Quiz-4

1. Question 1.

4 points

Suppose $f(x)$ is continuous at $x = c$. Which of the following is always continuous at $x = c$?

(A) $f(x)^2$

(B) $\dfrac{1}{1 + |f(x)|}$

(C) $f(x) + 2$

(D) $\sqrt{f(x)}$

Mark only one oval per row.

	Yes	No
A	✓	
B	✓	
C	✓	
D		✓

(A) $f(x)^2 = f(x)f(x)$ – product of two cont. fns. at c, hence cont.

(B) $1 + |f(x)|$ is cont. and nonvanishing at $x = c$. Thus, $\dfrac{1}{1+|f(x)|}$ is cont. at $x = c$.

(C) $f(x) + 2$ sum of 2 cont. fns. at $x = c$

(D) $\sqrt{f(x)}$ is cont. at $x = c$ $\iff$ $f(c) \geq 0$

In particular, if $f(c) < 0$, then $\sqrt{f(x)}$ is not cont. at $x = c$.

2. Question 2. 4 points

Let $f(x)$ be a function defined at $x = c$. Pick the correct statement.

(A) $f(x)^2$ is continuous at $x = c \implies f(x)$ is continuous at $x = c$

(B) $1/f(x)$ is continuous at $x = c \implies f(x)$ is continuous at $x = c$

(C) $f(x) + 2$ is continuous at $x = c \implies f(x)$ is continuous at $x = c$

(D) $e^{f(x)}$ is continuous at $x = c \implies f(x)$ is continuous at $x = c$

Mark only one oval per row.

	Yes	No
A	◯	✓
B	✓	◯
C	✓	◯
D	✓	◯

(A) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$

Then f is nowhere cont.
But $f(x)^2 = 1$, a constant. fn. hence cont.

(B) $1/f(x)$ cont. at $c \Rightarrow 1/f(c) \neq 0$
$\Rightarrow f(x) = 1/(1/f(x))$ is cont. at c.

(C) $f(x) = (f(x)+2) - 2$, hence cont. at c.

(D) Explanation given at the end.

3. Question 3. 4 points

$$\text{Evaluate: } \lim_{x\to 1} \frac{x + x^2 + x^3 + \cdots + x^{20} - 20}{x - 1}.$$

210 (Easy!)

4. Question 4. 4 points

$$\text{Evaluate: } \lim_{x\to\infty} \left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right).$$

1/2

$$\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}} = \frac{\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{x}\left(\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}}+1\right)}$$

$$= \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}}+1} \to \frac{1}{2} \quad \text{as } x \to \infty.$$

5. Question 5. 4 points

Which of the following functions is continuous everywhere on $\mathbb{R}$?

(A) $\ln(1+x)$

(B) $\ln(1+|x|)$

(C) $\dfrac{1}{2+\sin x}$

(D) $\dfrac{1}{1+\sin x}$

Mark only one oval per row.

	Yes	No
A		✓
B	✓	
C	✓	
D		✓

(A) Not defined for $x \le -1$

(B) $1+|x| \ge 1$ and cont. on $\mathbb{R}$, and $\ln$ is cont. on $[1,\infty)$.

(C) $2+\sin x > 0$ and cont. at all $x \in \mathbb{R}$, hence $\frac{1}{2+\sin x}$ is cont. on $\mathbb{R}$.

(D) Not defined at $x = (4n-1)\frac{\pi}{2}$, $n \in \mathbb{N}$.

6. Question 6. 4 points

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous everywhere. If $f(x) = \sin x$ for all $x \in \mathbb{Q}$, then $f(\pi) =$

0

Since f - cont. at π, $f(\pi) = \lim_{n\to\infty} f(x_n)$ for any $x_n \to \pi$.

$\exists$ a rational sequence $\{x_n\}_n$ with $x_n \to \pi$.

Thus by cont. of f and sine we have

$f(\pi) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin x_n = \sin\pi = 0.$

7. Question 7.

4 points

Suppose $f+g$ is continuous on $\mathbb{R}$. Pick the correct statement.

(A) If $f(x)$ is continuous at $x=c$, then so is $g(x)$

(B) If $f(x)$ is discontinuous at $x=c$, then so is $g(x)$

(C) It is possible that $f(x)$ is continuous at $x=c$ but $g(x)$ is not

(D) It is possible that both $f(x)$ and $g(x)$ are discontinuous at $x=c$.

Mark only one oval per row.

	Yes	No
A	✓	
B	✓	
C		✓
D	✓	

(A) $g(x) = (f(x)+g(x)) - f(x)$ hence cont. at c.

(B) If $g(x)$ is cont. at c, then by (A) $f(x)$ is also cont. at c

(C) Obviously not, by (A).

