भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

12th Lecture on Transform Techniques

(MA-2120)

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What will we learn today?

- Properties of Fourier Transform
- Fourier Cosine Transform
- Fourier Sine Transform

④ Theorem : (Modulation Theorem)

If $\mathcal{F}[f(t)] = F(\omega)$ and ω_0 is any real number, then

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\left[F(\omega+\omega_0) + F(\omega-\omega_0)\right]$$

and $$\mathcal{F}[f(t)\sin\omega_0 t] = \frac{i}{2}\left[F(\omega+\omega_0) - F(\omega-\omega_0)\right]$$

Proof: This can be proved by using the frequency shifting theorem.

Try it!

(5) Fourier transform of derivative :

Let $f(t)$ be continuous and $f^{(k)}(t)$, $k=1,2,\ldots$

n be piecewise continuous on every interval $[-l, l]$

and $\int_{-\infty}^{\infty} |f^{(k-1)}(t)|\,dt$, $k=1,2,\ldots,n$ converge.

Let $f^{(k)}(t) \to 0$ as $t \to \infty$ for $k=0,1,\ldots,$

$n-1$.

If $\mathcal{F}[f(t)] = F(\omega)$, then

$$\boxed{\mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(\omega).}$$

where $f(x)$ and all its derivatives vanish at infinity.

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)\, e^{-i\omega t}\, dt$$

$$= \left[f(t)\, e^{-i\omega t}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\omega f(t)\, e^{-i\omega t}\, dt$$

$$= i\omega F(\omega)$$

$$\boxed{\mathcal{F}[f''(t)] = -\omega^2 F(\omega)}$$

Example: Find the solution of the differential eqn —

$$y' - y = H(t)\, e^{-2t}, \quad -\infty < t < \infty.$$

using Fourier transform, where $H(t)$ is unit step function.

Solⁿ:

Applying Fourier transform —

$$\mathcal{F}[y'] - 2\,\mathcal{F}[y] = \mathcal{F}[H(t)\,e^{-2t}]$$

or $$i\omega Y(\omega) - 2Y(\omega) = \frac{1}{2+i\omega}$$

or $$Y(\omega) = -\frac{1}{(2+i\omega)(2-i\omega)} = -\frac{1}{4+\omega^2}$$

Take Inverse

$$\mathcal{F}^{-1}[Y(\omega)] = y(t) = \mathcal{F}^{-1}\left[-\frac{1}{4+\omega^2}\right] = -\frac{1}{4}e^{-2|t|}$$

The solution can be written as

$$\boxed{y(t) = \begin{cases} -\frac{1}{4} e^{2t}, & t < 0 \\ -\frac{1}{4} e^{-2t}, & t > 0 \end{cases}}$$

(6) Symmetry property of Fourier transform :

Let $\mathcal{F}[f(t)] = F(\omega)$. Then

$$\boxed{\mathcal{F}[F(t)] = 2\pi f(-\omega)}$$

Proof: From definition,

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{ist}\, ds .$$ Since ω is a dummy variable of integration.

Hence setting $t = -\omega$, we get

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(s)\, e^{-i\omega s}\, ds$$

$$= \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt$$

$$= \mathcal{F}[F(t)]$$

$$\boxed{\mathcal{F}[F(t)] = 2\pi f(-\omega)}$$ Duality Property.

Example: Find the Fourier transform of $f(t) = \dfrac{1}{5+it}$.

Sol^n: we know that $\dfrac{1}{5+i\omega}$ is the Fourier transform of $H(t)\,e^{-5t}$.

Let $g(t) = H(t)\,e^{-5t}$.

Then $\mathcal{F}[g(t)] = \mathcal{F}[H(t)\,e^{-5t}]$

$$= \frac{1}{5+i\omega} = G(\omega)$$

using symmetry property,

$$\mathcal{F}[G(t)] = 2\pi\, g(-\omega)$$

or, $\mathcal{F}\left[\frac{1}{5+it}\right] = 2\pi \cdot H(-\omega)\,e^{5\omega}$.

