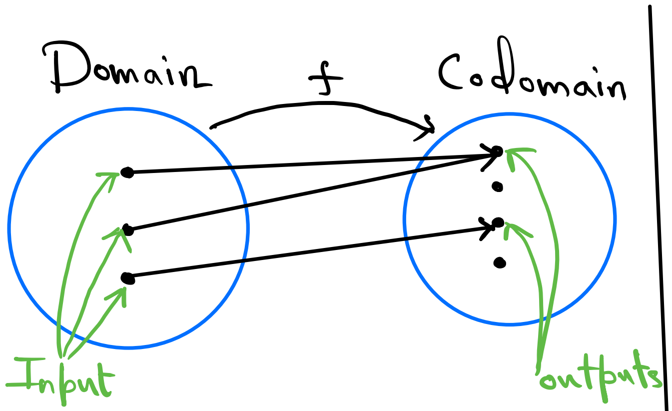
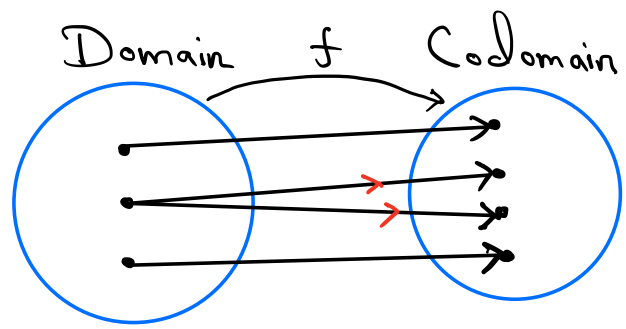
Functions



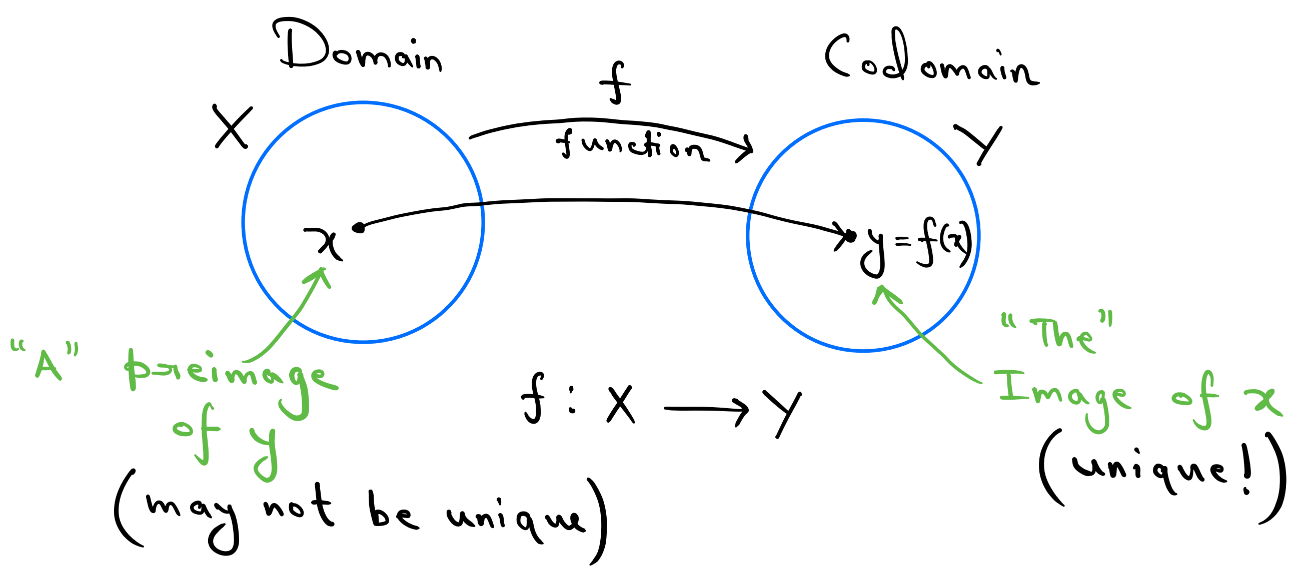
This is a function



Not a function

A $\boxed{\text{function}}$ is a rule that assigns to each input, a unique output.

Terminologies



$f : X \longrightarrow Y$

The Range of f $= \{ f(x) : x \in X \} \subseteq Y$

- How to check whether a given rule is a function?

Suppose $f : X \longrightarrow Y$ is a rule (not necessarily a fn.)

