Calculus 2: Multivariable Calculus

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Multivariable caclulus, as the name suggests, is a study calculus of functions $f : \mathbb{R}^m \to \mathbb{R}^n$ where $m > 1$ and $n \geq 1$. In this course however, we shall be working with real valued functions in two variables only, namely functions $f : \mathbb{R}^2 \to \mathbb{R}$. We shall look at sequences of elements in $\mathbb{R}^2$, continuity, limits, differentiability, partial derivative and local maxima/ minima of functions as mentioned above during course. Let us begin our study by introducing you to the notion of an m-dimensional Euclidean space and norm of elements in an m-dimensional Euclidean space.

1. Euclidean space and norms

Let m be a positive integer. An m-dimensional Euclidean space is the set $\mathbb{R}^m$ given by

$$\mathbb{R}^m := \{(x_1, \dots, x_m) : x_i \in \mathbb{R} \text{ for } i = 1, \dots, m\}.$$

If $m = 1$, then an element of $\mathbb{R}^1 := \mathbb{R}$ is said to be a **scalar** and if $m \geq 2$, then an element of $\mathbb{R}^m$ is called a **vector**. As mentioned above, we shall define the notions of Euclidean spaces, functions from subsets of $\mathbb{R}^m$ to $\mathbb{R}$, and we shall study their properties. However, in most cases, we shall restrict ourselves to the case when $m = 2$.

Let us first see some algebraic operations on vectors. For $\mathbf{x} := (x_1, \dots, x_m)$, $\mathbf{y} := (y_1, \dots, y_m) \in \mathbb{R}^m$ and $a \in \mathbb{R}$, we define

Sum of two vectors: $\mathbf{x} + \mathbf{y} := (x_1 + y_1, \dots, x_m + y_m)$

Scalar multiple of a vector: $a\mathbf{x} := (ax_1, \dots, ax_m)$

The two algebraic properties make $\mathbb{R}^m$ a *vector space*, a notion that you will be introduced to in your Linear Algebra course.

For $\mathbf{x} := (x_1, \dots, x_m) \in \mathbb{R}^m$, we define the **norm** of $\mathbf{x}$ by

$$||\mathbf{x}|| := \sqrt{x_1^2 + \cdots + x_m^2}.$$

It follows that when $m = 1$, then the norm of an element x is just its absolute value, i.e. $|\mathbf{x}|$. Note that, we have the following useful inequalities:

$$\max\{|x_1|, \dots, |x_m|\} \leq ||\mathbf{x}|| \leq |x_1| + \cdots + |x_m|. \tag{1}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||. \quad \text{(Triangle inequality)} \tag{2}$$

It follows rather trivially that for $\mathbf{x} \in \mathbb{R}^m$, $||\mathbf{x}|| = 0 \iff \mathbf{x} = \mathbf{0}$, where $\mathbf{)} = (0, \dots, 0)$. For $\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, we define the **dot product** of $\mathbf{x}$ and $\mathbf{y}$ as:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + \cdots + x_my_m.$$

Note that dot product of two vectors is a scalar. To this end, we have the following inequality:

$$||\mathbf{x} \cdot \mathbf{y}|| \leq ||\mathbf{x}||||\mathbf{y}||. \quad \text{(Cauchy-Schwarz inequality)} \tag{3}$$

We say that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ are **perpendicular** (to each other) or **orthogonal** if $\mathbf{x}\cdot\mathbf{y} = 0$. Furthermore, if $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$, then the angle between them is the unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}.$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. We define the **distance** between $\mathbf{x}$ and $\mathbf{y}$ to be $||\mathbf{x} - \mathbf{y}||$. When $m = 1$, this is just $|x - y|$. Lets fix a point $\mathbf{x}_0 \in \mathbb{R}^m$. For $r > 0$, the set

$$B(\mathbf{x}_0, r) := \{\mathbf{x} \in \mathbb{R}^m : ||\mathbf{x} - \mathbf{x}_0|| < r\}$$

is called the **neighbourhood** of $\mathbf{x}_0$ of radius r. For $m = 1$, and a point $x_0 \in \mathbb{R}$, the neighbourhood of x_0 of radius r is simply the open interval $(x_0 - r, x_0 + r)$. What happens when $m = 2$? If $\mathbf{x}_0 = (x_0, y_0)$, then

$$B(\mathbf{x}_0, r) := \{(x, y) \in \mathbb{R}_2 : (x - x_0)^2 + (y - y_0)^2 < r\}.$$

This object is called an **open disk** of radius r around $\mathbf{x}_0$. You may see that one could easily find a description of $B(\mathbf{x}_0, r)$ even for higher values of m.

