Convergence Tests

Let $S = \sum_n a_n = a_1 + a_2 + \cdots$

Step 1. Compute $\boxed{\lim_{n\to\infty} a_n}$.

If $\lim_{n\to\infty} a_n \neq 0$. Then stop!

Conclude S does not converge.

If $\lim_{n\to\infty} a_n = 0$, the procede to

$\sum_n \frac{1}{n^3 \sin^2 n}$

Step 2. Test for Absolute convergence.

Tests for series with positive terms

I. Integral Test.

Works when $\boxed{a_n = f(n)}$ & $f(x)$ satisfies

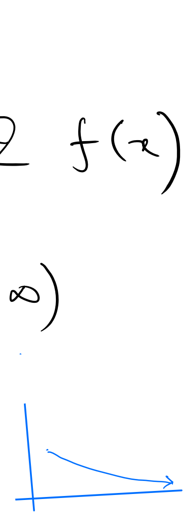
$S = \sum_n f(n)$

- $f(x) > 0 \quad \forall x \in [1, \infty)$
- f - monotone decreasing on $[1, \infty)$ with $\lim_{x\to\infty} f(x) = 0$

$\int_1^\infty f(x)\,dx$

- f - integrable on $[1, \infty)$.

(Think f - continuous)



$y = f(x)$

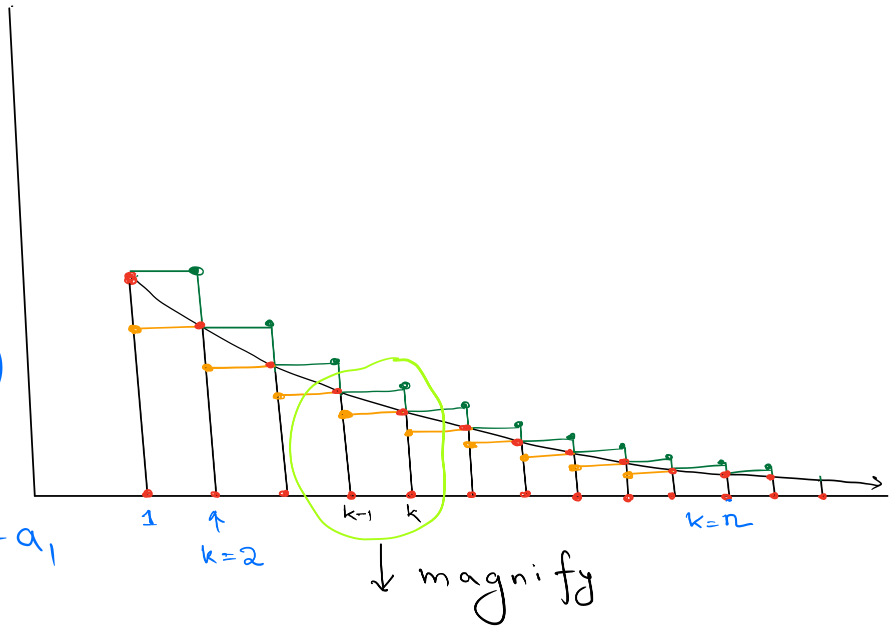
$$f(1) \geq \int_1^2 \geq f(2)$$

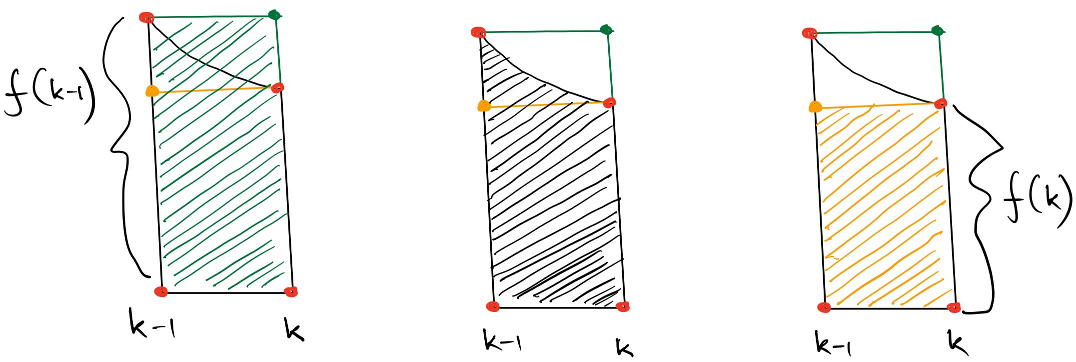
$$f(2) \geq \int_2^3 \geq f(3)$$

$$\vdots$$

$$f(n-1) \geq \int_{n-1}^{n} \geq f(n)$$

$$S_{n-1} \geq \int_1^n f\,dx \geq S_n - a_1$$





$$\text{Area} = f(k-1) \;\geq\; \text{Area} = \int_{k-1}^{k} f(x)\,dx \;\geq\; \text{Area} = f(k)$$

Thus, $$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\,dx \leq \sum_{k=2}^{n} f(k-1)$$

An useful observation

If A_n, B_n monotone increasing with $A_n \leq B_n \;\forall\, n \geq n_0$,
then B_n - convgs. $\Rightarrow$ A_n - convgs. & A_n-divgs $\Rightarrow$ B_n-divgs.

That is $\quad S_{n-1} \geq \int_1^n f(x)\,dx \geq S_n - a_1$

Let $\quad I_{n,f} = \int_1^n f(x)\,dx$

$\boxed{S_{n-1} \geq I_{n,f} \geq S_n - a_1 \quad \forall\, n.}$

and $\quad I_f = \int_1^\infty f(x)\,dx := \lim_{n\to\infty} I_{n,f}$

Case (i). $\boxed{I_f < \infty}$.

$\Rightarrow S_n - a_1 \leq I_f \quad \forall\, n \in \mathbb{N} \ (n>1)$

$\Rightarrow \quad S_n$ converges $\Rightarrow \boxed{S \text{ converges}}$