Then f is a function provided,

implication symbol

$$f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$$

if S_1 holds
then S_2 holds
$S_1 \Rightarrow S_2$
$n \in \mathbb{N} \Rightarrow n \in \mathbb{Z}$

$$\begin{tikzcd} x_1 \arrow[r, "f"] \arrow[d, phantom, "\neq"] & f(x_1) \arrow[d, phantom, "\neq"] \\ x_2 \arrow[r, "f"] & f(x_2) \end{tikzcd}$$

$$\begin{tikzcd} & f(x) \\ x \arrow[ur] \arrow[r] & f(x) \end{tikzcd}$$

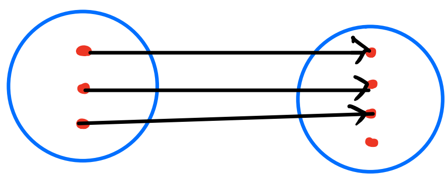
Alternatively,

$$\boxed{x_1 = x_2 \Rightarrow f(x_1) = f(x_2)}$$

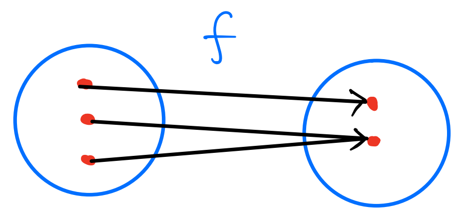
$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^2$$

$$x_1 = x_2 \Rightarrow x_1^2 = x_2^2$$

Injective (one-one), Surjective (onto) and bijective functions



Injective



Surjective

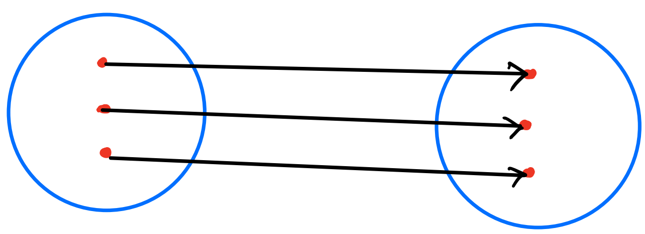
- $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

  equivalently,

  $\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

- Range = Codomain

  $y = f(x)$



Bijective = Inj. + Surj.

Examples.

- $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x+1$

  $2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$ inj. ✓

  $y = 2x+1$ surj. ✓

  $x = \frac{y-1}{2}$ bij. ✓

- $f: \mathbb{Z} \to \mathbb{Z}$, given by $f(x) = 2x+1$

  inj. ✓
  surj. ✗

- $f: \mathbb{R}^* \longrightarrow \mathbb{R}$, given by $f(x) = 1/x$

  $\mathbb{R} \setminus \{0\}$

  $\mathbb{R}^*$

  inj. ✓
  surj ✗

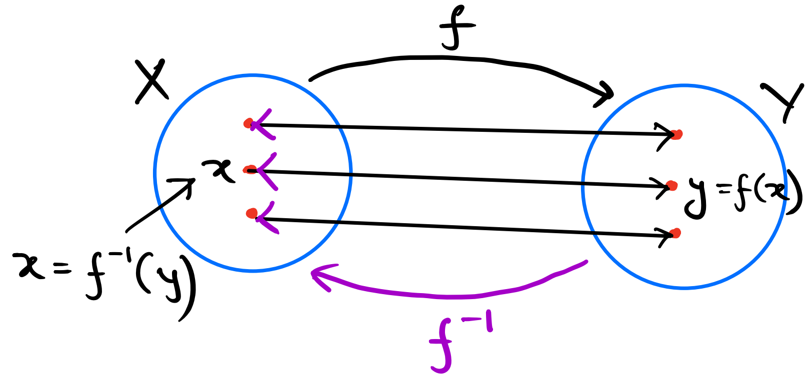
  $1/x = 0$

- $f: \mathbb{R} \longrightarrow [-1, 1]$, given by $f(x) = \sin x$

  surj. ✓
  inj. ✗

  $\sin \pi = \sin 0 = 0$

<u>The inverse of a bijective function</u>



- f^{-1} is a function
- f^{-1} is bijective

- $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x+1$

$$y = f(x) = 2x+1$$

$$x = ?$$

solving for x, we get

$$x = \frac{y-1}{2} \leftarrow f^{-1}(y)$$

Thus, the inverse fn. of $f(x) = 2x+1$ is $f^{-1}(x) = \frac{x-1}{2}$.

