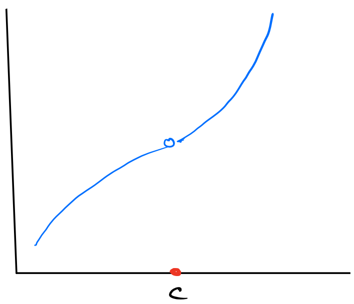
# Continuity at a point

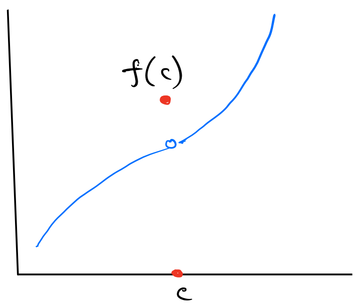
A function $f$ is said to be **continuous at c** if

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

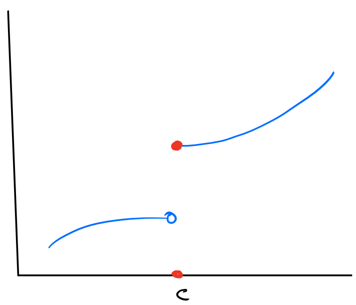
1. $f$ is **defined on a nbhd. of c** (in particular, $f$ is defined at $c$)
2. $\boxed{\lim_{x \to c} f(x)}$ exists **finitely**
3. $\boxed{\lim_{x \to c} f(x)} = f(c)$.




$f$ not defined at c but $\lim_{x \to c} f(x)$ exists



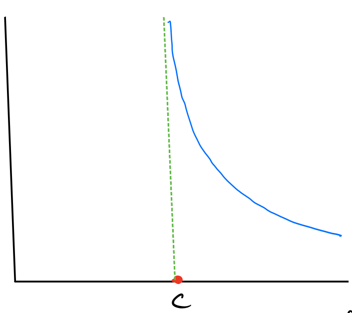

$f$ is defined at c, $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq f(c)$




$f$ is defined at c but $\lim_{x \to c} f(x)$ does not exists

But $\lim_{x \to c^+} f(x)$ exists and $= f(c)$, hence, $f(x)$ is continuous from the right **at c**.

But not cont. from left at c.




$f$ is not defined at c

$\lim_{x \to c} f(x)$ **is not finite**

## An $\varepsilon$-$\delta$ definition of continuity

$f$ is said to be continuous at $x=c$ if $f(x)$ is defined on a nbhd. of $c$, and for every $\varepsilon>0$ $\exists$ a $\delta>0$ s.t.

$$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$$

**Thm.** A function $f(x)$ is cont. at $x=c$

$$\Longleftrightarrow$$

$f(x_n)\to f(c)$ for every sequence $x_n\to c$.

**Proof.** "$\Rightarrow$" part.

Suppose $\lim\limits_{x\to c} f(x) = f(c)$

By Thm. A. of limits,

$f(x_n)\to f(c)$ for every sequence $x_n \to c$.

"$\Leftarrow$" part.

Suppose $f(x_n)\to f(c)$ for every sequence $x_n\to c$. Then by Thm. B of limits, $\lim\limits_{x\to c} f(x) = f(c)$

$\Rightarrow f$ is cont. at $x=c$.

## Examples

- The function $f(x) = x$ is cont. at every $c \in \mathbb{R}$.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

A (trivial) choice for $\delta$, is $\delta = \varepsilon$.

Obviously, $|x-c| < \varepsilon \Rightarrow |\underbrace{x}_{f(x)} - \underbrace{c}_{f(c)}| < \varepsilon$.

- $f(x) = \sin x$ is cont. at every $c \in \mathbb{R}$.

