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10th Lecture on Transform Techniques

(MA-2120)

What will we learn today?

- Convergence of Fourier Series
- Fourier Integral

Convergence of Fourier Series:

Now we will discuss the conditions under which Fourier Series expansion is possible and also find the function to which the series converges.

Convergence Theorem: (Dirichlet's theorem):

Let $f(x)$ be defined like that —

1. $f(x)$ is defined and continuous for all $x \in (-l, l)$ except at a finite number of points in $(-l, l)$.
2. $f(x)$ is periodic with a period $2l$.
3. $f(x)$ is piecewise continuous.
4. One sided derivatives of $f(x)$ exist and is finite at each point in $[-l, l]$,

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x+)}{h}$$ exist at each $[-l, l]$

This is Right derivative.

$$\lim_{h \to 0^+} \frac{f(x-) - f(x-h)}{h}$$ exist at each $[-l, l)$

This is Left derivative.

most importantly one sided limits $\lim_{x \to l^+} f(x)$ and $\lim_{x \to l^-} f(x)$ exist and finite

Then for each $x \in [-l, l]$, the Fourier Series

$$\frac{a_0}{2} + \sum_{n\geq 1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

converges to $\dfrac{f(x+) + f(x-)}{2}$ at a point of discontinuity and converges to $f(x)$ at a point of continuity.

where $f(x+)$ and $f(x-)$ are the right hand and left hand limit respectively.

Remark: If function is continuous, we will have uniformly convergence

Remark: At both the end points of the interval $[-l, l]$, the Fourier series converges to ——

$$\frac{1}{2}\left[f(-l+) + f(l-)\right].$$

The Series Converges to the Same number at l and $-l$, Since the Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

has the Same Value

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(-1)^n \cdot a_n.$$

at both the end points $x = l$ and $x = -l$.

Remarks: Let the sum upto j terms of the Fourier Series be denoted by

$$S_j = \frac{a_0}{2} + \sum_{n=1}^{j} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$j = 1, 2, 3 \ldots$

The partial sums S_j give successive approximation to $f(x)$. S_j gets closer closer to $f(x)$ as j increases.

Ex:

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ 0, & 0 \le x < \pi \end{cases}$$

Soln:

Fourier Series expansion of $f(x)$ is obtained as

$$f(x) = \pi/4 + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] - \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots\right]$$

At the point $x=0$, $f(x)$ is discontinuous.

Therefore the series converges to

$$\frac{1}{2}\left[f(0-)+f(0+)\right] = \frac{1}{2}\left[\pi+0\right] = \pi/2$$

Setting $x=0$, we have

$$\pi/2 = \pi/4 + \frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \underline{\underline{\pi^2/8}}.$

Ex: $f(x) = x^2, \quad -2 \le x \le 2.$

Fourier Series expansion of $f(x) = x^2$ $-2 \le x \le 2$ was obtained as —

$$f(x) = 4/3 + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{2}\right).$$

At the end points of the interval $[-2,2]$ the series converges to

$$\frac{1}{2}\left[f(-2^+)+f(2^-)\right] = \frac{1}{2}(4+4)$$
$$=4.$$

Setting $x=2$,

$$4 = 4/3 + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$
$$= 4/3 + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/16\left(4 - 4/3\right) = \pi^2/6.$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \pi^2/6.$$

At $x=0$, the given function is continuous

The series converges to $f(0)=0$.

Therefore $0 = 4/3 + \frac{16}{\pi^2}\sum_{n=1}^{\infty} (-1)^n/n^2$

$$\text{or } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

Complex form of the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2} a_n \left(e^{in\pi x/l} + e^{-in\pi x/l}\right) + \frac{1}{2i} b_n \left(e^{in\pi x/l} - e^{-in\pi x/l}\right)$$

$a_n \cos\frac{n\pi x}{l}$

$b_n \sin\frac{n\pi x}{l}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - i b_n) e^{in\pi x/l} + \frac{1}{2}(a_n + i b_n) e^{-in\pi x/l}\right]$$

Now define $c_0 = a_0/2$, $c_n = \frac{1}{2}(a_n - i b_n)$

and $c_{-n} = \frac{1}{2}(a_n + i b_n)$

we can write $f(x)$ as

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l} \right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} .$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\, dx.$$

$n = 0, \pm 1, \pm 2, \ldots$

Example: Find the complex Fourier series of the function $f(x) = e^{-x}$, $-\pi < x < \pi$.