2. Sequence in $\mathbb{R}^2$ and its limit

A **sequence** in $\mathbb{R}^2$ is a function from $\mathbb{N} \to \mathbb{R}^2$. We denote the n-th term of such a sequence by (x_n, y_n) and the sequence itself by $((x_n, y_n))$. A sequence $((x_n, y_n))$ in $\mathbb{R}^2$ is said to be **convergent** if both the *real sequences* (x_n) and (y_n) converge. In such a case we write,

$$(x_n, y_n) \to (x_0, y_0) \quad \iff \quad x_n \to x_0 \text{ and } y_n \to y_0 \quad \text{as } n \to \infty.$$

For a sequence $((x_n, y_y))$ in $\mathbb{R}^2$, we derive using inequality 1 that

$$\max\{|x_n - x_0|, |y_n - y_0|\} \leq ||(x_n, y_n) - (x_0, y_0)|| \leq |x_n - x_0| + |y_n - y_0|$$

for all $n \in \mathbb{N}$. In particular, this shows that

$$(x_n, y_n) \to (x_0, y_0) \quad \iff \quad ||(x_n, y_n) - (x_0, y_0)|| \to 0. \tag{4}$$

In particular, we have established that the convergence of sequences in $\mathbb{R}^2$ is equivalent to convergence of a sequence of real numbers. Keeping this mind, we shall not spend much time on sequences. We conclude with a remark that the definition of convergence of sequence in $\mathbb{R}^2$ can be easily generalized to convergence of sequences in $\mathbb{R}^m$ for arbitrary m.

3. Functions of several variables

Let $D \subset \mathbb{R}^2$ and let f be a real valued function defined on D, i.e. $f : D \to \mathbb{R}^2$. Let us introduce a few terminologies now.

Definition 3.1.

- The **graph of** f is defined as the set
$$\{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in D\}.$$
Geometrically, it is the *surface* $z = f(x, y)$ in $\mathbb{R}^3$.
- Let $c \in \mathbb{R}$. The **contour line** of f (corresponding to c) is the subset
$$\{(x, y, c) \in \mathbb{R}^3 : (x, y) \in D \text{ and } f(x, y) = c\}.$$
Geometrically, it is the intersection of the graph of f with the plane $z = c$ in $\mathbb{R}^3$.

- The **level curve of** f corresponding to c is the subset

$$\{(x, y) \in D : f(x, y) = c\}.$$

  Of course the level curve of f is a subset of $\mathbb{R}^2$ and in particular, a subset of the domain D of the function f. Remember that, in single variable calculus, we are used to with thinking about subsets of x- axis as the domain of the function. Keeping this in mind, it is natural to think about the level curves as a subset of xy plane. In this case, it is just the projection of the contour line on xy plane.

4. Continuity

Even though we chose not to spend more time on sequences in $\mathbb{R}^2$ (thanks to reducing the problem of their convergence to the problem of convergence of sequence of real numbers, that you are perhaps master of by now), it is very useful in general. In fact, in what follows, we shall define continuity of real valued functions in two variables in terms of convergence of sequences.

DEFINITION 4.1. Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. We say that f is **continuous at** (x_0, y_0) if

$$(x_n, y_n) \to (x_0, y_0) \ \ (\text{in } \mathbb{R}^2) \quad \Longrightarrow \quad f(x_n, y_n) \to f(x_0, y_0) \quad (\text{in } \mathbb{R}), \tag{5}$$

for every sequence $((x_n, y_n))$ in D.

We say that a function is **continuous in** D if f is continuous at all points of D and **discontinuous** if there exists a point in D where it is not continuous[1]. At any rate, there are a couple of ways in which you will claim that a function is discontinuous:

- Find one sequence $((x_n, y_n)) \to (x_0, y_0)$ such that $f(x_n, y_n) \not\to f(x_0, y_0)$.
- Two different sequences $((x_n, y_n))$ and $((z_n, w_n))$ converging to (x_0, y_0) but

$$\lim_{n\to\infty} f(x_n, y_n) \neq \lim_{n\to\infty} f(z_n, w_n).$$

  Remember the uniqueness of limit of a sequence?

Question that you should think about: What happens if there are no sequence $((x_n, y_n)) \to (x_0, y_0)$ in D other than an *eventually constant* sequence, i.e. a sequence $((x_n, y_n))$ such that for all sufficiently large n, we have $(x_n, y_n) = (x_0, y_0)$?

PROPOSITION 4.2 (Properties of continuous functions). *Let $D \subset \mathbb{R}$, $f, g : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. If f, g are continuous at (x_0, y_0), then*

(a) *$f + g$ is continuous at (x_0, y_0)*
(b) *fg is continuous at (x_0, y_0)*
(c) *if $g(x_0, y_0) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at (x_0, y_0).*

EXERCISE 4.3. Prove or disprove that

(a) If $f : D \to \mathbb{R}$ is continuous at $(x_0, y_0) \in D$, so is $|f|$.
(b) If $f, g : D \to \mathbb{R}$ are continuous at (x_0, y_0), then so are $\max\{f, g\}$ and $\min\{f, g\}$.

EXAMPLE 4.4. Using Proposition 4.2 we could show the following:

(a) A **bivariate polynomial** $f(x, y)$ is continuous on $\mathbb{R}^2$.
(b) If $f(x, y)$ and $g(x, y)$ are two bivariate polynomials, and $g(x_0, y_0) \neq 0$, then the **rational function** $\frac{f(x,y)}{g(x,y)}$ is continuous at (x_0, y_0).