Restricting Domains and bijections

- $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ (nonnegative reals)

$f(x) = x^2$ is not bijective

However, $f: \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$

$f(x) = x^2$ is bijective

- what is $f^{-1}(x)$?

Solving for x in

$$y = f(x) = x^2, \text{ gives}$$

$$x = \sqrt{y}$$

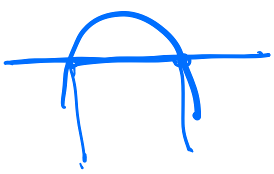
Thus, $f^{-1}(x) = \sqrt{x}$

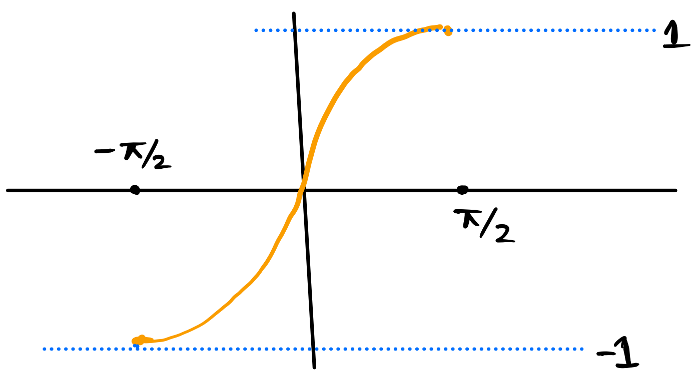
- $f : \mathbb{R} \longrightarrow [-1, 1]$

  $f(x) = \sin x$, is not bijective.

But $f : [-\pi/2, \pi/2] \longrightarrow [-1, 1]$

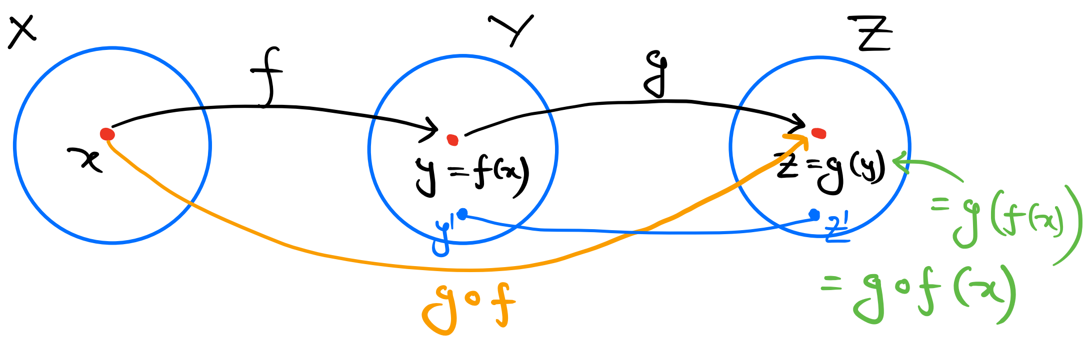
is a bijective





The inverse of $\sin x$ is defined as the $\arcsin x$ or $\sin^{-1} x$ with domain $[-1, 1]$.

Composition of functions.

