/n^2} \cdot \left(\frac{n}{e}\right)^{1/n} \cdot e^{\frac{1}{12n^3}}$$

- $(2\pi)^{1/n^2} \longrightarrow 1$
- $n^{1/n^2} \longrightarrow 1$
- $\frac{1}{n^2(12n+1)} \longrightarrow 0 \Rightarrow e^{\frac{1}{n(12n+1)}} \longrightarrow 1$
- similarly $e^{\frac{1}{12n^3}} \longrightarrow 1$
- $n^{1/n} \longrightarrow 1$
- $e^{1/n} \longrightarrow 1$

Thus,

$$\underbrace{a_n}_{\text{limit} = 1} \leq (n!)^{1/n^2} \leq \underbrace{b_n}_{\text{limit} = 1}$$

$$\Rightarrow (n!)^{1/n^2} \longrightarrow 1 \quad \text{(by Sandwich Thm.)}$$

13. Show that $n^{1/n} - 1$ is a null sequence.

$$0 < n^{1/n} - 1 = a_n + \frac{a_n^2}{2!} + \cdots \quad \text{where } a_n = \frac{\ln n}{n} < 1 \text{ for say, } n > 2.$$

$$< a_n\left(1 + a_n + a_n^2 + \cdots\right)$$

$$= \frac{a_n}{1 - a_n} = \underbrace{\underbrace{a_n}_{\to 0} \cdot \frac{1}{\underbrace{1 - a_n}_{\to 1}}}_{\to 0}$$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$

14. Suppose a sequence $\{a_n\}_n$ satisfies the property that $a_n \neq 0$ for all $n \in \mathbb{N}$, and that $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{2}$. Show that $a_n \to 0$. What would be your answer if $1/2$ is replaced by 1?

The idea is to bound $\{a_n\}$ by a null sequence.

$$\frac{a_{n+1}}{a_n} \longrightarrow \frac{1}{2}$$

For $\varepsilon = \frac{1}{4}$ (in fact, any $\varepsilon < \frac{1}{2}$ will do), $\exists$ a $n_0 \in \mathbb{N}$ s.t.

$$\left|\frac{a_{n+1}}{a_n} - \frac{1}{2}\right| < \frac{1}{4} \quad \forall\ n \geq n_0$$

$$\Rightarrow |a_{n+1}| < \frac{3}{4}|a_n| \quad \forall\ n \geq n_0$$

Let $n > n_0$, then $n = n_0 + k$ for some $k \in \mathbb{N}$.

Then $0 \leq |a_n| = |a_{k+n_0}| < \frac{3}{4}|a_{k+n_0-1}| < \left(\frac{3}{4}\right)^2 |a_{k+n_0-2}| < \cdots < \left(\frac{3}{4}\right)^k |a_{n_0}| = \left(\frac{3}{4}\right)^{n-n_0} |a_{n_0}|$

$$\Rightarrow 0 \leq |a_n| < \underbrace{\left(\frac{3}{4}\right)^n \cdot c}_{\to 0} \quad \text{where } c = \frac{|a_{n_0}|}{(3/4)^{n_0}}$$

$$\Rightarrow a_n \longrightarrow 0.$$

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then the statement does not hold.

e.g. if $a_n = n$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

but $\lim_{n\to\infty} n = \infty$.

15. Show that if $a_n \to 1$, then $a_n = 0$ for at most finitely many indices $n \in \mathbb{N}$.

For $\varepsilon = \frac{1}{2}$, $\exists\, n_0 \in \mathbb{N}$ s.t $|a_n - 1| < \frac{1}{2}$ $\forall\, n \geq n_0$

Thus, if $n \geq n_0$, then $|a_n| > \frac{1}{2}$

so that $a_n = 0 \Rightarrow n < n_0$, hence $a_n = 0$ for at most finitely man $n \in \mathbb{N}$.