[1]You will see this kind of a sentence in many parts of mathematics. If you have a function and if you are studying a P of the function at every point, you will say that the function has property P if it satisfies P at every point in the domain. Also, you will say that the function does not have the property P if there exists at least one point where the function fails to satisfy P

PROPOSITION 4.5. *Let $D \subset \mathbb{R}^2$ and $E \subset \mathbb{R}$. Suppose that $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are functions such that $f(D) \subset E$. If f is continuous at $(x_0, y_0) \in D$ and g is continuous at $f(x_0, y_0) \in E$, then the* ***composite function*** *$g \circ f : D \to \mathbb{R}$ is continuous at (x_0, y_0).*

EXAMPLE 4.6. Let $p(x, y)$ be a bivariate polynomial function, then $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = e^{p(x,y)}$$

is continuous on $\mathbb{R}^2$.

EXERCISE 4.7. Show that the following functions are continuous:

(a) $e^{\frac{x-y}{x^2+y^2+1}}$.
(b) $\ln(x^2y^2 + 1)$.

THEOREM 4.8 (ϵ-δ condition). *Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ and $f : D \to \mathbb{R}$ be a function. Then f is continuous at (x_0, y_0) if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x, y) - f(x_0, y_0)| < \epsilon$$

for all $(x, y) \in D$, whenever $||(x, y) - (x_0, y_0)|| < \delta$.

EXERCISE 4.9. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is continuous on $\mathbb{R}^2$. Use this fact to show that the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x, y) = \sqrt{x^2 + y^2}$$

is continuous on $\mathbb{R}^2$.

EXAMPLE 4.10. In our lecture videos we worked out the following examples:

(a) If
$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0), \end{cases}$$
then f is NOT continuous on $\mathbb{R}^2$.

(b) If
$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0), \end{cases}$$
then f is continuous on $\mathbb{R}^2$.

(c) If
$$f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0), \end{cases}$$
then f is NOT continuous on $\mathbb{R}^2$.

(d) If
$$f(x, y) = \begin{cases} \frac{x^3y}{x^4+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0), \end{cases}$$
then f is continuous on $\mathbb{R}^2$.

EXERCISE 4.11. Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ and $f : D \to \mathbb{R}$ be continuous at (x_0, y_0). If $f(x_0, y_0) > 0$, show that there exists $\delta > 0$ such that $f(x, y) > 0$ for $(x, y) \in D$ satisfying $||(x, y) - (x_0, y_0)|| < \delta$.

EXERCISE 4.12. *Discuss the continuity of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$.*

(a)
$$f(x,y) = \begin{cases} \frac{x^3y}{x^6+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases}$$

(b)
$$f(x,y) = \begin{cases} \frac{x^3y-xy^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases}$$

(c)
$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases}$$

5. Limits of functions

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$. Suppose that (x_0, y_0) is a **limit point**[2] of D. We say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists if there is $\ell \in \mathbb{R}$ such that $f(x_n, y_n) \to \ell$ for all sequences $((x_n, y_n))$ in D satisfying:

(a) $(x_n, y_n) \neq (x_0, y_0)$ for all $n \in \mathbb{N}$
(b) $(x_n, y_n) \to (x_0, y_0)$ as $n \to \infty$.

Note that, while defining continuity of a function $f : D \to \mathbb{R}$ at $(x_0, y_0) \in D$ we needed the function to be defined at (x_0, y_0). However, this is not required for defining limit of a function $f(x)$ as $(x, y) \to (x_0, y_0)$.

Example 5.1.

(a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = \sin(xy)$, then
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$
exists for all $(x_0, y_0) \in \mathbb{R}^2$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$
Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does NOT exist.

Now let us relate the concepts of limits and continuity of functions of two variables. Let $D \subset \mathbb{R}^2$. Suppose tht (x_0, y_0) be a limit point of D. Then

$f : D \to \mathbb{R}$ is continuous at (x_0, y_0) $\iff$ the limit $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and it is equal to $f(x_0, y_0)$.

Thus for f to be continuous at a limit point (x_0, y_0) of D, we need

(i) $f(x_0, y_0)$ to be defined
(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

[2] this means that every neighbourhood of (x_0, y_0) intersects D at at least one point other than (x_0, y_0). As such limit points do not need to be an element of the set.

Theorem 5.2 (Left as exercise). Let $D \subset \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$ a limit point and $f, g : D \to \mathbb{R}$ are real valued functions. Suppose that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell_f \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = \ell_g.$$

We have,

(a) $\lim_{(x,y)\to(x_0,y_0)} (f \pm g) = \ell_f \pm \ell_g$,

(b) $\lim_{(x,y)\to(x_0,y_0)} (f \cdot g) = \ell_f \cdot \ell_g$,

(c) if $\ell_g \neq 0$, then $\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{\ell_f}{\ell_g}$,

(d) if $f(x,y) \leq g(x,y)$ for all $(x,y) \neq (x_0,y_0)$ near (x_0,y_0), then $\ell_f \leq \ell_g$,

(e) (Sandwich Theorem) if $f(x,y) \leq g(x,y) \leq h(x,y)$ for all $(x,y) \neq (x_0,y_0)$ near (x_0,y_0) and $\ell_f = \ell = \ell_h$, then $\lim_{(x,y)\to(x_0,y_0)} h(x,y) = \ell$.

