भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

5th Lecture on Differential Equation

(MA-1150)

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

What have we learnt?

- Integrating Factors
- Method of solving non-exact first order and first degree ode:
    - ✓ Rule I
    - ✓ Rule II
    - ✓ Rule III
    - ✓ Rule IV

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

What will we learn today?

- Method of solving non-exact first order and first degree ode:
  - ✓ Rule V

- Linear ODE

- Non-linear ODE: Bernoulli Equation

For Finding Integrating Factor (Rule-V)

- **Rule-V:** If the differential equation $M\,dx + N\,dy = 0$ is of the form $x^{\alpha}y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1}y^{\beta_1}(m_1y\,dx + n_1x\,dy) = 0$, where $\alpha, \beta, \alpha_1, \beta_1, m, n, m_1, n_1$ are constants and $(mn_1 - m_1n) \neq 0$, then the integrating factor is $x^h y^k$, where h and k are constants given by

$$\left.\begin{aligned} \frac{h+1+\alpha}{m} &= \frac{k+1+\beta}{n} \\ \frac{h+1+\alpha_1}{m_1} &= \frac{k+1+\beta_1}{n_1} \end{aligned}\right\}$$

For Finding Integrating Factor (Rule-V)

Proof: The differential eq^n is

$$x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1} y^{\beta_1}(m_1 y\,dx + n_1 x\,dy) = 0. \quad \text{--- (1)}$$

we have to prove that $x^h y^k$ is an integrating factor of eq^n (1) satisfying two eq^{n}s

$$\frac{h+1+\alpha}{m} = \frac{k+1+\beta}{n}$$

For Finding Integrating Factor (Rule-V)

and $\dfrac{h+1+\alpha_1}{m_1} = \dfrac{k+1+\beta_1}{n_1}$

Let $x^h y^k$ be the integrating factor of eqn ①.

Multiplying eqn ① by $x^h y^k$, we have

$$\left(m x^{h+\alpha} y^{k+\beta+1} + m_1 x^{h+\alpha_1} y^{k+\beta_1+1}\right) dx + \left(n x^{h+\alpha+1} y^{\beta+k} + n_1 x^{h+\alpha_1+1} y^{\beta_1+k}\right) dy = 0.$$

For Finding Integrating Factor (Rule-V)

Now it is exact.

So $$\frac{\partial}{\partial y}\left\{m x^{h+\alpha} y^{k+\beta+1} + m_1 x^{h+\alpha_1} y^{k+\beta_1+1}\right\}$$

$$= \frac{\partial}{\partial x}\left\{n x^{h+\alpha+1} y^{\beta+k} + n_1 x^{h+\alpha_1+1} y^{\beta_1+k}\right\}$$

i.e, $$\boxed{\begin{aligned} &m(k+\beta+1)x^{h+\alpha}y^{k+\beta} + m_1(k+\beta_1+1)x^{h+\alpha_1}y^{k+\beta_1} \\ &= n(h+\alpha+1)x^{h+\alpha}y^{k+\beta} + n_1(h+\alpha_1+1)x^{h+\alpha_1}y^{k+\beta_1} \end{aligned}}$$

For Finding Integrating Factor (Rule-V)

Now comparing the coefficients $x^{h+\alpha} y^{k+\beta}$ and $x^{h+\alpha_1} y^{k+\beta_1}$, we get

$$m(k+\beta+1) = n(h+\alpha+1)$$

$$\Rightarrow \boxed{\frac{h+\alpha+1}{m} = \frac{k+\beta+1}{n}}$$

These are two equations.

and $m_1(k+\beta_1+1) = n_1(h+\alpha_1+1)$

$$\Rightarrow \boxed{\frac{h+\alpha_1+1}{m_1} = \frac{k+\beta_1+1}{n_1}}$$

![IIT Hyderabad logo]

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Examples

Ex ① $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$

Rearranging the terms, we write the equation in the form

$$y^2(ydx + 2xdy) - x^2(ydx + xdy) = 0$$

This is of the form

$$x^\alpha y^\beta(mydx + nxdy) + x^{\alpha_1}y^{\beta_1}(m_1ydx + n_1xdy) = 0$$

Here $\alpha = 0, \beta = 2, m = 1, n = 2, \alpha_1 = 2, \beta_1 = 0, m_1 = -2, n_1 = -1$.

Hence the relation gives

$$\frac{h+1}{1} = \frac{k+3}{2} \quad \text{and} \quad \frac{h+3}{-2} = \frac{k+1}{-1}$$

Solving for h and k, we get $h = k = 1$.

Then IF is xy.

