भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

3rd Lecture on Transform Techniques

(MA-2120)

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What have we learnt so far?

- Function of Exponential Order
- Conditions for existence of Laplace Transform
- Laplace Transform of some elementary functions
- Basic properties of Laplace Transform

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What will we learn today?

- Linearity Property of Laplace Transform
- Some Examples
- First Shifting or Translation Theorem
- Second Translation Theorem
- Some Examples

② Properties of Laplace Transforms:

① **Linearity:** If $f(t)$ and $g(t)$ are two functions whose Laplace transform exist, then for any two constants α and β, we have

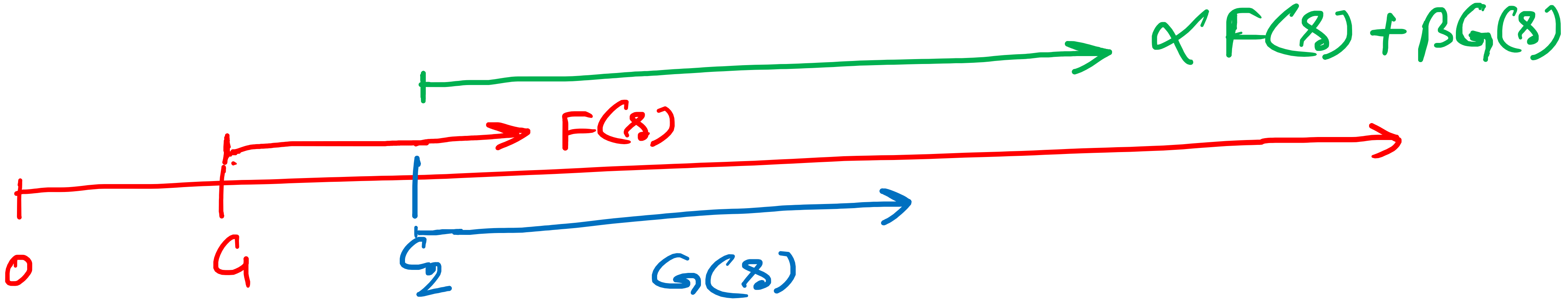
$$\boxed{\mathcal{L}\left[\alpha f(t) + \beta g(t)\right] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)] = \underbrace{\alpha F(s) + \beta G(s)}_{\text{region of convergence?}}}$$

Consider $F(s)$ has region of convergence $s > c_1$ ($c_1 > 0$)

and $G(s)$ has region of convergence $s > c_2$ ($c_2 > 0$)

Then the region of convergence for

$\alpha F(s) + \beta G(s)$ is $s > \max\{c_1, c_2\}$



Examples: $f(t) = \cosh t, \quad t \geq 0$

$$= \frac{e^{t} + e^{-t}}{2}, \quad t \geq 0$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\cosh t] = \mathcal{L}\left[\frac{e^{t} + e^{-t}}{2}\right]$$

$$= \frac{1}{2}\mathcal{L}[e^{t}] + \frac{1}{2}\mathcal{L}[e^{-t}] \quad \left[\begin{array}{l}\text{By Linearity} \\ \text{Property}\end{array}\right]$$

$$= \frac{1}{2}\underbrace{\frac{1}{s-1}}_{s>1} + \frac{1}{2}\underbrace{\frac{1}{s+1}}_{s>-1}, \quad s > \underbrace{\max\{-1, 1\}}_{=1}$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right] = \frac{s}{s^2-1}, \quad s>1$$

$$\boxed{\mathcal{L}\left[\cosh t\right] = \frac{s}{s^2-1}, \quad s>1}$$

Ex. $\mathcal{L}[\sinh at] = \dfrac{a}{s^2-a^2},\ s>a$

Sol^n We have $\sinh at = \dfrac{e^{at}-e^{-at}}{2}$

using the linearity property, we get

$$\mathcal{L}[\sinh at] = \mathcal{L}\left[\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\right]$$

$$= \frac{1}{2}\mathcal{L}[e^{at}] - \frac{1}{2}\mathcal{L}[e^{-at}]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2},\ s>a.$$

Similarly $\mathcal{L}[\cosh at]$

$$= \mathcal{L}\left[\frac{e^{at}}{2} + \frac{e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2-a^2}, \quad s > a.$$

Ex: $\mathcal{L}[\sin at]$

$\sin at = \dfrac{e^{iat} - e^{-iat}}{2i}$.

$= \mathcal{L}\left[\dfrac{e^{iat} - e^{-iat}}{2}\right]$

$= \dfrac{a}{s^2+a^2}$ [Similarly by Linearity property].

$$\boxed{\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}}, s>0.$$

Ex: $\mathcal{L}[\cos at] = \dfrac{s}{s^2+a^2}, \; s>0$

Try it! (Homework).

