Hint Page

1. Show that sequence is Cauchy

2. (f) Rewrite the given expression as

$$\frac{2^n (n!)^2}{2n!}.$$

Now, use Ratio Test.

(i) Note that $\frac{\sin x}{x} \sim 1$ when $x \sim 1$

$\Rightarrow \frac{\sin \frac{1}{n}}{\frac{1}{n}} \sim 1$ when n suff. large

$\Rightarrow \frac{\sin \frac{1}{n}}{\frac{1}{n}} > \frac{1}{2}$ for n suff. large.

(m) Use Stirling's estimate for $m!$ (refer to Q.12, Worksheet 2)

(n) Integral Test.

3. Apply Ratio / comparison Test.

4. Apply Root Test

6. Use the result of Q. 7.
Correctly identify a_n and b_n, and see if they satisfy the required conditions.

7. This one is hard!

8. Show that $\{S_n\}_n$ is Cauchy.
What is $|S_m - S_n|$ in terms of E_n and E_m?

9. $a_n \to 0 \Rightarrow a_n < 1$ for $n \geq n_0$

10. If $\ell < 1$, find a dominant convergent geometric series.
If $\ell > 1$, then find a smaller divergent geometric series.

WORKSHEET - 3

1. For $n \in \mathbb{N}$, define $a_n = \displaystyle\int_1^n \frac{\cos x}{x^2} dx$. Show that $\{a_n\}_n$ is convergent. (May require some basic knowledge of integration)

We show that $\{a_n\}$ is Cauchy.

Let $m > n$. Then

$$|a_m - a_n| = \left| \int_n^m \frac{\cos x}{x^2} dx \right|$$

$$\leq \int_n^m \frac{|\cos x|}{x^2} dx$$

$$\leq \int_n^m \frac{1}{x^2} dx$$

$$= \frac{1}{n} - \frac{1}{m}$$

Given $\varepsilon > 0$, $\exists$ n_0 s.t $\frac{1}{n} < \varepsilon$ $\forall$ $n \geq n_0$

(by the Arch. Prop.)

Now, for $m \geq n \geq n_0$

$$|a_m - a_n| \leq \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon.$$

Hence $\{a_n\}_n$ is Cauchy.

2. Determine whether the series is convergent/divergent in each of the following cases. It might be a good idea to try to apply all the five tests (namely, integral, comparison, ratio, root and alternating series test whenever applicable) and find out which ones work.

(a) $\sum_n \frac{n}{n^2+4}$ Clearly $\frac{n}{n^2+4} = \frac{1}{n} \cdot \frac{1}{1+\frac{4}{n^2}} \to 0$

Integral Test.

The function $\frac{x}{x^2+4}$ is +ve, mono. decreasing & $\lim_{x\to\infty} \frac{x}{x^2+4} = 0$.

Now, $\int_1^n \frac{x}{x^2+4}\,dx = \left[\frac{1}{2}\ln(x^2+4)\right]_1^n$

$= \frac{1}{2}\ln(n^2+4) - \frac{1}{2}\ln 5 \to \infty$

Hence, the series not convergent.

Comparison Test.

Observe that if $a \geq b > 1$ are integers, then

$ab = a + (b-1)a \geq 2a \geq a+b$.

So, $n^2+4 \leq 4n^2 \quad \forall\, n \geq 2$.

Now,

$$\frac{1}{4n} = \frac{n}{4n^2} \leq \frac{n}{n^2+4}$$

Since $\sum_n \frac{1}{4n}$ diverges, hence $\sum_n \frac{n}{n^2+4}$ diverges.

Root & Ratio Test:

Easy to check that both tests are inconclusive in this case.

(b) $\sum_n \frac{n}{(n^2+4)^2}$ ← Convergent (Do it yourself)

(c) $\sum_n \frac{1}{\sqrt{n^2+4}}$ ← Divergent (Do it yourself)

(d) $\sum_n \frac{(\sin n)^3}{n^2}$ ← Absolutely convergent since

$$\frac{|\sin n|^3}{n^2} \leq \frac{1}{n^2} \quad \forall\, n$$ (now use the comparison test)

(e) $\sum_n \frac{n}{e^n}$ ← Perfect candidate for Root test.

$$\lim_{n\to\infty} \left(\frac{n}{e^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{e} = \frac{1}{e} < 1,$$ hence convergent

(f) $\sum_n \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

$2 \quad 4 \quad 6 \quad 2n$

Multiplying the denominator & numerator by

$$2\cdot 4\cdot 6\cdots 2n = 2^n\cdot n!,$$

$$\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{2^n (n!)^2}{(2n)!} = \frac{2^n}{\binom{2n}{n}}.$$

We directly proceed to the Ratio Test

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!\,(n+1)!}{(2n+2)!}\cdot\frac{2n!}{2^n\, n!\, n!}$$

$$= \frac{n+1}{2n+1} \longrightarrow \frac{1}{2}$$

Thus, the series converges.