Therefore,

$$\mathcal{F}[f(t)] = F(\omega) = \begin{cases} 2\pi e^{5\omega}, & \omega \leq 0 \\ 0, & \omega > 0 \end{cases}$$

(7) Derivative of the Fourier transform:

Theorem: Let $f(t)$ be piecewise continuous on every interval $[-l, l]$. Let $\int_{-\infty}^{\infty} |t^n f(t)|\, dt$ converge. Then $\mathcal{F}[t^n f(t)] = i^n F^{(n)}(\omega)$.

Proof: From the definition, we have

$$F'(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{d}{d\omega}\left(f(t)\,e^{-i\omega t}\right) dt$$

[assuming that the integration and differentiation can be interchanged]

$$= -i\int_{-\infty}^{\infty} [t f(t)]\, e^{-i\omega t}\,dt$$

$$= -i\,\mathcal{F}[t f(t)].$$

$$\mathcal{F}[t f(t)] = -\frac{1}{i} F'(\omega) = i F'(\omega)$$

Similarly $F''(\omega) = (-i)^2 \mathcal{F}[t^2 f(t)]$

$$\Rightarrow \mathcal{F}[t^2 f(t)] = -F''(\omega).$$

By induction, $\boxed{\mathcal{F}[t^n f(t)] = i^n F^{(n)}(\omega).}$

Inverse: $\mathcal{F}^{-1}[F^{(n)}(\omega)] = (-i)^n t^n f(t).$

(8) Fourier Transform of an integral:

Theorem: Let $f(t)$ be piecewise continuous on every interval $[-l, l]$ and $\int_{-\infty}^{\infty} |f(t)|\, dt$ converge. Let $\mathcal{F}[f(t)] = F(\omega)$ and $F(\omega)$ satisfies $F(0) = 0$. Then

$$\mathcal{F}\left[\int_{-\infty}^{t} f(\tau)\, d\tau\right] = \frac{1}{i\omega} F(\omega).$$

Inverse:

$$\mathcal{F}^{-1}\left[\frac{F(\omega)}{\omega}\right] = i\int_{-\infty}^{t} f(\tau)\,d\tau .$$

Ex: Find the inverse Fourier transform of

$$\frac{\sqrt{\pi}\,\omega\, e^{-\omega^2/8}}{4\sqrt{2}\,i}.$$

Solⁿ: we have $\frac{\sqrt{\pi}}{4\sqrt{2}\,i}\,\omega\, e^{-\omega^2/8}$

$$= -\frac{\sqrt{\pi}}{\sqrt{2}\,i}\,\frac{d}{d\omega}\left(e^{-\omega^2/8}\right)$$

Therefore $\mathcal{F}^{-1}\left[\frac{\sqrt{\pi}\,\omega}{4\sqrt{2}\,i}\cdot e^{-\omega^2/8}\right]$

$$= -\frac{\sqrt{\pi}}{i\sqrt{2}}\,\mathcal{F}^{-1}\left[\frac{d}{d\omega}\left(e^{-\omega^2/8}\right)\right]$$

Now $\mathcal{F}^{-1}\left[\frac{d}{d\omega}\left(e^{-\omega^2/8}\right)\right] = \mathcal{F}^{-1}\left[\frac{dF(\omega)}{d\omega}\right]$

where $F(\omega) = e^{-\omega^2/8} = \mathcal{F}^{-1}[f(t)]$

$$\mathcal{F}^{-1}\left[\frac{dF}{d\omega}\right] = -it\, f(t) = -it\,\mathcal{F}^{-1}[F(\omega)]$$

$$\mathcal{F}^{-1}\left[\frac{d}{d\omega}\left(e^{-\omega^2/8}\right)\right] = -it \cdot \mathcal{F}^{-1}[F(\omega)]$$

$$= -it \cdot \frac{\sqrt{2}}{\sqrt{\pi}} e^{-2t^2}.$$

by using the result

$$\mathcal{F}\left[e^{at^2}\right] = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{-\omega^2/4a}.$$

$$\therefore \boxed{\mathcal{F}^{-1}\left[\frac{\sqrt{\pi}\,\omega}{4\sqrt{2}\,i}\, e^{-\omega^2/8}\right] = t\, e^{-2t^2}.}$$