Ex. $\mathcal{L}[\cos^2(at)]$

we have $\cos^2(at) = (1+\cos 2at)/2$

Therefore, $\mathcal{L}[\cos^2 at]$

$= \mathcal{L}\left[\frac{1}{2} + \frac{1}{2}\cos 2at\right]$

$= \frac{1}{2}\mathcal{L}[1] + \frac{1}{2}\mathcal{L}[\cos 2at]$ [using linearity property]

$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+4a^2} = \frac{s^2+2a^2}{s(s^2+4a^2)}$, $s>0$

Ex. $\mathcal{L}\left[(\sin t-\cos t)^2\right]$

$$(\sin t-\cos t)^2 = 1-\sin 2t$$

$$\mathcal{L}[1-\sin 2t] = \mathcal{L}[1]-\mathcal{L}[\sin 2t]$$

$$= 1/s - \frac{2}{s^2+4}, \quad s>0$$

$$= \frac{(s^2-2s+4)}{s(s^2+4)}, \quad s>0$$

Ex: $\mathcal{L}[\sin 2t \cos 3t]$

Sol^n:

$$\sin 2t \cos 3t = \frac{1}{2}(\sin 5t - \sin t)$$

$$\mathcal{L}[\sin 2t \cos 3t] = \mathcal{L}\left[\frac{1}{2}(\sin 5t - \sin t)\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1}\right], s>0$$

$$= \frac{2(s^2-5)}{(s^2+25)(s^2+1)}, \underline{s>0}$$

Ex. $\mathcal{L}[\sin^3 2t]$

Sol^n: Since $\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

$$\therefore \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\mathcal{L}[\sin^3 2t] = \mathcal{L}\left[\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right]$$

$$= \frac{48}{(s^2+4)(s^2+36)}, \quad s>0.$$

Ex. $\mathcal{L}[\sinh^2 at]$

Solⁿ: since $\cosh 2\theta = 1 + 2\sinh^2\theta$

$\Rightarrow \sinh^2\theta = \frac{1}{2}(\cosh 2\theta - 1)$

$$\mathcal{L}[\sinh^2 at] = \mathcal{L}\left[\frac{1}{2}\cosh 2at - \frac{1}{2}\right]$$

$$= \frac{2a^2}{s(s^2-4a^2)}, \quad s > 2|a|$$

Similarly,

$$\mathcal{L}[\cosh^2 at] = \frac{s^2-2a^2}{s(s^2-4a^2)}, \quad s > |2a|$$

by using $\cosh 2\theta = 2\cosh^2\theta - 1$

Ex: Find the Laplace transformation of the piecewise continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 2 \\ k, & t \geq 2, \end{cases} \quad k \text{ constant.}$$

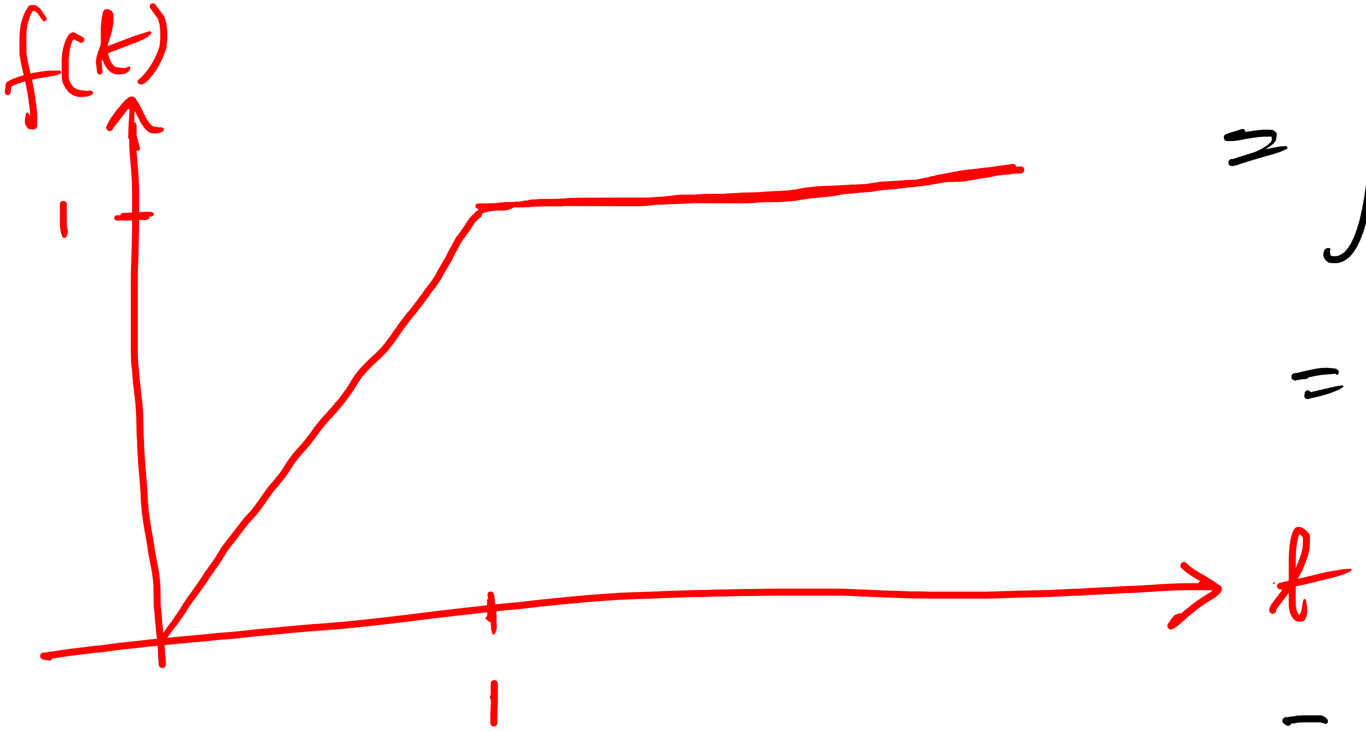
Sol^n:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_2^\infty k e^{-st}\, dt = \frac{k e^{-2s}}{s}, \quad s > 0$$

