In testing differentiability of a fn. $f(x)$ at $c$, we will always try to compute the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

**WORKSHEET - 6**

**1.** Discuss the differentiability of the function $f(x) = |x^2 - 4|$ at $x = 2$.

To compute: $\displaystyle \lim_{h \to 0} \frac{|(2+h)^2 - 4| - 0}{h} = \lim_{h \to 0} \frac{|h(2+h)|}{h}$

Since, we are interested in $h$ close to $0$, hence $2+h > 0$

Thus, the limit is

$$\lim_{h \to 0} \frac{|h|}{h} \cdot (2+h)$$

Now,

$$\lim_{h \to 0^+} \frac{|h|}{h} \cdot (2+h) = \lim_{h \to 0^+} (2+h) = 2$$

$$\lim_{h \to 0^-} \frac{|h|}{h} (2+h) = \lim_{h \to 0^-} -(2+h) = -2$$

Thus, the limit $\displaystyle \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ does <u>not</u> exist.

**2.** Determine whether or not $f(x)$ is differentiable at $x = 0$.

(a) $f(x) = \sqrt[3]{x}$

$$\lim_{h \to 0^+} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt[3]{h^2}} = \infty$$

$$\lim_{h \to 0^-} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0^-} \frac{\sqrt[3]{-h}}{-h} = \lim_{h \to 0^+} \frac{\sqrt[3]{h}}{h} = \infty$$

Thus $\lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \infty$,

hence the fn. is $\underline{\underline{\text{not}}}$ differentiable at 0.

(b) $f(x) = \sqrt{|x|}$

$$\lim_{h \to 0^+} \frac{\sqrt{|h|}}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

$$\lim_{h \to 0^-} \frac{\sqrt{|h|}}{h} = \lim_{h \to 0^-} \frac{\sqrt{|h|}}{-|h|} = \lim_{h \to 0^+} -\frac{\sqrt{h}}{h} = -\infty.$$

Thus, the function is not diff. at 0.

(c) The function $f(x)$ defined as

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } \quad x \neq 0 \\ 0 & \text{if } \quad x = 0. \end{cases}$$

$$\lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

which does not exist.

Hence <u>not</u> diff. at 0.

(d) $f(x) = x|x|$.

$$\lim_{h \to 0} \frac{h|h|}{h} = \lim_{h \to 0} |h| = 0$$

hence the derivative exists at 0.

**3.** Discuss the differentiability of the function $f(x)$ defined below at $x = 0$.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

$$\lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

hence diff. at 0 with derivative $= 0$.

**4.** Let $f(x)$ be differentiable at $x = a$ and $n$ a positive integer. Compute the limit

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a}.$$

$$\frac{a^n f(x) - x^n f(a)}{x-a} = \frac{a^n\big(f(x) - f(a)\big)}{x-a} - \frac{f(a)\,(x^n - a^n)}{x-a}$$

Thus,

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x-a} = a^n f'(a) - n\,a^{n-1} f(a)$$

**5.** Show that the equation $e^x = 1 + x$ has a unique real solution. Do the same for $x = \cos x$.

Let $f(x) = e^x - x - 1$

So that $f(x)$ is diff. (and hence cont.) everywhere on $\mathbb{R}$.

Easy to see that
$$f(0) = 0$$

To show: $f(x) \neq 0 \quad \forall\ x \neq 0$.

Supp. $f(x_1) = 0$ with $x_1 \neq 0$.

By MVT applied on $[0, x_1]$ or $[x_1, 0]$ (depending whether $x_1 > 0$), $\exists$ $c$ between $0$ and $x_1$ s.t.
$$|x_1 - 0|\,|f'(c)| = |f(x_1) - f(0)| = 0$$
$$\Rightarrow f'(c) = 0.$$

That is, $e^c - 1 = 0$
$\Rightarrow c = 0$, impossible.

So, we are done!

Next consider $f(x) = x - \cos x$.

If $f(c) = 0$, then

$$c = \cos c \Rightarrow c \in [-1, 1].$$

If $d$ is another point s.t. $f(d) = 0$, then $d \in [-1,1]$, and since $c \neq d$,

by MVT, $\exists\ x \in (c, d)$ or $x \in (d, c)$

s.t.

$$0 = \frac{f(c) - f(d)}{c - d} = f'(x)$$

$$\Rightarrow 1 - \sin x = 0$$

$$\Rightarrow \sin x = 1 \Rightarrow x = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}.$$

But then

$$|x| = |4n+1| \frac{\pi}{2} \geq \frac{\pi}{2} > 1$$

which contradicts the fact that $x \in (c, d) \subseteq [-1, 1]$

**6.** Suppose $f(x)$ is continuous on $[a,b]$, and differentiable on $(a,b)$ except possibly at $c \in (a,b)$. Assume that

$$\lim_{x\to c} f'(x) = \ell$$

exists (finitely). Show that $f(x)$ is differentiable at $x = c$, and that $f'(c) = \ell$.

