WORKSHEET - 4 - KEY

1. Examine the following limits by working from the definition.

(a) $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}} = 0$

To show: given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$0 < x < \delta \Rightarrow \left| \frac{\sin x}{\sqrt{x}} \right| < \varepsilon$

Now, for $x > 0$,

$$\frac{|\sin x|}{\sqrt{x}} \le \frac{x}{\sqrt{x}} = \sqrt{x} < \varepsilon$$

provided, $x < \varepsilon^2$

Take $\delta = \varepsilon^2$.

Then

$$0 < x < \delta \Rightarrow \frac{|\sin x|}{\sqrt{x}} < \varepsilon.$$

(b) $\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} = 0$

To show: given $\varepsilon > 0$, $\exists$ $M > 0$ s.t.

$$x > M \Rightarrow \frac{|\sin x|}{\sqrt{x}} < \varepsilon$$

But, $\frac{|\sin x|}{\sqrt{x}} \le \frac{1}{\sqrt{x}} < \varepsilon$

whenever $x > \frac{1}{\varepsilon^2}$

Take $M = \frac{1}{\varepsilon^2}$

Then $x > M \Rightarrow \frac{|\sin x|}{\sqrt{x}} < \varepsilon$.

(c) $\lim_{x\to 1} \frac{x^2}{x-1} = \infty$

To show: given $M > 0$, $\exists$ a $\delta > 0$ s.t. $|x-1| < \delta \Rightarrow \left|\frac{x^2}{x-1}\right| > M$.

We work with x satisfying

$$|x-1| < \frac{1}{2}$$

$$\Rightarrow x > \frac{1}{2}$$

Thus $\frac{x^2}{|x-1|} > \frac{1}{4|x-1|} > M$

whenever $|x-1| < \frac{1}{4M}$

Take $\delta = \min\left\{\frac{1}{2}, \frac{1}{4M}\right\}$.

Then $|x-1| < \delta \Rightarrow \frac{x^2}{|x-1|} > M$.

2. Evaluate the following limits in any way you want.

(a) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$ $= 1$

Do it yourself.

(b) $\lim_{x\to\infty} \frac{x+\sin x}{\sqrt{x^2+1}}$ $= 1$

Do it yourself.

(c) $\lim_{x\to 1} \frac{(1-x)(1-\sqrt[2]{x})(1-\sqrt[3]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^n}$ $= n!$

For any $m \in \mathbb{N}$,

$$1 - y^m = (1-y)(1+y+y^2+\cdots+y^{m-1})$$

Thus,

$$1-x = (1-\sqrt[m]{x})(1+\sqrt[m]{x}+\sqrt[m]{x}^2+\cdots+\sqrt[m]{x}^{m-1})$$

$$\Rightarrow \lim_{x\to 1} \frac{1-\sqrt[m]{x}}{1-x} = \lim_{x\to 1} (1+\sqrt[m]{x}+\cdots+\sqrt[m]{x}^{m-1})$$

$$= m.$$

Therefore,

$$\lim_{x\to 1} \frac{(1-x)(1-\sqrt[2]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^n}$$

$$= \left(\lim_{x\to 1} \frac{1-\sqrt[2]{x}}{1-x}\right)\left(\lim_{x\to 1} \frac{1-\sqrt[3]{x}}{1-x}\right)\cdots\left(\lim_{x\to 1} \frac{1-\sqrt[n]{x}}{1-x}\right)$$

$$= 2\cdot 3\cdot \cdots \cdot n = n!$$

3. Suppose $f(x)$ is defined on a deleted neighbourhood of c, and $\lim_{x\to c} f(x) = \ell$. Show that $\lim_{x\to c} |f(x)| = |\ell|$.

To show: given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow \big|\, |f(x)| - |\ell| \,\big| < \varepsilon.$$

It is given that $\lim_{x\to c} f(x) = \ell$.

Thus, given $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

Next, observe that

$$|f(x)| = |f(x) - \ell + \ell| \leq |f(x) - \ell| + |\ell|$$

$$\Rightarrow |f(x)| - |\ell| \leq |f(x) - \ell|$$

similarly, $|\ell| - |f(x)| \leq |f(x) - \ell|$

Thus, $\big|\,|f(x)| - |\ell|\,\big| \leq |f(x) - \ell|$

Therefore, if $|x-c| < \delta$, then

$$\big|\,|f(x)| - |\ell|\,\big| < |f(x) - \ell| < \varepsilon.$$

4. Define the function $f(x)$ as
$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$
Prove that $\lim_{x \to 0} f(x) = 0$.

