MA1140 Elementary linear algebra

March 28 to May 02, 2022 (1-2 segment)

Assignment 3 (Due date: 22.04.2022, 11:59 PM)

Rules:

- Answer all questions.
- Provide complete answers with full justification and all steps worked out.
- The deadline is strict and even a one minute late submission cannot be accepted. Late submissions receive 0 marks.
- Only three grades are possible for each question: 0 for a wrong answer, 1.5 for a partially correct answer or 3 for a fully correct answer.

Questions:

1. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)$.

2. Show that the transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined by
$$T(a + bx + cx^2) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix},$$
is a linear transformation (here the coefficients $a, b, c \in \mathbb{R}$).

3. For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 2b - c \\ 3a - b + 2c \\ a + b + 2c \end{bmatrix}$$
calculate the preimages of
$$(a) \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad (b) \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}.$$

4. Is the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined below injective? Justify your answer.
$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2a + b + c \\ -a + 3b + c - d \\ 3a + b + 2c - 2d \end{bmatrix}$$

5. Find the eigenvalues and eigenvectors of
$$(a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (b) \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

6. Find all eigenvalues and eigenvectors of an idempotent matrix A. (Note: an idempotent matrix A obeys the relation $A^2 = A$). Verify your results for $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.