conclusion

Case (ii). $\boxed{I_f = \infty}$.

Then $\quad S_{n-1} \geq I_{n,f} \longrightarrow \infty$

$\Rightarrow \quad S_n \longrightarrow \infty.$

$U_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$\Rightarrow \quad \boxed{S \text{ diverges to } \infty}.$

conclusion

$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \ln n$

$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \int_1^n \frac{dx}{x} = 1 + \ln n$

$1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$

$\Rightarrow \ln(n+1) \leq U_n \leq 1 + \ln n$

$\frac{\ln(n+1)}{\ln n} \leq \frac{U_n}{\ln n} \leq 1 + \frac{1}{\ln n}$

$\Rightarrow \frac{U_n}{\ln n} \longrightarrow 1$

Examples.

- $S = \sum_n \frac{1}{n^p}$, $p \in \mathbb{R}$, fixed.

  - $\boxed{p < 0} \Rightarrow \frac{1}{n^p} = n^{-p} \longrightarrow \infty$

    so not convergent.

  - $\boxed{p = 0} \Rightarrow \frac{1}{n^p} = 1$

    so not convergent.

  - $p > 0$. $f(x) = \frac{1}{x^p}$

$$I_{n,f} = \int_1^n \frac{dx}{x^p} = \begin{cases} \ln n & \text{if } p = 1 \\ \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) & \end{cases}$$

Thus, $I_f = \lim_{n \to \infty} I_{n,f} = \begin{cases} \infty & \text{if } p \leq 1 \\ \frac{1}{1-p} & \text{if } p > 1 \end{cases}$

Conclusion: $\boxed{\sum_n \frac{1}{n^p} \text{ converges if and only if } p > 1}$

$\sum \frac{1}{\sqrt{n}}$, $\sum_n \frac{1}{n^2}$, $\sum_n \frac{1}{n^{1.5}}$, $\sum_n \frac{1}{n^{1.0001}}$

- $\boxed{S = \sum_n \frac{1}{1+n^2}}$.

  - $\frac{1}{1+n^2} \longrightarrow 0$

  - $f(x) = \frac{1}{1+x^2}$ decreasing with $\lim_{x\to\infty} \frac{1}{1+x^2} = 0$

$$I_f = \int_1^\infty f(x)\,dx = \left[\tan^{-1}x\right]_1^\infty = \pi/4$$

hence, $\underline{S}$ $\underline{\text{converges}}$.