Theorem 5.3 (ϵ-δ condition). *Let $D \subset \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$ a limit point and $f : D \to \mathbb{R}$ a real valued function. Let ℓ be a real number. Then*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$$

iff the following ϵ-δ condition is satisfied: For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $(x,y) \in D$

$$0 < ||(x,y) - (x_0,y_0)|| < \delta \qquad \Longrightarrow \qquad |f(x,y) - \ell| < \epsilon.$$

Example 5.4. Let $f, g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x^2 y}{x^2 + y^2} \quad \text{and} \quad g(x,y) = y \sin \frac{1}{x^2 + y^2}.$$

Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = \lim_{(x,y)\to(0,0)} g(x,y).$$

When you learnt limits of functions in one variable, you saw that $\lim_{x\to c} f(x)$ exists iff $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ exist and they are equal. Here is a higher dimensional analogue for the same.

Proposition 5.5 (Two path test for nonexistence of a limit). *If a function $f(x,y)$ has different limits along two different paths as (x,y) approaches (x_0,y_0), then the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.*

The above proposition can be proved easily using convergence of sequences, but we skip the proof. But here is an example of how to use it.

Example 5.6. The function

$$f(x,y) = \frac{2x^2 y}{x^4 + y^2}$$

has no limits as (x,y) approches $(0,0)$. To see this we let (x,y) approach $(0,0)$ along the parabola $y = kx^2$, with $x \neq 0$. Note that

$$f(x, kx^2) = \frac{2kx^4}{x^4 + k^2 x^4} = \frac{2k}{1 + k^2}.$$

Therefore, the limit when we let (x,y) approach $(0,0)$ along the parabolas $y = kx^2$ for different values of k, changes with k. The above proposition implies that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

This apparantly new technique was not discussed in class. In principle, if you know this vaguely you could produce two sequences $((x_n, y_n)), ((z_n, w_n)) \to (x_0, y_0)$ such that

$$\lim_{(x_n,y_n)\to(x_0,y_0)} f(x_n, y_n) \neq \lim_{(z_n,w_n)\to(x_0,y_0)} f(z_n, w_n).$$

Let us try to illustrate this fact with the above example. Instead of letting (x, y) approach $(0, 0)$ along the parabolas $y = kx^2$, we take a sequence $(x_n, y_n) = (\frac{1}{n}, \frac{k}{n^2})$ on the parabola $y = kx^2$. Clearly, $(x_n, y_n) \to (0, 0)$, but

$$f(x_n, y_n) = \frac{2k}{1 + k^2}.$$

Thus $f(x_n, y_n)$ is a constant sequence $2k/(1+k^2)$ and hence converge to the same quantity. Note that for different values of k we have produced sequences converging to $(0, 0)$ but the sequence of functional values converging to different quantities. Hence the limit does not exist.

Exercise 5.7. *Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ be defined as in the following exercises. In each of the exercises, find $\lim_{(x,y)\to(0,0)} f(x, y)$ or show that the limit does not exist.*

(a) $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$.
(b) $f(x, y) = \frac{x^4-y^2}{x^4+y^2}$.
(c) $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$.
(d) $f(x, y) = \frac{x+y}{2+\cos x}$.
(e) $f(x, y) = \frac{x}{\sqrt{x^2+y^2}}$.
(f) $f(x, y) = \frac{x^3-xy^2}{x^2+y^2}$.
(g) $f(x, y) = \frac{x^2+\sin^2 y}{2x^2+y^2}$.

6. Partial Derivatives

As mentioned in our lectures, introduction to partial derivatives of functions in two variables mark our first baby-step towards understanding the notion of differentiability of real valued functions in two variables.

Definition 6.1. Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. A function $f : D \to \mathbb{R}$ iis said to have a **partial derivative with respect to** x at (x_0, y_0) if

$$\lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

exists, and in such a case it is denoted by $f_x(x_0, y_0)$ or by $\frac{\partial f}{\partial x}(x_0, y_0)$.

The above definition makes sense if there exists $r > 0$ such that the horizontal line segment

$$\{(x, y_0) : x \in (x_0 - r, x_0 + r)\} \subset D.$$

Geometric meaning of partial derivative with respect to x: Let C_{y_0} denote the curve obtained by intersecting the graph of f by the plane $y = y_0$. Then $f_x(x_0, y_0)$ is the **slope of the tangent** to the curve C_{y_0} at (x_0, y_0).

Computationally, $f_x(x_0, y_0)$ can be obtained by differentiating f with respect to x at x_0 treating y as the constant y_0.

