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2^{nd} Lecture on Differential Equation

(MA-1150)

What have we learnt?

- Classification on differential equation
- Applications of differential equation
- Order and degree of ODE
- Definition of linear and non-linear ODE
- Notion of solutions to ODE

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What will we learn today?

- Nature of solutions: Explicit and Implicit Solutions
- First Order and First Degree ODE
- Separation of Variables
- Homogeneous Equations

Explicit Solution

- **Explicit Solution:**

$$\boxed{y = y(x)}$$

Ex. $\frac{dy}{dx} = x \Rightarrow y = x^2/2 + c$

This is the explicit sol^n.

So y can be written as a function of x.

Here you can get only one explicit form.

Ex. $\frac{dy}{dx} = y$ $\Rightarrow$ $y = e^x + c$,

This is explicit sol^n

Implicit Solutions:

Now you see the implicit solution which can be written in the form

$F(x,y)=0$

➢ **Implicit Solution:**

$\boxed{F(x,y)=0} \Rightarrow$ This gives more explicit solutions.

Examples

Solⁿ → $\boxed{y\sqrt{x^2-1}+x\sqrt{y^2-1}=0}$

Solution → $\boxed{x^2+y^2=a}$, $a>0$

⟹ These are the implicit solutions

Nature of the Solutions:

For implicit solutions:

(i) Most of the times, you can't be able to find its explicit form like $y = f(x)$.

(ii) If you get Explicit solution, then it will be more than one.

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For example, the differential equation $x + y\frac{dy}{dx} = 0$ has a particular solution $x^2+y^2=1$ as an implicit solution on the interval $(-1, 1)$.

because the derivative of $y(x)$ does not exist at $x = \pm 1$, since the vertical tangent exists at that points.

See it in next page.

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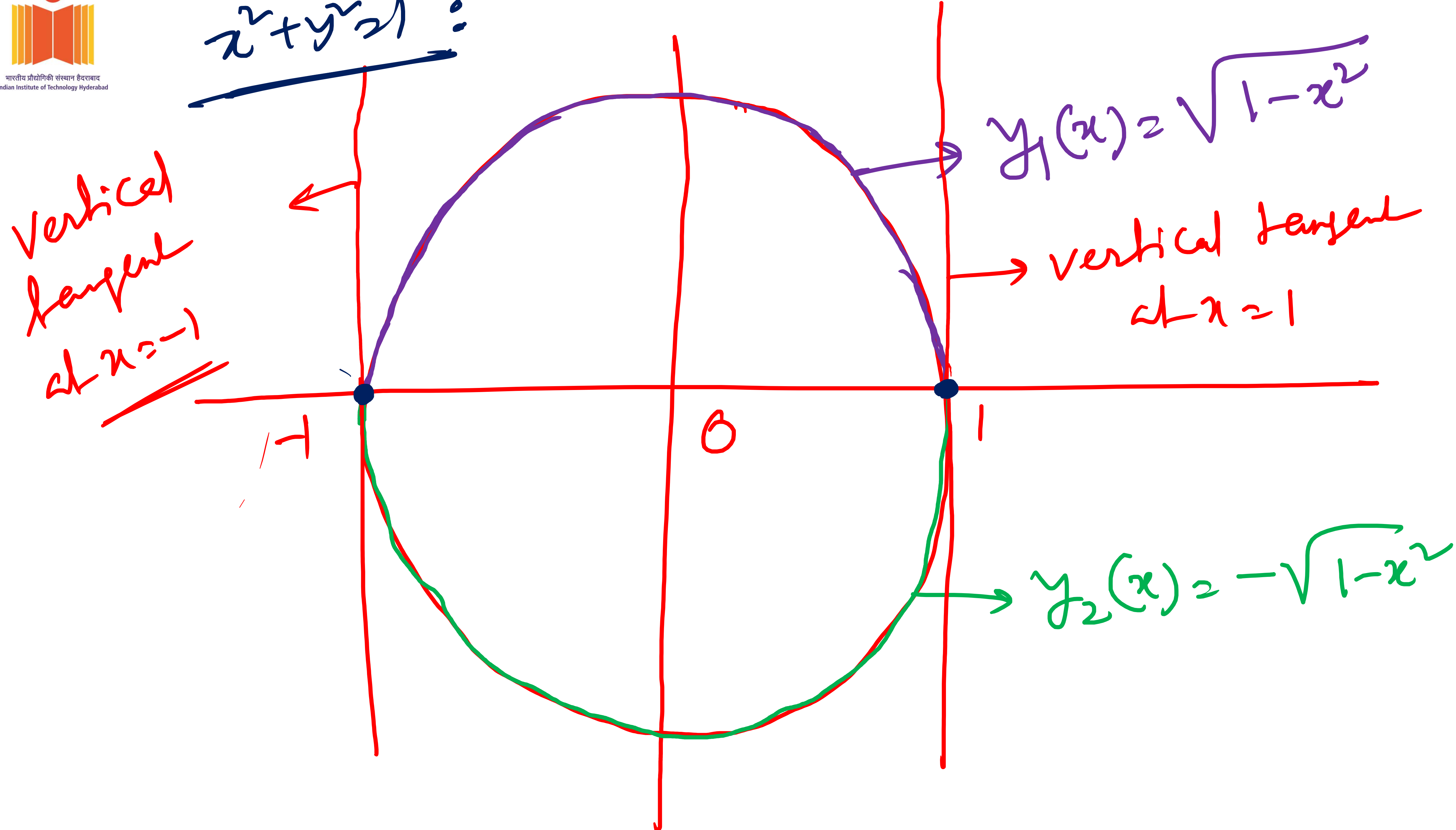
Explicitly we can write $x^2+y^2=1$ as
$y = \pm\sqrt{1-x^2}$, if $|x| \le 1$ for all real values of x.

if we consider $|x|>1$, then
$\sqrt{1-x^2}$ is no longer real function and it is complex function.