Ex. Find $\mathcal{L}[f(t)]$ where $f(t)$ is defined by

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$$



$\mathcal{L}[f(t)]$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\, t\,dt + \lim_{R\to\infty} \int_1^R e^{-st}\,dt$$

$$= \frac{t e^{-st}}{-s}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt + \lim_{R\to\infty} \frac{e^{-st}}{-s}\Big|_1^R$$

$$= \frac{1-e^{-s}}{s^2} \quad (\mathrm{Re}(s)>0)$$

Ex. Find the Laplace transformation of the function

$$f(t) = \sum_{k=0}^{n} a_k t^k = a_0 + a_1 t + \dots + a_n t^n$$

is a polynomial of degree n.

Sol^n:

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\sum_{k=0}^{n} a_k t^k\right]$$

$$\left[\text{i.e. } \mathcal{L}[a_0] + \mathcal{L}[a_1 t] + \mathcal{L}[a_2 t^2] + \dots + \mathcal{L}[a_n t^n]\right]$$

$$= \sum_{k=0}^{n} a_k \mathcal{L}[t^k]$$

$$= \sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}$$

Here Laplace transformation term by term is possible for finite series

Remark: For a finite series $\sum_{k=0}^{n} a_k t^k$, it is possible to obtain Laplace transformation of series by taking Laplace transform term by term.

In general, the similar is not possible for infinite series

- For an infinite series, $\sum_{n=0}^{\infty} a_n t^n$, in general it is not possible to obtain the Laplace transform of the series by taking the transform term by term.

See the example in next page.

Example: Find the Laplace transform of

$$f(t) = e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$ [Infinite Series]

Sol^n:

Taking the Laplace transform term by term, we have

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{-t^2}] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \mathcal{L}[t^{2n}]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{(2n)!}{s^{2n+1}}$$

$$= \frac{1}{s} \sum_{n=0}^{\infty} (-1)^n \cdot \underbrace{\frac{(2n)\cdots(n+2)(n+1)}{s^{2n}}}_{a_n}$$

Let $a_n = \dfrac{2n\cdots(n+2)(n+1)}{s^{2n}}$

Applying the ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2(2n+1)}{|s|^2}$

$= \infty$

and so the series diverges for all values of s.

However $\mathcal{L}\left[e^{-t^2}\right] = \displaystyle\int_0^\infty e^{-st} e^{-t^2}\,dt$ exists since e^{-t^2} is continuous and bounded on $[0,\infty)$.

This imply that for this infinite series, term by term Laplace transform is not possible.

Q. So when can we guarantee obtaining the Laplace transform of an infinite series by term by term computation?

Ans: If a series is convergent before taking the Laplace transform as well as after taking the Laplace transform, then it is possible to obtain the Laplace transform of the series by taking the Laplace transform the transform term by term.

Example: find $\mathcal{L}[\sin\sqrt{t}]$

Sol^n: $\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(\sqrt{t})^{2n+1}}{(2n+1)!}$$

Since this is convergent series, we can apply the Laplace transform term by term.

$$\mathcal{L}\left[\sin\sqrt{t}\right]$$

$$= \mathcal{L}\left[t^{1/2}\right] - \frac{1}{3!}\mathcal{L}\left[t^{3/2}\right] + \frac{1}{5!}\mathcal{L}\left[t^{5/2}\right]$$

$$- \frac{1}{7!}\mathcal{L}\left[t^{7/2}\right] + \cdots$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3!\, s^{5/2}} + \frac{\Gamma\left(\frac{7}{2}\right)}{5!\, s^{7/2}} - \frac{\Gamma\left(\frac{9}{2}\right)}{7!\, s^{9/2}} + \cdots$$

$$s > 0$$

$$\left[\mathcal{L}\left[t^{\alpha}\right] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1\right]$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\,\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \cdots$$

$$\left[\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \text{ and } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s} \qquad (s>0)$$

$\Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$,

$\Gamma(\frac{5}{2}) = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

Ex:

$$\mathcal{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\pi/s}\; e^{-1/4s}, \quad s>0$$

Similarly, you solve it.

Example:

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$$

Try!

(2) First Shifting Theorem: (Translation Theorem)

Translation on the s-axis or shifting on the s-axis:

Let, $\mathcal{L}[f(t)] = F(s)$ for $Re(s) > \alpha$ and a be a real constant. Then

$$\boxed{\mathcal{L}[e^{at}f(t)] = F(s-a), \quad Re(s) > a+\alpha}$$

$a \in \mathbb{R}$.