To show: Let $c \in \mathbb{R}$. Then given $\varepsilon > 0$ $\exists\ \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |\sin x - \sin c| < \varepsilon.$$

Now,

$$|\sin x - \sin c| = 2\left|\sin\left(\frac{x-c}{2}\right)\right|\left|\cos\left(\frac{x+c}{2}\right)\right|$$
$$\leq |x - c| \quad \text{—— } (*)$$

Take $\delta = \varepsilon$. Then by $(*)$,

$$|x-c| < \varepsilon \Rightarrow |\sin x - \sin c| \leq |x-c| < \varepsilon.$$

• $f(x) = e^x$ is cont. at every $c \in \mathbb{R}$.

$$|e^x - e^c| = e^c \, |e^{(x-c)} - 1|$$

Now, $|e^y - 1| = \left| y + \frac{y^2}{2!} + \cdots \right|$

$$\leq |y| + \frac{|y|^2}{2!} + \cdots$$

$$\leq |y| + \frac{|y|^2}{2} + \frac{|y|^3}{2^2} + \cdots$$

$$= \frac{|y|}{1 - \frac{|y|}{2}} \quad \text{if } |y| < 2$$

Next, $\dfrac{|y|}{1 - \frac{|y|}{2}} < \dfrac{\varepsilon}{M}$

if $|y| < \dfrac{2\varepsilon}{2M + \varepsilon}$

Thus, $|e^y - 1| < \dfrac{\varepsilon}{M}$

whenever $|y| < \min\left\{ 1, \dfrac{2\varepsilon}{2M+\varepsilon} \right\}$

Taking $M = e^c$, we get

$$|e^{(x-c)} - 1| < \frac{\varepsilon}{e^c}$$

provided $|x-c| < \underbrace{\min\left\{1, \frac{2\varepsilon}{2e^c+\varepsilon}\right\}}_{\delta}$

Thus, if $|x-c| < \delta$, then

$$|e^x - e^c| = e^c |e^{(x-c)} - 1|$$
$$< e^c \cdot \frac{\varepsilon}{e^c} = \varepsilon.$$

Remark: $\delta$ depends on

- $\varepsilon$
- $f$
- $c$

# Algebraic Properties of Continuous Functions

**Thm.** If $f$ and $g$ are cont. at $x=c$, then

(i) $f+g$ is cont. at $c$

(ii) $k\cdot f$ is cont. at $c$

(iii) $f\cdot g$ is cont. at $c$

(iv) $f/g$ is cont at $c$, provided $g(c)\neq 0$.

**Corollary.** Every polynomial is cont. at every $c\in\mathbb{R}$.

$x^2+x+1$

**Corollary.** If $f$ is cont. at $c$, then the following functions are cont. at $c$.

$\dfrac{x+1}{(x^2-4)}$

- $g(x) = f(x) + r$ , $r\in\mathbb{R}$ is fixed.
- $h(x) = f(x)^m$ , $m\in\mathbb{N}$
- $h(x) = \dfrac{1}{f(x)^m}$ , $m\in\mathbb{N}$, provided $f(c)\neq 0$.

# Continuity of composition

When is $g \circ f(x)$ cont. at $x = c$?

$$g \circ f(c) = g(f(c)).$$

If $f$ is cont. at $c$ and $g$ is cont. at $f(c)$, then $g \circ f$ is cont. at $x = c$.

Proof. Suffices to show that if $x_n \to c$,

then $$g \circ f(x_n) = g(f(x_n)) \to \underset{g \circ f(c)}{\underset{\parallel}{g(f(c))}}.$$

Suppose $x_n \to c$

By the continuity of $f$ at $c$,

$$f(x_n) \to f(c)$$

Now, by cont. of $g$ at $f(c)$,

$$g(f(x_n)) \to g(f(c))$$

Since $\{x_n\}_n$ is arbitrary sequence converging to $c$,

it follows from ~~Thm B~~ of limits that $g \circ f$ is cont. at $x = c$.

# Examples

- $g(x) = |f(x)|$ is cont. at $x=c$ if $f$ is cont. at $x=c$. (1.1 of $f$)

- $g(x) = f(|x|)$ is cont. at $x=c$ if $f$ is cont. at $x=|c|$. ($f$ o 1.1)

- If $f(c) > 0$, then $g(x) = \sqrt{f(x)}$ is cont. at $x=c$. if $f$ is cont. at $c$.

- $g(x) = f(\sqrt{x})$ is cont. at $c$, if ($c \geq 0$) $f$ is cont. at $x = \sqrt{c}$

# Continuity on Intervals

## Continuity on $[a,b]$ (Closed Interval)

Let $f$ be defined on $[a,b]$.

Then $f$ is said to be continuous on $[a,b]$ if

- $f(x)$ is continuous at any $c \in (a,b)$
- $f(x)$ is left-continuous at $b$
- $f(x)$ is right-continuous at $a$

**Thm.** Suppose $f$ is continuous on $[a,b]$, and let $E = \{f(x) : x \in [a,b]\}$.

Then

- $E$ is closed
- $E$ is bounded

Proof. E is closed.

