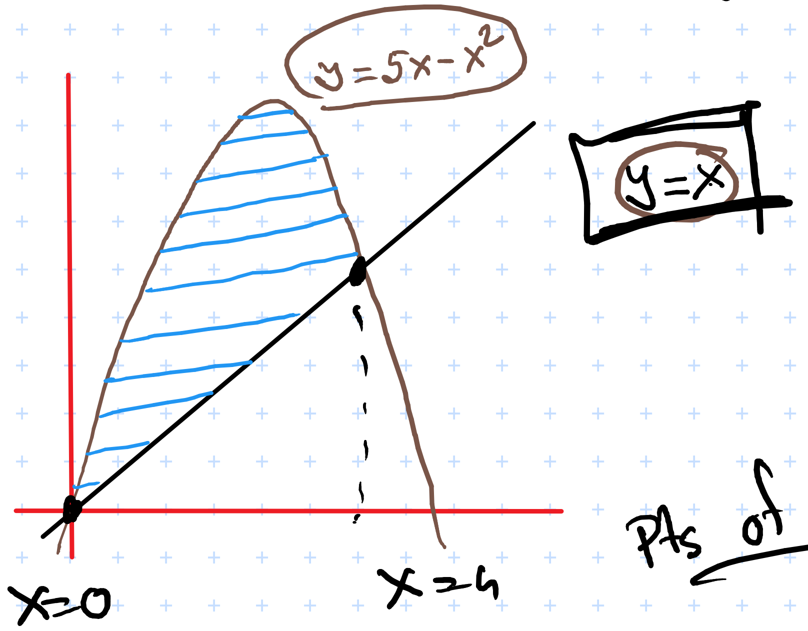
Find the area of the shaded region:



$y = f(x) = 5x - x^2$

$y = g(x) = x$

Pts of intersection

$f(x) = g(x)$

$\iff 5x - x^2 = x$

$\iff x^2 - 4x = 0$

$\iff x(x-4) = 0 \iff (x,y) = (0,0)$ or $(4,4)$

In the region $0 \le x \le 4$, $\boxed{f(x) \geq g(x)}$

The area of the shaded region

$$= \int_0^4 (f(x) - g(x))\,dx$$

$$= \int_0^4 (5x - x^2 - x)\,dx$$

$$= \int_0^4 (4x - x^2)\,dx \overset{FTC②}{=} \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= 2\cdot 4^2 - \frac{4^3}{3}$$

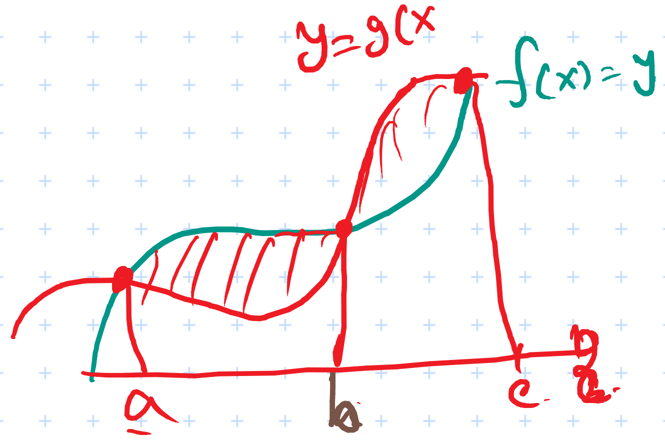
$$= 4^2\left(2 - \frac{4}{3}\right) = \frac{4^2\cdot 2}{3} = \frac{32}{3}$$

$5x - x^2 \geq x$

$\iff x^2 \le 4x$

$\iff x(x-4) \le 0$

$\iff \boxed{0 \le x \le 4}$



$\int_a^b f(x) - g(x)$

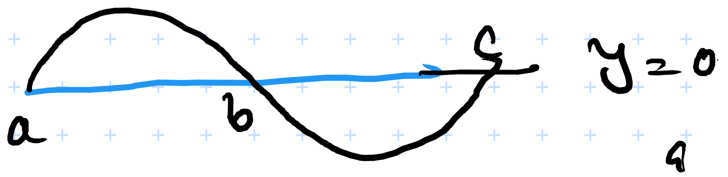
$\int_a^b (f(x) - g(x))\,dx + \int_b^c (g(x) - f(x))\,dx$

ⓐ To compute the points of intersection.