Difference quotient of $f$ at $c$ is

$$\varphi_f(t) = \frac{f(t) - f(c)}{t - c}$$

To show: $\lim_{t\to c} \varphi_f(t)$ exists and equal to $\ell$.

Suff. to show that $t_n \to c \Rightarrow \varphi_f(t_n) \to \ell$.

By MVT, for each $t_n$, $\exists$ $u_n \in (t_n, c)$ or $(c, t_n)$ (depending on whether $t_n < c$) s.t.

$$\varphi_f(t_n) = f'(u_n)$$

Note that

$$0 \le |c - u_n| < |c - t_n|$$

By Sandwich Thm. $|c - u_n| \to 0$

$\Rightarrow u_n \to c$

Since $\lim_{x\to c} f'(x) = \ell$,

hence $\lim_{n\to\infty} f'(u_n) = \ell$

$\Rightarrow \lim_{n\to\infty} \varphi_f(t_n) = \ell$,

as required.

**7.** Suppose $f(x)$ is differentiable everywhere on $\mathbb{R}$. Let $c \in \mathbb{R}$. Show that for every $h$, there is a $\theta$ satisfying $0 < \theta < 1$ such that

$$f(c+h) = f(c) + hf'(c+\theta h).$$

Next, set $f(x) = \dfrac{1}{1+x}$. Determine $\lim_{h \to 0} \theta$, for $c > 0$.

Apply MVT on $f$ on $[c, c+h]$ if $h>0$ and on $[c+h, c]$ if $h<0$.

to get that $\exists$ $x \in (c, c+h)$ or $(c+h, c)$ s.t.

$$\frac{f(c+h) - f(c)}{h} = f'(x) \quad \text{—} (*)$$

Since $x \in (c, c+h)$ or $(c+h, c)$

$\exists$ a $0 < \theta < 1$ s.t

$$x = c + \theta h$$

Consequently, from $(*)$, we get

$$f(c+h) = f(c) + hf'(c+\theta h). \quad \text{—} (**)$$

Next, suppose that $f(x) = \dfrac{1}{1+x}$.

Then, from $(**)$ for each $h$, $\exists$ a $\theta \in (0,1)$ s.t.

$$\frac{1}{1+c+h} = \frac{1}{1+c} - \frac{h}{(1+c+\theta h)^2}$$

Rearranging,

$$\frac{h}{(1+c+\theta h)^2} = \frac{h}{(1+c)(1+c+h)}$$

i.e., $(1+c+\theta h)^2 = (1+c)(1+c+h)$

Since $c>0$, $1+c>0$.

Thus by choosing $h$ small enough, we have

$1+c+h>0$ and $1+c+\theta h>0$.

Therefore, $\quad 1+c+\theta h = \sqrt{(1+c)(1+c+h)}$

$$\Rightarrow \theta = \frac{\sqrt{(1+c)(1+c+h)} - (1+c)}{h}$$

Thus, $\displaystyle\lim_{h\to 0} \theta = \lim_{h\to 0} \frac{\sqrt{(1+c)(1+c+h)} - (1+c)}{h}$

$$= \sqrt{1+c} \lim_{h\to 0} \frac{\sqrt{1+c+h} - \sqrt{1+c}}{h}$$

$$= \sqrt{1+c} \lim_{h\to 0} \frac{1}{\sqrt{1+c+h} + \sqrt{1+c}}$$

$$= \sqrt{1+c} \cdot \frac{1}{2\sqrt{1+c}} = \frac{1}{2}.$$

**8.** Use differential calculus to establish the following inequalities.

(a) $\ln(1+x) \leq x$ for any $x \geq 0$.