If $E' = \emptyset$, then done.

Otherwise, let $\ell \in E'$

To show: $\ell \in E$, i.e. $\exists$ a $c \in [a,b]$ s.t. $\ell = f(c)$.

$\ell \in E' \Rightarrow \exists$ a sequence $\{y_n\}_n$
with $y_n \in E$ s.t. $y_n \to \ell$.

$y_n \in E \Rightarrow y_n = f(x_n)$ where $x_n \in [a,b]$

Now, $\{x_n\}_n$ is a bounded sequence.

By Bolzano-Weierstrass Thm.,
$\{x_n\}_n$ has a convergent subsequence,
Say $\{x_{n_k}\}_k$ where $x_{n_k} \to c$.

Note that $c$ is a limit point
of Range $\{x_{n_k} : k \in \mathbb{N}\} \subseteq [a,b]$

$\Rightarrow c \in [a,b]$ since $[a,b]$ is closed.

Next, since $f$ is cont. on $[a,b]$,

$$x_{n_k} \to c \Rightarrow f(x_{n_k}) \to f(c)$$

But $f(x_n) \to \ell \Rightarrow f(x_{n_k}) \to \ell$

By the uniqueness of the limit of a sequence, deduce that

$$\ell = f(c).$$

E is bdd.

Proof by contradiction.

Suppose that E is not bounded.

Then for each $n \in \mathbb{N}$, $\exists$ a $y_n \in E$ s.t. $|y_n| > n$.

Let $y_n = f(x_n)$ where $x_n \in [a,b]$.

As before, $\{x_n\}_n$ has a convg. subsequence, $\{x_{n_k}\}_k$ with

$$x_{n_k} \to c \in [a,b]$$

By continuity, $f(x_{n_k}) \to f(c)$

Since convergent sequences are bdd.,

$\{f(x_{n_k})\}_k$ is bdd.

But $|f(x_{n_k})| = |y_{n_k}| > n_k \ \forall k$

is then a contradiction

since $n_1 < n_2 < \cdots < n_k < \cdots$ is an unbounded sequence.

$\square$

**Corollary.** Let $f$ be **cont.** on $[a,b]$, and let $E = \{f(x) : x \in [a,b]\}$. Then $M = \sup E$ and $m = \inf E$ **exists**. Furthermore, $m, M \in E$.

Proof. By the previous Thm., $E$ is bounded. Hence,

$M = \sup E$ & $m = \inf E$ exists by the lub & glb properties of $\mathbb{R}$.

Next, since $E$ is closed, we claim that $m, M$ are in $E$. For instance, let us show that $M \in E$.

If $M \in E$ already, then done.

If $M \notin E$, from the definition of a supremum, for every $\varepsilon > 0$, $\exists$ a $x \in E$ s.t.

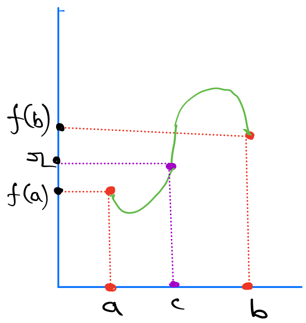
$$M - \varepsilon < x < M$$

$\Rightarrow$ every deleted nbhd. of $M$ contains an element of $E$.

$\Rightarrow M \in E'$ $\Rightarrow M \in E$ since $E$ is closed.

# Intermediate Value Property (IVP) of a continuous function

**Thm. (IVP)** Suppose $f$ is cont. on $[a,b]$




If $r$ is a real no. lying between $f(a)$ and $f(b)$ (either $f(a) < r < f(b)$, or $f(b) < r < f(a)$),

then $\exists$ a $c \in (a,b)$ s.t.

$$r = f(c).$$

Proof. We assume $f(a) < r < f(b)$

(the case $f(b) < r < f(a)$ can be handled similarly).

Consider the function

$$g(x) = f(x) - r$$

Then $g$ is cont. on $[a,b]$ with

$$g(a) = f(a) - r < 0$$

and $g(b) = f(b) - r > 0$

It suffices to show that $\exists\ c \in [a,b]$ s.t. $g(c) = 0$ (then $f(c) = r$).