Observe that

$$|f(x)| \le |x|$$

Thus, $|x| < \varepsilon \Rightarrow |f(x) - f(0)|$

$$= |f(x)| \le |x| < \varepsilon.$$

Work out the details.

5. Define the function $f(x)$ as

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 1 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

Prove that $\lim_{x\to 0} f(x)$ does not exists although $\lim_{n\to\infty} f(1/n) = 0$.

It is clear that

$$\lim_{n\to\infty} f\left(\tfrac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

On the other hand,

$$\frac{1}{n\sqrt{2}} \longrightarrow 0$$

but $f\left(\frac{1}{n\sqrt{2}}\right) = 1$

$$\Rightarrow \quad f\left(\frac{1}{n\sqrt{2}}\right) \longrightarrow 1$$

If $\lim_{x\to 0} f(x)$ did exist, then by Thm. A. of limits, we should have had

$$\lim_{n\to\infty} f\left(\tfrac{1}{n}\right) = \lim_{n\to\infty} f\left(\tfrac{1}{n\sqrt{2}}\right)$$

But $= 0$ and $= 1$.

Thus,

$\lim_{x\to 0} f(x)$ does not exist.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function. That is, there is a constant $a > 0$ such that $f(x + a) = f(x)$ for all $x \in \mathbb{R}$. Suppose $\lim_{x\to\infty} f(x) = \ell$ where ℓ is a real number. Show that $f(x)$ must be a constant function. Deduce that $\lim_{x\to\infty} \sin x$ does not exist.

$\lim_{x\to\infty} f(x) = \ell \Rightarrow$ given $\varepsilon > 0$, $\exists$ $M > 0$

s.t. $x > M \Rightarrow |f(x) - \ell| < \frac{\varepsilon}{2}$.

Let $x_1, x_2 \in \mathbb{R}$, be arbitrarily chosen.

$\exists$ $n_1, n_2 \in \mathbb{N}$ s.t. (by Arch. Prop.)

$y_1 = x_1 + n_1 a > M$, and

$y_2 = x_2 + n_2 a > M$.

Note that $f(y_1) = f(x_1)$ & $f(y_2) = f(x_2)$.

Now, $y_i > M$

$\Rightarrow |f(y_i) - \ell| < \frac{\varepsilon}{2}$

Thus, $|f(y_1) - f(y_2)|$

$= |(f(y_1) - \ell) - (f(y_2) - \ell)|$

$\leq |f(y_1) - \ell| + |f(y_2) - \ell|$

$< \varepsilon$.

$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ —— $(*)$

Since $(*)$ holds $\forall$ $\varepsilon > 0$, deduce that

$|f(x_1) - f(x_2)| = 0$

$\Rightarrow f(x_1) = f(x_2)$

Thus, f is constant.
(since x_1 & x_2 are two arbitrary points of $\mathbb{R}$.

Since $\sin x$ is not constant and bounded, hence $\lim_{x\to\infty} \sin x$ does not exist.

7. Prove the Sandwich Theorem for limits. Namely that if f, g and h are functions defined on a deleted neighbourhood I of c with $f(x) \le g(x) \le h(x)$ for all $x \in I$. If $\lim_{x\to c} f(x) = \ell$ and $\lim_{x\to c} h(x) = \ell$, then $\lim_{x\to c} g(x) = \ell$.

Let $x_n \to c$

$\lim_{x\to c} f(x) = \ell \Rightarrow f(x_n) \to \ell$

and $\lim_{x\to c} h(x) = \ell \Rightarrow h(x_n) \to \ell$

Next, $\exists$ a n_0 s.t. $x_n \in I \ \forall n \ge n_0$

Thus $f(x_n) \le g(x_n) \le h(x_n) \ \forall n \ge n_0$

$\Rightarrow g(x_n) \to \ell$ by Sandwich Thm.

Since, $x_n \to c$ is an arbitrary sequence, deduce by Thm. B. of Limits that

$\lim_{x\to c} g(x) = \ell$.

8. Suppose f and g are real valued functions with domain $\mathbb{R}$. We write $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. We also write $f(x) = o(g(x))$ (small "oh") if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. Establish the following.

(a) $x^3 + ax^2 + bx + c \sim x^3$ where a, b and c are constants.

$$\lim_{x\to\infty} \frac{x^3+ax^2+bx+c}{x^3} = \lim_{x\to\infty} 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}$$

$$= \lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{a}{x} + \lim_{x\to\infty} \frac{b}{x^2} + \lim_{x\to\infty} \frac{c}{x^3}$$

$$= 1.$$

(b) $x^3 + ax^2 + bx + c = o(x^4)$.

Do it yourself.