(Remark: By an exercise problem $a_n \to 0$, a step we skipped)

(g) $\sum_n \frac{1}{n\sqrt{n}}$ $\leftarrow$ convergent

(h) $\sum_n \frac{(-1)^{n+1}}{\sqrt{n}}$ $\leftarrow$ $= \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$

convergent by Alternating Series Test.

(i) $\sum_n \sin\left(\frac{1}{n}\right)$ $\leftarrow$ Divergent

Note that $\sin\frac{1}{n} \longrightarrow 0$

Observation: $\frac{\sin x}{x} \sim 1$ if $x \sim 0$

$\Rightarrow \frac{\sin\frac{1}{n}}{\frac{1}{n}} \sim 1$ if n-large

$\Rightarrow \frac{\sin\frac{1}{n}}{\frac{1}{n}} > \frac{1}{2}$ if n-large.

Now, $\frac{\sin x}{x} = \underbrace{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots}$

want to be $> \frac{1}{2}$

$\Rightarrow$ want $\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots < \frac{1}{2}$

Next, $\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots \leq \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots < \frac{x^2}{2^2} + \frac{x^4}{2^4} + \cdots$

$= \frac{x^2}{4-x^2} < \frac{1}{2}$

if $x < \frac{4}{3}$ (Solve)

Thus, $\frac{\sin x}{x} > \frac{1}{2}$

provided $x < \frac{4}{3}$

$\Rightarrow \frac{\sin\frac{1}{n}}{\frac{1}{n}} > \frac{1}{2}$ provided, $\frac{1}{n} < \frac{4}{3}$ or $n > \frac{3}{4}$

$\Rightarrow \sin\frac{1}{n} > \frac{1}{2n} \ \forall n \Rightarrow \sum_n \sin\frac{1}{n}$ diverges.

(j) $\sum_n \sin\left(\frac{1}{n^2}\right)$

$\sin\frac{1}{n^2} \leq \frac{1}{n^2}$, hence the series is convergent.

(k) $\sum_n \frac{n^{100}}{1.001^n}$

Root test:

$$\lim_{n\to\infty}\left(\frac{n^{100}}{1.001^n}\right)^{1/n} = \lim_{n\to\infty}\frac{n^{\frac{100}{n}}}{1.001} = \frac{1}{1.001} < 1,$$

hence the series is convergent.

(l) $\sum_n (-1)^n \frac{n^2}{n!}$

Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2} = \frac{n+1}{n^2} \to 0,$$ hence convergent.

(m) $\sum_n \frac{2n!}{4^n(n!)^2}$

Ratio Test is inconclusive (Check!)

Root Test also fails (after some hard work)

Let us use Stirling's formula:

$$\sqrt{2\pi m}\left(\frac{m}{e}\right)^m e^{\frac{1}{12m+1}} < m! < \sqrt{2\pi m}\left(\frac{m}{e}\right)^m e^{\frac{1}{12m}}$$

Thus $$\frac{2n!}{4^n\,(n!)^2} > \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}\cdot e^{\frac{1}{24n+1}}}{4^n\,(2\pi n)\left(\frac{n}{e}\right)^{2n} e^{\frac{1}{6n}}}$$

$$= \frac{e^{\frac{1}{24n+1}}}{\sqrt{\pi}\sqrt{n}\,e^{1/6n}} > \frac{1}{6\sqrt{n}}$$

$e^{\frac{1}{24n+1}} > 1$

$e^{1/6n} < e < 3$

$\sqrt{\pi} < 2$

hence the series diverges.

This series starts from n=2 → (n) $\sum_{n} \frac{1}{n(\ln n)^3}$ A candidate for Integral Test since $\frac{1}{x(\ln x)^3}$ satisfies the criteria and the fn. is easy to integrate

$$\int_2^n \frac{dx}{x(\ln x)^3} = \int_2^n \frac{d(\ln x)}{(\ln x)^3} = \left[\frac{-1}{2(\ln x)^2}\right]_2^n = \frac{1}{2}\left(\frac{1}{(\ln 2)^2} - \frac{1}{(\ln n)^2}\right) \to \frac{1}{2(\ln 2)^2}.$$

hence convergent.

