Quiz-2

1. Question

3 points

Which of the following sets is *not* open?

Check all that apply.

$(0,1) \cup (1,2)$

union of 2 open sets

- [x] Option 1

$[0,1] \cup (1,2)$

$= [0,2)$

no nbhd. of 0 is contained inside $[0,2)$

- [x] Option 2

$(0,1] \cup (1,2)$

$= (0,2)$

open

- [x] Option 3

$(0,1] \cup (1,2]$

$= (0,2]$

no nbhd. of 2 is contained inside $(0,2]$

- [x] Option 4

2. Question

2 points

How many limit points does the set $E = \{1, 1/2, 4/3, 1/4, 6/5, 1/6, 8/7, \ldots\}$ have?

Check all that apply.

$E = \left\{\frac{1}{2n} : n \in \mathbb{N}\right\} \cup \left\{\frac{2n+2}{2n+1} : n \in \mathbb{N}\right\} \cup \{1\}$

limit pt. 0

limit pt. 1

no limit pt.

- [ ] 0
- [ ] 1
- [x] 2
- [ ] infinity

3. Question 3 points

Which of the following sets has a limit point?

Check all that apply.

$\{\sin x : x \in [0, 1]\}$

infinite & bounded

- [x] Option 1

$\{\sin x : x \in \{\ 1, 2, 3, 4, 5\}\}$

finite set

- [ ] Option 2

$\{\sin n : n \in \mathbb{N}\}$

infinite & bounded

- [x] Option 3

$\left\{\frac{n}{100} : n \in \mathbb{N}\right\}$

Pretty much similar to the case of IN.

- [ ] Option 4

4. Question

3 points

Evaluate: $\lim_{n\to\infty} \frac{\lfloor 2^{n+0.5} \rfloor}{2^n}$, where $\lfloor\cdot\rfloor$ is the floor function.

Check all that apply.

0	1
☒ Option 1	☒ Option 2
$1/\sqrt{2}$	$\sqrt{2}$
☒ Option 3	☑ Option 4

Use that

$x-1 < \lfloor x \rfloor \leq x$

to get that

$2^n\sqrt{2} - 1 < \lfloor 2^n\sqrt{2} \rfloor \leq 2^n\sqrt{2}$

$\Rightarrow \sqrt{2} - \frac{1}{2^n} < \frac{\lfloor 2^n\sqrt{2} \rfloor}{2^n} \leq \sqrt{2}$

By Sandwich Thm., limit $= \sqrt{2}$

5. Question

2 points

The limit of the sequence $\frac{2^n 5^n}{n!}$ is

Check all that apply.

- ☑ 0
- ☐ 1
- ☐ 2
- ☐ infinity

These are < 1

$$0 < \frac{2^n \cdot 5^n}{n!} = \frac{10^n}{n!} < \frac{10^{10}}{10!} \cdot \left(\frac{10}{11}\right)\left(\frac{10}{12}\right)\cdots\left(\frac{10}{n}\right)$$

$$< c \cdot \frac{10}{n} \quad \text{where } c = \frac{10^{10}}{10!}$$

By Sandwich Theorem,

$$\lim_{n\to\infty} \frac{2^n \cdot 5^n}{n!} = 0.$$

6. Question 2 points

Suppose $\{a_n\}_n$ is a null sequence. Which of the following sequences always converge?

Check all that apply.

