भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

7th Lecture on Transform Techniques

(MA-2120)

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What did we learn in previous class?

- Inverse Laplace Transform
- Properties of Inverse Laplace Transform

What will we learn today?

- Applications of the Laplace Transform

Applications of Laplace Transform :

Solution of ODE:

Initial value problem (IVP):

Ex: Solve the initial value problem

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 6$$

Sol^n: We have $y'' + 4y = 0$

Taking Laplace Transform —

$$\mathcal{L}[y''] + 4\mathcal{L}[y(t)] = 0$$

or $\left[s^2\mathcal{L}[y(t)] - sy(0) - y'(0)\right]$

$$+ 4\mathcal{L}[y(t)] = 0$$

Let $\mathcal{L}[y(t)] = Y(s)$.

$$\Rightarrow s^2Y(s) - s - 6 + 4Y(s) = 0$$

or, $(s^2+4)Y(s) = s+6$

$$Y(s) = \frac{s+6}{s^2+4}$$

$$= \frac{s}{s^2+4} + \frac{6}{s^2+4}$$

Taking Inverse Laplace Transform

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + 3\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= \cos 2t + 3 \sin 2t.$$

Ex: Solve the initial value Problem

$$y'' + 2y' - 3y = 3, \quad y(0) = 4, \quad y'(0) = -7,$$

Solve:

$$y'' + 2y' - 3y = 3$$

Taking Laplace Transform of the given ODE, we have

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] - 3\mathcal{L}[y] = 3\mathcal{L}[1]$$

or, $$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] - 3Y(s) = 3/s$$

$$\text{or, } s^2Y(s) - 4s + 7 + 2sY(s) - 8 - 3Y(s) = 3/s$$

$$\text{or, } (s^2 + 2s - 3)Y(s) = 3/s + 4s + 1$$

$$Y(s) = \frac{3}{s(s-1)(s+3)} + \frac{4s+1}{(s-1)(s+3)}$$

$$= \frac{4s^2 + s + 3}{s(s-1)(s+3)}$$

$$Y(s) = -\frac{1}{s} + \frac{2}{s-1} + \frac{3}{s+3}.$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[-\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{2}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{3}{s+3}\right]$$

$$\Rightarrow \boxed{y(t) = -1 + 2e^{t} + 3e^{-3t}}$$

Solution

Ex:

Solve the IVP

$$y'' - 5y' + 4y = e^{2t}, \quad y(0) = 19/12$$
$$y'(0) = 8/3.$$

Sol^n:

Taking the Laplace Transform —

$$\left[s^2 Y(s) - s y(0) - y'(0)\right] - 5\left[s Y(s) - y(0)\right] + 4Y(s) = \frac{1}{s-2}$$

$$\Rightarrow \quad Y(s) = \frac{\frac{1}{s-2} + \frac{19s}{12} - \frac{63}{12}}{(s^2 - 5s + 4)}$$

$$Y(s) = -\frac{1}{2(s-2)} + \frac{14}{9(s-1)} + \frac{19}{36(s-4)}$$

$$\boxed{\therefore\ y(t) = -\frac{1}{2}e^{2t} + \frac{14}{9}e^{t} + \frac{19}{36}e^{4t}.}$$

Ex. Find solution of the IVP -

$$y'' + 4y' + 4y = 12t^2 e^{-2t}, \quad y(0) = 2, \quad y'(0) = 1$$

Try! $\boxed{y(t) = (2+5t+t^4)e^{-2t}}$ Sol^n.

Ex: Consider $y'' + y = e^{-t}$ and let $y(0) = y_0$, $y'(0) = y_1$ be unspecified. Find solution $y(t)$.