## Proof by contradiction

Suppose, $g(x) \neq 0 \;\; \forall \; x \in (a,b)$.

$$\Rightarrow \quad g(x) \neq 0 \;\; \forall \; x \in [a,b].$$

Let $X = \{ x \in [a,b] : g(x) < 0 \} \subseteq [a,b]$

and $Y = \{ x \in [a,b] : g(x) > 0 \} \subseteq [a,b]$.

Then $X \cup Y = [a,b]$, $X \cap Y = \phi$.

Note that $X$ is bounded, and $X \neq \phi$ (since $a \in X$)

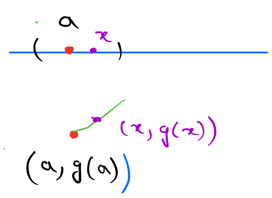
$\Rightarrow \sup X$ exists

Let $L = \sup X$

$L \in [a,b]$ since $[a,b]$ is closed.

We will show that $g(L) = 0$, thereby proving our assertion.

**Claim 1:** $L \in (a,b)$, i.e., $L \neq a$ and $L \neq b$.

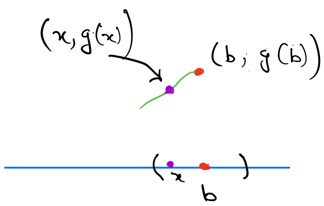



if $L = a$, then $x > L$ with $g(x) < 0$ is impossible

Now, $g$ - cont. from right at $x = a$

$\Rightarrow$ for $\varepsilon = -\frac{g(a)}{2} > 0$, $\exists$ a $\delta_0 > 0$

s.t. $a \leq x < a + \delta_0 \Rightarrow |g(x) - g(a)| < -\frac{g(a)}{2}$




if $L=b$, then by cont. of $g$ $g(x)>0$ in a nbhd. of b. But by sup property, $\exists$ a $x \in X$ in this nbhd. $\Rightarrow g(x)<0$, impossible.

Make $\delta_0$ small enough so that

$$[a, a+\delta_0) \subseteq [a,b)$$

Thus, for any $x$ satisfying

$a \le x < a+\delta_0$, we have

$$g(x) < g(a) - \frac{g(a)}{2} = \frac{g(a)}{2} < 0$$

Note that any such $x$ is in $[a,b]$

Thus, $\exists$ $x \in [a,b]$ with $x > a$ s.t. $g(x) < 0$.

$$\Rightarrow x \in X$$

$$\Rightarrow L \ge x > a \Rightarrow L \ne a.$$

Next, we show that $L \ne b$.

Using the right cont. of $g$ at $b$, for $\varepsilon = \frac{g(b)}{2}$, $\exists$ $\delta_1 > 0$ s.t.

$$b-\delta_1 < x \le b \Rightarrow |g(x) - g(b)| < \frac{g(b)}{2}$$

Take $\delta_1$ small enough so that

$$(b-\delta_1, b] \subseteq [a,b].$$

Now, if $x \in (b-\delta_1, b]$, then

$$|g(x) - g(b)| < \frac{g(b)}{2}$$

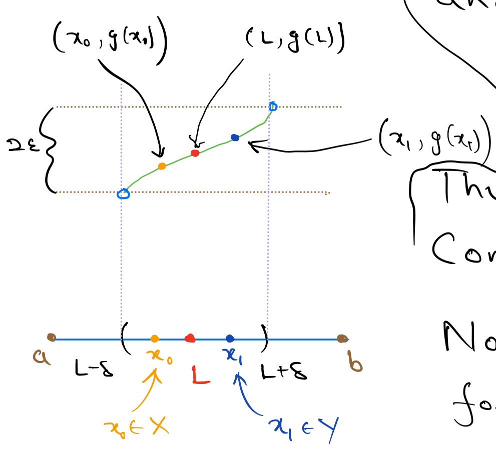
$$\Rightarrow \quad g(x) > g(b) - \frac{g(b)}{2} = \frac{g(b)}{2} > 0 \quad (*)$$

But if $L = b$, then from the supremum property of $L$, $\exists$ a $x \in X$ s.t.

$$b - \delta_1 < x \leq L$$

This is a contradiction since on one hand $g(x) < 0$ since $x \in X$

and $g(x) > 0$ by $(*)$ since $x \in (b-\delta_1, b]$.




