Higher Derivatives

If $f(x)$ is differentiable at $x=c$, and $f'(x)$ is cont. at $x=c$, then we say that $f(x)$ is continuously differentiable at $x=c$.

If $f(x)$ is continuously differentiable on $(a-h, b+h)$ $(h>0)$ then

(FTC) $$f(x) = \int_a^x f'(t)\,dt + f(a) \quad \forall\, x \in [a,b]$$

$f(x)$ is called the antiderivative of $f'(x)$.

If $f'(x)$ is differentiable at $x=c$, then we say that $f(x)$ has a derivative of order 2 at $x=c$. Denoted by

$$f''(c) \quad \text{or} \quad f^{(2)}(c).$$

Similarly, if for a given $n \in \mathbb{N}$, $f^{(n-1)}(x)$ is differentiable at $x=c$, then $f(x)$ has a derivative of order n at $x=c$. Denoted by $f^{(n)}(x)$.

In this case, we say $f(x)$ is n-smooth

A function is called **smooth at $x=c$** if it is **differentiable indefinitely** at $x=c$.

i.e. $f^{(n)}(c)$ exists $\forall\ n=1,2,\dots$

Fact. If a function is smooth at $x=c$, then it is smooth on a nbhd. of c.

Examples.

- **Smooth Functions**

  - Any **polynomial** is **smooth** everywhere

    $$f(x) = x^3+2x+1$$
    $$f'(x) = 3x^2+2$$
    $$f''(x) = 6x$$
    $$f^{(3)}(x) = 6$$
    $$f^{(n)}(x) = 0 \quad \forall\ n\geq 4.$$

  - $f(x) = e^x$

    $$f^{(n)}(x) = e^x \quad \forall\ n$$

  - $f(x) = \sin x$

    $$f^{(n)}(x) = \sin\left(x+\frac{n\pi}{2}\right)$$

  - $f(x) = \ln x$ is **smooth at** $x>0$

    $$f^{(n)}(x) = \frac{(-1)^{n-1}}{(n-1)!\,x^n}$$

- A function that is n-smooth but not $(n+1)$-smooth.

  - $f(x) = x^{n+\frac{1}{3}}$

    This is n-smooth at 0 with

    $f^{(n)}(x) = n!\, x^{1/3}$

    but $f^{(n)}(x)$ is not diff. at $x=0$ (although cont. at 0).

  - $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then $f'(0) = 0$

Does $f''(0)$ exist?

$$f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 \end{cases}$$

Claim: $f'(x)$ is not cont. at $x=0$.

$\lim_{x\to 0} 2x \cdot \sin\frac{1}{x} = 0$ but $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist.

- $f(x) = \int_0^x |t|\, dt$, $x \in \mathbb{R}$

By FTC, $f'(x) = |x|$

$\Rightarrow f''(0)$ does not exist.

However, $f'(x)$ is cont. at $x=0$.

Taylor's Theorem. (Generalization of MVT)

Suppose, f is n-smooth on a nbhd. I of a. Then for every $b \in I$, $\exists$ a c between a and b s.t.

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

$$\boxed{\begin{array}{l}\text{if } n=1\\ f(b)-f(a) = f'(c)(b-a)\end{array}}$$

Alternatively,

If f is n-smooth on $(a-\delta, a+\delta)$, then for every h with $0<|h|<\delta$ $\exists$ a θ with $0<\theta<1$ s.t.

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1} + \frac{f^{(n)}(a+\theta h)}{n!}h^n$$

Moral of the story: A n-smooth function can be approximated by a polynomial of degree $\le n-1$.

Examples.

- The function $f(x) = e^x$ is smooth everywhr.
  So, $f(x)$ is differentiable in every nbhd. of every pt.