ⓑ $\{x \mid f(x) \geq g(x)\} = [a', b']$

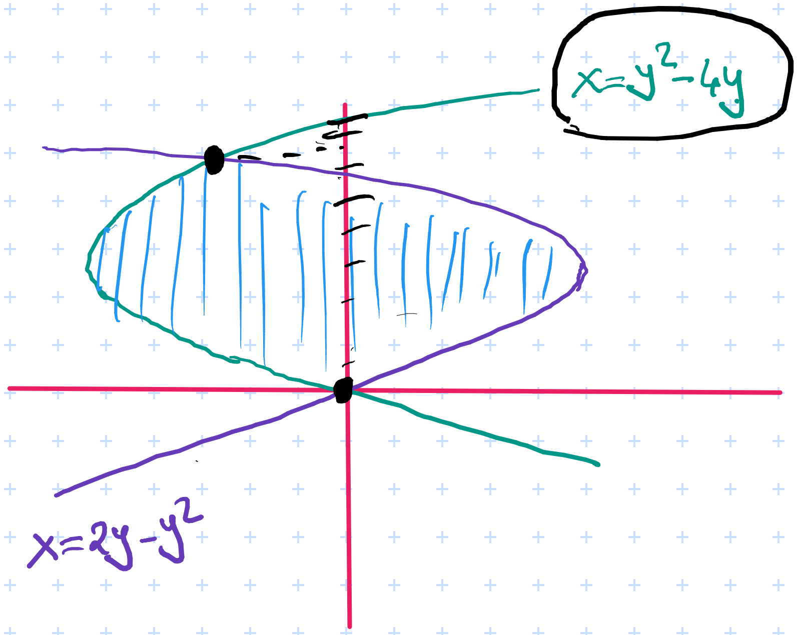
$\int_{a'}^{b'} f(x) - g(x)$

$\int g(x) - f(x)$



$\int_a^b f(x) + \int_b^c -f(x)$

(2)



Points of intersection

$x=y^2-4y$

$x=2y-y^2$

$y^2-4y=2y-y^2$

$\Leftrightarrow 2y^2-6y=0$

$\Leftrightarrow y=0$ or $y=3$

$x=0$ $\quad$ $x=-3$

$(0,0)$ $\quad$ $(-3,3)$.

$x=f(y)$

$x=g(y)$

$y=0$ $\quad$ $y=3$

If $f(y)=\underline{y^2-4y}$

$g(y)=2y-y^2$,

then. $f(y)\leq g(y)$

$\Leftrightarrow 2y^2\leq 6y$

$\Leftrightarrow y(y-3)\leq 0$

$\Leftrightarrow 0\leq y\leq 3$

Thus the area of the region is

$$\int_0^3 (g(y)-f(y))\,dy$$

$$=\int_0^3 \left((2y-y^2)-(y^2-4y)\right)dy$$

$$=\int_0^3 (6y-2y^2)\,dy.$$

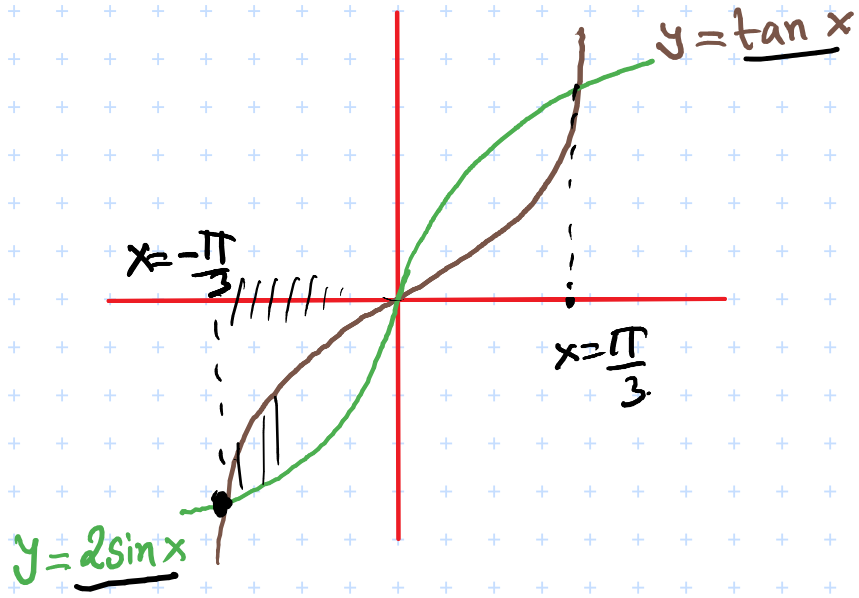
$$\overset{FTC(2)}{=} \left.3y^2-\frac{2}{3}y^3\right]_0^3$$