Sol^n: Taking L.T.,

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s+1}$$

or $$(s^2+1)Y(s) - s y_0 - y_1 = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s^2+1)} + \frac{s y_0}{(s^2+1)}$$

$$+ \frac{y_1}{s^2+1}$$

$$= \frac{1}{2}\frac{1}{s+1} - \frac{\frac{1}{2}(s-1)}{s^2+1} + \frac{y_0 s}{s^2+1} + \frac{y_1}{s^2+1}$$

$$= \frac{1}{2}\frac{1}{s+1} + \left(y_0 - \frac{1}{2}\right)\frac{s}{s^2+1} + \left(y_1 + \frac{1}{2}\right)\frac{1}{s^2+1}$$

Taking inverse L. T.,

$$y(t) = \frac{e^{-t}}{2} + \left(y_0 - \frac{1}{2}\right)\cos t$$
$$+ \left(y_1 + \frac{1}{2}\right)\sin t.$$

Since y_0, y_1 take on all possible values, the general solution to the problem is given by —

$$\boxed{y(t) = c_0 \cos t + c_1 \sin t + \frac{e^{-t}}{2}}$$

$-\infty < t < \infty.$

Example: Solve $y''' + y'' = e^t + t + 1$

$y(0) = y'(0) = y'''(0) = 0.$

Sol^n: Taking Laplace Transform of both sides, we have

$$\left[s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)\right] + \left[s^2 Y(s) - s y(0) - y'(0)\right] = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}$$

$$s^3 Y(s) + s^2 Y(s) = \frac{2s^2 - 1}{s^2(s-1)}$$

$$\text{or} \quad Y(s) = \frac{2s^2-1}{s^4(s-1)(s+1)}.$$

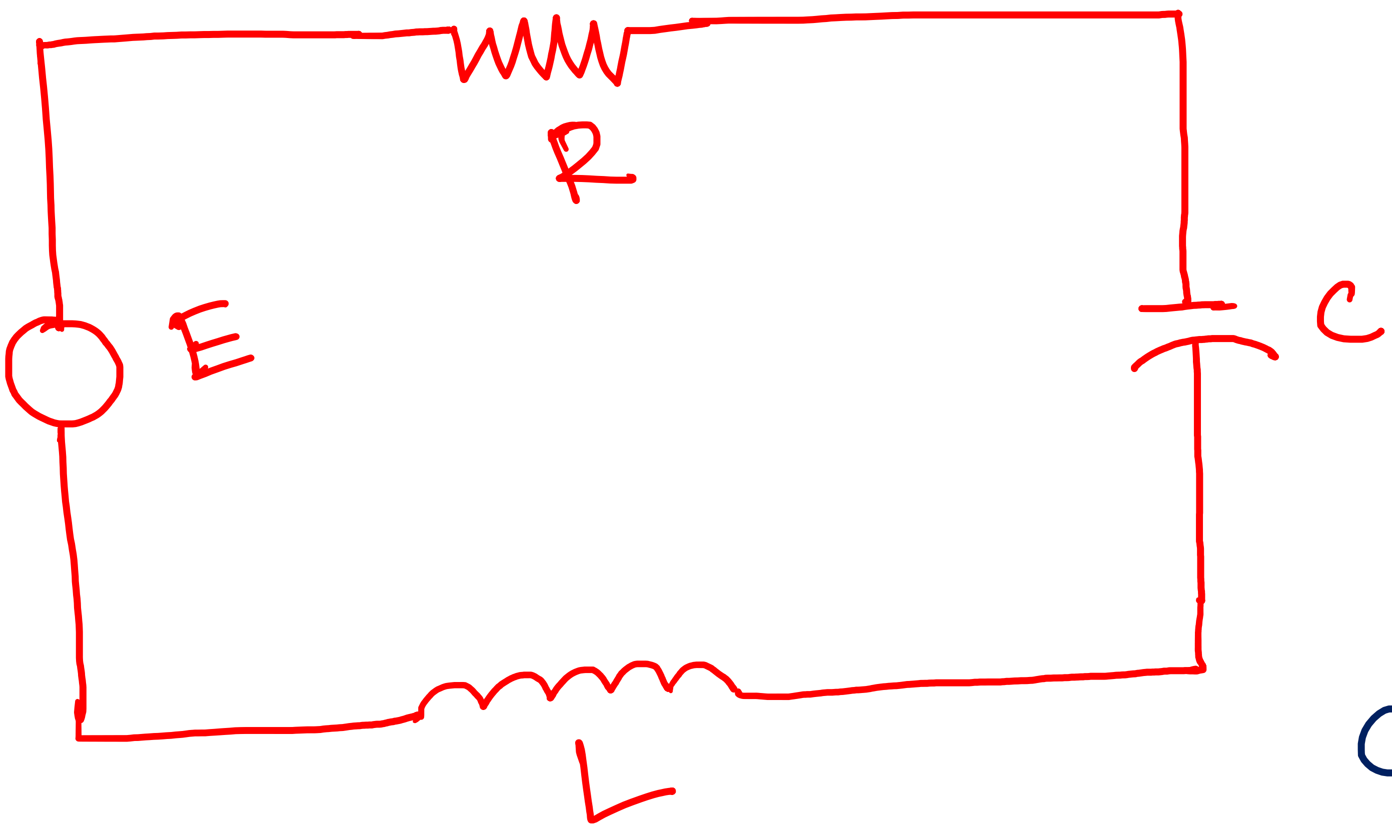
$$= -\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