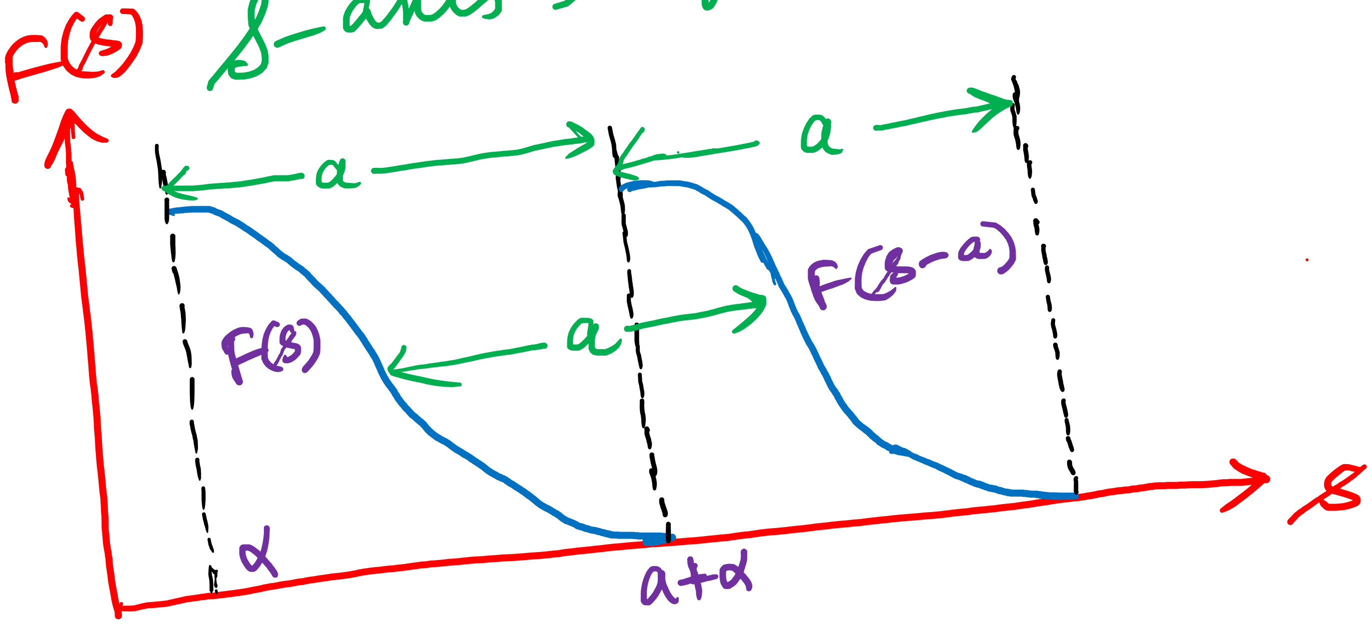
Proof: By definition, we have

$$\mathcal{L}\left[e^{at} f(t)\right] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a), \quad Re(s) > a+\alpha$$

The translation on the s-axis is given by



Example :

(a) $\mathcal{L}[e^{at}\cos bt]$

Solⁿ :

$$\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2} = F(s), \quad Re(s) > 0$$

$$\mathcal{L}[e^{at}\cos bt] = F(s-a) = \frac{s-a}{(s-a)^2+b^2}, \quad Re(s) > a$$

(b) $\mathcal{L}[e^{\alpha t}t^3] \Rightarrow \mathcal{L}[t^3] = \frac{3!}{s^4} = F(s), \quad Re(s) > 0$

$$\mathcal{L}[e^{\alpha t}t^3] = F(s-\alpha) = \frac{3!}{(s-\alpha)^4}. \quad Re(s) > \alpha.$$

(c) $\mathcal{L}[e^{-2t}\sin 4t] = F(s+2) = \dfrac{4}{(s+2)^2+16}$

$$= \frac{4}{s^2+4s+20}$$

$Re(s) > -2$.

(d) $\mathcal{L}[(t+3)^2 e^t]$

$\mathcal{L}[(t+3)^2] = \mathcal{L}[t^2+6t+9]$

$$= \frac{2!}{s^3} + 6\cdot\frac{1}{s^2} + \frac{9}{s}$$

$$= \frac{2+6s+9s^2}{s^3} = F(s),$$

$Re(s)>0$

$\mathcal{L}[e^t(t+3)^2] = F(s-1)$

$$= \frac{2+6(s-1)+9(s-1)^2}{(s-1)^3}, \quad Re(s)>1$$

ⓔ $\mathcal{L}\left[e^{-t}\cos^2 t\right]$

<u>Soln:</u>

$$\mathcal{L}\left[\cos^2 t\right] = \mathcal{L}\left[\frac{1}{2}(1+\cos 2t)\right]$$

$$= \frac{s^2+2}{s(s^2+4)} = F(s), \quad Re(s) > 0$$

$$\mathcal{L}\left[e^{-t}\cos^2 t\right] = F(s+1) = \frac{(s+1)^2+2}{(s+1)\{(s+1)^2+4\}}.$$

$$= \frac{s^2+2s+3}{(s+1)(s^2+2s+5)} \quad Re(s) > -1$$

Try: (f) $\mathcal{L}\left[e^{-t}(3\sinh t - 5\cosh 2t)\right]$

$$= \frac{1-5s}{s^2+2s-3} \text{ (Ans).}$$

(g) $\mathcal{L}\left[(1+te^{-t})^3\right] = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$

(Ans).