3. For what values of x, does the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converge?

$\frac{x^n}{n!} \to 0$ for any x (passes the defining sequence test)

If $x=0$, then the series $=1$

If $x \neq 0$ consider $1 + |x| + \frac{|x|^2}{2!} + \cdots$

The ratio limit $= \lim_{n\to\infty} \left(\frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n}\right) = \lim_{n\to\infty} \frac{|x|}{n+1} \to 0$

hence absolutely convergent by Ratio Test $\forall x \neq 0$.

Thus, the series converges everywhere.

4. For which x and p does the series $\sum_{n} \frac{x^n}{n^p}$ converge?

The series $=0$ if $x=0$. So assume $x \neq 0$.

Root Test for absolute convergence:

$$\lim_{n\to\infty} \left(\frac{|x|^n}{n^p}\right)^{1/n} = \lim_{n\to\infty} \frac{|x|}{n^{p/n}} = |x|$$

$\Rightarrow$ The series converges for every p and $\forall x$ with $|x|<1$

and the series diverges for every p and $\forall x$ with $|x|>1$

Next assume $|x|=1$.

If $x=1$, then (proved in class) the series converges $\forall$ p satisfying $0<p<1$, and divgs. otherwise

If $x=-1$, the series is conditionally convg. $\forall p>0$

and absolutely convg. $\forall p>1$.

Finally if $x=-1$ and $p<0$, then the series is not convergent as its defining sequence is not a null sequence.

5. Suppose $S = \sum_n a_n$ and $T = \sum_n b_n$ are convergent series. Answer the following.

(a) Prove that $\sum_n (a_n \pm b_n) = S \pm T$.

Let S_n and T_n be the partial sums, then $S_n \pm T_n$ is the nth partial sum of $\sum_n (a_n \pm b_n)$

Since $S_n \to S$ and $T_n \to T$, by a result on sequences, $S_n \pm T_n \longrightarrow S \pm T$.

(b) Prove that $\sum_n (ca_n) = cS$ for any constant c.

Do it yourself.

(c) How would you define the product series of S and T so that the resulting series converges to ST?

ST is the limit of the sequence $\{S_n T_n\}_n$.

$$\text{Now, } S_n T_n = (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)$$
$$= a_1 b_1 + a_1 b_2 + \cdots\cdots + a_n b_n$$
$$= \sum_{i+j \leq 2n} a_i b_j$$

Thus, the product series is

$$ST = \sum_{n=1}^{\infty} \sum_{i+j \leq 2n} a_i b_j$$

After Rearranging

Warning: Rearrangement can be done only if one of S and T is absolutely convergent. But we did not talk about this!

$$ST = \underbrace{\sum_{n=1}^{\infty} \sum_{i+j=n} a_i b_j}_{}$$

called the Cauchy Product of series.

6*. Does the series $\sum_n \frac{\sin n}{n}$ converge?

We make use of Abel's Theorem from Q.7.

Set up. Set $b_n = \sin n$ & $a_n = \frac{1}{n}$.

To apply Abel's Theorem, need

- $a_n \to 0$

  (since $\frac{1}{n} \to 0$, so fine)

- $\sum_n |a_{n+1} - a_n|$ convergent

  ($\sum_n \left|\frac{1}{n+1} - \frac{1}{n}\right| = \sum_n \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ since this is a telescoping series)

- $|b_1 + b_2 + \cdots + b_n| \leq M \quad \forall n$ for some fixed $M > 0$.

if you have not seen this formula before, just look it up!

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\sin\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$\longrightarrow$ $$\sin 1 + \sin 2 + \cdots + \sin n = \frac{\sin\left(\frac{n+1}{2}\right)\sin\left(\frac{n}{2}\right)}{\sin\left(\frac{1}{2}\right)}$$

Thus,

$$|\sin 1 + \sin 2 + \cdots + \sin n| \leq \frac{1}{\sin\frac{1}{2}} \quad \forall n.$$

Therefore, all the conditions for Abel's Thm. are satisfied, hence

$$\sum_n \frac{\sin n}{n} = \sum_n a_n b_n \text{ converges.}$$

7*. (Abel's Theorem) Suppose $\{a_n\}_n$ and $\{b_n\}_n$ are sequences satisfying the following data:

(1) The sequence of partial sums $\{S_n\}_n$ for the series $\sum_n b_n$ is bounded. That is, there is a constant $M > 0$ such that

$$|S_n| = |b_1 + b_2 + \cdots + b_n| \leq M \quad \forall \quad n \in \mathbb{N}.$$

(2) $\lim_{n\to\infty} a_n = 0$.