- $\boxed{S = \sum \frac{\ln n}{n}}$

  - $\frac{\ln n}{n} \longrightarrow 0$

$\frac{\ln x}{x} \longrightarrow 0$ as $x \longrightarrow \infty$

$\uparrow$ monotone decreasing on $[1,\infty)$

$$I_{n,f} = \int_1^n \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln n)^2 \longrightarrow \infty$$

hence $\underline{S = \infty}$

II. Comparison Test

- Suppose $S = \sum_n a_n$ & $S' = \sum_n b_n$ with $a_n, b_n \geq 0 \ \forall n \in \mathbb{N}$.

Bigger converges $\Downarrow$ Smaller converges

Smaller diverges $\Downarrow$ Bigger diverges

Further suppose that

$$a_n \leq b_n \quad \forall \quad n \geq N.$$

s' dominates s, eventually

Then

- $S = \infty \Rightarrow S' = \infty$
- $S' < \infty \Rightarrow S < \infty$.

Proof. Let $A = a_1 + a_2 + \cdots + a_{N-1}$

$A' = b_1 + b_2 + \cdots + b_{N-1}$

Given condition implies that

$S_n - A = a_N + a_{N+1} + \cdots + a_n$

$S_n' - A = b_N + b_{N+1} + \cdots + b$

$$S_n - A \leq S_n' - A' \quad \forall \ n \geq N.$$

- $S_n' \geq S_n - A + A'$

$\Rightarrow \quad S' \geq S - A + A'$

Thus, $S = \infty \Rightarrow S' = \infty$.

- $S_n \leq S_n' - A' + A$

$\Rightarrow \quad S \leq S' - A' + A$

Thus $S' < \infty \Rightarrow S < \infty$.

Examples.

- $\boxed{S = \sum_n \frac{1}{n!}}$

$$n! \geq 2^n \quad \forall \; n \geq 2$$

$$\Rightarrow \quad \frac{1}{n!} \leq \frac{1}{2^n} \quad \forall \; n \geq 2$$

$$\Rightarrow \quad \underline{S < \infty}.$$

- $\boxed{S = \sum_n \frac{\ln n}{n^2}}$.

$$\frac{\ln n}{n^2} \leq \frac{n}{n^2} = \frac{1}{n}, \quad \sum_n \frac{1}{n} = \infty$$

so won't work.

$$\frac{\ln n}{n^2} \geq \frac{1}{n^2},$$

Although $\sum_n \frac{1}{n^2} < \infty$, this says nothing about $\sum_n \frac{\ln n}{n^2}$.

So, want

$$\frac{\ln n}{n^2} \leq \frac{1}{n^{1.5}} \quad \forall \; n \geq N.$$

i.e.

$$\ln n \leq \sqrt{n} \quad \forall \; n \geq N$$

This can achieved using calculus or use the following estimate

$$e^{2\sqrt{n}} = 1 + 2\sqrt{n} + 2n + \cdots$$
$$> n \qquad \forall\ n \in \mathbb{N}.$$

$$\Rightarrow \quad \ln n < 2\sqrt{n}$$

$$\Rightarrow \quad \frac{\ln n}{n^2} < \frac{2}{n^{3/2}}$$

$$\Rightarrow \quad \sum_n \frac{\ln n}{n^2} \leq 2 \sum_n \frac{1}{n^{3/2}} < \infty.$$

- Let $x = 0.d_1 d_2 \cdots \qquad d_n \in \{0,1,2,\dots,9\}$.

  why is $x \leq 1$?

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=1}^{\infty} \frac{9}{10^n} = 9 \cdot \sum_{n=1}^{\infty} \frac{1}{10^n}$$
$$= 1.$$

- Suppose $S = \sum_n a_n < \infty$ where $a_n > 0$ $\forall n$.

  Let $S' = \sum_n e_n a_n$ where $|e_n| \leq 1$.

  Then $\boxed{S' < \infty}$.

$$|e_n a_n| = |e_n| a_n \leq a_n.$$

III. Ratio Test.

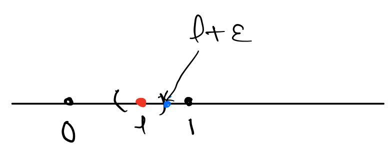
$$S = \sum_n a_n \quad , \quad a_n > 0 \quad \forall \, n.$$

Let $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$. (if exists!)