16. Let $a_n = \sqrt{n+1} - \sqrt{n}$. Show that the sequences $\{a_n\}_n$, $\{na_n\}_n$ and $\{\sqrt{n}a_n\}_n$ are all convergent by computing their limits.

$$0 < a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \to 0$$

$$\Rightarrow a_n \to 0$$

$$n a_n = \frac{n}{\sqrt{n+1} + \sqrt{n}} \quad \text{so that,}$$

$$\underbrace{\frac{n+1}{4\sqrt{n+1}}}_{\to\infty} \leq \frac{n}{2\sqrt{n+1}} < n a_n < \underbrace{\frac{n}{\sqrt{n}}}_{\to\infty} \quad \forall\, n \in \mathbb{N}$$

hence $n a_n \to \infty$.

$$\sqrt{n}\, a_n = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1+\frac{1}{n}} + 1} \to \frac{1}{2}$$

or,

$\sqrt{n+1}$

17. Let $\{a_n\}_n$ be a sequence satisfying

$$|a_{n+1} - a_n| < \frac{1}{2^n} \quad \forall \quad n \in \mathbb{N}.$$

Show that $\{a_n\}_n$ is convergent.

We begin by showing that $\{a_n\}$ is bdd.

$$|a_{n+1} - a_n| < \frac{1}{2^n} \ \forall\ n \in \mathbb{N} \Rightarrow |a_{n+1}| < |a_n| + \frac{1}{2^n} \ \forall\ n \in \mathbb{N}.$$

Thus, $|a_n| < |a_{n-1}| + \frac{1}{2^{n-1}} < |a_{n-2}| + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} < \cdots < |a_1| + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

hence $\{a_n\}$ is bdd. $< |a_1| + 1 \ \forall\ n \in \mathbb{N}$.

By Bolzano-Weierstrass, $\{a_n\}$ has a convergent subsequence, say $\{a_{n_k}\}_k \to a$

We show that $a_n \to a$.

$a_{n_k} \to a \Rightarrow$ given $\varepsilon > 0$, $\exists\ k_0 \in \mathbb{N}$ s.t

$$|a_{n_k} - a| < \varepsilon/2 \ \forall\ k \geq k_0.$$

Now, if $n > n_k$, then

$$\begin{aligned}
|a_n - a_{n_k}| &= |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots (a_{n_k+1} - a_{n_k})| \\
&\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{n_k+1} - a_{n_k}| \\
&< \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^{n_k}} \\
&= \frac{1}{2^{n_k}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-n_k-1}}\right) < \frac{1}{2^{n_k-1}}
\end{aligned}$$

By AP, $\exists$ a $k_1 \in \mathbb{N}$ s.t $\frac{1}{2^{n_k-1}} < \varepsilon/2 \ \forall\ k \geq k_1$

Now, take $k_2 \geq \max\{k_0, k_1\}$ and $n \geq n_{k_2}$. Then

$$|a_n - a| \leq |a_n - a_{n_{k_2}}| + |a_{n_{k_2}} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $a_n \to a$

Note: $k_2 \geq k_1 \Rightarrow |a_n - a_{n_{k_2}}| < \frac{1}{2^{n_{k_2}-1}} < \frac{\varepsilon}{2}$

$k_2 \geq k_0 \Rightarrow |a_{n_{k_2}} - a| < \frac{\varepsilon}{2}$.

18. Show that the following recurrent sequences converge.

(a) $a_1 = 1$, and for $n > 1$, a_n is defined recursively by $a_n = \dfrac{2a_{n-1}+3}{4}$.

If the sequence converge, then $\ell = \dfrac{2\ell+3}{4}$ gives $\ell = 3/2$.

first few terms: $1, \frac{5}{4}, \frac{11}{8}, \ldots$

perhaps $\{a_n\}$ is monotone increasing & bdd.

- $a_n \le \frac{3}{2}$ $\forall n \in \mathbb{N}$. Use induction.