Thus, $L \neq b$.

Consequently, $a < L < b$.

Now, $f$ is cont. at $x = L$ means for every $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t.

$$x \in (L-\delta, L+\delta) \Rightarrow |g(x) - g(L)| < \varepsilon. \quad (**)$$

Take $\delta$ small enough so that $(L-\delta, L+\delta) \subseteq (a, b)$.

The supremum property of $L$ again implies that $\exists$ $x_0 \in X$ s.t. $x_0 \in (L-\delta, L]$.

Thus, by $(**)$,

$$|g(x_0) - g(L)| < \varepsilon$$

$$\Rightarrow \quad g(L) < g(x_0) + \varepsilon < \varepsilon$$

since $g(x_0) < 0$.

Also, if $x \in (L, L+\delta)$, then

$$g(x) > 0$$

else, $g(x) < 0 \Rightarrow x \in X$

but then $x \leq L$ since

$$L = \sup X$$

$$\Rightarrow x \notin (L, L+\delta).$$

Pick any $x_1 \in (L, L+\delta)$.

Then by $(**)$,

$$|g(x_1) - g(L)| < \varepsilon$$

$$\Rightarrow \quad g(L) > g(x_1) - \varepsilon > -\varepsilon$$

since $g(x_1) > 0$.

Thus,

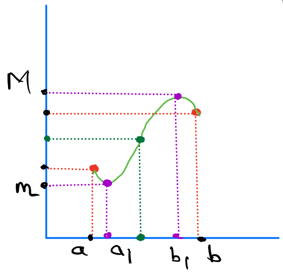
$$-\varepsilon < g(L) < \varepsilon \quad \forall \; \varepsilon > 0$$

i.e.,

$$|g(L)| < \varepsilon \quad \forall \; \varepsilon > 0$$

$$\Rightarrow \quad g(L) = 0, \text{ as required.}$$

**Corollary.** Let $f$ be cont. on $[a,b]$.




Let $E = \{ f(x) : x \in [a,b] \}$, $= f([a_1,b_1])$

$m = \inf E$ and $M = \sup E$.

Then $E = [m, M]$.

**Proof.** It is clear that $E \subseteq [m, M]$

Since $E$ is closed,

$m \in E$ and $M \in E$.

$\Rightarrow \exists \; a_1, b_1 \in [a,b]$ s.t.

$f(a_1) = m$, $f(b_1) = M$.

Let $x \in (m, M)$.

By IVP of cont. functions,

$\exists \; c \in (a_1, b_1)$ s.t.

$f(c) = x$

$\Rightarrow x \in E$

$\Rightarrow [m, M] \subseteq E$.

Since $E \subseteq [m, M]$, it follows that $E = [m, M]$. $\square$

## Applications.

- Let $f(x) = x^3 + ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$.

Then $\exists\ c \in \mathbb{R}$ s.t. $f(c) = 0$.

*Proof.* For any $n \in \mathbb{N}$, we have

$$f(-n) = -n^3 + an^2 - bn + c$$
$$= -n^3\left(1 - \frac{a}{n} + \frac{b}{n^2} - \frac{c}{n^3}\right)$$

Since $\frac{a}{n} \to 0$, $\frac{b}{n^2} \to 0$ & $\frac{c}{n^3} \to 0$, $\exists$ a $n_0$ s.t. if $n \geq n_0$, then

$$1 - \frac{a}{n} + \frac{b}{n^2} - \frac{c}{n^3} > \frac{1}{2}$$

$$\left| \frac{-a}{n} + \frac{b}{n^2} - \frac{c}{n^3} \right| \leq \frac{|a|}{n} + \frac{|b|}{n^2} + \frac{|c|}{n^3} < \varepsilon \quad \forall n \geq n_0$$

$$\Rightarrow -\varepsilon < -\frac{a}{n} + \frac{b}{n^2} - \frac{c}{n^3} < \varepsilon \quad \forall n \geq n_0$$

$$\Rightarrow f(-n) < -\frac{n^3}{2} < 0 \quad \forall\ n \geq n_0$$

In particular, $f(-n_0) < 0$

Similarly, $\exists\ n_1 \in \mathbb{N}$ s.t.

$$f(n_1) > \frac{n_1^3}{2} > 0.$$

Since, $f$ is cont. on $[-n_0, n_1]$, hence by IVP, $\exists\ c \in (-n_0, n_1)$ s.t. $f(c) = 0$.

