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3rd Lecture on Differential Equation

(MA-1150)

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What have we learnt?

- Nature of solutions: Explicit and Implicit Solutions
- First Order and First Degree ODE
- Separation of Variables
- Homogeneous Equations

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What will we learn today?

- Exact First Order Differential Equation
- Methods of solving Exact differential equations
- Exact Homogeneous Equations
- Integrating Factors

Exact First Order Differential Equation

Consider a first order differential equation as

$$\boxed{M\,dx + N\,dy = 0} \quad \text{--- (1)}$$

where M and N are functions of x and y. Now the equation (1) is said to be exact if there exists a differentiable function $f(x,y)$ such that

$$\boxed{M\,dx + N\,dy = d f(x,y)} \quad \text{--- (2)} \quad \forall (x,y) \in D$$

Rectangular domain,

OR,

$$\boxed{M\,dx + N\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy} \quad \text{--- (3)}$$

So, that means for exactness of differential eq^n, eq^n (1) can be represented by $\boxed{df(x,y) = 0} \Rightarrow \boxed{f(x,y) = C}$ → General solution

where C is arbitrary constant

Exact First Order Differential Equation

Note: For a function $f(x,y)$ of two variables, we know that if $f(x,y)$ has continuous first order partial derivatives, then the total or exact differential of $f(x,y)$ is expressed as

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}, \quad \forall\, (x,y) \in D,$$

D is the rectangular Domain'

*** From equation (3), we can obtain

$$\boxed{\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)}$$

Exact First Order Differential Equation

- **Theorem:** The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Note: The above condition exist if $M(x,y)$ and $N(x,y)$ are defined and have continuous partial derivatives in the region in which the given differential equation is valid.

i.e., Here M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.

Exact First Order Differential Equation

Proof: First we prove that the condition is necessary.

Let the given ODE be exact. Then there exists a function $f(x,y)$ such that

$$\boxed{M\,dx + N\,dy = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

Equating coefficients of dx and dy, we have

$$M = \frac{\partial f}{\partial x} \quad \text{and} \quad N = \frac{\partial f}{\partial y}.$$

Exact First Order Differential Equation

Assuming f to be continuous upto second order partial derivatives, we have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

This is necessary condition.

Exact First Order Differential Equation

2nd Part: Now we will prove that the given condition is sufficient.

we assume $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and show that $M\,dx + N\,dy$ is exact.

That means we need to find a function $f(x,y)$ such that $\boxed{df = M\,dx + N\,dy}$

Exact First Order Differential Equation

Let $F(x,y) = \int M\,dx$ (where the integration is performed on M by keeping y as constant) be the partial integral of M

Such that $\frac{\partial F}{\partial x} = M$.

we will show that $\left(N - \frac{\partial F}{\partial y}\right)$ is a function of y only.

Exact First Order Differential Equation

we have $\dfrac{\partial F}{\partial x} = M$

$$\frac{\partial}{\partial x}\left(N - \frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 F}{\partial x \partial y}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2 F}{\partial y \partial x}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

Exact First Order Differential Equation

$$\frac{\partial}{\partial x}\left(N-\frac{\partial F}{\partial y}\right)=\frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

$$=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=0$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\frac{\partial}{\partial x}\left(N-\frac{\partial F}{\partial y}\right)=0 \Rightarrow \boxed{N-\frac{\partial F}{\partial y}=g(y)}$

Exact First Order Differential Equation

So $\left(N - \frac{\partial F}{\partial y}\right)$ is the function of y alone.

Now consider

$$f = F(x,y) + \int \left(N - \frac{\partial F}{\partial y}\right) dy$$

we will show that $\boxed{df = M\,dx + N\,dy}$

Exact First Order Differential Equation

$$f = F(x,y) + \int \left(N - \frac{\partial F}{\partial y}\right) dy$$

$$\text{or} \quad df = dF + d\int g(y)\,dy$$

$$\text{or} \quad df = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial}{\partial x}\left(\int g(y)\,dy\right) dx$$

$$+ \frac{\partial}{\partial y}\left(\int g(y)\,dy\right) dy$$

Exact First Order Differential Equation

$$\text{as } df = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \cancel{\frac{\partial}{\partial x}\left(\int g(y)\,dy\right)dx} \to 0$$

$$+ \frac{\partial}{\partial y}\left(\int g(y)\,dy\right)dy$$

$$\text{as } df = M\,dx + \frac{\partial F}{\partial y}dy + g(y)\,dy$$

$$= M\,dx + \frac{\partial F}{\partial y}dy + \left(N - \frac{\partial F}{\partial y}\right)dy$$

$$= M\,dx + \frac{\partial F}{\partial y}dy + N\,dy - \frac{\partial F}{\partial y}dy$$

Exact First Order Differential Equation

$$\boxed{df = M\,dx + N\,dy}$$

This shows that $M\,dx + N\,dy$ is exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

(Proved)

Exact First Order Differential Equation

Now we will learn three methods for finding the general Solution of the exact first order ordinary differential eqn. One can use anyone of these three methods. But the first method is the best one.