  If $a_{n-1} \le \frac{3}{2}$, then $a_n = \dfrac{2a_{n-1}+3}{4} \le \frac{6}{4} = \frac{3}{2}$.

- $a_{n+1} \ge a_n$ $\forall n \in \mathbb{N}$.

  $a_{n+1} - a_n = \dfrac{2a_n+3}{4} - a_n = \dfrac{3-2a_n}{4} = \frac{1}{2}\left(\frac{3}{2} - a_n\right) \ge 0$

  by the last part.

Thus, $\{a_n\}$ is convergent.

(b) $a_1 = \sqrt{2}$, and for $n > 1$, a_n is defined recursively by $a_n = \sqrt{2 + a_{n-1}}$.

If $\lim a_n$ exists, and $= \ell$, then $\ell = \sqrt{2+\ell}$ or $\ell^2 - \ell - 2 = 0$, $\ell = 2$ or -1. So, $\ell = 2$ since $a_n > 0$ $\forall n \in \mathbb{N}$

first few terms, $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

perhaps, $\{a_n\}$ is monotone increasing & bdd.

- $\{a_n\}$ is bdd.

  Proof by induction.

  Suppose $a_{n-1} \le 2$, then

  $a_n = \sqrt{2+a_{n-1}} \le \sqrt{2+2} = 2$ ($\sqrt{2+x}$ is an increasing fn.)

- $\{a_n\}$ is monotone increasing.

  $a_{n+1} - a_n = \sqrt{2+a_n} - a_n = \dfrac{2+a_n-a_n^2}{\sqrt{2+a_n}+a_n}$ $\sqrt{2+a_n}+a_n > 0$

  suff. to show that $2 + a_n - a_n^2 \ge 0$.

  But $(2+a_n-a_n^2) = \underbrace{(2-a_n)}_{\ge 0}\underbrace{(1+a_n)}_{>0} \ge 0$.

19. Let $\{a_n\}_n$ be a bounded sequence such that its range E has a *unique* limit point. Does this guarantee that $\{a_n\}_n$ is convergent?

No! Consider the sequence $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{6}, \cdots$.

i.e. $a_n = \begin{cases} 1 & \text{if } n\text{-odd} \\ \frac{1}{n} & \text{if } n\text{-even} \end{cases}$

Here, $E = \{\frac{1}{n} : n \in \mathbb{N}\}$, has a unique limit pt. 0 but $\{a_n\}$ does not converge.

20. What extra conditions can one impose on the previous problem to assure the convergence?

Extra condition. The range of every subsequence is infinite.

We would like to show now that $\{a_n\}$ convgs.

In fact, we will prove that $a_n \to \ell$ where ℓ is the unique limit pt. of E.

What if $a_n \not\to \ell$?

Then there is some nbhd. of ℓ outside which there are ∞-ly terms of $\{a_n\}$. This means there is a whole subsequence $\{a_{n_k}\}_k$ outside this nbhd. Let this nbhd. of ℓ be $(\ell-\varepsilon, \ell+\varepsilon)$.

Let E_1 = Range of $\{a_{n_k}\}_k$

Then E_1 is infinite by the Extra hypothesis and E_1 is bdd. since $E_1 \subseteq E$ and E-bdd.

$\Rightarrow$ E_1 has a limit pt. (by Bolzano-Weierstrass), say, ℓ'.

But a limit pt. of E_1 is a limit pt. of E, and E has a unique limit pt. ℓ.

So $\ell = \ell'$. But this is impossible since $(\ell-\varepsilon, \ell+\varepsilon)$ contains no element of E_1.

21. For each $n \in \mathbb{N}$, let $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then it is easy to see that $U_1 \supset U_2 \supset U_3 \supset \cdots$. Show that $\bigcap_{n=1}^{\infty} U_n = \{0\}$.

Easy to see that $0 \in U_n \ \forall\ n \in \mathbb{N}$.