$$y(t) = -t + t^3/6 - \frac{1}{2}e^{-t} + e^t/2 \quad \underline{([illegible])}$$

Integro-differential eqⁿ :

In many problems of electrical engineering, we need to solve the integro-differential eqⁿs. Consider a series of electric circuit. Using the Kirchoff's second law, we obtain that the flow of current i satisfies the integro-differential eqⁿ –

$$\boxed{L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,d\tau = E_0 \cos \omega t}$$



Series of Circuit

C : capacitance

E : impressed voltage

R : resistance

L : inductance

Ex: Find the solution of the initial value problem —

$$y' + 3y + 2\int_0^t y(\tau)\,d\tau = t, \quad y(0) = 0$$

Sol^n:

$$y' + 3y + 2\int_0^t y(\tau)\,d\tau = t$$

Taking the Laplace Transform

$$sY(s) - y(0) + 3Y(s) + \frac{2}{s}Y(s) = \frac{1}{s^2}$$

where $\mathcal{L}[y(t)] = Y(s)$.

a) $(s^2+3s+2)Y(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{s}F(s)$$

where $F(s) = \frac{1}{(s+1)} \cdot \frac{1}{s+2} = \frac{1}{s+1} - \frac{1}{s+2}$

$$\Rightarrow \mathcal{L}^{-1}[F(s)] = f(t) = e^{-t} - e^{-2t}.$$

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right]$$

$$= \int_0^t f(\tau)\,d\tau$$

$$y(t) = \int_0^t \left(e^{-\tau} - e^{-2\tau}\right) d\tau$$

$$= \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2} \cdot \quad (\text{sol}^n)$$

Ex: $$y'(t) - y(t) = \sin t + 6\int_0^t y(\tau)\, d\tau,$$

$$y(0) = 2$$

Try! **Ans:** $$y(t) = -\frac{7}{50}\cos t - \frac{1}{50}\sin t + \frac{63}{50}e^{3t} + \frac{22}{25}e^{-2t}.$$

Ex: Find the sol^n of IVP (First shifting Property)

$$y'' + 4y' + 13y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 2.$$

Sol^n: Taking L.T., we have

$$\left[s^2 Y(s) - s(0) - 2\right] + 4\left[sY(s) - 0\right] + 13Y(s) = \frac{1}{s+1}$$

$$\text{or } Y(s) = \frac{2s+3}{(s+1)(s^2+4s+13)}$$

$$Y(s) = \frac{1}{10}\left[\frac{1}{s+1} - \frac{s-17}{s^2+4s+13}\right]$$

$$= \frac{1}{10}\left[\frac{1}{s+1} - \frac{s+2}{(s+2)^2+3^2} + \frac{19}{(s+2)^2+3^2}\right]$$

Taking Inverse Laplace Transform —

$$y(t) = \frac{1}{10}\left[e^{-t} - e^{-2t}\cos 3t + \frac{19}{3}e^{-2t}\sin 3t\right]$$

[Here First shifting theorem is applied.]