Definition 6.2. Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. A function $f : D \to \mathbb{R}$ iis said to have a **partial derivative with respect to** y at (x_0, y_0) if

$$\lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{h}$$

exists, and in such a case it is denoted by $f_y(x_0, y_0)$ or by $\frac{\partial f}{\partial y}(x_0, y_0)$.

The above definition makes sense if there exists $r > 0$ such that the horizontal line segment

$$\{(x_0, y) : y \in (y_0 - r, y_0 + r)\} \subset D.$$

Geometric meaning of partial derivative with respect to y: Let C_{x_0} denote the curve obtained by intersecting the graph of f by the plane $x = x_0$. Then $f_y(x_0, y_0)$ is the **slope of the tangent** to the curve C_{x_0} at (x_0, y_0).

Computationally, $f_y(x_0, y_0)$ can be obtained by differentiating f with respect to y at y_0 treating x as the constant x_0.

The rules for computing partial derivatives of sums, products, quotients and compositions of functions of two variables are similar to that for derivatives of functions in one variable. So we omit the discussion here.

DEFINITION 6.3. If the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ of f exists at (x_0, y_0), then

$$(\nabla f)(x_0, y_0) := (f_x(x_0, y_0), f_y(x_0, y_0))$$

is called the **gradient** of f at (x_0, y_0).

EXAMPLE 6.4. The following examples were worked out in the lecture videos:

(a) Let $f(x, y) = x^2 + y^2$. Then $f_x(x_0, y_0) = 2x_0$ and $f_y(x_0, y_0) = 2y_0$.

(b) If $f(x, y) = \sqrt{x^2 + y^2}$, then $f_x(0, 0)$ and $f_y(0, 0)$ do not exist. However, the f is continuous.

(c) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. But we have seen that f is not continuous. This shows that existence of both the partial derivatives of a function is not sufficient for the function to be continuous.

EXERCISE 6.5. Discuss the continuity and existence of partial derivatives of the following functions at the point $(0, 0)$:

(a)

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(b)

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(c)

$$f(x, y) = \begin{cases} x \sin(1/x) + y \sin(1/y) & \text{if } x \neq 0, y \neq 0 \\ x \sin(1/x) & \text{if } x \neq 0, y = 0 \\ y \sin(1/y) & \text{if } x = 0, y \neq 0 \\ 0 & \text{if } x = 0, y = 0 \end{cases}$$

7. Higher order partial derivatives

- Let $D \subset \mathbb{R}^2$ and $f : D \to R$. Suppose that $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to x at (x_0, y_0), then it is denoted by $f_{xx}(x_0, y_0)$ or $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.
- Let $D \subset \mathbb{R}^2$ and $f : D \to R$. Suppose that $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to y at (x_0, y_0), then it is denoted by $f_{xy}(x_0, y_0)$ or $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.
- Let $D \subset \mathbb{R}^2$ and $f : D \to R$. Suppose that $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to x at (x_0, y_0), then it is denoted by $f_{xx}(x_0, y_0)$ or $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.
- Let $D \subset \mathbb{R}^2$ and $f : D \to R$. Suppose that $f_y(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_y : D \to \mathbb{R}$ has a partial derivative with respect to y at (x_0, y_0), then it is denoted by $f_{yy}(x_0, y_0)$ or $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$.

EXAMPLE 7.1. This example shows that, in general, the **mixed partial derivatives** $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ may not be equal. Let

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

We can easily see that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

The following theorem, often referred to as Clairaut's Theorem, gives us a sufficient condition for the mixed partial derivatives to be equal.

THEOREM 7.2 (Mixed partials theorem). *Suppose that $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ and $f : D \to \mathbb{R}$. If $f, f_x, f_y, f_{xy}, f_{yx}$ are*

- *defined in an open neighbourhood of (x_0, y_0) and*
- *are continuous at (x_0, y_0),*

then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

One could go on and define even higher order partial derivatives, but for the purpose of this course, we shall not discuss such issues here.

8. Differentiability

8.1. Differentiation and total derivatives.

DEFINITION 8.1. Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ be an interior point of D and $f : D \to \mathbb{R}$. We say that f is **differentiable at** (x_0, y_0) if there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2 + k^2}} = 0.$$

In such a case, (α, β) is called the **total derivative of** f **at** (x_0, y_0).

- Letting $(h, k) \to (0, 0)$ along the x-axis, we see that $\alpha = f_x(x_0, y_0)$.
- Letting $(h, k) \to (0, 0)$ along the x-axis, we see that $\beta = f_y(x_0, y_0)$.

As a consequence of the above, we see that the total derivative of f at (x_0, y_0) is given by

$$(\alpha, \beta) = (f_x(x_0, y_0), f_y(x_0, y_0)) = \nabla f(x_0, y_0).$$

Let us now consider a two dimensional analogue of Carathéodory's Lemma.