  Also, $f^{(n)}(0) = 1 \ \forall \ n = 1, 2, \cdots$

  Thus, taking $a=0$ and $b=x$ in Taylor's Thm. we see that $\exists$ a $c \in (0,x)$ or $(x,0)$ s.t.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^c}{n!}$$

Note that for a fixed x,

$$\frac{|c|^n}{n!} < \frac{|x|^n}{n!} \longrightarrow 0$$

Thus, by taking n to be sufficiently big,

$$\frac{|c|^n}{n!} < \varepsilon$$

whence

$$e^x \approx 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} \quad (\text{with error} < \varepsilon)$$

Leibniz Differentiation Formula

Let f and g be n-smooth on (a,b) and set $h(x) = f(x)g(x)$.

Then $h(x)$ is also n-smooth on (a,b), and

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x) \quad \forall x \in (a,b)$$

$$= f^{(n)}(x)\, g(x) + \binom{n}{1} f^{(n-1)}(x)\, g'(x) + \binom{n}{2} f^{(n-2)}(x)\, g^{(2)}(x) + \cdots + \cdots + \binom{n}{n-1} f'(x)\, g^{(n-1)}(x) + f(x)\, g^{(n)}(x).$$

Example.

Let $h(x) = e^{x} \sin x$. Compute $h^{(5)}(x)$.

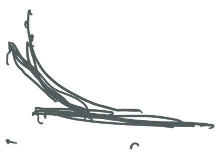
$$h^{(5)}(x) = \sum_{k=0}^{5} \binom{5}{k} \left(e^{x}\right)^{(5-k)} \cdot \left(\sin x\right)^{k}$$

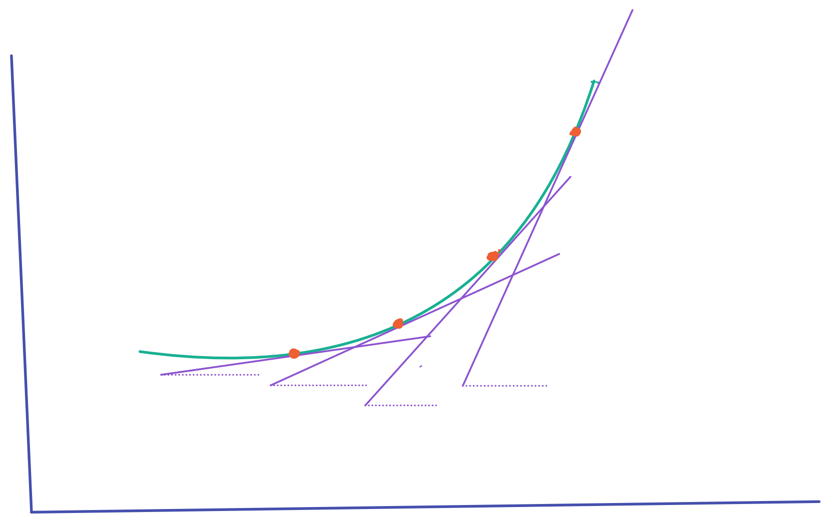
$$= \sum_{k=0}^{5} \binom{5}{k} \cdot e^{x} \cdot \sin\left(x + \frac{k\pi}{2}\right)$$

$$= e^{x} \cdot \sum_{k=0}^{5} \binom{5}{k} \sin\left(x + \frac{k\pi}{2}\right).$$

$$= e^{x} \left(\sin x + 5\cos x - 10 \sin x - 10 \cos x + 5 \sin x + \cos x \right).$$

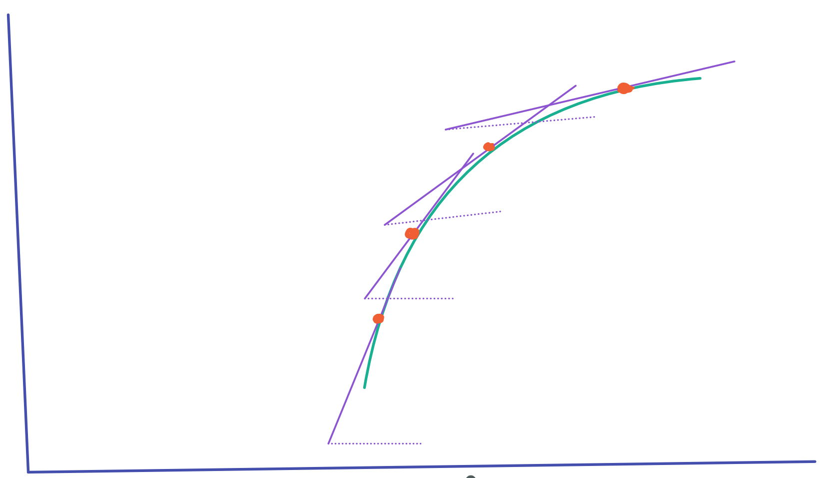
The 2nd derivative & the geometry of a function





