IIT Hyderabad

Assignment 2

Submitted by:
Abhishek Ghosh
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Submitted to:
Dr. Vishnu Unni

Problem 1

1 Given two arbitrary locations on the earth's surface, find the range of velocity vectors with which a projectile needs to be launched to transport it from one location to another. Ignore atmospheric drag effects for this problem.

Ignoring the rotation of Earth, We study the ballistic missile problem of launching an unpowered object to hit a specific spot on Earth. For a given initial velocity v_0, there are typically two initial firing angles β_0 that match the target condition. If only one angle exists, the minimum initial velocity is used.

For suborbital flights in an inverse-squared gravity field, the initial velocity v_0 must be less than Earth's escape velocity. The desired range S is related to the semirange angle φ by:

$$\varphi = \frac{1}{2}\frac{S}{R_0}$$

The semirange angle φ is related to initial launch states r_0 and v_0 using orbit elements a and e. With $\varphi = \pi - f$, we have:

$$\cos\varphi = -\cos f = \frac{1}{e}\left(1 - \frac{p}{R_0}\right)$$

The initial position and velocity vectors are:

$$r_0 = R_0 J_r, \quad v_0 = v_0 \sin\beta_0 r + v_0 \cos\beta_0 J_\theta$$

The angular momentum h and semilatus rectum p are:

$$h = R_0 \sin\theta, \quad p = R_0 v_0^2 \cos^2\beta_0$$

The semimajor axis a and eccentricity e are:

$$a = \frac{R_0}{2 - v_0^2}, \quad e = \sqrt{1 - v_0^2(2 - v_0^2)\cos^2\beta_0}$$

The semirange angle φ is expressed as:

$$\tan\varphi = \frac{v_0^2 \tan\beta_0}{1 - v_0^2 + \tan^2\beta_0}$$

The optimal initial launch angle $\beta_{0,opt}$ for maximum range is:

$$\beta_{0,opt} = \cos^{-1}\left(\pm\sqrt{\frac{1}{2 - v_0^2}}\right)$$

The maximum semirange angle $\varphi_{\max}$ is:

$$\tan\varphi_{\max} = \frac{v_0^2}{2\sqrt{1 - v_0^2}}$$

For a given v_0 and φ, the initial launch angles β_0 are:

$$\tan\beta_0 = \frac{v_0^2 \pm \sqrt{v_0^4 - 4(1-v_0^2)\tan^2\varphi}}{2\tan\varphi}$$

The minimum velocity $v_{0,\text{min}}$ to achieve φ is:

$$v_{0,\text{min}}^2 = 2\tan^2\varphi\left(\frac{1}{\sin\varphi} - 1\right)$$

Classical Solutions

To determine the angular orbit position at any time, we use Kepler's equation. The angular momentum h is constant, and we have:

$$h = r^2\dot{\theta} = r^2 f$$

Rearranging and integrating, we get:

$$\sqrt{\frac{\mu}{p^3}}(t_1 - t_0) = \int_{f_0}^{f} \frac{df}{(1 + e\cos f)^2}$$

For $v_0^2 < 1$ there are always two possible trajectories for a given velocity, a "high" and a "low" path. However, once $v_0^2 > 1$ the two possible trajectories each lie on opposite sides of the Earth.