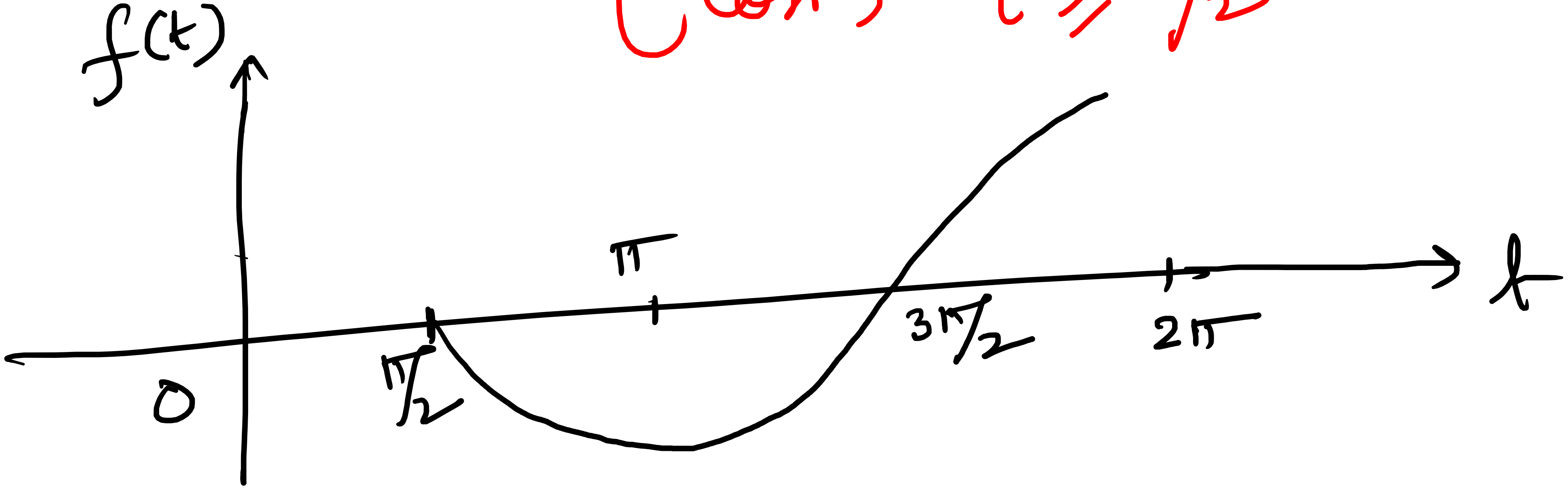
Ex: Find the solution of the IVP —

Second shifting property

$$y' + y = f(t), \quad y(0) = 2$$

where $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \cos t, & t \ge \pi/2 \end{cases}$

Sol^n:



The function $f(t)$ can be represented by unit step function.

we have $f(t) = \cos t \cdot H(t - \pi/2)$

So the given ODE can be written as

$$y' + y = \cos t \cdot H(t - \pi/2)$$

Taking L.T., we have -

a) $sY(s) - 2 + Y(s) = \mathcal{L}\left[(\cos t)\, H(t - \pi/2)\right]$

$$= e^{-\pi/2\, s} \cdot \mathcal{L}\left[\cos(t + \pi/2)\right]$$

$$= -e^{-\pi/2\, s}\, \mathcal{L}\left[\sin t\right]$$

$$= -e^{-\pi/2\, s} \cdot \frac{1}{s^2+1}$$

$$Y(s) = \frac{2}{s+1} - \frac{e^{-\pi/2 \cdot s}}{(s+1)(s^2+1)}$$

$$Y(s) = \frac{2}{s+1} - \frac{1}{2}\left[\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] e^{-\frac{\pi s}{2}}$$