(3) The series $\sum_n |a_{n+1} - a_n|$ is convergent.

Prove that the series $\sum_n a_n b_n$ is convergent (This is not the product series).

As usual, we show that the sequence of partial sums is Cauchy!

The following clever argument is due to Abel!

Consider the partial sum

$$T_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Now, $T_m - T_n = \sum_{k=n}^{m} a_k b_k$

Next write $b_k = S_k - S_{k-1}$ (involving S_k is imperative)

Then

$$T_m - T_n = \sum_{k=n}^{m} a_k S_k - \sum_{k=n}^{m} a_k S_{k-1}$$

$$= \sum_{k=n}^{m} a_k S_k - \sum_{k=n-1}^{m-1} a_{k+1} S_k \quad \text{(relabelling the index)}$$

$$= \sum_{k=n-1}^{m-1} (a_k S_k - a_{k+1} S_k) + a_n S_n - a_m S_m$$

Thus, $$|T_m - T_n| \leq \sum_{k=n-1}^{m-1} |S_k| |a_k - a_{k+1}| + |a_n S_n - a_m S_m|$$

$$\leq M \sum_{k=n-1}^{m-1} |a_k - a_{k+1}| + |a_n S_n - a_m S_m|$$

It is given that $\sum_{k=1}^{\infty} |a_k - a_{k+1}|$ is convg.

Let A_n = partial sum $\sum_{k=1}^{n} |a_k - a_{k+1}|$

Then $\{A_n\}_n$ is convg. and hence Cauchy.

$\Rightarrow$ given $\varepsilon > 0$ $\exists$ n_0 s.t.

$$|A_m - A_n| < \varepsilon/2M \quad \forall \; m, n \geq n_0$$

Next,

$$0 \leq |a_n S_n| \leq M |a_n| \longrightarrow 0$$

$$\Rightarrow a_n S_n \longrightarrow 0$$

$\Rightarrow$ $\{a_n S_n\}$ is Cauchy

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ n_1 s.t.

$$|a_n S_n - a_m S_m| < \varepsilon/2 \quad \forall \; m, n > n_1$$

Thus, if $m \geq n \geq \max\{(n_0 + 1), n_1\}$, then

note this

$$|T_m - T_n| \leq M \sum_{k=n-1}^{m-1} |a_k - a_{k+1}| + |a_n S_n - a_m S_m|$$

$$= M |A_{m-1} - A_{n-1}| + |a_n S_n - a_m S_m|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow$ $\{T_n\}_n$ is Cauchy. $\square$

8. Let $S = \sum_n a_n$ (do not assume that the series converges). Recall that $S = S_n + E_n$ where $S_n = a_1 + a_2 + \cdots + a_n$ is the n-th partial sum and $E_n = a_{n+1} + a_{n+2} + \cdots$ is the n-th tail part of S. Suppose it is given that $\{E_n\}_n$ is a null sequence. Prove that $S < \infty$.

Once again, we show that $\{S_n\}_n$ is Cauchy.

If $m > n$, then

$$|S_m - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_m|$$
$$= |E_n - E_m|$$

Since $\{E_n\}_n$ is Cauchy (convergent), it follows that $\{S_n\}_n$ is Cauchy!

9. Suppose $\{a_n\}_n$ is a sequence of *positive* terms such that $\sum_n a_n$ is convergent. What can you say about the convergence of the series $\sum_n a_n^2$? Does your conclusion hold if the terms of $\{a_n\}_n$ are not positive?

$\sum_n a_n$ convergent $\Rightarrow a_n \longrightarrow 0 \Rightarrow \exists\, n_0$ s.t $a_n < 1$ $\forall n \geq n_0$

$\Rightarrow a_n^2 < a_n \quad \forall n \geq n_0$

$\Rightarrow \sum_n a_n^2$ converges by comparison test.

10. Prove the conclusion of the Root Test.

Find a proof online!