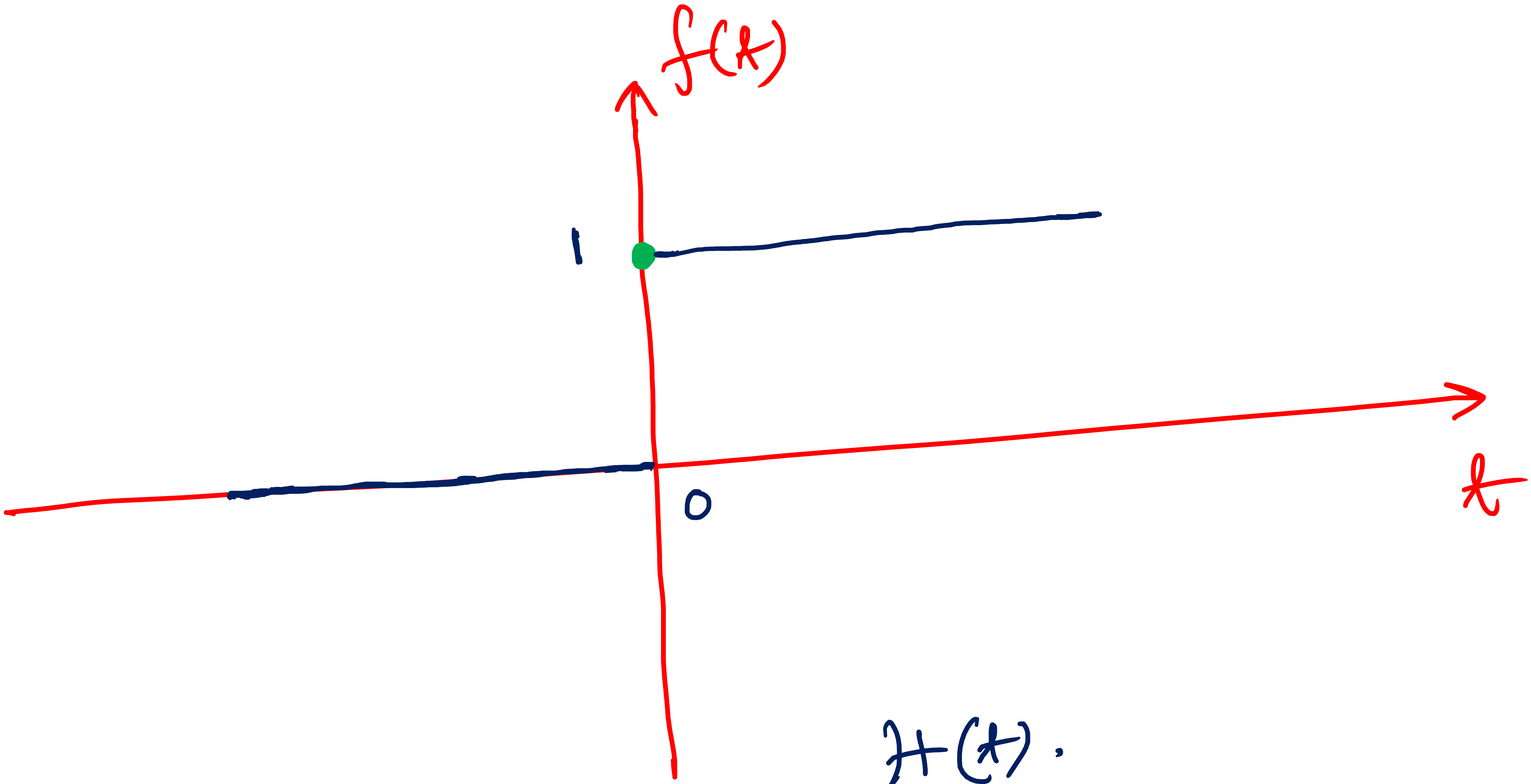
Home work!

Heaviside function or unit step function

This function is an important function occurring in electrical systems.

The Heaviside function or unit step function is defined by

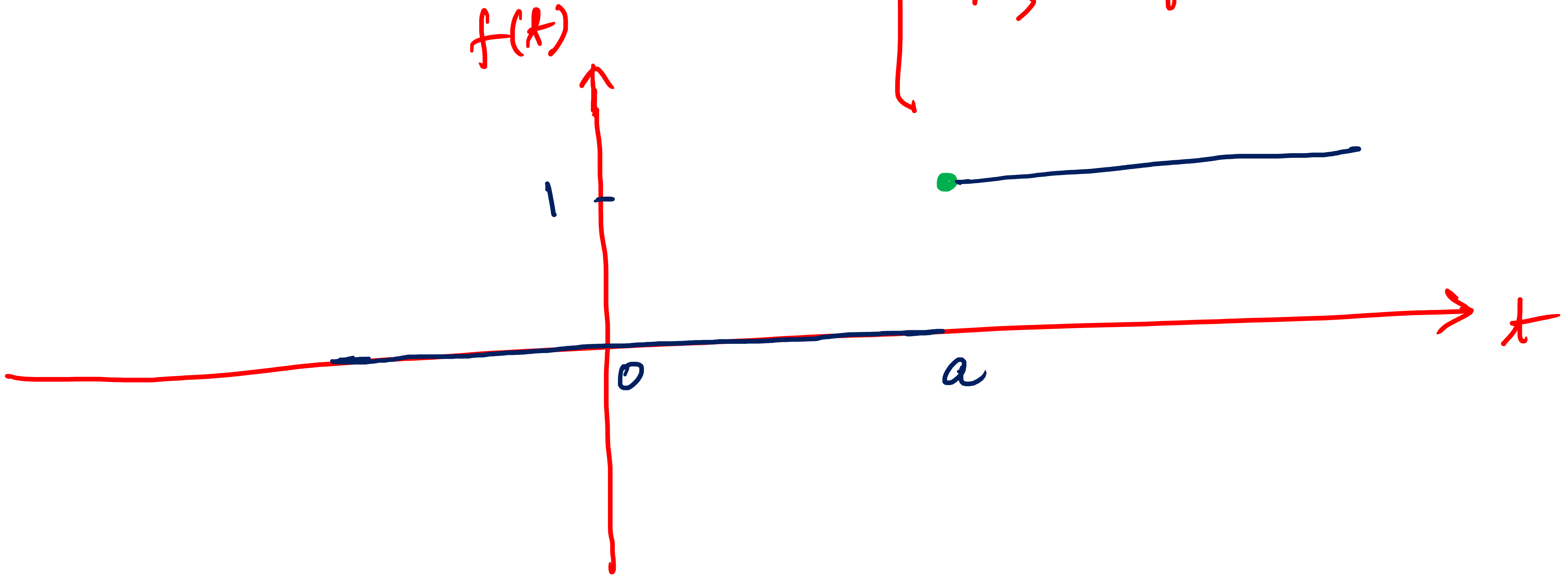
$$\boxed{\begin{aligned} H(t) &= 0, \quad \text{if } t < 0 \\ &= 1, \quad \text{if } t \geq 0 \end{aligned}}$$