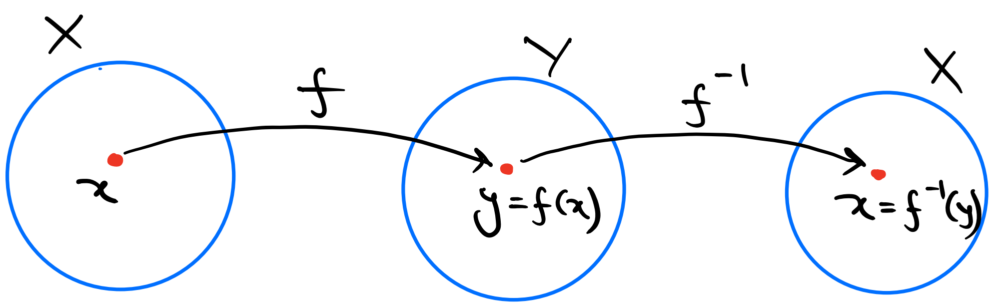


Domain of gof = Domain of f

Range of gof $\subseteq$ Range of g

Fundamental Assumption is that

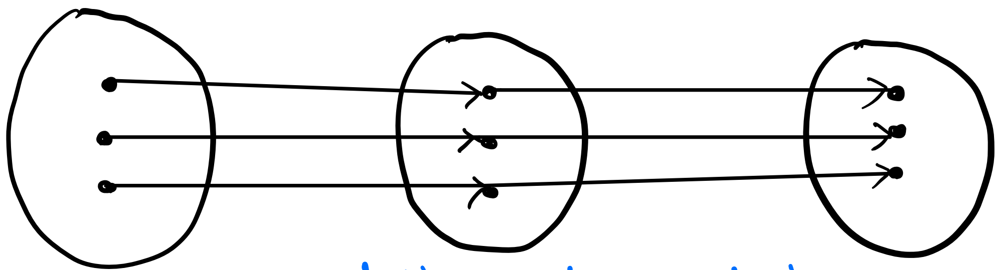
Range of f $\subseteq$ Domain of g.



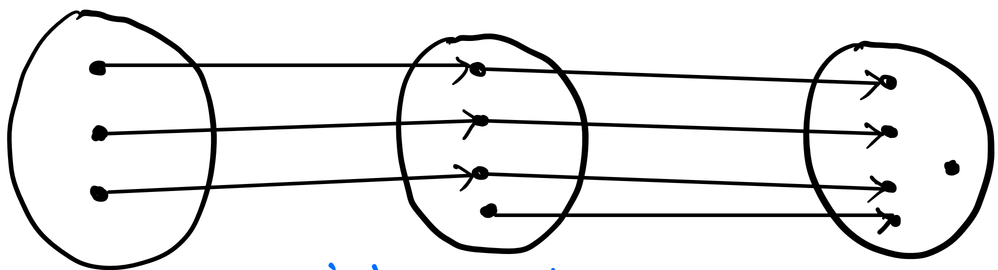
$f^{-1} \circ f$ = Identity on X

similarly $f \circ f^{-1}$ = Identity on Y.

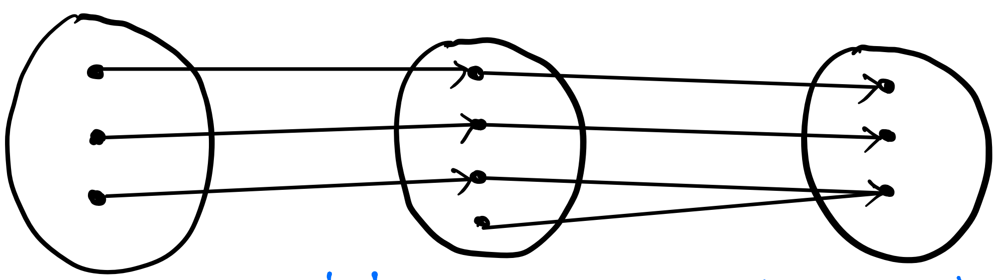
Behavior of injection/surjection/bijection under composition



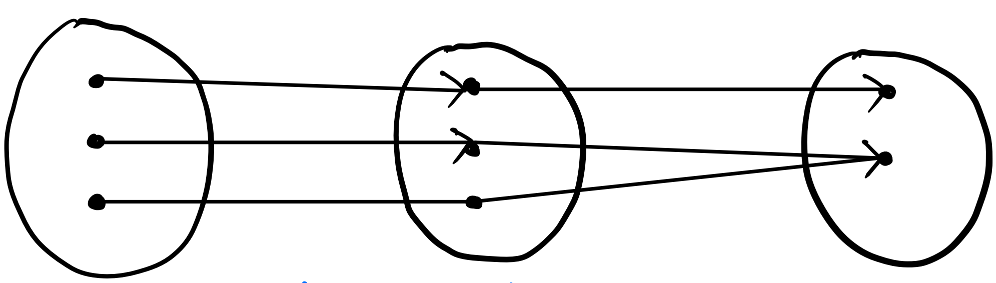
bij. o bij = bij.



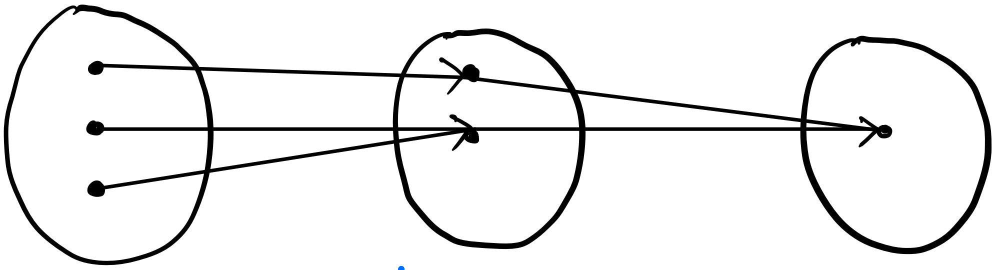
inj. o inj. = inj.



inj. o surj. = inj. & surj. = bij.