Examples

Multiplying by I.F, we get -

$$xy^4\,dx + 2x^2y^3\,dy - 2x^3y^2\,dx - x^4y\,dy = 0$$

or, $\frac{1}{2}d(x^2y^4) - \frac{1}{2}d(x^4y^2) = 0$ [By applying grouping method]

Integrating, we have

$$\frac{x^2y^4}{2} - \frac{x^4y^2}{2} = \frac{c}{2}$$

or, $\boxed{x^2y^4 - x^4y^2 = c}$ — This is the general solution

Examples

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Ex. 2 $3y\,dx - 2x\,dy + x^2y^{-1}(10y\,dx - 6x\,dy) = 0$

Solⁿ: If $x^h y^k$ is an IF, then

$$x^h y^k(3y\,dx - 2x\,dy) + x^{h+2}y^{k-1}(10y\,dx - 6x\,dy) = 0 \quad \text{--- (1)}$$

Now eqⁿ (1) is exact and we have

$\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$ for exactness of the differention equation $M\,dx + N\,dy = 0$.

Hence $\dfrac{\partial}{\partial y}\left\{3y^{k+1}x^h + 10x^{h+2}y^k\right\} = \dfrac{\partial}{\partial x}\left\{-2x^{h+1}y^k - 6x^{h+3}y^{k-1}\right\}$

Examples

or, $3(k+1)x^h y^k + 10k\,x^{h+2}y^{k-1} = -2(h+1)x^h y^k - 6(h+3)x^{h+2}y^{k-1}$,

$\forall\, x, y.$

Now comparing coefficient of $x^h y^k$, we get

$$3(k+1) = -2(h+1) \Rightarrow \boxed{2h+3k=-5} \quad \text{— (i)}$$

Comparing coefficient of $x^{h+2}y^{k-1}$, we get

$$10k = -6(h+3) \Rightarrow \boxed{6h+10k=-18} \quad \text{— (ii)}$$

Solving eq^{n}s (i) and (ii) for h and k, we get $h=2$, $k=-3$

So, x^2y^{-3} is an IF of the given eq. multiplying by this IF,

Examples

we have,

$$x^2y^{-3}(3y\,dx - 2x\,dy) + x^4y^{-4}(10y\,dx - 6x\,dy) = 0$$

$$3x^2y^{-2}\,dx - 2x^3y^{-3}\,dy + 10x^4y^{-3}\,dx - 6x^5y^{-4}\,dy = 0$$

or, $(3x^2y^{-2}\,dx - 2x^3y^{-3}\,dy) + 2(5x^4y^{-3}\,dx - 3x^5y^{-4}\,dy) = 0.$

or, $d(x^3y^{-2}) + 2d(x^5y^{-3}) = 0$ [By applying grouping method]

Integrating, we have-

or $\boxed{x^3y^{-2} + 2x^5y^{-3} = c}$ — This is the general solution

Linear Equations

Linear First order ODE:

A first order ODE is linear in the dependent variable y and the independent variable x if it is or can be written as

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)}$$

, This is the general form of linear ODE of first order,

Linear Equations

Here $P(x)$ and $Q(x)$ are either functions of x or constants (including zero).

* If P and Q are both constants, then the variables are separable and it is also true if P or Q is zero.

Linear Equations

Note: If $Q(x)=0$, the linear ODE $\frac{dy}{dx}+P(x)y=0$ is known as Homogeneous linear first order ODE.

If $Q(x)\neq 0$, the linear ODE $\frac{dy}{dx}+P(x)y=Q(x)$ is called as non-homogeneous linear first order ODE.

Linear Equations

Theorem: The linear differential eqn $\frac{dy}{dx} + P(x)y = Q(x)$ has an integrating factor as $e^{\int P(x)dx}$ and the general solution is

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} Q(x)\, dx + c\right]$$

This is one-parameter family of solutions.

Linear Equations

Proof: we have linear first order ODE

$$\frac{dy}{dx} + P(x)\,y(x) = Q(x).$$

we need to find the I.F.

It is easily seen that the linear ODE has the integrating factor of function x only.

Linear Equations

Let us consider the integrating factor as $f(x)$.

Now the given ODE can be rewritten as

$$(P(x)y - Q(x))\,dx + dy = 0$$ which is the equation in the form

$$\underline{M\,dx + N\,dy = 0}.$$

Linear Equations

Here $M(x, y) = P(x)y(x) - Q(x)$

$N(x,y) = 1$

Since $\frac{\partial M}{\partial y} = P(x)$ and $\frac{\partial N}{\partial x} = 0$.

The given ODE is exact if $P(x) = 0$.

and it is easily solved by Separation of variables.

Linear Equations

But if $P(x) \neq 0$, then

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

So the ODE is not exact.

Now multiply the given ODE by IF to make it exact ODE.

Linear Equations

Here for $P(x) \neq 0$, we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x) \left[\text{function of } x \text{ only}\right]$$

So apply the Rule-II to find the I.F.