$H(t)$.

If the jump discontinuity at $t=a$, then we define

$$H_a(t) = H(t-a) = \begin{cases} 0, & \text{if } t < a, \\ 1, & \text{if } t \geq a \end{cases}$$



Here the jump is of magnitude 1.

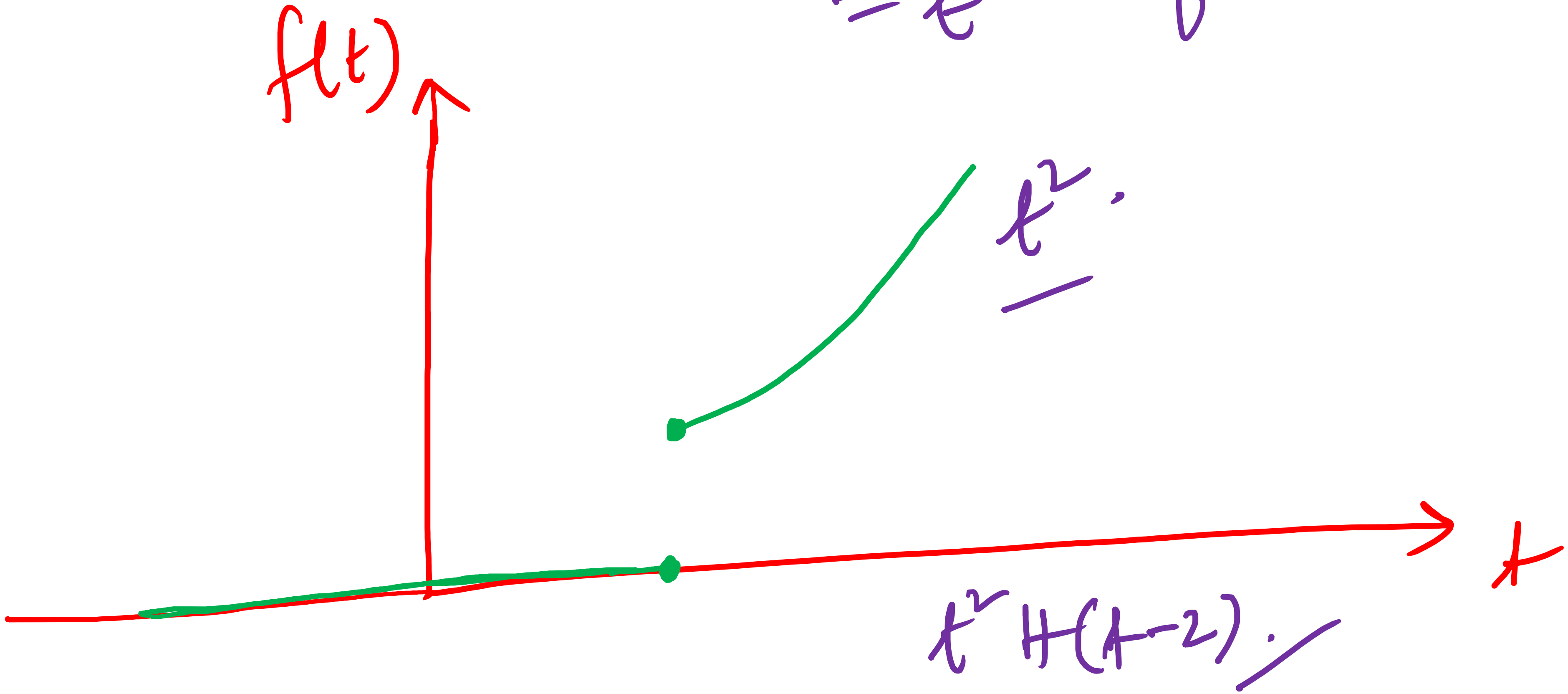
This function $\mathcal{H}(t-a)$ delays its output until $t=a$ and then assumes a constant value of one unit.

Example:

$f(t) = t^2 H(t-2)$ is can be written

by $f(t) = 0$, if $t < 2$

$= t^2$ if $t \geq 2$.