Let $f(x) = x - \ln(1+x)$

$f'(x)$ exists on $[0, \infty)$, and

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0 \quad \forall\, x \geq 0$$

$\Rightarrow$ $f(x)$ is monotone increasing for $x \geq 0$

$\Rightarrow$ $f(x) \geq f(0) = 0 \quad \forall\, x \geq 0.$

Alternatively, if $x > 0$, then by MVT,
$\exists$ a $c \in (0, x)$ s.t.

$$f(x) - f(0) = x f'(c)$$

i.e., $$x - \ln(1+x) = x \cdot \frac{c}{1+c} > 0$$

$$\Rightarrow x > \ln(1+x) \quad \forall\, x > 0.$$

(b) $x - \frac{x^3}{6} < \sin x < x$ for any $0 < x \leq \frac{\pi}{2}$

Let $f(x) = x - \sin x$

and $g(x) = \sin x - x + \frac{x^3}{6}$

To show: both $f$ & $g$ are increasing on $(0, \frac{\pi}{2}]$.

$$f'(x) = 1 - \cos x > 0 \quad \forall\, x \in (0, \tfrac{\pi}{2}]$$

Let $x \in (0, \pi/2]$, by MVT, $\exists\, c \in (0, x)$ s.t.

$$x - \sin x = x \cdot (1 - \cos c) > 0 \quad \left(\begin{array}{l}\text{since } c>0 \\ \cos c < 1\end{array}\right)$$

$$\Rightarrow x > \sin x \quad \forall\, x \in (0, \pi/2]$$

For, $g(x)$ with $x > 0$, we have

$$\sin x - x + \frac{x^3}{6} = x \cdot \left(\cos c - 1 + \frac{c^2}{2}\right)$$

Claim: $\cos x > 1 - \frac{x^2}{2} \quad \forall\, x > 0.$

Let $h(x) = \cos x - 1 + \frac{x^2}{2}$.

By MVT, $\exists\ d \in (0, x)$ s.t.

$$\cos x - 1 + \frac{x^2}{2} = x \cdot \left(-\sin x + x\right)$$

$> 0$ by the last part, hence done.

Thus, $$\sin x - x + \frac{x^3}{6} = x\left(\cos c - 1 + \frac{c^2}{2}\right)$$

$> 0 \quad \forall\, x \in \left(0, \frac{\pi}{2}\right).$

**9.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume that there is a constant $\beta > 0$ such that

$$|f(x) - f(y)| \leq |x - y|^{1+\beta} \quad \forall \quad x, y \in \mathbb{R}.$$

Show that $f(x)$ is a constant function.

To show: • $f'(x)$ exists $\forall\, x$

• $f'(x) = 0 \quad \forall\, x$

Now,

$$0 \leq |\varphi_f(t)| = \frac{|f(t) - f(c)|}{|t - c|} \leq |t - c|^{\beta}$$

By Sandwich Thm.,

$$\lim_{t \to c} |\varphi_f(t)| = 0$$

$$\Rightarrow \lim_{t \to c} \varphi_f(t) = 0$$

$$\Rightarrow f'(c) = 0.$$

**10.** Show that the function $f(x) = x + \sin x$ is strictly increasing in $[1, \infty)$.

Let $1 \le x < y < \infty$.

Then by MVT applied to $f(x)$ on $[x, y]$,

$\exists$ a $c \in (x, y)$ s.t.

$$f(y) - f(x) = f'(c)(y-x)$$
$$= (1+\cos c)(y-x)$$
$$\ge 0.$$

Thus, $f(x)$ is monotone increasing.

To show that $f(x)$ is strictly increasing, we note that for $h > 0$,

$$f(x+h) - f(x) = h + \sin(x+h) - \sin x$$
$$= h + 2\sin\frac{h}{2} \cdot \cos\left(x+\frac{h}{2}\right)$$

Thus, $f(x+h) - f(x) = 0$ implies

$$h + 2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right) = 0$$

$$\Rightarrow \quad h = 2\left|\sin\frac{h}{2}\right|\left|\cos\left(x+\frac{h}{2}\right)\right| \quad \left(\begin{array}{c}\text{Thus,}\\ \left|\cos\left(x+\frac{h}{2}\right)\right| > 0\end{array}\right)$$

$$< 2\cdot\frac{h}{2}\left|\cos\left(x+\frac{h}{2}\right)\right|$$

since $h>0$

$$\leq h\left|\cos\left(x+\frac{h}{2}\right)\right| \leq h$$

$\Rightarrow \quad h < h$ , a contradiction.

Here, we have used that

$$|\sin x| < x \quad \forall\, x > 0.$$

If $x > 1$, then done since $|\sin x| \leq 1$

If $x \in (0,1]$, then $|\sin x| = \sin x$

and then $\sin x < x$ by 8.(b)

since $(0,1] \subset (0, \frac{\pi}{2}]$.