Here IF is $e^{\int P(x)dx}$ (Proved)

OR you can follow the alternative way to find PI.

Linear Equations

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

② Alternative way to find PI without any help of theorem on Rule II : (You may skip this)

Since we have considered $f(x)$ as the integrating factor, multiplying it to the given linear ODE, we have ———

Linear Equations

$$f(x)\left[P(x)y(x) - Q(x)\right]dx + f(x)\,dy = 0$$

or, $$\left[f(x)\,P(x)y(x) - f(x)Q(x)\right]dx$$

$$+ f(x)\,dy = 0.$$

Now the reduced ODE is exact

if and only if

Linear Equations

$$\frac{\partial}{\partial y}\left[f(x)P(x)y(x) - f(x)Q(x) \right]$$

$$= \frac{\partial}{\partial x}\left[f(x) \right]$$

as $f(x)P(x) = \dfrac{df}{dx}$, This differential eqⁿ is Separable.

Linear Equations

or $\dfrac{df}{f(x)} = p(x)\,dx$

Integrating we have

$\ln|f(x)| = \int p(x)\,dx$

or, $f(x) = e^{\int p(x)\,dx}$, $\underline{f(x) > 0.}$

Linear Equations

So the integrating factor (IF) is $e^{\int P(x)\,dx}$. (Proved)

Now we want to find the general solution.

Multiplying the given linear ODE by IF $e^{\int P(x)\,dx}$, we have

Linear Equations

$$e^{\int p(x)\,dx} \frac{dy}{dx} + e^{\int p(x)\,dx} p(x)\, y(x) = Q(x)\, e^{\int p(x)\,dx}.$$

as $$\frac{d}{dx}\left[e^{\int p(x)\,dx} \cdot y(x) \right] = Q(x)\, e^{\int p(x)\,dx}$$

Integrating we have —

Linear Equations

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$$e^{\int P(x)\,dx}\, y(x) = \int e^{\int P(x)\,dx} Q(x)\, dx + c$$

$$\boxed{y(x) = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx} Q(x)\, dx + c\right]}$$

This is the general soln, where c is an arbitrary constant.

Examples

Ex. $x\cos x \dfrac{dy}{dx} + y(x\sin x + \cos x) = 1$

or, $\dfrac{dy}{dx} + \dfrac{x\sin x + \cos x}{x\cos x}\, y = \dfrac{1}{x\cos x}$

It is linear eqn in y.

$$IF = e^{\int \frac{x\sin x + \cos x}{x\cos x}dx} = e^{\int \frac{\sin x}{\cos x}dx + \int \frac{dx}{x}}$$

$$= e^{-\log|\cos x| + \log|x|}$$

$$= x/\cos x, \quad x > 0$$

Examples

Multiplying by I.F $\Rightarrow$ we obtain

$$\frac{d}{dx}\left(y \cdot \frac{x}{\cos x}\right) = \frac{1}{x\cos x} \cdot \frac{x}{\cos x} = \sec^2 x$$

or, $$y \cdot \frac{x}{\cos x} = \int \sec^2 x \, dx + c$$

$$= \tan x + c$$

or, $$\boxed{xy = \sin x + c\cos x}$$ — This is general sol^n.

where c is an arbitrary constant.

Examples

Ex ② $y^2 dx + (x - \frac{1}{y}) dy = 0$ — This is nonlinear eqn in y.

or, $\frac{xy-1}{y} \frac{dy}{dx} = -y^2$

or, $\frac{dx}{dy} + \frac{1}{y^2} \cdot x = \frac{1}{y^3}$, which is a linear eqn in x.

Now IF $= e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$.

multiplying by IF, we have

$$\frac{d}{dy}\left(x e^{-\frac{1}{y}}\right) = e^{-1/y} \cdot \frac{1}{y^3}$$

$\Rightarrow x e^{-1/y} = e^{-1/y}\left(1 + \frac{1}{y}\right) + c$, or, $\boxed{x = 1 + \frac{1}{y} + c e^{1/y}}$

This is G.S.

where c is arbitrary constant.

Application of Linear ODE

③ Electric circuit Problem:

The initial value problem governing the current I flowing in a series RL circuit when a voltage $E(t) = t$ is applied, is given by

$$\boxed{L\frac{dI}{dt} + RI = E}, \quad t \geq 0, \text{ with}$$

initial condition $I(0) = 0$, where R and L are constants. Find the current $I(t)$ at time t.

electromotive force produces the voltage.