(3) Second translation theorem or second shifting theorem: (translation on the t-axis)

Let $\mathcal{L}[f(t)] = F(s)$, $Re(s) > \alpha$ and $a \geq 0$ be a real number. Then

$$\boxed{\begin{aligned}\mathcal{L}[f(t-a)H(t-a)] &= e^{-as}F(s), \quad Re(s) > \alpha, \\ &= e^{-as}\mathcal{L}[f(t)] \qquad a \geq 0\end{aligned}}$$

Proof:

$$\mathcal{L}\left[\mathcal{H}(t-a)\, f(t-a)\right]$$

$\mathcal{H}(t-a)$
$= 0, \text{ if } t<a$
$= 1, \text{ if } t \geq a$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Set $\tau = t-a$.

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau$$

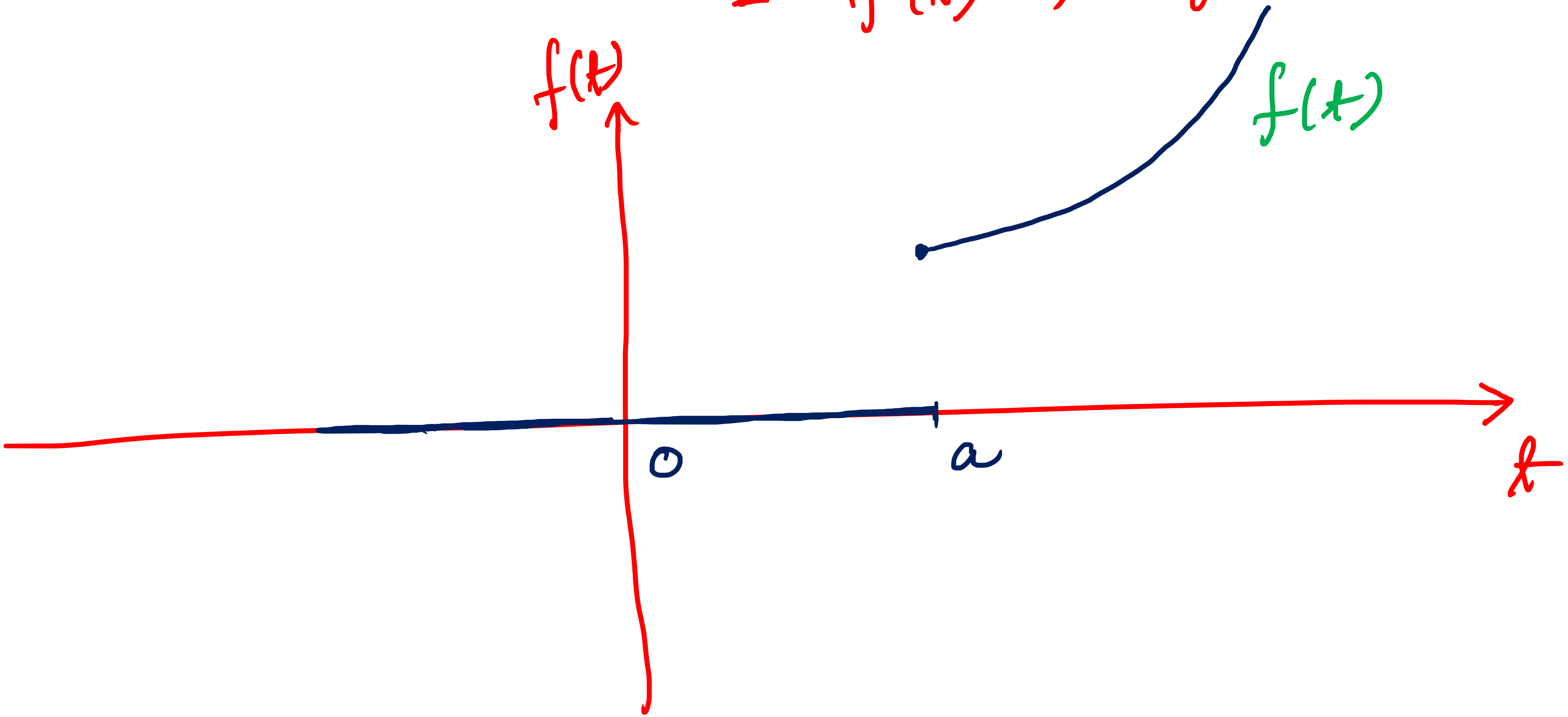
$$= e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= e^{-as} F(s),$$