Method I: The General Solution for First Order Exact ODE

- If the given differential equation is exact, then $f(x,y)$ can be determined using

$$\boxed{\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N}$$

After getting ~~When we can get~~ the function $f(x,y)$, we can get the general solution as

$f(x,y) = c$, for the given ODE

$M\,dx + N\,dy = 0$. [$M\,dx + N\,dy = df = 0$, for exactness $\Rightarrow f = c$]

Now our aim is to find the function $f(x,y)$.

Method I: The General Solution for First Order Exact ODE

Therefore, Integrating $\frac{\partial f}{\partial x} = M$, we have

as $\boxed{f(x,y) = \int M(x,y)\,dx + g(y),}$ where $g(y)$ is function of y only.

Note that f is a function of two variables and $\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x. Therefore, the integration on the right hand side of the above eqn is partial integration, that is y is taken as a constant in $M(x,y)$. The second term on the right hand side, $g(y)$, play the role of an arbitrary constant of integration.

Method I: The General Solution for First Order Exact ODE

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we have

$f(x,y) = \int M(x,y)\,dx + g(y)$, set $F(x,y) = \int M(x,y)\,dx$ (assuming y to be constant)

Diff. w.r.t. y, we obtain —

$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial y} + g'(y)$$

$$\text{or, } N(x,y) = \frac{\partial F}{\partial y} + g'(y) \quad \left[\frac{\partial f}{\partial y} = N,\ g'(y) = \frac{dg}{dy}\right]$$

$$\text{or, } g'(y) = N(x,y) - \frac{\partial F}{\partial y}$$

$$\text{or, } \boxed{\frac{dg}{dy} = N - \frac{\partial F}{\partial y}}$$

where N and $\frac{\partial F}{\partial y}$ are known to us.

Method I: The General Solution for First Order Exact ODE

If we integrate $\boxed{\frac{dg}{dy} = N - \frac{\partial F}{\partial y}}$, then we can obtain $g(y)$. After getting $g(y)$, we can obtain $f(x,y)$ from the following the eq^n

$$\boxed{f(x,y) = \int M\,dx + g(y)}$$

This is the best technique for finding $f(x,y)$

After having $f(x,y)$, the general solution ~~is~~ for the exact first order ODE is — $\boxed{f(x,y) = c}$, where c is an arbitrary constant.

Method II: The General Solution for First Order Exact ODE

- Alternative process, for finding the function $f(x,y)$ or the general solution of the exact first order ODE $\underline{M\,dx + N\,dy = 0}$

The general solⁿ is we have: $\boxed{F(x,y) = \int M\,dx}$

$$\boxed{\int M\,dx \,(y \text{ is constant}) + \int \underbrace{\left(N - \frac{\partial F}{\partial y}\right)}_{\text{functions of } y \text{ only}} dy = c}$$

, where c is an arbitrary constant

OR, one can write it in another form –

$$\boxed{\int M\,dx \,(y \text{ is constant}) + \int (\text{term of } N \text{ not containing } x)\, dy = c}$$

(*) One can use it for finding G.S. of exact eqⁿ.

Method III: The General Solution for First Order Exact ODE

Or one can write it also in following way –

$$\begin{cases} f_1(x,y) = \int M\,dx \ (y \text{ is constant}), & \text{(i)} \\ f_2(x,y) = \int N\,dy \ (x \text{ is constant but delete those terms already in (i)}) \end{cases}$$

and the general solution is

$$\boxed{f_1(x,y) + f_2(x,y) = c}$$

, where c is an arbitrary constant.

This process is used a lot in our classnotes.

Example (Method I)

Ex: ①

Solve $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$, $\forall\,(x,y)\in D$.

$M = \cos y + y\cos x$, $N = \sin x - x\sin y$

we know, $\frac{\partial f}{\partial x} = M$, $\Rightarrow f(x,y) = \underbrace{\int M\,dx}_{y \text{ is constant}} + g(y)$ — ⊛

we have

$\int M\,dx$ (y to be constant)

$= x\cos y + y\sin x$.