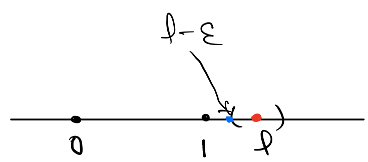
Then

- $\ell < 1 \Rightarrow S < \infty$
- $\ell > 1 \Rightarrow S = \infty$
- $\ell = 1$, Test is inconclusive!

Proof.



If $\ell < 1$, then $\exists$ a $\varepsilon > 0$ s.t
$\ell + \varepsilon < 1$



If $\ell > 1$, then $\exists$ a $\varepsilon > 0$ s.t.
$\ell - \varepsilon > 1$

- $\ell < 1$ $\exists$ $n_0 \in \mathbb{N}$

s.t. $\left|\dfrac{a_{n+1}}{a_n} - \ell\right| < \varepsilon \quad \forall\, n \geq n_0$

$\Rightarrow \quad a_{n+1} < (\ell+\varepsilon)\, a_n \quad \forall\, n \geq n_0$

$\Rightarrow \quad a_{n_0+1} < (\ell+\varepsilon)\, a_{n_0}$

$a_{n_0+2} < (\ell+\varepsilon)\, a_{n_0+1}$
$\quad < (\ell+\varepsilon)^2 a_{n_0}$

$\vdots$

- $\ell > 1$, $\exists$ $n_0 \in \mathbb{N}$

s.t. $\left|\dfrac{a_{n+1}}{a_n} - \ell\right| < \varepsilon$
$\forall\, n \geq n_0$

$\Rightarrow \quad a_{n+1} > (\ell-\varepsilon)\, a_n$

$\Rightarrow \quad a_{n_0+1} > (\ell-\varepsilon)\, a_{n_0}$

$\vdots$

$a_{n_0+k} > (\ell-\varepsilon)^k a_{n_0}$

$a_n < (\ell+\varepsilon)^{n-n_0} \cdot a_{n_0}$ $\quad a_{n_0+k} < (\ell+\varepsilon)^k a_{n_0}$

$a_n < \boxed{\dfrac{a_{n_0}}{(\ell+\varepsilon)^{n_0}}} (\ell+\varepsilon)^n$ $\Rightarrow$ $\sum_n a_n$ is eventually dominated by $\sum_n (\ell+\varepsilon)^n \dfrac{a_{n_0}}{(\ell+\varepsilon)^{n_0}}$ const.

by Comparison Test

$$\sum_n a_n < \infty$$

$\Rightarrow \sum_n a_n$ dominates $\sum_n (\ell-\varepsilon)^{n-n_0} a_{n_0}$

$= \text{const.} \sum_n (\ell-\varepsilon)^n$

$\Rightarrow \sum_n a_n$ divgs.

- If $\ell = 1$, the test is inconclusive.

- $\sum_n \frac{1}{n}$, $\quad \dfrac{\frac{1}{n+1}}{\frac{1}{n}} = \dfrac{n}{n+1} \longrightarrow 1$

  but $\sum_n \frac{1}{n} = \infty$

- $\sum_n \frac{1}{n^2}$, $\quad \dfrac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \left(\dfrac{n}{n+1}\right)^2 \longrightarrow 1$

  but $\sum_n \frac{1}{n^2} < \infty$

Examples.

- $\sum_n \dfrac{1}{\binom{2n}{n}}$

$$\frac{a_{n+1}}{a_n} = \frac{\binom{2n}{n}}{\binom{2n+2}{n+1}} = \frac{2n!}{n!\,n!} \cdot \frac{(n+1)!\,(n+1)!}{(2n+2)!}$$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)}$$

Thus, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}$

$\Rightarrow \sum_n \frac{1}{\binom{2n}{n}} < \infty$.

- $\sum_n \frac{10^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{10}{n+1} \to 0$$

$\Rightarrow \sum_n \frac{10^n}{n!} < \infty$.

IV. Root Test.

$S = \sum_n a_n$ with $a_n > 0$.

Let $\rho = \lim_{n\to\infty} a_n^{1/n}$. (if it exists!)

Then

- If $\rho < 1$, then $S < \infty$
- If $\rho > 1$, then $S = \infty$
- If $\rho = 1$, then the test is inconclusive.