Solⁿ: Here $l = \pi$. The Fourier coefficients are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x} \cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(1+in)x}\, dx$$

$$= -\frac{1}{2\pi(1+in)}\left[e^{-(1+in)x}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{2\pi(1+in)}\left[e^{-(1+in)\pi} - e^{(1+in)\pi}\right]$$

$$= -\frac{(1-in)}{2\pi(1+n^2)}\left[e^{-\pi}(\cos n\pi - i\sin n\pi) - e^{\pi}(\cos n\pi + i\sin n\pi)\right]$$

$$= -\frac{(1-in)}{2\pi(1+n^2)}\left[-\cos n\pi\,(e^{\pi} - e^{-\pi})\right]$$

$$= \frac{\sinh \pi}{\pi(1+n^2)}(1-in)\cos n\pi$$

Now the complex form of the Fourier Series is

$$f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1-in}{1+n^2}\right) e^{inx}.$$

Ex: Find the complex Fourier Series of the function

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$$

Try it!

Ans: $f(x) = \sum_{n=-\infty}^{\infty} -\frac{i}{n\pi}(1-\cos n\pi)\, e^{in\pi x}$

Fourier Integral:

If we have any periodic function of period $2l$ on $[-l, l]$, it can be easily represented by a Fourier Series.

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)}$$

on $[-l, l]$.

Q. What will it be if $l \to \infty$ i.e, if we consider the interval $(-\infty, \infty)$? or what will it be if the function is non periodic?

Ans: For this case, the function $f(x)$ can not be represented by Fourier Series. we can represent it by Fourier integral

Let us consider the nonperiodic function $f(x)$ on $(-\infty, \infty)$

Let $$f_l(x) = \frac{a_0}{2} + \sum_{n\geq 1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$\boxed{f(x) = \lim_{l\to\infty} f_l(x)}$$

$$f_l(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n\geq 1}^{\infty}\left[\left(\frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt\right)\cos\frac{n\pi x}{l} + \left(\frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt\right)\sin\frac{n\pi x}{l}\right]$$

$$f_l(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\left[\left(\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt\right)\cos\frac{n\pi x}{l} + \left(\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt\right)\sin\frac{n\pi x}{l}\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}(t-x)\,dt$$

Set $\omega_n = \frac{n\pi}{l}$ and $\Delta\omega = \omega_n - \omega_{n-1}$

Discrete variable

$$= \frac{\pi}{l}(n-n+1)$$

$$= \pi/l.$$

$$f_l(x) = \frac{\Delta\omega}{2\pi}\int_{-l}^{l} f(t)\,dt + \frac{1}{\pi}\cdot\Delta\omega\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\left(n\Delta\omega(t-x)\right)dt$$

Let $l\to\infty$, $\Delta\omega\to 0$.

$$f(x) = \lim_{l\to\infty} f_l(x) = \lim_{\Delta\omega\to 0}\left[\frac{\Delta\omega}{2\pi}\int_{-l}^{l} f(t)\,dt + \Delta\omega\sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-l}^{l} f(t)\cos\left(n\Delta\omega(t-x)\right)dt\right]$$

$$f(x) = \lim_{\Delta\omega \to 0} \left[\frac{\Delta\omega}{2\pi} \int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty} \Delta\omega \cdot F(n\Delta\omega) \right]$$

where $F(n\Delta\omega) = \frac{1}{\pi} \int_{-l}^{l} f(t) \cos\left(n\Delta\omega\,(t-x)\right) dt$