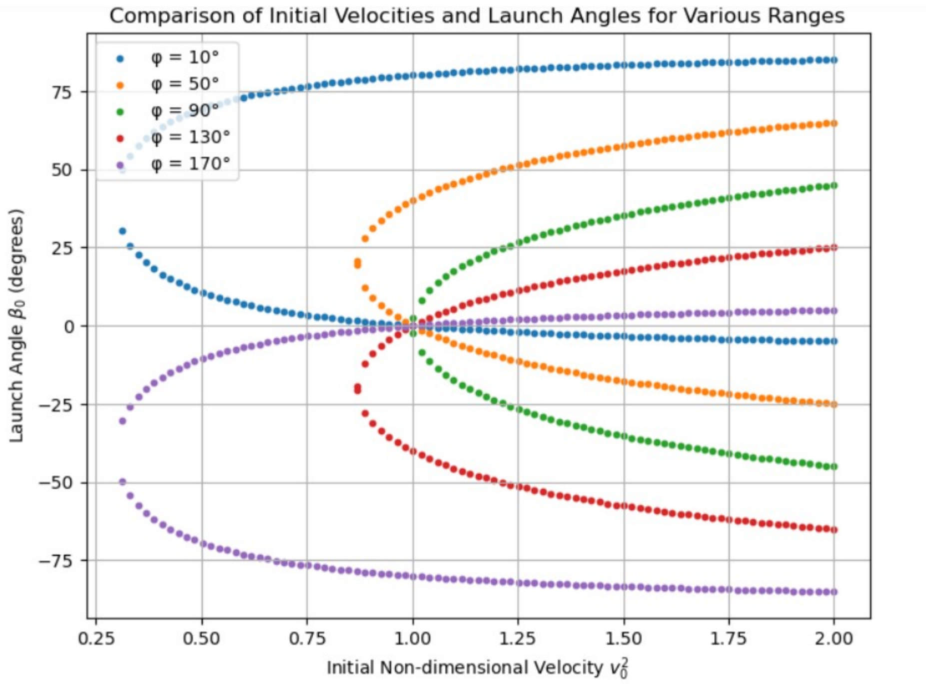

Figure 1: Comparison of Initial Velocities and Launch Angles for various ranges

At any given $\phi = \pi - f$ the tangent at leftmost point gives minimum velocity at corresponding β. The maximum velocity from graph is always taken less than the escape velocity of Earth.

2 Control of pendulum on a cart.

2.1 State-Space Representation

The state variables for the system are:

$$x = \text{Cart position} \tag{1}$$
$$\dot{x} = \text{Cart velocity} \tag{2}$$
$$\theta = \text{Pendulum angle} \tag{3}$$
$$\dot{\theta} = \text{Pendulum angular velocity} \tag{4}$$

The state-space representation is given by:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \tag{5}$$

where the system matrices are (b=1 means pendulum up, d means damping) :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{M} & \frac{bmg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{bd}{ML} & -\frac{b(M+m)g}{ML} & 0 \end{bmatrix}, \tag{6}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{b}{ML} \end{bmatrix}. \tag{7}$$

2.2 Nonlinear Equations of Motion

The equations of motion derived from Newton's Second Law are:

$$\ddot{x} = \frac{mL\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta - d\dot{x}}{M + m\sin^2\theta}, \tag{8}$$

$$\ddot{\theta} = \frac{-mL\dot{\theta}^2 \sin\theta\cos\theta + (M+m)g\sin\theta + d\dot{x}\cos\theta}{L(M + m\sin^2\theta)}. \tag{9}$$

2.3 Numerical Integration

To numerically solve these equations, we define the state vector:

$$\mathbf{y} = \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T. \tag{10}$$

Using the function `odeint`, we integrate the system over time.

2.4 Controller Equations

A state-feedback controller is used to stabilize the system (u is force to move cart):

$$u = -K\mathbf{x}, \tag{11}$$

where K is the gain matrix. The closed-loop system dynamics become:

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x}. \tag{12}$$

The controller gain K is determined using techniques such as pole placement or Linear Quadratic Regulator (LQR):

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}. \tag{13}$$

Using the function `odeint`, we integrate the system over time.

2.5 Kinematic Relations

Velocity:

$$v_x = \dot{x} \tag{14}$$
$$v_\theta = \dot{\theta} \tag{15}$$

Acceleration:

$$a_x = \frac{dv_x}{dt} = \ddot{x} \tag{16}$$
$$a_\theta = \frac{dv_\theta}{dt} = \ddot{\theta} \tag{17}$$