(9) Scaling property:

Theorem: If $\mathcal{F}[f(t)] = F(\omega)$, then

$$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\omega/a\right)}, \quad a \neq 0$$

(10) Convolution :

Theorem: Let $f(t)$, $g(t)$ be piecewise continuous on every interval $[-l, l]$ and let $\int_{-\infty}^{\infty} |f(t)|\,dt$, $\int_{-\infty}^{\infty} |g(t)|\,dt$ converge.

Then $\mathcal{F}[f(t)] = F(\omega)$ and $\mathcal{F}[g(t)] = G(\omega)$.

Then $\mathcal{F}[f * g(t)]$

$$= F(\omega)\, G(\omega) \cdot$$

[Convolution w.r.t to time]

$$\mathcal{F}[f(t)\, g(t)] = \frac{1}{2\pi}[F * G](\omega)$$

[Convolution w.r.t. frequency]

Inverse:

$$\mathcal{F}^{-1}\left[F(\omega)\,G(\omega)\right] = f * g\,(t)$$

$$\mathcal{F}^{-1}\left[F * G(\omega)\right] = 2\pi f(t)\,g(t).$$

Ex: Find $\mathcal{F}^{-1}\left[\dfrac{1}{12+7i\omega-\omega^2}\right]$

Sol^n: we have $12+7i\omega-\omega^2$

$$= (4+i\omega)(3+i\omega)$$

Therefore $\mathcal{F}^{-1}\left[\dfrac{1}{12+7i\omega-\omega^2}\right]$

$$= \mathcal{F}^{-1}\left[\dfrac{1}{(4+i\omega)(3+i\omega)}\right]$$

$$= \mathcal{F}^{-1}\left[\frac{1}{4+i\omega}\right] * \mathcal{F}^{-1}\left[\frac{1}{3+i\omega}\right]$$

$$= e^{-4t} H(t) * e^{-3t} H(t)$$

$$= \int_{-\infty}^{\infty} e^{-4\tau} H(\tau)\, e^{-3(t-\tau)} H(t-\tau)\, d\tau$$

$$= e^{-3t} \int_{-\infty}^{\infty} e^{-\tau} H(\tau) H(t-\tau)\, d\tau$$

$$H(\tau)\,H(t-\tau) = \begin{cases} 0, & \tau < 0 \text{ and } \tau > t \\ 1, & \text{for } 0 < \tau < t \end{cases}$$

$$\Rightarrow \mathcal{F}^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right] = e^{-3t}\int_0^t e^{-\tau}\,d\tau$$

$$= e^{-3t}\left[1-e^{-t}\right], \quad t \geq 0$$

Fourier Cosine Transform :

The Fourier Cosine transform of $f(t)$ is defined as —

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty f(t)\cos\omega t\, dt = F_c(\omega)$$

Inverse Fourier Cosine transform :

$$\mathcal{F}_c^{-1}\left[F_c(\omega)\right] = f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega t\, d\omega$$

Fourier Cosine integral $\Rightarrow$ $f(t) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos\omega t\, d\omega$

$A(\omega) = 2\int_0^\infty f(t)\cos\omega t\, dt.$

Example: Find the Fourier cosine transform of $f(t)$, where

$$f(t) = \begin{cases} t, & 0 \le t \le l \\ 0, & t > l \end{cases}$$

Sol^n:

$$\mathcal{F}_c\left[f(t)\right] = \int_0^{\infty} f(t) \cos \omega t \, dt$$

$$= \int_0^{l} t \cos \omega t \, dt$$

$$= \frac{l}{\omega} \sin(\omega l) + \frac{1}{\omega^2}\left[\cos(\omega l) - 1\right].$$

Fourier Sine Transform :