Application of Linear ODE

Sol^n:

The first order linear ODE is

$$L\frac{dI}{dt} + RI = E = t$$

$$\Rightarrow \boxed{\frac{dI}{dt} + \frac{R}{L} I = t/L}, \text{ linear eq}^n \text{ in } I$$

$$IF = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

Multiplying the given ODE by IF and integrating we have -

Application of Linear ODE

$$I\, e^{Rt/L} = \frac{1}{L}\int e^{Rt/L}\, t\, dt + c$$

or $\boxed{I(t) = \frac{1}{R}\left(t - L/R\right) + c\, e^{-Rt/L}}$.

Applying the initial condition $I(0)=0$, we have — $c = L/R^2$.

So the current $I(t)$ at time t is given by — $I(t) = t/R + \frac{L}{R^2}\left[e^{-Rt/L} - 1\right]$.

Non-linear Equation: Bernoulli Equation

- The Bernoulli eqⁿ is

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n},$$

where n is any real number. If $n=0$ or 1, then the equation is linear. For all other values of n, the equation is non-linear.

The given non-linear equation can be reduced to the linear form by change of the dependent variable.

We can rewrite the eqⁿ as

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \quad \left[\text{dividing by } y^n\right]$$

Examples

Let $z = y^{1-n}$, Then $\frac{dz}{dx} = (1-n) y^{-n} \frac{dy}{dx}$.

So, the eqn becomes

$$\frac{1}{1-n} \frac{dz}{dx} + Pz = Q$$

or, $$\boxed{\frac{dz}{dx} + (1-n) P z = (1-n) Q}$$

which is a linear eqn in z. Its IF is

$$e^{(1-n)\int P(x)\,dx}$$

Multiplying by IF, we obtain —

Examples

$$\frac{d}{dx}\left\{z \cdot e^{(1-n)\int P\,dx}\right\} = Q(1-n)\, e^{(1-n)\int P\,dx}$$

Integrating,

$$y^{1-n} = e^{-(1-n)\int P\,dx}\left[C + \int\left\{Q(1-n)\, e^{(1-n)\int P\,dx}\right\} dx\right]$$

This is the general solution.

Ex. $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

or, $\frac{dy}{dx} - xy = -e^{-x^2} y^3$

or, $\frac{1}{y^3}\frac{dy}{dx} - x \cdot \frac{1}{y^2} = -e^{-x^2}$. which is non-linear eq.

Examples

Let $z = \frac{1}{y^2}$ & $\frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$

$\frac{dz}{dx} + 2xz = 2e^{-x^2}$, This eqn is linear in z and .

IF is $e^{\int 2x\,dx} = e^{x^2}$

Multiplying by IF, we have

$$\frac{d}{dx}\left\{ze^{x^2}\right\} = 2$$

Integrating, $ze^{x^2} = 2x + c$

$\Rightarrow \boxed{\frac{1}{y^2} = (2x+c)e^{-x^2}}$ This is general solution, where c is an arbitrary constant.

Examples

Ex ② $(x^2y^3 + 2xy)\,dy = dx$

or, $\frac{dx}{dy} - 2y.x = y^3x^2$, which is non-linear in x.

or, $\frac{1}{x^2}\frac{dx}{dy} - 2y.\frac{1}{x} = y^3$

Put $\frac{1}{x} = z$ and the eqn reduces to

$$\frac{dz}{dy} + 2yz = -y^3$$, which is linear in z

$$IF = e^{\int 2y\,dy} = e^{y^2}$$

Examples

Multiplying by IF and integrating we have

$$z e^{y^2} = -\int y^3 e^{y^2} dy \quad (\text{Put } y^2 = u)$$

$$= -\frac{1}{2} e^{y^2}(y^2-1) + c$$

or, $$\boxed{\frac{1}{x} = \frac{1}{2}(1-y^2) + c e^{-y^2}}$$ → This is the general solution.

Application of Bernoulli Equation

Population Growth model (Logistic Equation):

The logistic eqn in the form of nonlinear Bernoulli eqn is given by—

$$\boxed{\frac{dy}{dx} = Ay - By^2}$$, where A and B are constants.

$$\text{or } \frac{dy}{dx} - Ay = -By^2$$

$$\text{or } -\frac{1}{y^2}\frac{dy}{dx} + A\frac{1}{y} = +B.$$

Application of Bernoulli Equation

Let $z = \frac{1}{y}$. $\Rightarrow \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

Therefore the eq^n becomes —

$$\boxed{\frac{dz}{dx} + Az = B} \rightarrow \text{linear in } z$$

$IF = e^{\int A\,dx} = e^{Ax}$.

Multiplying by the IF, we have

Application of Bernoulli Equation

$$z e^{Ax} = \frac{B}{A} e^{Ax} + c$$

or $$\frac{1}{y} = B/A + c\, e^{-Ax}$$

or $$\boxed{y = \frac{1}{c\, e^{-Ax} + B/A}}$$

This is the general solution.