- **Fixed Point Thm.**

Supp. $f : [0,1] \xrightarrow{\text{cont.}} [0,1]$.

Then $\exists\ c \in [0,1]$ s.t.

$$f(c) = c.$$

Proof: Without loss of any generality, assume that $f(0) \neq 0$ and $f(1) \neq 1$. Else, if $f(0) = 0$ or $f(1) = 1$, then we are done.

Thus, can assume

$$0 < f(0) \leq 1$$
$$0 \leq f(1) < 1$$

Consider the function.

$$g(x) = f(x) - x$$

Then $g$ is cont. on $[0,1]$.

Thus, $g(0) = f(0) - 0 = f(0) > 0$

and $g(1) = f(1) - 1 < 0$

By IVP of $g$, $\exists\ c \in (0,1)$

s.t. $g(c) = 0$

so that $f(c) = c$.

$\square$

## Continuous Functions on Open Intervals

**Defn.** $f$ is said to be cont. on $(a,b)$ if $f$ is cont. at every pt. on $(a,b)$.

**Remarks.** Supp. $f$ is cont. on $(a,b)$.

- $f$ need not be bounded

  e.g. $f(x) = \dfrac{1}{(x-a)(x-b)}$

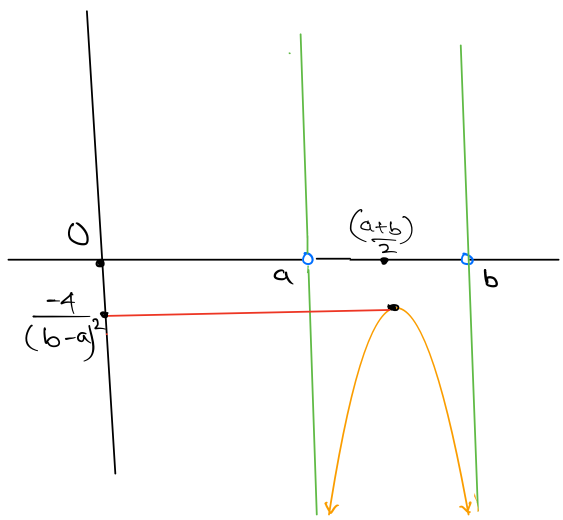
We compute:

$$\lim_{x \to a^+} \frac{1}{(x-a)(x-b)} = -\infty$$

$$\lim_{x \to b^-} \frac{1}{(x-a)(x-b)} = -\infty$$

Also, $f(x)$ attains its maximum at $x = \frac{a+b}{2}$, and

$$f\left(\frac{a+b}{2}\right) = \frac{-4}{(b-a)^2}.$$




In this case, $E = \{ f(x) : x \in (a,b) \}$

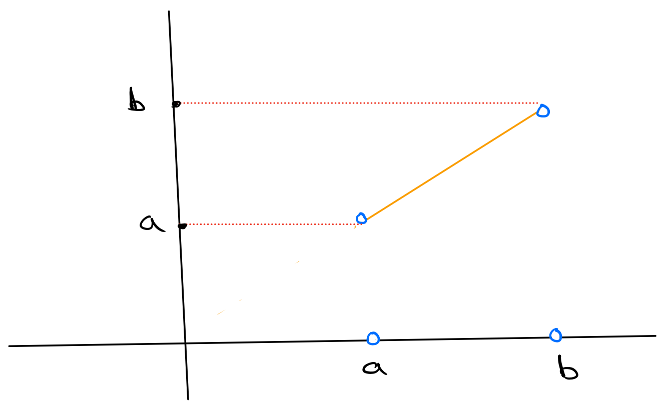
$$\left(-\infty, \frac{-4}{(b-a)^2}\right]$$

which is not bdd.

This example also shows that $E = \{ f(x) : x \in (a,b) \}$ need not be open either, since

$\left(-\infty, \frac{-4}{(b-a)^2}\right]$ is <u>not</u> open.

- $E = \{ f(x) : x \in (a, b) \}$ need not be closed.




$f(x) = x$

$E = (a, b)$ not closed.