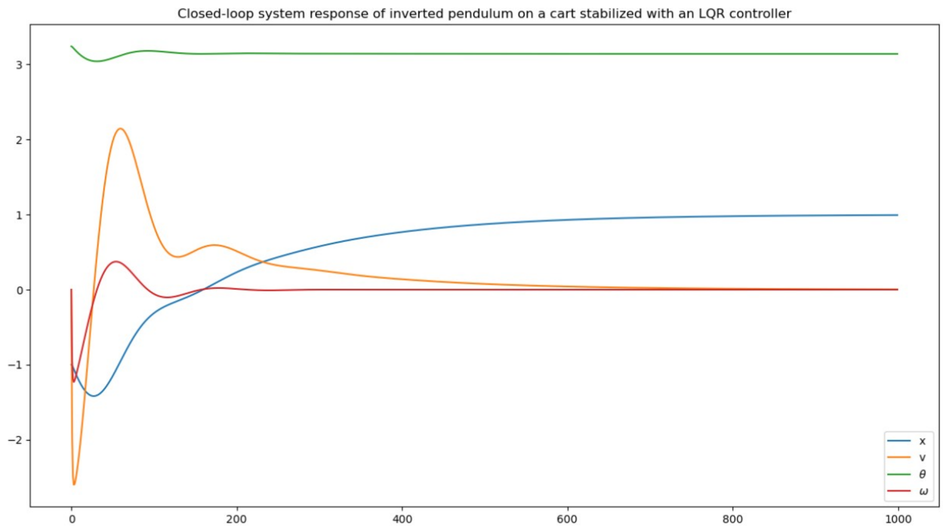

Figure 2: Closed loop response of Inverted Pendulum on a cart with LQR Controller

Q is penalty on x, R is penalty on u

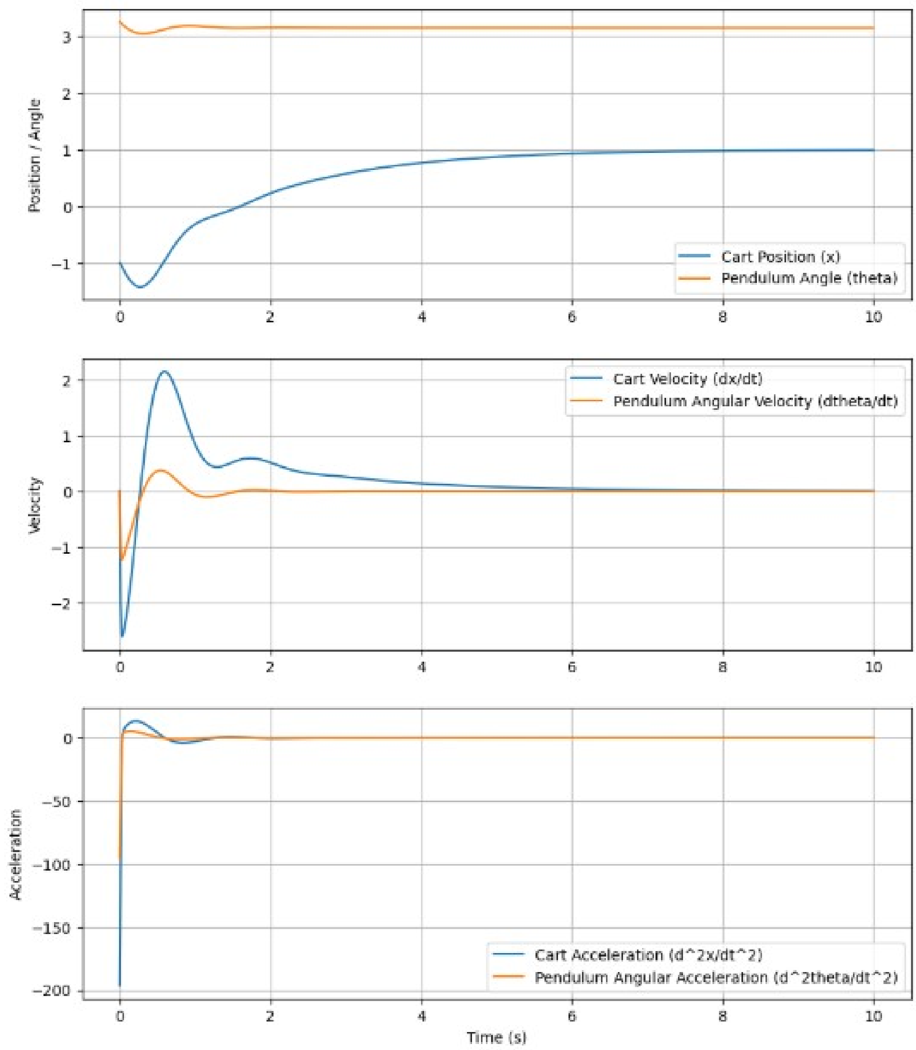

Figure 3: Plot of Position, Velocity and Acceleration of Cart and Pendulum Vs Time

3 A ball being launched from a spinning massless tube. Refer Problem 2.4 from Schaub and Junkins, 4th edition. Find the initial conditions (Theta and Omega) to ensure that the ball enters a specific a hole on the ground after being pushed out of the pipe.