PROPOSITION 8.2 (Increment Lemma). *Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ be an interior point of D and $f : D \to \mathbb{R}$. Then f is differentiable at (x_0, y_0) iff there exist functions $f_1, f_2 : D \to \mathbb{R}$ such that f_1, f_2 are continuous at (x_0, y_0) and for all $(x, y) \in D$,*

$$f(x,y) - f(x_0,y_0) = (x - x_0)f_1(x,y) + (y - y_0)f_2(x,y). \tag{6}$$

A pair of functions (f_1, f_2) as in (6) is called a **pair of increment functions** associated with the function f and the point (x_0, y_0). Unlike the increment function that appears in the Carathéodory's Lemma, such a pair of increment functions is not unique. Indeed, if $g : D \to \mathbb{R}$ is any function that is continuous at (x_0, y_0), the pair (g_1, g_2) is also a pair of increment functions, where

$$g_1(x,y) := f_1(x,y) + (y - y_0)g(x,y) \quad \text{and} \quad g_2(x,y) := f_2(x,y) - (x - x_0)g(x,y).$$

COROLLARY 8.3. *Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ be an interior point of D and $f : D \to \mathbb{R}$. If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).*

PROPOSITION 8.4 (Algebraic properties). *Let $D \subset \mathbb{R}^2$, (x_0, y_0) an interior point of D and suppose that $f, g : D \to \mathbb{R}$ are differentiable at (x_0, y_0). Then $f \pm g$ an fg are differentiable at (x_0, y_0). Moreover,*

$$\nabla(f \pm g)(x_0,y_0) = \nabla f(x_0,y_0) \pm \nabla g(x_0,y_0) \tag{7}$$

$$\nabla(fg)(x_0,y_0) = f(x_0,y_0)\nabla g(x_0,y_0) + g(x_0,y_0)\nabla f(x_0,y_0) \tag{8}$$

Furthermore, if $g(x_0, y_0) \neq 0$, then $\frac{f}{g}$ is differentiable at (x_0, y_0) and

$$\nabla\left(\frac{f}{g}\right)(x_0,y_0) = \frac{g(x_0,y_0)\nabla f(x_0,y_0) - f(x_0,y_0)\nabla g(x_0,y_0)}{g(x_0,y_0)^2}. \tag{9}$$

PROPOSITION 8.5 (Chain Rule). Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ be an interior point of D and $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0).

(a) Let $E \subset \mathbb{R}$ such that $f(D) \subset E$ and let $z_0 := f(x_0, y_0)$ be an interior point of E. If $g : E \to \mathbb{R}$ is differentiable at z_0, then the function $h := g \circ f : D \to \mathbb{R}$ is differentiable at (x_0, y_0). In such a case,

$$h_x(x_0,y_0) = g'(z_0)f_x(x_0,y_0) \quad \text{and} \quad h_y(x_0,y_0) = g'(z_0)f_y(x_0,y_0) \tag{10}$$

(b) Let $E \subset \mathbb{R}$ and $t_0 \in E$ an interior point of E. If $x, y : E \to \mathbb{R}$ are differentiable at t_0 and if $(x(t), y(t)) \in D$ for all $t \in E$ and $(x(t_0), y(t_0)) := (x_0, y_0)$, then the function $\phi : E \to \mathbb{R}$ defined by $\phi(t) := f(x(t), y(t))$ for $t \in E$ is differentiable at t_0. In such a case

$$\phi'(t_0) = f_x(x_0,y_0)x'(t_0) + f_y(x_0,y_0)y'(t_0). \tag{11}$$

(c) Let $E \subset \mathbb{R}^2$ and let (u_0, v_0) be an interior point of E. If $x, y : E \to \mathbb{R}$ are differentiable at (u_0, v_0), and if $(x(u, v), y(u, v)) \in D$ for all $(u, v) \in E$ and $(x(u_0, v_0), y(u_0, v_0)) = (x_0, y_0)$, then the function $F : E \to \mathbb{R}$ defined by $F(u, v) := f((x(u, v), y(u, v))$ is differentiable at (u_0, v_0). In such a case,

$$F_u(u_0,v_0) = f_x(x_0,y_0)x_u(u_0,v_0) + f_y(x_0,y_0)y_u(u_0,v_0) \tag{12}$$

and

$$F_v(u_0,v_0) = f_x(x_0,y_0)x_v(u_0,v_0) + f_y(x_0,y_0)y_v(u_0,v_0). \tag{13}$$

It is often helpful to write the identities (10), (11), (12) and (13) in the following informal but suggestive way :

(a) If $z = f(x,y)$ and $w = g(z)$, then w is a function of (x,y) and
$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y}.$$
(b) If $z = f(x,y)$ and if $x = x(t), y = y(t)$, then z is a function of t and
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
(c) If $z = f(x,y)$ and if $x = x(u,v)$, $y = y(u,v)$, then z is a function of u and v, and
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

It should be noted that the above identities should are valid when the concerned (partial) derivatives are evaluated at relevant points and when the required conditions as mentioned in the corresponding parts of Proposition 8.5 are satisfied.

8.2. Directional derivatives and differentiability. Let $D \subset \mathbb{R}^2$, (x_0, y_0) an interior point of D and let $\mathbf{u} := (u_1, u_2)$ be a unit vector, i.e. $||\mathbf{u}|| = 1$.