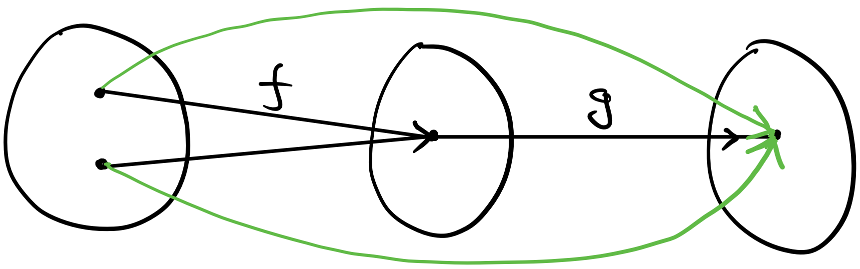
inj. o surj. ≠ inj.



surj. o surj. = surj.

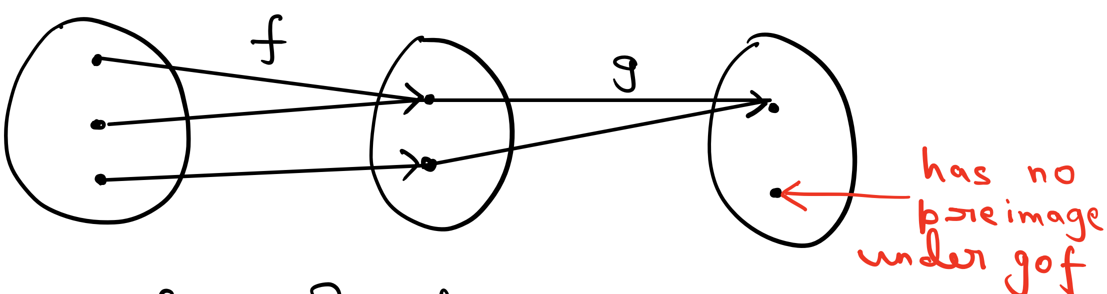
If $g \circ f$ is Injective, can we say something about g and/or f?

f must be injective



If $g \circ f$ is Surjective, can we say something about g and/or f?

g must be surjective



If $g \circ f$ is Bijective, can we say something about g and/or f?

$g \circ f$ - bijective $\Rightarrow$ f - injective & g - surjective

Cardinality of Sets

Let $A \subseteq \mathbb{R}$,

The cardinality of A is roughly the no. of elements in A.

Denoted by $|A|$.

Examples.

$$|\phi| = 0$$

$$|\{1, 2, \dots, n\}| = n$$

Set of Natural numbers

$$|\mathbb{N}| = \infty$$

$$|\mathbb{R}| = \infty$$

Question: Is $|\mathbb{N}| = |\mathbb{R}|$?

Ans. No!

$\mathbb{N}$ is countably infinite

$\mathbb{R}$ is uncountably infinite

Notation: If $f : X \xrightarrow{\text{bijection}} Y$, then we write $X \sim Y$.

Remark:
- $X \sim X$
- $X \sim Y \Rightarrow Y \sim X$
- $X \sim Y$ & $Y \sim Z \Rightarrow X \sim Z$

Defn. Two sets A and B are said to have the <u>same</u> cardinality if $A \sim B$, that is, $\exists$ a bijection $f : A \to B$.

Defn. An infinite set A is countably infinite if $A \sim \mathbb{N}$.

$\mathbb{Q} \sim \mathbb{N}$

$\mathbb{R} \not\sim \mathbb{N}$

<u>Examples</u>

- $A = \{ 1/n : n \in \mathbb{N} \}$

  Then $A \sim \mathbb{N}$ (countably infinite)

  $f : \mathbb{N} \to A$

  $f(n) = 1/n$

- $\mathbb{Z} \sim \mathbb{N}$

  $f : \mathbb{N} \to \mathbb{Z}$

  $$f(n) = \begin{cases} n/2 & \text{if } n\text{-even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n\text{-odd} \end{cases}$$

  So,

  $2 \to 1$

  $4 \to 2$

  $6 \to 3$

  and so on

  $1 \to 0$

  $3 \to -1$

  $5 \to -2$

  and so on