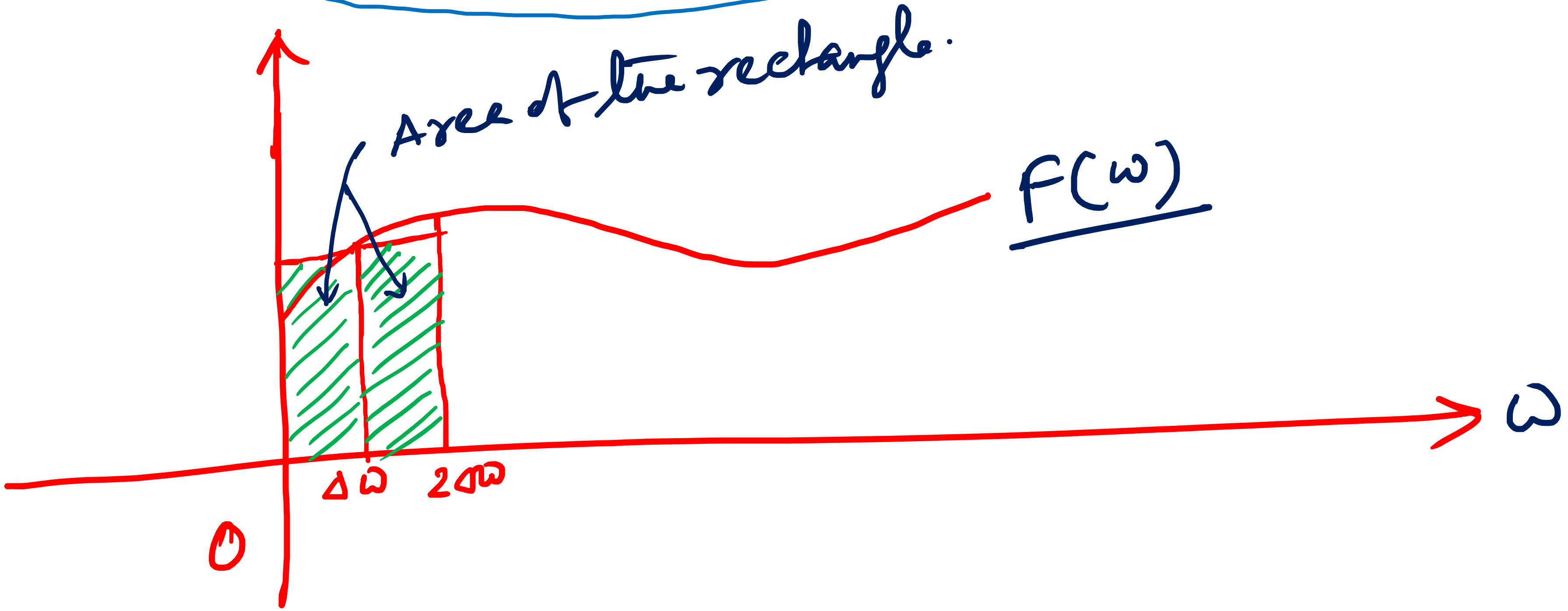
Now if we assume that $f(x)$ is absolutely integrable i.e., $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges, then

$$\boxed{\lim_{\Delta\omega \to 0} \frac{\Delta\omega}{2\pi} \int_{-l}^{l} f(t)\,dt = 0.}$$

Now we have

$$f(x) = \boxed{\lim_{\Delta\omega \to 0} \sum_{n=1}^{\infty} \Delta\omega \cdot F(n\Delta\omega)}$$

Riemann sum of a definite integral.



$$f(x) = \int_0^\infty F(\omega)\, d\omega$$

[Here discrete variable ω_n changes to continuous variable ω]

$\uparrow$ $n\pi/L$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(t-x)\, dt\, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right] d\omega$$