$\boxed{\begin{array}{c} Re(s) > \alpha \\ \text{and } a \geq 0 \end{array}}$

Here $f(t)\,H(t-a) = 0, \quad \text{if } t < a$

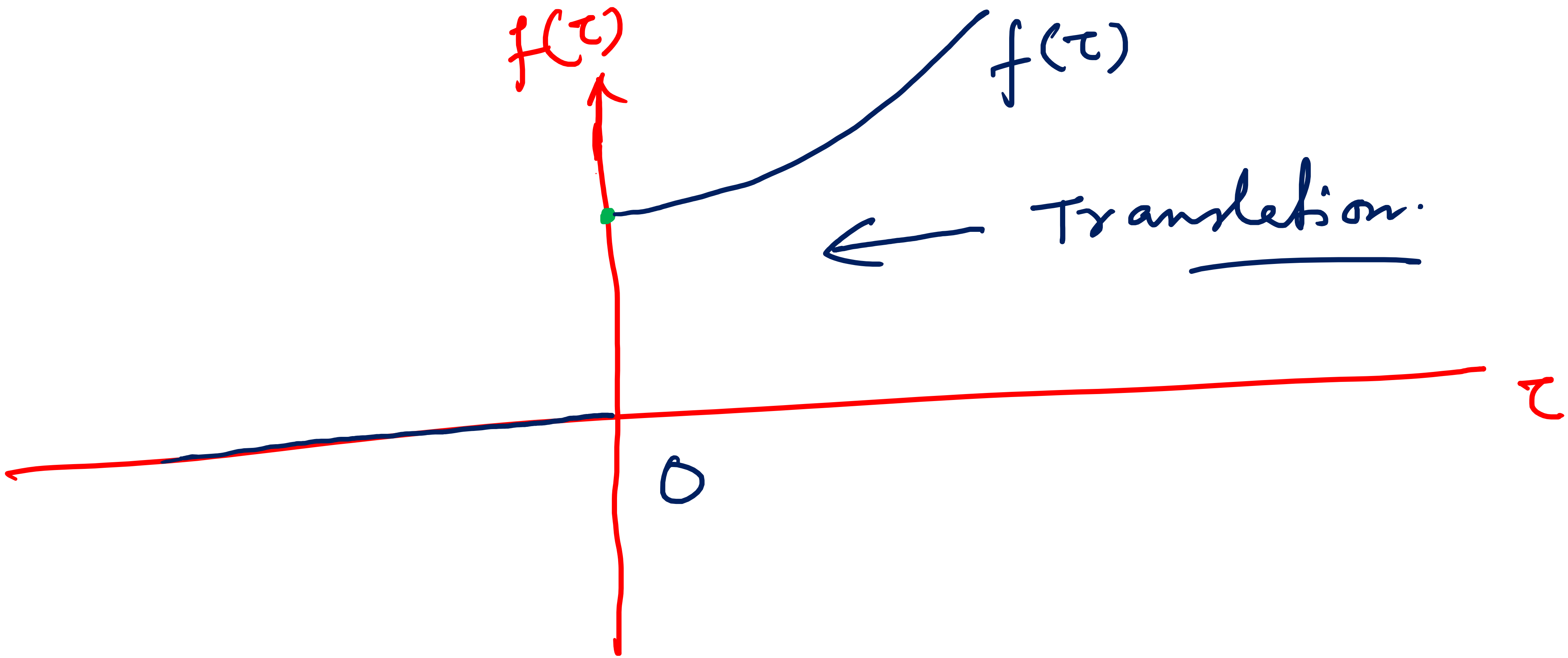
$= f(t), \quad \text{if } t \geq a.$



Let $\tau = t - a$

So, $f(\tau)H(\tau) = 0, \text{ if } \tau < 0$

$= f(\tau), \text{ if } \tau \geq 0.$



Example:

$$\mathcal{L}\left[H(t-a)\right]$$

$$= \mathcal{L}\left[1 \cdot H(t-a)\right]$$

$$= e^{-as} \cdot \mathcal{L}\left[f(t)\right]$$

$$= e^{-as} \cdot \mathcal{L}[1] = \frac{e^{-as}}{s},$$

$$\mathrm{Re}(s) > 0.$$

Note: For some case, we may get the problem as find the Laplace transform of $f(t)H(t-a)$. [i.e the function $f(t)$ does not admit the translation].

$$\mathcal{L}\left[f(t)H(t-a)\right]$$

$$= \mathcal{L}\left[f(t+a-a)H(t-a)\right]$$

$$= e^{-as}\mathcal{L}\left[f(t+a)\right]$$

<u>Example:</u> Find the Laplace transform of the function

$$g(t) = \begin{cases} 0, & 0 \le t < 3 \\ (t-3)^2, & t \geq 3 \end{cases}$$

Note that $g(t)$ is a function $f(t) = (t-3)^2$ delayed by 3 unit of time.

Sol^n: $\mathcal{L}[g(t)]$

$$g(t) = (t-3)^2 H(t-3).$$

$$= f(t-3)\,H(t-3) \quad \text{where} \quad f(t) = t^2$$

Here $a=3$.

$$\mathcal{L}[g(t)] = e^{-3s} \cdot \mathcal{L}[f(t)]$$

$$= e^{-3s} \cdot \mathcal{L}[t^2] = \frac{2e^{-3s}}{s^3}.$$

Example: $\mathcal{L}\left[t^2 H(t-3)\right]$

Here $g(t) = t^2 H(t-3) = f(t) H(t-3)$

Here $f(t) = t^2$

$$\mathcal{L}\left[f(t) H(t-3)\right] = e^{-3s}\mathcal{L}\left[f(t+3)\right]$$

$$= e^{-3s}\mathcal{L}\left[(t+3)^2\right]$$

$$= e^{-3s}\mathcal{L}\left[t^2+6t+9\right]$$

$$= e^{-3s}\cdot\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right].$$

Ex, $\mathcal{L}[G(t)]$ where

$$G(t) = \begin{cases} \sin(t - \pi/3), & t > \pi/3 \\ 0, & t < \pi/3. \end{cases}$$

Ans: $\dfrac{e^{-\pi s/3}}{s^2+1}$, $s > 0$.

Try it (Homework)

Ex: Find the Laplace Transform of the following problem –

(a) $$g(t) = \begin{cases} 1, & 0<t<1 \\ -1, & 1<t<2 \\ 0, & t>2 \end{cases}$$

Home work!

(b) $$g(t) = \sin t \, H(t-\pi).$$