concave up

$f'(x)$ is increasing with x



concave down

$f'(x)$ is decreasing with increasing x..

Defn. A function is said to be concave up (resp. down) on (a,b) if $f'(x)$ is strictly increasing (resp. decreasing) on (a,b).

Thm. Supp. $f(x)$ is cont. on $[a,b]$ and $f''(x)$ exists throughout on (a,b). Then the function $f(x)$ is concave up (resp. down) on (a,b) if $f''(x) > 0$ (resp. $f''(x) < 0$) $\forall\, x \in (a,b)$.

Proof. Let $a < x < y < b$.

Applying MVT $f'(x)$ on $[x,y]$,

$\exists\ c \in (x,y)$ s.t.

$$f'(y) - f'(x) = f''(c)\,(y-x) > 0$$

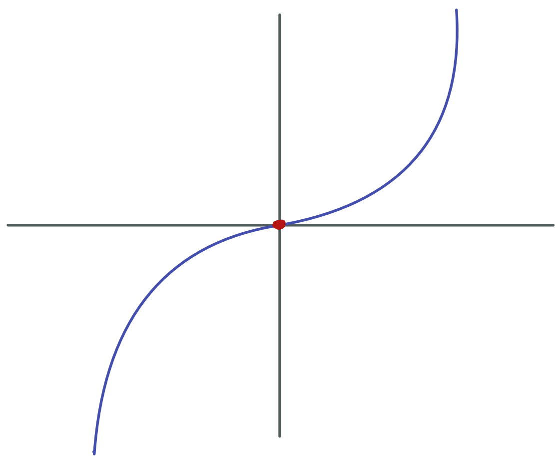
$\Rightarrow f'$ is strictly increasing on (a,b)

hence f is concave up.

Inflection pt.

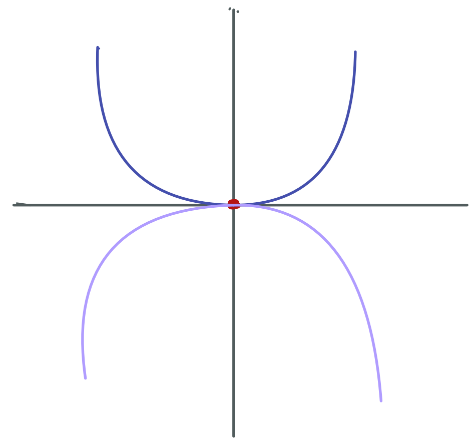
Let f be cont. on $[a,b]$ and diff. on (a,b).

A pt. $c \in (a,b)$ is called an inflection pt. of f if f is concave up (resp. down) on (a,c) and concave down (resp. up) on (c,b).



An inflection pt.

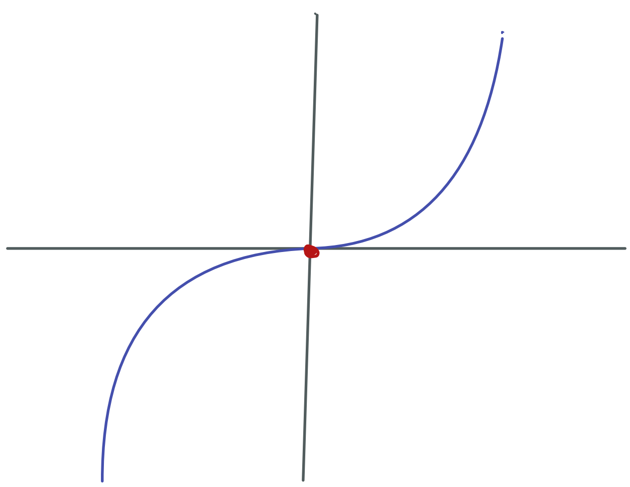
$f(x) = x^3$



Not an inflection pt.