(D) f be the Dirichlet's fn. and $g(x) = -f(x)$. Then f and g are not cont. but $f(x)+g(x)=0$, hence cont at c.

8. Question 8.

4 points

Let I be an interval, and $f:[0,1]\to I$ be continuous. Which I among the following ensures that $f(x)$ always has a fixed point?

(A) $I=[0,1]$

(B) $I=(0,1)$

(C) $I=[0,1)$

(D) $I=(0,1]$

Mark only one oval per row.

	Yes	No
A	✓	
B	✓	
C	✓	
D	✓	

(A) Possible by IVP

(B) Note that $f(0)>0$ and $f(1)<1$

set $g(x) = f(x) - x$

$g(0) = f(0) > 0$

and $g(1) = f(1) - 1 < 0$

$\Rightarrow \exists\, c \in (0,1)$ s.t. (by IVP)

$g(c) = 0$

$\Rightarrow f(c) = c$.

(C) If $f(0)=0$, then done, else proceed as in (B).

(D) If $f(1)=1$, then done, else proceed as in (B).

9. Question 9.

4 points

Which of the following sets can be the continuous image of a closed interval $[a, b]$?

(A) $\{1\}$ ← constant function.

(B) $\{0, 1\}$ ← Impossible by the IVP.

(C) $[1, \infty)$ ← Impossible since not bounded above.

(D) $[1, 3]$ ← Always possible.

Mark only one oval per row.

	Yes	No
A	✓	
B		✓
C		✓
D	✓	

10. Question 10.

4 points

Define the function $f : [1, \infty) \to \mathbb{R}$ by

$$f(x) = \sum_{1 \le n \le x} \frac{1}{n^2}.$$

For instance, $f(1) = 1, f(1.5) = 1, f(3.5) = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$ etc.

Pick the correct statement.

(A) $f(x)$ is continuous at $x = 100$

(B) $f(x)$ is continuous at 130.5

(C) $f(x)$ is bounded on [redacted] $[1,\infty)$

(D) $\lim_{x\to\infty} f(x)$ exists

Mark only one oval per row.

	Yes	No
A		✓
B	✓	
C	✓	
D	✓	

[redacted]

(A) $\lim_{x \to 100^-} f(x) = 1 + \frac{1}{4} + \cdots + \frac{1}{99^2}$

$\lim_{x \to 100^+} f(x) = 1 + \frac{1}{4} + \cdots + \frac{1}{100^2}$

hence not cont. at 100.

(B) $f(x)$ is constant on a small nbhd. of 130.5, say $(130.25, 130.75)$

On this nbhd., $f(x) = 1 + \frac{1}{4} + \cdots + \frac{1}{130^2}$.

(C) Let $S = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} + \cdots < \infty$.

Then $0 < f(x) < S \ \forall x$

(D) See explanation below.

10. (D) We show that $\lim_{x \to \infty} f(x) = S$.

Let S_n denote the nth partial sum of S.

Then given $\varepsilon > 0$, $\exists\, n_0$ s.t. $|S_n - S| < \varepsilon \ \forall n \ge n_0$.

We show that if $x \ge n_0$, then $|f(x) - S| < \varepsilon$ which will prove our result.

By Arch. Prop. $\exists\, n \in \mathbb{N}$ s.t. $n \le x \le n+1$.

Then $f(x) = S_n$

Now, $x \ge n_0 \Rightarrow n \ge n_0 \Rightarrow |S_n - S| < \varepsilon \Rightarrow |f(x) - S| < \varepsilon$.

Explanation for 2.(D).

If $c>0$, then $\ln x$ is cont. at c.

To see this,

$$\ln(c+h) - \ln c = \ln\left(1+\frac{h}{c}\right)$$

Now, $|\ln(1+x)| = \left|x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right|$, for $|x|<1$

$$\leq |x|\left(1 + \frac{|x|}{2} + \frac{|x|^2}{3} + \cdots\right)$$

$$< |x|\left(1+|x|+\cdots\right)$$

$$= \frac{|x|}{1-|x|} < \varepsilon$$

if $|x| < \frac{\varepsilon}{1+\varepsilon}$

Take $|h|<c$,

so that $\left|\ln\left(1+\frac{c}{h}\right)\right| < \varepsilon$ provided $|h| \leq c\varepsilon$

Thus, setting $\delta = c\varepsilon$,

we have

$|h|<\delta \Rightarrow |\ln(c+h) - \ln c| < \varepsilon$, proving our result.

Thus, if $g(x)$ is cont. at c & $g(c)>0$, then $\ln g(x)$ is cont. at $x=c$.

Now, it is given that $e^{f(x)}$ is cont. at c and we know that $e^{f(c)}>0$ for any c.

Hence $f(x) = \ln\left(e^{f(x)}\right)$ is cont. at $x=c$.