Definition 8.6. A function $f : D \to \mathbb{R}$ is said to have a **directional derivative along u** at (x_0, y_0) if
$$\lim_{t\to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$
exists, and in such a case it is denoted by $D_{\mathbf{u}}f(x_0, y_0)$.

This definition also makes sense if there is $r > 0$ such that the (slanted) line segment
$$\{(x_0 + tu_1, y_0 + tu_2) : t \in (-r, r)\} \subset D.$$

If $\mathbf{u} = (1,0)$, then $D_{\mathbf{u}}f(x_0,y_0) = f_x(x_0,y_0)$ and similarly if $\mathbf{u} = (0,1)$, then $D_{\mathbf{u}}f(x_0,y_0) = f_y(x_0,y_0)$.

Example 8.7.

(a) Let $f(x,y) = x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$. Then for any $(x_0,y_0) \in \mathbb{R}^2$ and any unit vector $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ we have
$$D_{\mathbf{u}}f(x_0,y_0) = 2x_0u_1 + 2y_0u_2.$$
One readily checks that $D_{\mathbf{u}}f(x_0,y_0) = \mathbf{u} \cdot \nabla f(x_0,y_0)$.

(b) Let $f : \mathbb{R}^2 \to R$ be given by
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$
We have seen that $D_{\mathbf{u}}f(0,0)$ exists iff $u_1 = 0$ or $u_2 = 0$.

(c) Let $f : \mathbb{R}^2 \to R$ be given by
$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$
It was proved in the lectures that $D_{\mathbf{u}}f(0,0) = 0$ if $u_2 = 0$ and u_1^2/u_2 if $u_2 \neq 0$. One checks that $D_{\mathbf{u}}f(x_0,y_0) = \mathbf{u} \cdot \nabla f(x_0,y_0)$ iff $u_1 = 0$ or $u_2 = 0$.

Theorem 8.8. *If $f : D \to \mathbb{R}$ is differentiable at (x_0,y_0), then $D_{\boldsymbol{u}}f(x_0,y_0)$ exists for all unit vectors $\boldsymbol{u} \in \mathbb{R}^2$. In such a case, we have $D_{\boldsymbol{u}}f(x_0,y_0) = \boldsymbol{u} \cdot \nabla f(x_0,y_0)$.*

Necessary conditions for differentiability: Here is a summary of all the necessary conditions for differentiability to hold. In particular, we should keep an eye on these conditions while testing for differentiability of a function. For example, if one of the following conditions fail, then we deduce that the function at hand is not differentiable. To this end, as always, let $D \subset \mathbb{R}^2$, (x_0, y_0) an interior point of D and $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0). Then

(1) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.
(2) $D_{\mathbf{u}} f(x_0, y_0)$ exists for every unit vectors $\mathbf{u}$.
(3) $D_{\mathbf{u}} f(x_0, y_0) = \mathbf{u} \cdot \nabla f(x_0, y_0)$ every unit vectors $\mathbf{u}$.
(4) f is continuous at (x_0, y_0).

Finally, we give a sufficient condition for differentiability.

THEOREM 8.9. *Let $D \subset \mathbb{R}^2$, (x_0, y_0) an interior point of D and $f : D \to \mathbb{R}$. If the partial derivatives f_x and f_y exist on an open set containing (x_0, y_0) and they are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).*

EXERCISE 8.10. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ below, determine whether the directional derivatives $D_{\mathbf{u}} f(0, 0)$ exists for every unit vector $\mathbf{u}$ in $\mathbb{R}^2$. If it does, then check whether $D_{\mathbf{u}} f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u}$ for every unit vector $\mathbf{u}$ in $\mathbb{R}^2$. Finally determine whether f is differentiable at $(0, 0)$.

(a)
$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

(b)
$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

(c)
$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

EXERCISE 8.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2 + y^2} & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0. \end{cases}$$

Show that

(a) f is continuous at $(0, 0)$.
(b) $f_x(0, 0)$ and $f_y(0, 0)$ exist.
(c) $D_{\mathbf{u}} f(0, 0)$ exists for every unit vector $\mathbf{u}$ in $\mathbb{R}^2$.
(d) f is not differentiable at $(0, 0)$.

EXERCISE 8.12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} (x^2 + y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Show that

(a) f is continuous at $(0, 0)$.
(b) $f_x(0, 0)$ and $f_y(0, 0)$ exist, but are unbounded in every neighbourhood of $(0, 0)$.
(c) f is differentiable at $(0, 0)$.

9. Maxima and Minima

Let $D \subset \mathbb{R}^2$. We say that D is **bounded** if there exists $\alpha > 0$ such that $||(x, y)|| \le \alpha$ for all $(x, y) \in D$. The subset $D \subset \mathbb{R}^2$ is said to be **closed** if every sequence in D that converges in $\mathbb{R}^2$ converges in D. Namely, D is closed if the following holds:

Let $((x_n, y_n))$ be a sequence in D, i.e. $(x_n, y_n) \in D$ for all $n \in \mathbb{N}$. Then

$$(x_n, y_n) \to (x_0, y_0) \implies (x_0, y_0) \in D.$$

We look at a two-dimensional analogue of the *extreme value property* of real valued continuous functions on a closed and bounded interval, that you may have come across in Calculus 1.