Fourier Integral

where $A(\omega) = \int_{-\infty}^\infty f(t)\cos\omega t\, dt$

$B(\omega) = \int_{-\infty}^\infty f(t)\sin\omega t\, dt.$

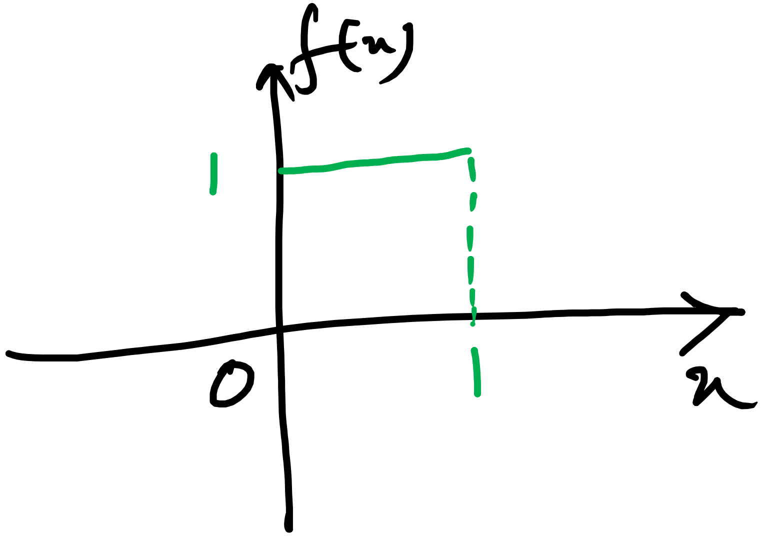
Theorem: Assume that $f(x)$ is piecewise continuous and one sided derivatives of $f(x)$ exist at each point on every finite interval on the x-axis and let $f(x)$ be absolutely integrable on $-\infty < x < \infty$ i.e. $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges.

Then for each x on the entire real axis

$$\boxed{\frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\cos \omega x + B(\omega)\sin \omega x\right] d\omega = \frac{f(x+)+f(x-)}{2}}$$

Ex: Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$



Hence show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x/2)\,dx}{x} = \pi/2$

Sol^n: Here $f(x)$ is non periodic function and we can represent it by Fourier integral.

$$A(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt$$

$$= \int_{0}^{1} \cos \omega t \, dt = \frac{\sin \omega}{\omega}.$$

$$B(\omega) = \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt = \int_{0}^{1} \sin \omega t \, dt$$

$$= \frac{1}{\omega}(1 - \cos \omega)$$

Therefore

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\left[\sin\omega\cos\omega x + (1-\cos\omega)\sin\omega x\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2}{\omega}\sin\frac{\omega}{2}\left[\cos\frac{\omega}{2}\cos\omega x + \sin\frac{\omega}{2}\cdot\sin\omega x\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}\sin\frac{\omega}{2}\cos\omega\left(x-\frac{1}{2}\right) d\omega.$$

This is the Fourier integral representation of the given function.

Let $x = \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = 1$

Hence

$$1 = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega} \sin\left(\frac{\omega}{2}\right) d\omega$$

$$\text{or} \int_0^\infty \frac{1}{\omega} \sin\left(\frac{\omega}{2}\right) d\omega = \pi/2$$

$$\Rightarrow \int_0^\infty \frac{\sin x/2}{x}\, dx = \pi/2.$$

Ex: Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0, & x<0 \\ x, & 0<x<1 \\ 0, & x>1 \end{cases}$$

Try it!

Determine the convergence of the integral at $x=1$.

Ans: Here $A(\omega) = \frac{1}{\pi}\left[\frac{\cos\omega + \omega\sin\omega - 1}{\omega^2}\right]$

and $B(\omega) = \frac{1}{\pi}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right)$

The function is not defined at $x=1$.

$\frac{f(1-)+f(1+)}{2} = \frac{1}{2}$

$= \int_0^\infty \frac{1-\cos\omega}{\pi\omega^2}\, d\omega$