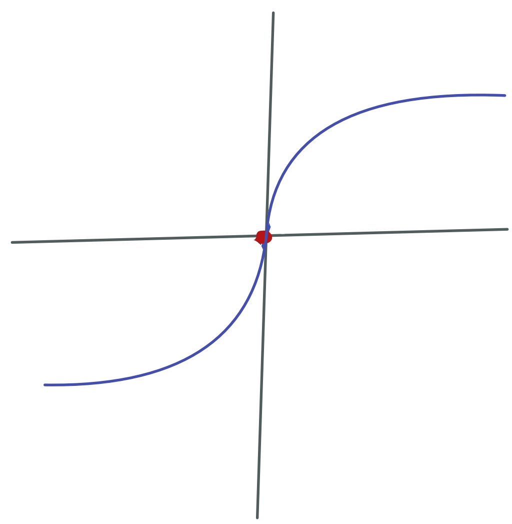
$f(x) = x^4$

$f(x) = -x^4$



An inflection pt.

$f(x) = x^{5/3}$



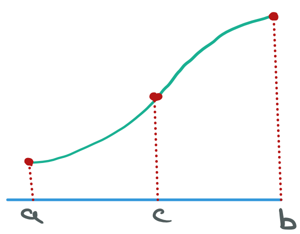
An inflection pt. $f(x) = x^{1/3}$

Thm. Let **c** be an **inflection point of $f(x)$ on (a,b)**. Then **either** $f''(c) = 0$

or $f''(c)$ does **not exist**.

Proof. If $f''(c)$ exists, then

$$\frac{f'(t)-f'(c)}{t-c} \qquad \lim_{t\to c^-} \varphi_{f'}(t) = f''(c) = \lim_{t\to c^+} \varphi_{f'}(t)$$



If say, the concavity changes from up to down at c, then

$$\varphi_{f'}(t) > 0 \quad \text{for } t\in(a,c)$$

$$\text{and } \varphi_{f'}(t) < 0 \quad \text{for } t\in(c,b)$$

so that,

$$0 \le \lim_{t\to c^-} \varphi_{f'}(t) = f''(c) = \lim_{t\to c} \varphi_{f'}(t) \le 0$$

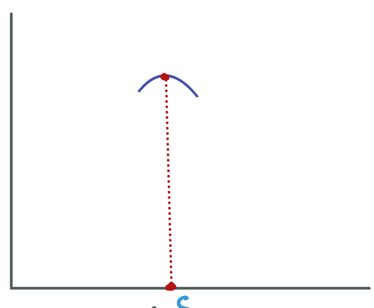
$$\Rightarrow f''(c) = 0.$$

Thm. (Second derivative test)

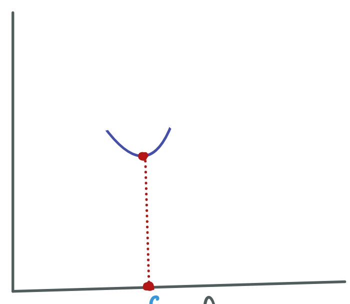
Let f be cont. on $[a,b]$ and twice differentiable on (a,b) with f'' cont. on (a,b). Let $c \in (a,b)$ be a critical point of $f(x)$ (i.e. $f'(c)=0$). Then

- $f''(c) < 0 \Rightarrow c$ is a local max.
- $f''(c) > 0 \Rightarrow c$ is a local min.
- $f''(c) = 0 \Rightarrow$ Test is inconclusive.

Proof.



Local max.
concave down on a nbhd. of c



Local min.
concave up on a nbhd. of c.