A massless tube of length L rotates in a horizontal plane about a fixed point. A ball inside the tube is pushed outward by a force and exits with a velocity v. The objective is to determine the initial angle θ and angular velocity ω such that the ball lands at a specific hole located at coordinates (x_h, y_h).

3.1 Kinematic Equations

At the moment of release, the ball has the following initial conditions:

3.1.1

Position The position of the ball at the instant of release is given by:

$$x_0 = L\cos\theta \tag{18}$$
$$y_0 = L\sin\theta \tag{19}$$

3.1.2 Velocity Components

The velocity components of the ball are determined by both the radial velocity v and the tangential velocity due to rotation:

$$v_x = v\cos\theta - L\omega\sin\theta \tag{20}$$
$$v_y = v\sin\theta + L\omega\cos\theta \tag{21}$$

3.2 Projectile Motion Equations

After the ball exits the tube, it follows projectile motion under gravity. The horizontal and vertical positions at time t are:

$$x_f = x_0 + v_x t \tag{22}$$
$$y_f = y_0 + v_y t - \frac{1}{2}gt^2 \tag{23}$$

To ensure the ball lands at (x_h, y_h), we solve for t using the quadratic equation:

$$y_h = L\sin\theta + (v\sin\theta + L\omega\cos\theta)t - \frac{1}{2}gt^2 \tag{24}$$

This yields a positive root t_f.

Substituting t_f into the horizontal equation:

$$x_h = L\cos\theta + (v\cos\theta - L\omega\sin\theta)t_f \tag{25}$$

We solve this equation numerically to determine θ and ω.

3.3 Numerical Solution

Since the above equations are nonlinear, we use numerical methods such as the Newton-Raphson method or an optimization routine to find the values of θ and ω that satisfy both equations.

3.4 Sample Case

For a specific case where:

$$\begin{aligned} L &= 1.0 \text{ m}, \\ v &= 5.0 \text{ m/s}, \\ g &= 9.81 \text{ m/s}^2, \\ x_h &= 2.5 \text{ m}, \\ y_h &= 0 \text{ m}, \end{aligned}$$

the computed initial conditions are:

$$\theta = 35.85^\circ,$$
$$\omega = 4.89 \text{ rad/s}.$$

These values ensure that the ball lands exactly at the hole.

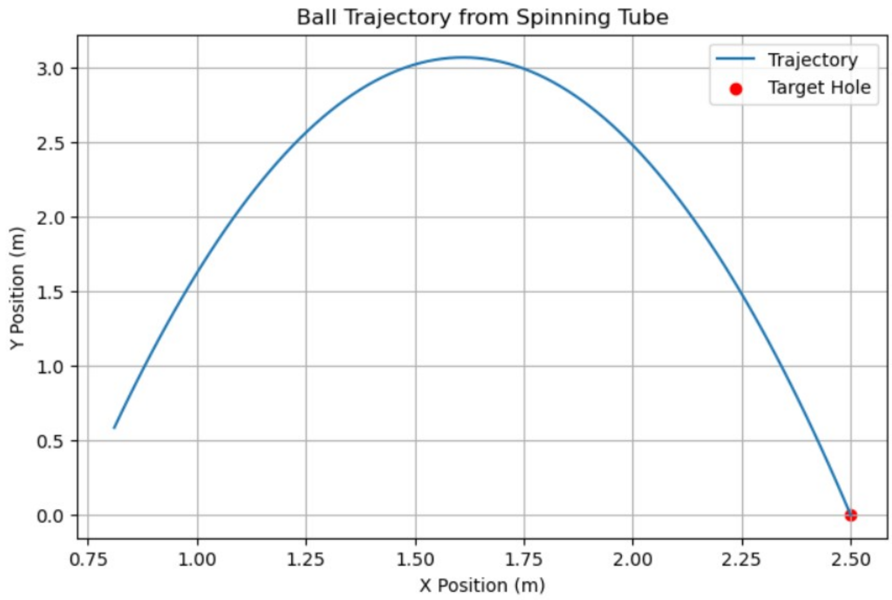

Figure 4: Ball Trajectory from Spinning Tube

References

1 The code used to generate plots is in the ME21BTECH11001.ipynb file.

2 The photos and GIFs for the animation are in the images Folder.

3 The code has been generated with the help of Chat GPT 4-o model and Example 9.6 from Schaub and Junkins, 4th edition has been used as reference for Problem 1.