THEOREM 9.1. *Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a continuous function. If D is a nonempty closed and bounded subset, then f is bounded and f attains its bounds in D. That is, there exist $m, M \in \mathbb{R}$ such that $m \le f(x, y) \le M$ for all $(x, y) \in D$. Furthermore, there exist $(x_1, y_1), (x_2, y_2) \in D$ such that $f(x_1, y_1) = m$ and $f(x_2, y_2) = M$.*

DEFINITION 9.2. Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(a, b) \in D$.

(a) f is said to have a **local maxima** at (a, b) if there exists $r > 0$ such that

$$f(a, b) \ge f(x, y) \quad \text{for all} \quad (x, y) \in D \cap B((a, b), r).$$

(b) f is said to have a **local minima** at (a, b) if there exists $r > 0$ such that

$$f(a, b) \le f(x, y) \quad \text{for all} \quad (x, y) \in D \cap B((a, b), r).$$

(c) f is said to have a **global maxima** at (a, b) if

$$f(a, b) \ge f(x, y) \quad \text{for all} \quad (x, y) \in D.$$

(d) f is said to have a **global minima** at (a, b) if

$$f(a, b) \le f(x, y) \quad \text{for all} \quad (x, y) \in D.$$

One readily verifies that if f has a global maxima (resp. minima) at (a, b), then f has a local maxima (resp. minima) at (a, b). In what follows, we shall only be interested in local maxima/minima of a function. We begin with a necessary condition for a function to have a local maxima/ minima at a point (a, b).

PROPOSITION 9.3. *Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(a, b) \in D$. Suppose that f has a local maxima/minima at (a, b). If $f_x(a, b)$ and $f_y(a, b)$ exist, then*

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

The above proposition implies that if f has a local maxima/minima at (a, b), then there are two possibilities:

- $\nabla f(a, b)$ does not exist or
- $\nabla f(a, b)$ exists and $\nabla f(a, b) = (0, 0)$.

A point $(a, b) \in D$ satisfying one of those conditions is called a **critical point**.

However, as we have seen, the converse of the Proposition 9.3 is not true. In particular, we have the following example:

EXAMPLE 9.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2$. We have seen that $(0, 0)$ is a critical point of f but it is neither a local maxima, nor a local minima of f. Such a point is called a **saddle point** of f.

THEOREM 9.5 (Second derivative test). *Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(a, b) \in D$ an interior point of D. Suppose f satisfies the following conditions:*

(a) *f_x and f_y exist at (a, b) and both are continuous in an open neighbourhood of (a, b),*

(b) $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ *exist at* (a, b) *and are continuous in an open neighbourhood of* (a, b), *and*
(c) $\nabla f(a, b) = (0, 0)$.

We define $H(a, b) = f_x x(a, b) f_y y(a, b) - (f_{xy}(a, b))^2$. *We have*
(i) *if* $H(a, b) > 0$ *and* $f_{xx}(a, b) < 0$, *then* f *has a local maxima at* (a, b).
(ii) *if* $H(a, b) > 0$ *and* $f_{xx}(a, b) > 0$, *then* f *has a local minima at* (a, b).
(iii) *if* $H(a, b) < 0$, *then* f *has a saddle point at* (a, b).

Exercise 9.6. Find all the local maxima, local minima and saddle points of the functions given below:
(a) $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$.
(b) $f(x, y) = x^2 + xy + 3x + 2y + 5$.
(c) $f(x, y) = x^3 - y^3 - 2xy + 6$.
(d) $f(x, y) = 9x^3 + y^3/3 - 4xy$.
(e) $f(x, y) = 4xy - x^4 - y^4$.

Exercise 9.7. Determine all the values of k such that the second derivative test guarantees that the function $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at $(0, 0)$.

Theorem 9.8 (Lagrange's Multiplier Theorem). *Let* $D \subset \mathbb{R}^2$ *and* $(a, b) \in D$ *be an interior point of* D. *Suppose that* $f, g : D \to \mathbb{R}$ *have continuous partial derivatives in a neighbourhood of* (a, b). *Define* $C := \{(x, y) \in D : g(X, y) = 0\}$. *If (i)* $(a, b) \in C$, *(ii)* $\nabla g(a, b) = (0, 0)$ *and (iii)* f *restricted to* C *has a local maxima/minima at* (a, b), *then there exists* $\lambda_0 \in \mathbb{R}$ *(called* ***Lagrange's multiplier****) such that*
$$\nabla f(a, b) = \lambda_0 \nabla g(a, b).$$

Exercise 9.9.
(a) Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its maximum or minimum values.
(b) Find the maximum values of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.
(c) Find the minimum value of xy subject to the constraints $x + y = 16$.
(d) Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.
(e) Find the length and height of a rectangle with maximum area that can be inscribed within the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides of the rectangle parallel to the coordinate axes.

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