$\frac{1}{\sqrt{n}} \to 0$

But $n \cdot \frac{1}{\sqrt{n}} = \sqrt{n} \to \infty$

$\{na_n\}_n$

☒ Option 1

$a_n \to 0$ & $\frac{1}{n} \to 0$

$\Rightarrow \frac{a_n}{n} \to 0$ by the product rule of limits

$\{a_n/n\}_n$

☑ Option 2

$|\sin x| \leq |x| \;\; \forall x \in \mathbb{R}$

Thus, $0 \leq |\sin a_n| \leq |a_n|$ $\forall n \in \mathbb{N}$

$\{\sin(a_n)\}_n$

Now, $a_n \to 0$

$\Rightarrow \sin a_n \to 0$

☑ Option 3

by option 3, $\sin(a_n/2) \to 0$

$\Rightarrow \cos a_n = 1 - 2\sin^2(a_n/2)$

$\to 1$

$\{\cos(a_n)\}_n$

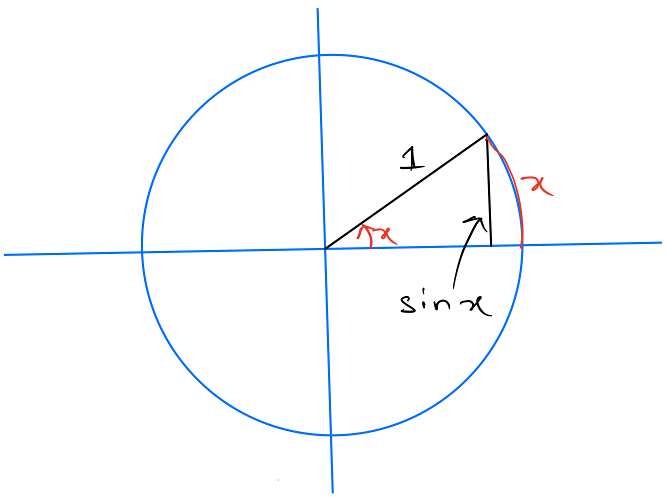
☑ Option 4

$-\sin -x \leq x$

Pf. of $|\sin x| \leq |x|$ $\to$ If $|x| \geq 1$, then

$|\sin x| \leq 1 \leq |x|$

If $|x| < 1$, then consider the following



$\sin x \leq x \;\; \forall x \geq 0 \Rightarrow -\sin -x \leq -x \;\; \forall x \geq 0 \Rightarrow |\sin x| \leq |x| \;\; \forall x \in (-1, 1).$

(*) $a_n \le \frac{\sqrt{5}+1}{2}$ by induction (do it!)

$a_{n+1} - a_n = 1+2a_n - a_n - a_n^2 = 1 + a_n - a_n^2$

$= \left(\frac{\sqrt{5}+1}{2} - a_n\right)\left(a_n + \frac{\sqrt{5}-1}{2}\right) \ge 0$, since $0 \le a_n \le \frac{\sqrt{5}+1}{2}$

$\Rightarrow a_n$ converges (since monotone & bdd.)

7. Question

3 points

Suppose $\{a_n\}_n$ satisfies the recurrence relation $a_1 = 1$, and $a_{n+1} = \dfrac{1+2a_n}{1+a_n}$. (*)

Then $\lim_{n\to\infty} a_n = ?$

The limit "l" satisfies

$l = \frac{1+2l}{1+l} \Rightarrow l = \frac{1 \pm \sqrt{5}}{2}$. Since $a_n > 0$ $\forall n \in \mathbb{N}$, deduce that $l = \frac{1+\sqrt{5}}{2}$

Check all that apply.

$$\frac{\sqrt{5}+1}{2}$$

- [x] Option 1

$$\frac{\sqrt{5}-1}{2}$$

- [ ] Option 2

$$\frac{1-\sqrt{5}}{2}$$

- [ ] Option 3

None of the above

- [ ] Option 4

8. Question

2 points

Evaluate: $\lim_{n\to\infty} (n^2+2)^{1/n}$

1

By AM $\ge$ GM,

$n^2+2 \ge 2\sqrt{2}\,n$ $\forall n \in \mathbb{N}$

Also, $n^2+2 \le 2n^2$ $\forall n \in \mathbb{N}$

Thus

$\underbrace{(2\sqrt{2}\,n)^{1/n}}_{\to 1} \le (n^2+2)^{1/n} \le \underbrace{(2n^2)^{1/n}}_{\to 1}$, hence by Sandwich Thm.

$(n^2+2)^{1/n} \to 1$.

9. Question

2 points

The sequence $\{\sin n\}_n$ is convergent.

Mark only one oval.

- [ ] True
- [x] False

Supp. $\sin n \to \ell$,
then $\sin 2n \to \ell$, $\sin 3n \to \ell$ and $\sin 4n \to \ell$
Now, $\sin 4n = 2 \sin 2n \cos 2n = 2 \sin 2n (1 - 2\sin^2 n)$
Taking limits $\ell = 2\ell(1 - 2\ell^2) \Rightarrow \ell^2 = 1/4$ — (1)
Next, $\sin 3n = 3 \sin n - 4 \sin^3 n$
taking limits $\ell = 3\ell - 4\ell^3$
$\Rightarrow \ell^2 = \frac{1}{2}$ — (2)
(1) and (2) contradict!

10.

3 points

There is a sequence of natural numbers $n_1 < n_2 < \cdots$ such that $\{\sin n_k\}_k$ is convergent.

Mark only one oval.

- [x] True
- [ ] False

$\{\sin n\}_n$ is bdd.
hence, by Bolzano-Weierstrass Thm.,
$\{\sin n\}_n$ has a convergent subsequence.

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