So, $f(x,y) = x\cos y + y\sin x + g(y)$

Diff. w.r.t y

or $\frac{\partial f}{\partial y} = -x\sin y + \sin x + \frac{dg}{dy}$

or $\sin x - x\sin y = -x\sin y + \sin x + \frac{dg}{dy}$ $\left[\text{Since } N = \frac{\partial f}{\partial y}\right]$

or $\frac{dg}{dy} = 0 \Rightarrow g(y) = c_1$, where c_1 is an arbitrary constant.

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Examples

The function $f(x,y)$ is obtained from (*) as

$$\boxed{f(x,y) = x\cos y + y\sin x + C_1}$$

Now the general solution is

$f(x,y) = C_2$, where C_2 is an arbitrary constant

$$\hookrightarrow \boxed{x\cos y + y\sin x = C_2 - C_1 = C}$$

where $C = C_2 - C_1$ is an arbitrary constant.

This is the best technique for finding the g.s. of exact eqn.

Same example (Method II)

Ex: ① Solve $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$, $(x,y) \in D$.

Sol^n: $\dfrac{\partial M}{\partial y} = -\sin y + \cos x = \dfrac{\partial N}{\partial x}$, Hence the equation is exact.

Now, $\int M\,dx = \int (\cos y + y\cos x)\,dx$ (assuming y to be constant)

$= x\cos y + y\sin x$

$\int N\,dy$ (term of N not containing x or term of N containing y only)

$= \int 0\,dy = 0$

The general solution is $\boxed{x\cos y + y\sin x = 0}$

Same example (Method III)

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Ex: 1 Solve $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$

Sol^n: $\dfrac{\partial M}{\partial y} = -\sin y + \cos x = \dfrac{\partial N}{\partial x}$, Hence the equation is exact.

Now, $\int M\,dx = \int (\cos y + y\cos x)\,dx$ (assuming y to be constant)

$= x\cos y + y\sin x$

$\int N\,dy = \int (\sin x - x\sin y)\,dy$ (assuming x to be constant)

$= y\sin x + x\cos y$

Both the terms in $\int N\,dy$ are already obtained in $\int M\,dx$.

So, the general solution is - $\boxed{x\cos y + y\sin x = c}$

Examples (Method - I)

Ex 2: $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy e^{xy^2} - 3y^2)\,dy = 0$

$M = y^2 e^{xy^2} + 4x^3$, $N = 2xy e^{xy^2} - 3y^2$,

we know, $\frac{\partial f}{\partial x} = M$, or, $f(x,y) = \int M\,dx$ (y is constant) $+ g(y)$ function of y only

Now, we have

$\int M\,dx = \int (y^2 e^{xy^2} + 4x^3)\,dx$ (y as constant)

$= e^{xy^2} + x^4$

So, $f(x,y) = e^{xy^2} + x^4 + g(y)$ —— (*)

Diff. w.r.t. y, we get $\frac{\partial f}{\partial y} = x.e^{xy^2}.2y + \frac{dg}{dy}$

Examples

$\frac{\partial f}{\partial y} = x \cdot e^{xy^2} \cdot 2y + \frac{dg}{dy} \Rightarrow N = e^{xy^2} \cdot 2xy + \frac{dg}{dy}$

$\Rightarrow 2xye^{xy^2} - 3y^2 = 2xye^{xy^2} + \frac{dg}{dy}$

or, $\frac{dg}{dy} = -3y^2 \Rightarrow g(y) = -y^3 + c_1$, where c_1 is an arbitrary constant.

Now the function $f(x,y)$ is obtained from ⊛ as

$$\boxed{f(x,y) = e^{xy^2} + x^4 - y^3 + c_1}$$

The general solution is $f(x,y) = c_2$, where c_2 is an arbitrary constant

$$\Rightarrow \boxed{e^{xy^2} + x^4 - y^3 = c}, \text{ where } c = c_2 - c_1.$$

This is the best approach.

Same example (Method II)

Ex: ② $(y^2e^{xy^2}+4x^3)\,dx+(2xye^{xy^2}-3y^2)\,dy=0$

$\frac{\partial M}{\partial y}=2ye^{xy^2}+2xy^3e^{xy^2}=\frac{\partial N}{\partial x}$, the eqn is exact.