$$=27-\frac{2}{3}\cdot 27$$

$$=9.$$

Find the area of the region enclosed by the given curves:

$y = \tan x \qquad y = 2\sin x \qquad \boxed{-\frac{\pi}{3} \le x \le \frac{\pi}{3}}$



Points of intersection:

$\tan x = 2\sin x.$

$\Longleftrightarrow \sin x = 2\sin x \cos x$

$\Longleftrightarrow \sin x\,(2\cos x - 1) = 0$

$\Longleftrightarrow \sin x = 0$ or $\cos x = \frac{1}{2}$.

$\Longleftrightarrow x = 0$ or $x = \pm\frac{\pi}{3}$

$f(x) = \tan x$

$g(x) = 2\sin x$

If $x \in [-\frac{\pi}{3}, 0]$, then $g(x) \le f(x)$.

If $x \in [0, \frac{\pi}{3}]$, then $f(x) \le g(x)$.

$$\int_{-\pi/3}^{0} (\tan x - 2\sin x)\,dx + \int_{0}^{\frac{\pi}{3}} (\cancel{\ }2\sin x - \tan x)\,dx$$

$$\int_{-\frac{\pi}{3}}^{\pi/3} \underbrace{(|\tan x - 2\sin x|)}_{\substack{F(x) \\ \| \\ F(-x)}}\,dx$$

$$= 2\int_{0}^{\frac{\pi}{3}} (2\sin x - \tan x)\,dx$$

The area $= \displaystyle\int_{-\frac{\pi}{3}}^{\pi/3} |f(x) - g(x)|\,dx$

Thomas' Calculus

Stewart's Calculus

$$= \int_{-\frac{\pi}{3}}^{\pi/3} |\tan x - 2\sin x|\,dx.$$

$$= 2\int_{0}^{\pi/3} |2\sin x - \tan x|\,dx$$

$$= 2\int_{0}^{\pi/3} (2\sin x - \tan x)\,dx$$

$F(x) = |\tan x - 2\sin x|$

$F(-x)$

$= |\tan(-x) - 2\sin(-x)|$

$= |-\tan x + 2\sin x|$

$= |-(\tan x - 2\sin x)|$

$= F(x).$

even function.

FTC(2)

$$2\left[-2\cos x - \log|\sec x|\right]_{0}^{\frac{\pi}{3}}$$

$$= 2(1 - \log 2).$$

$y = x^3 = f(x)$

$y = x = g(x)$

Points of intersection:

$y=f(x)$ and $y=g(x)$ meet at (a,b)