Proof: Exercise. (Procede as in the Ratio Test)

- $\sum_n \frac{n^3}{2^n}$

$$a_n^{1/n} = \frac{n^{3/n}}{2} \longrightarrow \frac{1}{2}$$

Hence $\sum_n \frac{n^3}{2^n} < \infty$

- $\sum_n \frac{\ln n}{n^2}$

$$a_n^{1/n} = \frac{(\ln n)^{1/n}}{n^{2/n}}$$

Now $\ln 2 \leq \ln n \leq n \quad \forall n \geq 2$

$$\Rightarrow \underbrace{(\ln 2)^{1/n}}_{\to 1} \leq (\ln n)^{1/n} \leq \underbrace{n^{1/n}}_{\to 1} \quad \forall n \geq 2$$

By Sandwich Thm., $(\ln n)^{1/n} \to 1$

$$\Rightarrow a_n^{1/n} \to 1$$

So the test is inconclusive.

• $S = \sum_n \frac{(-1)^n}{n^{1+n}}$, $S^+ = \sum_n \frac{1}{n^{1+n}}$

$$a_n^{1/n} = \frac{1}{n^{1+\frac{1}{n}}} \longrightarrow 0$$

$\Rightarrow \quad S < \infty$.

Step. 3 Alternating Series Test for certain conditionally convergent sequences.

Let $S = a_1 - a_2 + a_3 - a_4 + \cdots$

where

- $a_n > 0 \quad \forall \; n$
- $a_1 \geq a_2 \geq a_3 \geq \cdots\cdots$
  i.e $a_n \geq a_{n+1}$
- $a_n \longrightarrow 0$

$(-1)^{n-1} a_n \longrightarrow 0$

Then $\boxed{S < \infty}$.

Example. $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

here, • $a_n = \frac{1}{n} > 0$
• $\frac{1}{n} > \frac{1}{n+1}$
• $\frac{1}{n} \longrightarrow 0$

$\Rightarrow S < \infty$

Proof. Investigate the partial sums.

$$S_{2m} = \underbrace{(a_1 - a_2)}_{\geq 0} + \underbrace{(a_3 - a_4)}_{\geq 0} + \cdots + \underbrace{(a_{2m-1} - a_{2m})}_{\geq 0}$$

$\Rightarrow S_{2m} \geq 0.$

Now, $S_{2m+2} = S_{2m} + \underbrace{(a_{2m+1} - a_{2m+2})}_{\geq 0}$
$\geq S_{2m}$

Thus, • $S_2 \leq S_4 \leq S_6 \leq \cdots\cdots$
• $S_{2m} \geq 0$

Furthermore,

$$S_{2m} = a_1 \overbrace{(-a_2 + a_3)}^{\leq 0} \overbrace{(-a_4 + a_5)}^{\leq 0} - \overbrace{(-a_{2m-2} + a_{2m-1})}^{\leq 0} \underbrace{- a_{2m}}_{\leq 0}$$

$\leq a_1$

Thus $\{S_{2m}\}_m$ is convergent.

Let $S_{2m} \longrightarrow L$.

Next, $S_{2m+1} = \underbrace{S_{2m}}_{\to L} + \underbrace{a_{2m+1}}_{\to 0}$ $\leftarrow$ $a_n \to 0 \Rightarrow a_{2m+1} \to 0$

$\Rightarrow S_{2m+1} \longrightarrow L$

$\Rightarrow S_n \longrightarrow L$ (why?).

proof of

given $\varepsilon > 0$, $\exists$ $n_0, n_1 \in \mathbb{N}$ s.t.

$$|S_{2m} - L| < \varepsilon \quad \forall\ n \geq n_0$$

and $$|S_{2m+1} - L| < \varepsilon \quad \forall\ n \geq n_1$$

Let $n_2 = \max\{n_0, n_1\}$.

Then if $n \geq n_2$, then

$$|S_n - L| < \varepsilon$$

$$\Rightarrow S_n \longrightarrow L.$$

$\square$

The following series are conditionally convergent.

① $\displaystyle\sum_n \frac{(-1)^n}{\sqrt{n}}$

② $\displaystyle\sum_n \frac{(-1)^n \ln n}{n}$