The Fourier Sine transform of $f(t)$ is defined as

$$\boxed{\mathcal{F}_s\left[f(t)\right] = \int_0^{\infty} f(t)\, \sin\omega t\, dt = F_s(\omega).}$$

Inverse Fourier Sine transform :

$$\mathcal{F}_s^{-1}\left[F_s(\omega)\right] = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\, \sin\omega t\, d\omega.$$

Fourier Sine integral $\Rightarrow$ $f(t) = \frac{1}{\pi}\int_0^{\infty} B(\omega)\, \sin\omega t\, d\omega$

where $B(\omega) = 2\int_0^{\infty} f(t)\, \sin\omega t\, dt$

Ex: Find the Fourier sine transform of

$$f(t) = \begin{cases} t, & 0 \le t \le l \\ 0, & t > l. \end{cases}$$

Sol^n: $$\mathcal{F}_s[f(t)] = \int_0^\infty f(t)\sin\omega t \, dt$$

$$= \int_0^l t \sin\omega t \, dt$$

$$= \frac{\sin\omega l}{\omega^2} - \frac{l\cos\omega l}{\omega}.$$

Fourier cosine and sine transforms of Derivatives:

- Let $f(t)$ and $f'(t)$ be continuous on $[0,\infty)$. Let $f(t) \to 0$, $f'(t) \to 0$ as $t \to \infty$ and $f''(t)$ be Piecewise continuous on every subinterval $[0,l]$.

Then —

$$\boxed{\mathcal{F}_c[f''(t)] = -\omega^2 \mathcal{F}_c[f(t)] - f'(0)}$$

and

$$\mathcal{F}_s[f''(t)] = -\omega^2 \mathcal{F}_s[f(t)] + \omega f(0)$$

Proof:

From the def^n, we have —

$$F_c[f''(t)] = \int_0^\infty f''(t)\cos\omega t\,dt$$

$$= \left[f'(t)\cos\omega t + \omega f(t)\sin\omega t\right]\Big|_0^\infty - \omega^2\int_0^\infty f(t)\cos\omega t\,dt$$

$$= -\omega^2 F_c[f(t)] - f'(0)$$

Similarly try for $\mathcal{F}_s[f''(t)]$.

Remark: Let $f(t)$ be continuous and $f'(t)$ be piece-wise continuous on $[0,\infty)$. Let $f(t) \to 0$ as $t \to \infty$. Then

$$\boxed{\begin{aligned} \mathcal{F}_c[f'(t)] &= \omega\, \mathcal{F}_s[f(t)] - f(0) \\ \text{and} \quad \mathcal{F}_s[f'(t)] &= -\omega\, \mathcal{F}_c[f(t)] \end{aligned}}$$

Try to prove it.

Fourier Transform of the Dirac delta function:

$$\delta(t) = \lim_{K\to 0} \frac{1}{K}\left[H(t) - H(t-K)\right]$$

where $H(t)$ is the unit step function.

we have $H(t) - H(t-K) = \begin{cases} 0, & t<0 \\ 1, & 0\le t<K \\ 0, & t\geq K \end{cases}$.

$$\mathcal{F}\left[\delta(t)\right] = \lim_{K\to 0}\left\{\frac{1}{K}\mathcal{F}\left[H(t) - H(t-K)\right]\right\}$$

$$= \lim_{K\to 0}\left\{\frac{1}{K}\int_0^K e^{-i\omega t}\,dt\right\} = \lim_{K\to 0}\frac{1-e^{-i\omega K}}{i\omega K}$$

$$= 1.$$

By using the theorem on shifting t axis (time shift theorem)

$$\boxed{\mathcal{F}[\delta(t-a)] = e^{-ia\omega}F(\omega) = e^{-ia\omega}. \text{ since } F(\omega) = \mathcal{F}[\delta(t)] = 1.}$$

By using symmetry property,

$$\boxed{\mathcal{F}[1] = 2\pi\,\delta(-\omega) = 2\pi\,\delta(\omega).}$$