Taking Inverse Laplace Transform —

$$y(t) = 2e^{-t} - \frac{1}{2}\left[e^{-(t-\pi/2)} - \cos(t-\pi/2) + \sin(t-\pi/2)\right] H(t-\pi/2)$$

$$y(t) = 2e^{-t} - \frac{1}{2}\left[e^{-(t-\pi/2)} - \sin t - \cos t\right] H(t - \pi/2).$$

$$y(t) = \begin{cases} 2e^{-t}, & 0 \le t < \pi/2 \\ 2e^{-t} - \frac{1}{2}\left[e^{-(t-\pi/2)} - \sin t - \cos t\right], & t \geq \pi/2. \end{cases}$$

$\boxed{\text{Here second shifting property is applied}}$

Ex: Find the solution of the IVP

$$y'' + ty' - 2y = 6 - t, \quad y(0) = 0, \quad y'(0) = 1$$

given that $\mathcal{L}[y(t)]$ exists.

Sol^n: Taking the Laplace Transform —

$$\mathcal{L}[y''] + \mathcal{L}[ty'] - 2\mathcal{L}[y] = \mathcal{L}[6-t]$$

$$\Rightarrow s^2 Y(s) - 1 + \left[-\frac{d}{ds}\left\{ sY(s) - 0 \right\} \right] - 2Y(s) = 6/s - \frac{1}{s^2}$$

or $s^2 Y(s) - [sY'(s) + Y(s)] - 2Y(s) = 1 + \frac{6}{s} - \frac{1}{s^2}$

or $\boxed{Y'(s) + \left(\frac{3}{s} - s\right) Y(s) = -\left(\frac{1}{s} + \frac{6}{s^2} - \frac{1}{s^3}\right)}$ — (1)

This is first order linear ODE in $Y(s)$

IF $= e^{\int \left[3/s - s\right] ds} = s^3 e^{-s^2/2}$

Multiplying $s^3 e^{-s^2/2}$ to the ODE (1), we have

$$s^3 e^{-s^2/2} Y(s) = -\int s^3 e^{-s^2/2} \left(\frac{1}{s} + \frac{6}{s^2} - \frac{1}{s^3}\right) ds + C$$

a) $s^3 e^{-s^2/2} Y(s)$

$$= -\int s^3 e^{-s^2/2}\left(\frac{1}{s}+\frac{6}{s^2}-\frac{1}{s^3}\right) ds + c$$

where c is arbitrary constant.

Consider $-s^2/2 = u$

$-s\,ds = du$

$$= \int e^{-s^2/2} ds - 6\int s e^{-s^2/2} ds - \int s^2 e^{-s^2/2} ds + c$$

$$= \int e^{-s^2/2} ds + 6 e^{-s^2/2} - \left[-s e^{-s^2/2} + \int e^{-s^2/2} ds\right] + c$$

$$s^3 e^{-s^2/2} Y(s) = (6+s) e^{-s^2/2} + c$$

$$\boxed{Y(s) = \frac{6}{s^3} + \frac{1}{s^2} + \frac{c}{s^3} e^{s^2/2}}$$

Since $\mathcal{L}[y(t)]$ exist, $\lim_{s \to \infty} Y(s) = 0$

$\Rightarrow \quad c = 0$

$\therefore Y(s) = \frac{6}{s^3} + \frac{1}{s^2} \Rightarrow y(t) = 3t^2 + t.$

Note: Consider an ODE —

$$\boxed{y' - 2ty = 0, \quad y(0) = 1}$$

This IVP has $y(t) = e^{t^2}$ as its solution.

but this function does not possess a Laplace transform.

So $\mathcal{L}[y(t)]$ does not exist.

Ex: Find the sol^n to the IVP –

$$ty'' + 2ty' + 2y = 2,$$

$y(0) = 1$, $y'(0)$ is arbitrary.

$$\boxed{y(t) = 1 - (4 - c_1)\, t e^{-2t}}$$

Try it!

Note: $y'(0) = c_1 - 4$. where c_1 is an arbitrary constant.

Here sol^n is valid for arbitrary value of $y'(0)$.