Thus, $0 \in \bigcap_{n=1}^{\infty} U_n$.

Claim: if $x \neq 0$, then $x \notin \bigcap_{n=1}^{\infty} U_n$

(so that $\bigcap_{n=1}^{\infty} U_n = \{0\}$).

Suppose $x \in U_n \ \forall\ n \in \mathbb{N}$, then

$$-\frac{1}{n} < x < \frac{1}{n} \quad \forall\ n \in \mathbb{N}.$$

By Sandwich Thm., $x = 0$.

22. For each $n \in \mathbb{N}$, let $U_n = \left(0, \frac{1}{n}\right)$. Then it is easy to see that $U_1 \supset U_2 \supset U_3 \supset \cdots$. Show that $\bigcap_{n=1}^{\infty} U_n = \emptyset$. Try to identify the part of the proof of the Nested Interval Property that fails in this case.

If $\bigcap_{n=1}^{\infty} U_n \neq \phi$, then let $x \in \bigcap_{n=1}^{\infty} U_n$

$\Rightarrow x \in U_n \;\; \forall \; n \in \mathbb{N}$

$\Rightarrow 0 < x < \frac{1}{n} \;\; \forall \; n$

$\Rightarrow x = 0$ by Sandwich Thm.

But $0 \notin U_n$, hence a contradiction.

In the pf. of NIP, we had

$a_1 \leq a_2 \leq \cdots$
$b_1 \geq b_2 \geq \cdots$ where $I_n = [a_n, b_n]$

Then we argued that

$$a_n \to a = \sup\{a_n\}_n, \quad b_n \to a = \inf\{b_n\}_n$$

so that $a_n \leq a \leq b_n \;\; \forall \; n \in \mathbb{N}$
and as such $a \in [a_n, b_n]$.

But does $a_n \leq a \leq b_n \;\; \forall \; n \in \mathbb{N}$
imply $a \in (a_n, b_n) \;\; \forall \; n$?

Answer is no!

This is precisely what is happening in the problem in hand.

Namely, here $a_n = 0 \quad \forall\, n \in \mathbb{N}$.

$b_n = \frac{1}{n} \quad \forall\, n \in \mathbb{N}$

and $a = 0$

So that $\underset{0}{\underset{\parallel}{a_n}} \leq \underset{0}{\underset{\parallel}{a}} \leq \frac{1}{n}$

but $0 \in \left(0, \frac{1}{n}\right)$

If we had a $\left[0, \frac{1}{n}\right)$ instead, then the argument of NIP holds again and

$$\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right) = \{0\}.$$

10. a) (Extra bit)

To show: $$\frac{\ln n}{1 + \ln n + \frac{(\ln n)^2}{2}} \to 0$$

$\downarrow$

For $n > 1$, $$= \frac{1}{\frac{1}{\ln n} + 1 + \frac{\ln n}{2}}$$

Now, $\frac{1}{\ln n} + 1 + \frac{\ln n}{2} > \frac{\ln n}{2} \quad \forall\, n > 1$

Thus

$$0 < \frac{1}{\frac{1}{\ln n} + 1 + \frac{\ln n}{2}} < \frac{2}{\ln n} \quad \forall \ n > 1.$$

Claim! $\frac{1}{\ln n} \to 0$ (and then we are done! by Sandwich Thm.)

For a given $\varepsilon > 0$,

$$\frac{1}{\ln n} < \varepsilon \quad \text{if} \quad \ln n > \frac{1}{\varepsilon}$$

$$\text{or if} \quad n > e^{1/\varepsilon}$$

Let $n_0 = \lceil e^{1/\varepsilon} \rceil$, then

$$\ln n > \frac{1}{\varepsilon} \quad \forall \ n \geq n_0$$

$$\Rightarrow \quad \frac{1}{\ln n} < \varepsilon \quad \forall \ n \geq n_0$$

$$\Rightarrow \quad \frac{1}{\ln n} \to 0,$$ as required.