Lecture 2

# Fundamental Theorem of Calculus

**FTC 1:** If $f$ is continuous on $[a,b]$, then

$$\left(\int_a^x f(t)\,dt\right)' = f(x) \qquad \forall x \in [a,b]$$

**FTC 2:** If $f'$ exists and $f'$ is integrable in $[a,b]$, then

$$\int_a^b f' = f(b) - f(a).$$

## Examples:

(1) Compute the derivative of $g(x) = \int_1^x \sqrt{t^3+4t}\,dt$. at $x=2$.

$\boxed{g'(x) = \sqrt{x^3+4x}}$

**Solution:** $\int_a^x f(t)\,dt$.

FTC 1

$f(t)$ continuous.

$\left(\int_a^x f(t)\,dt\right)' = f(x)$

$f(t) = \sqrt{t^3+4t}$.

$f : [1,3] \to \mathbb{R}$ is continuous on $[1,3]$.

Thus FTC (1) applies and we obtain

$$g'(2) = f(2) = \sqrt{2^3+4\cdot 2} = 4.$$

$2=1$

$x^2 = x\cdot x$

$= \underbrace{x+x+\cdots+x}_{x \text{ times}}$

$2x = \underbrace{1+1+\cdots+1}_{x \text{ times}}$

$\Rightarrow 2x = x$

$\Rightarrow 2=1$

$\lim\limits_{h\to 0}$

② Determine the derivative of $g(x) = \int_{-\frac{\pi}{2}}^{x} \sqrt{\sin^2 t + 2}\, dt$.

at $\boxed{x = \frac{\pi}{6}}$

Define $f(t) = \sqrt{\sin^2 t + 2}$

$x \in [a, b]$

$f$ is continuous on $[-\frac{\pi}{3}, \pi]$.

Thus FTC ① applies and $g'(\frac{\pi}{6}) = f(\frac{\pi}{6}) = \sqrt{\sin^2(\frac{\pi}{6}) + 2}$.

$$= \sqrt{\frac{1}{4} + 2}$$

$$= \frac{3}{2}.$$

3. Find the derivative of: $g(x) = \int_1^{x^3} t^2 \, dt$.

We define $f(t) = t^2$.

Since $f$ is continuous on $[1, a]$ for some sufficiently large value of $a$, FTC applies, and we obtain that

$F(x) = \int_1^x t^2 \, dt$ is differentiable and $F'(x) = x^2$.

Now $g(x) = F(x^3)$.

(chain rule) $\Rightarrow$ $g'(x) = \frac{d}{dx^3} F(x^3) \cdot \frac{dx^3}{dx} = 3x^2 \, F'(x^3)$

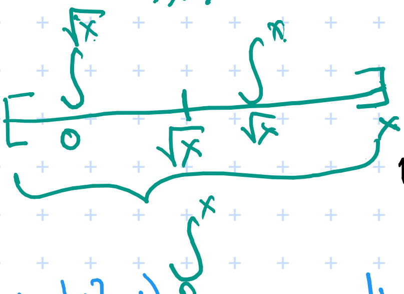
$= 3x^2 \cdot x^6$

$= 3x^8$.

4. Compute the derivative of $g(x)=\int_{\sqrt{x}}^{x}(t^2-t)\,dt$ at $x=2$

$\sqrt{x}<x$

$\sqrt{x}>x$



(domain additivity) $= \int_0^x (t^2-t)\,dt - \int_0^{\sqrt{x}}(t^2-t)\,dt$

$= F(x) - F(\sqrt{x})$

where $F(x)=\int_0^x (t^2-t)\,dt$.

$f(t)=(t^2-t)$ is continuous on $[0,a]$ for $a$ sufficiently large.

Thus FTC ① applies and yields $F'(x)=x^2-x$.

Thus $g'(x)=F'(x)-F'(\sqrt{x})$

$$=(x^2-x)-\frac{1}{2\sqrt{x}}(x-\sqrt{x})$$

⑤ Find the derivative of $g(x) = \int_{x^2}^{x^3} t\,dt$

$g(x) = F(x^3) - F(x^2)$.

$F(x) = \int_0^x t\,dt$

$\Rightarrow g'(x) = 3x^2 F'(x^3)$

$- 2x F'(x^2)$.

FTC(1) $\Longrightarrow F'(x) = x$.

$= 3x^2 \cdot x^3 - 2x \cdot x^2$.

$= 3x^5 - 2x^3$.

⑥ Compute $\int_0^1 (\sqrt[3]{t} - \sqrt{t})\, dt$.

$$t^{1/3} = \frac{t^{4/3}}{4/3}$$

Let $f(t) = \dfrac{t^{4/3}}{4/3} - \dfrac{t^{3/2}}{3/2}$.

FTC 2

Then $f'(t) = \sqrt[3]{t} - \sqrt{t}$ exists and it is continuous on $[0,1]$. Thus FTC 2 applies and we have

$$\int_0^1 (\sqrt[3]{t} - \sqrt{t})\,dt = \left.\frac{t^{4/3}}{4/3} - \frac{t^{3/2}}{3/2}\right]_0^1$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}.$$

7. Compute $\int_{-2}^{1} |x^2-1|\,dx$.

$$|x^2-1| = \begin{cases} x^2-1 & \text{if} \quad x>1 \text{ or } x<-1 \\ 1-x^2 & \text{if} \quad -1 \leq x \leq 1 \end{cases}$$

$$\int_{-2}^{1} |x^2-1|\,dx = \int_{-2}^{-1} \underline{(x^2-1)\,dx} + \int_{-1}^{1} \underline{(1-x^2)\,dx}$$

(Domain additivity)

$$= \frac{8}{3}$$

## Integration by parts:

Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ be continuous functions such that

ⓐ $f$ is differentiable on $[a,b]$ and $f'$ is integrable on $[a,b]$.

ⓑ $g$ is integrable on $[a,b]$ with antiderivative $G$ on $[a,b]$.

Then $$\int_a^b f(x)\,g(x)\,dx = f(b)G(b) - f(a)G(a) - \int_a^b f'(x)\,G(x)\,dx$$

**Example:**

$$\int_0^4 x e^{-x}\,dx$$

$f(x) = x$
$g(x) = e^{-x}$ } ← continuous

① $f$ differentiable; $f'(x) = 1$ integrable on $[0,4]$.

② $g = e^{-x}$ is continuous and hence integrable with an antiderivative $G(x) = -e^{-x}$ on $[0,4]$

$$\int_0^4 f(x)\,g(x)\,dx = f(4)\,G(4) - f(0)\,G(0) - \int_0^4 f'(x)\,G(x)\,dx$$
$$= (4)\cdot(-e^{-4}) - 0\cdot(-e^{0}) - \int_0^4 1\cdot(-e^{-x})\,dx$$
$$= -4e^{-4} + \int_0^4 e^{-x}\,dx$$
$$= -4e^{-4} - e^{-4} + 1$$
$$= 1 - 5e^{-4}.$$

$$\int_0^4 e^{-x}\,dx = -e^{-x}\Big]_0^4 = -e^{-4} + 1$$

$f(x) = e^{-x}$ integrable.
$F(x) = -e^{-x}$ is an antiderivative of $e^{-x}$ on $[0,4]$
FTC ② applies

**Next Session:** Method of substitution and Riemann sum (Lectures 3 and 4).