$\int M\,dx=\int(y^2e^{xy^2}+4x^3)\,dx=e^{xy^2}+x^4$ (assuming y as constant)

$\int N\,dy$ (terms of y only) $=\int -3y^2\,dy=-y^3$ (Here $N=2xye^{xy^2}-3y^2$)

Therefore, the required general solution is

$$\boxed{e^{xy^2}+x^4-y^3=c}$$

$\left(\int M\,dx+\int N\,dy=c\right)$

Examples (without applying any method)

Ex. ① $(x^2-y)\,dx + (y^2-x)\,dy = 0, \quad \forall (x,y)\in D.$

- $y\,dx + x\,dy = d(xy).$

$M = x^2-y$ and $N = y^2-x.$

$\Rightarrow \dfrac{\partial M}{\partial y} = -1 = \dfrac{\partial N}{\partial x}$; The given differential eqn is exact in the domain D.

$x^2dx - y\,dx + y^2dy - x\,dy = 0$

or, $x^2dx + y^2dy - (y\,dx + x\,dy) = 0$

or, $\frac{1}{3}\,d(x^3+y^3) - d(xy) = 0$

or, $d\left(\dfrac{x^3+y^3}{3} - xy\right) = 0$

Integrating, or $\dfrac{x^3+y^3}{3} - xy = c_1$

$\Rightarrow \boxed{x^3+y^3-3xy = c}$ $(c = 3c_1)$

This is the general solution.

This is the Method of Grouping

Directly, we can find $f(x,y)$.

Equations which are both homogeneous and exact

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Theorem: Let M and N be homogeneous functions of x and y of degree $n (\neq -1)$. Let $M\,dx + N\,dy = 0$ be exact. Then the general solution is $Mx + Ny = c$.

Proof: Let $f = Mx + Ny$.

Let M and N be homogeneous functions of x and y of degree $n (\neq -1)$

Equations which are both homogeneous and exact

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Let $M\,dx + N\,dy = 0$ be exact.

Now from Euler's theorem on homogeneous functions M and N, we have

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM$$

and $$x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN$$

Equations which are both homogeneous and exact

Since the given eq^n is exact,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Now
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(Mx + Ny)$$
$$= M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}$$
$$= M + x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}$$

Equations which are both homogeneous and exact

$$\frac{\partial f}{\partial x} = M + nM = (n+1)M.$$

Similarly $\frac{\partial f}{\partial y} = (n+1)N$

Now we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= (n+1)(Mdx + Ndy)$$

Equations which are both homogeneous and exact

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or, $M\,dx + N\,dy = \frac{1}{(n+1)}\,df$

or, $M\,dx + N\,dy = \frac{1}{(n+1)}\,d\{Mx+Ny\}$.

if $n \neq -1$.

So, $M\,dx + N\,dy = 0$

$\Rightarrow d(Mx+Ny) = 0$.

$\Rightarrow \boxed{Mx+Ny = c} \Rightarrow$ This is the general solⁿ.

Equations which are both homogeneous and exact

Since $M(x,y)$ is the homogeneous function of degree n,

Note:

$$\boxed{M(x,y) = x^n \varphi(y/x).}$$

How to get Euler's Theorem?

$$\frac{\partial M}{\partial x} = n x^{n-1} \varphi(y/x) + x^n \varphi'(y/x) \cdot \left(-y/x^2\right)$$

$$= n x^{n-1} \varphi(y/x) - x^{n-2} y \varphi'(y/x)$$

$$\boxed{x \frac{\partial M}{\partial x} = n x^n \varphi(y/x) - x^{n-1} y \varphi'(y/x)}$$

$$\frac{\partial M}{\partial y} = x^n \cdot \varphi'(y/x) \cdot \frac{1}{x} \Rightarrow \boxed{y \frac{\partial M}{\partial y} = x^{n-1} y \varphi'(y/x)}$$

Equations which are both homogeneous and exact

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Now we have

$$\boxed{x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = n x^{n}\phi(y/x) = nM}$$

Similarly we have for homogeneous function $N(x,y)$,

$$\boxed{x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN}$$

Examples

(1) $(x^3+3y^2x)\,dx + (y^3+3x^2y)\,dy = 0$

Sol^n:

Here $M = x^3+3y^2x$,

$N = y^3+3x^2y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6xy \quad (\text{Exact}).$$

The eq^n is exact. M and N are both homogeneous of degree 3.

Examples

Now by theorem on exact homogeneous differential eqn, the general solⁿ is

$$Mx + Ny = c$$

$$\Rightarrow \boxed{x^4 + 6x^2y^2 + y^4 = c}$$

where c is an arbitrary constant.