$\iff a^3 = a$

$\iff a = 0, 1, -1.$

$f(x) \geqslant g(x)$

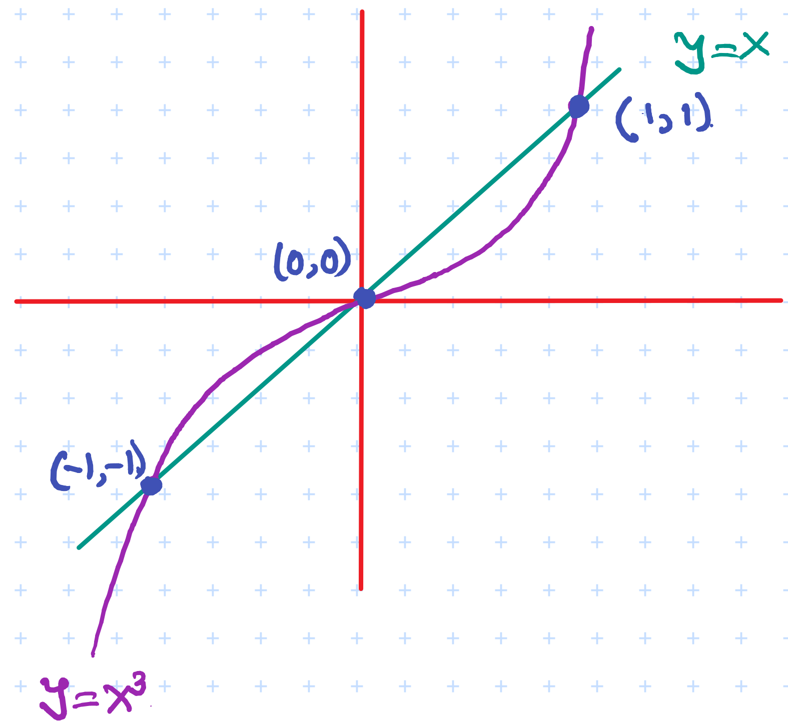
$\iff x^3 \geqslant x$

$\iff x(x^2-1) \geqslant 0$

$\iff \left(x \geqslant 0 \text{ and } x^2 \geqslant 1 \right)$ or $\left(x \leq 0 \text{ and } x^2 \leq 1 \right)$

$\iff (x \geqslant 0, x \geqslant 1)$ or $(x \leq 0, -1 \leq x \leq 1)$

$\iff -1 \leq x \leq 0$



Thus the area is given by

$$\int_{-1}^{1} |x^3 - x| \, dx$$

$$= 2 \int_0^1 (x - x^3) \, dx$$ (Since $|x^3 - x|$ is an even function)

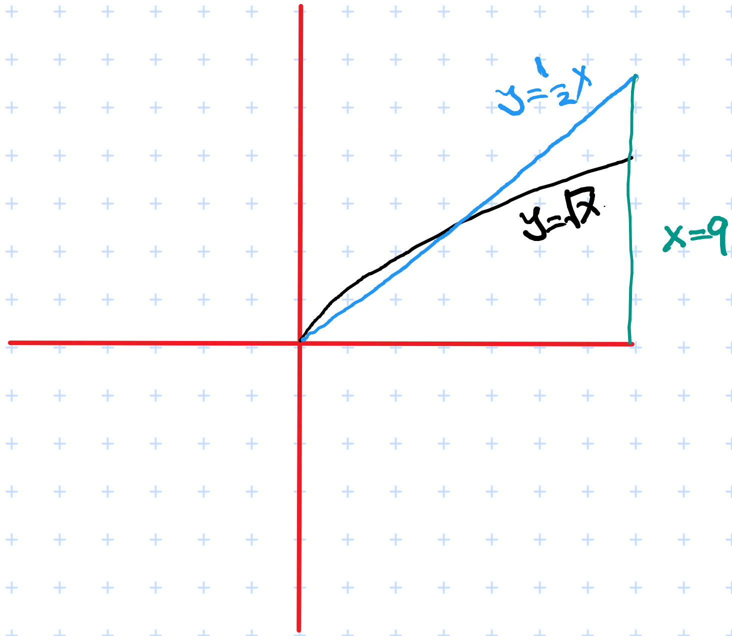
FTC $$= x^2 - \frac{x^4}{2} \Big]_0^1$$

$$= \frac{1}{2}.$$

$y = \frac{1}{2}x = f(x)$

$x = 9$

$y = \sqrt{x} = g(x)$.



$y=f(x)$ and $y=g(x)$ intersect at (a,b).

$\Leftrightarrow \frac{1}{2}a = \sqrt{a}$.

$\Leftrightarrow a^2 = 4a$

$\Leftrightarrow a = 0$ or $a = 4$.

$f(x) = \frac{1}{2}x \geq \sqrt{x} = g(x)$.

$\Leftrightarrow x^2 > 4x$.

$\Leftrightarrow x(x-4) > 0$

$\Leftrightarrow x < 0$ or $x > 4$.

The area

$$= \int_0^9 |f(x) - g(x)|\,dx.$$

$$= \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right)dx + \int_4^9 \left(\frac{1}{2}x - \sqrt{x}\right)dx.$$

$$\overset{FTC(2)}{=} \frac{59}{12}.$$