$x^2+y^2=1$:


vertical tangent at $x=-1$
$y_1(x) = \sqrt{1-x^2}$
vertical tangent at $x=1$
-1
0
1
$y_2(x) = -\sqrt{1-x^2}$

$y(x) = \pm\sqrt{1-x^2}$

are the two explicit solutions as $y_1(x)$ and $y_2(x)$ of the differential eqn

$$\boxed{x + y\frac{dy}{dx} = 0}$$ on $(-1,1)$.

Since $y_1(x) = \sqrt{1-x^2}$ and $y_2(x) = -\sqrt{1-x^2}$ clearly satisfy the given ODE on the interval $(-1,1)$.

So, $y^2+x^2=1$ is an implicit solutions of the given differential eqn.

But $x^2+y^2=-1$ is not an implicit solutions of the given diff. eqn

Since $y=\pm\sqrt{-1-x^2}$ attains the complex values for any real values of x.

So, it does not define any real function on any interval.

Hence $y^2+x^2=-1$ is only a formal solution because it satisfies the given diff. eq^n. but it is not implicit solutions.

First Order and First Degree ODE

➢Definition:

An **ordinary differential equation of first order and first degree** can be written as

$$\frac{dy}{dx} = f(x,y) \quad \text{--- (1)} \qquad \forall (x,y) \in D.$$

D is the domain.

or in the more symmetric form —

$$M\,dx + N\,dy = 0 \quad \text{--- (2)}$$

where $M = M(x,y)$ and $N = N(x,y)$ are functions of x and y. Here geometrically, $f(x,y)$ defines a slope at every point (x,y) at which the function $f(x,y)$ is defined.

Equations Solvable by Separation of Variables

- If the differential eqn is written in the form -

only function of x

$$f_1(x)\,dx + f_2(y)\,dy = 0$$

only function of y.

Then we can say variables are separable in the given differential eqn.

Now, its general solution is

$$\boxed{\int f_1(x)\,dx + \int f_2(y)\,dy = c}$$

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Ex.

$$\frac{dy}{dx} = y \quad \text{(by using separation of variable)}$$

$$\Rightarrow \frac{dy}{y} = dx$$

$$\Rightarrow \log|y| = x + c \Rightarrow |y| = ce^{x}$$

$$\Rightarrow y = ce^{x}.$$

By using the next discussed theorem

This is the general soln.

Since $|y| = ce^{x}$

$$\Rightarrow |ye^{-x}| = c$$

This is continuous fn

$$\Rightarrow ye^{-x} = c$$

$$\Rightarrow y = ce^{x}$$

where c is an arbitrary constant.

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be continuous function and $|f(x)|$ is constant. Then $f(x)$ is constant.

Examples

Ex. (1) $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} = 0$ a, $\boxed{\sin^{-1}x + \sin^{-1}y = c}$ general solⁿ

(2) $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

$\Rightarrow \dfrac{x\,dx}{\sqrt{1-x^2}} + \dfrac{y\,dy}{\sqrt{1-y^2}} = 0$ a, $\boxed{\sqrt{1-x^2} + \sqrt{1-y^2} = c}$ general solⁿ

(3) Exercise: $(x^2+1)(y^2-1)\,dx + xy\,dy = 0$

Solⁿ: $\boxed{y^2 = \dfrac{c\,e^{-x^2}}{x^2} + 1}$

Home work.

(4) $\dfrac{dy}{dx} = e^{x-2y} + x^2 e^{-2y}$

Solⁿ: $\boxed{e^{2y} = 2e^{x} + \tfrac{2}{3}x^3 + c}$

Home work

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Note: $\sin^{-1}y + \sin^{-1}x = 1$ is an explicit solution since y can be written in the function of x only:

$$\therefore\ y = \underbrace{\sin\left[1 - \sin^{-1}x\right]}_{f(x)\ \leftarrow\ \text{Explicit form}} \text{ on } x \in (-1, 1)$$

Equations Reducible to Separable Form

$$\frac{dy}{dx} = f(ax+by+c)$$

let $ax+by+c = t$, $\Rightarrow a + b\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t)$$

$$\Rightarrow \frac{dt}{dx} = a + bf(t) \Rightarrow \frac{dt}{a+bf(t)} = dx$$

$$\Rightarrow \int \frac{dt}{a+bf(t)} = x + c$$

$\rightarrow$ general solⁿ

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Examples

Ex: 1 $\quad \frac{dy}{dx} = \sin(x+y)$, Let $x+y=v$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$$\frac{dv}{1+\sin v} = dx$$

or, $\frac{(1-\sin v)\,dv}{1-\sin^2 v} = dx$

or, $\int (\sec^2 v - \sec v \tan v)\,dv = \int dx + c$

$\Rightarrow \tan v - \sec v = x + c$

$\Rightarrow \boxed{\tan(x+y) - \sec(x+y) = x + c}$

Examples

Ex. 2 $\frac{dy}{dx} = \sec(x+y)$, Let $x+y = v$

Home work.

Solⁿ: $\boxed{y = \tan\left(\frac{x+y}{2}\right) + c}$

Homogeneous Equations

- A differential equation of first order and first degree is said to be homogeneous if it is of the form or can be put in the form —

$$\boxed{\frac{dy}{dx} = f(y/x)}$$

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The first order and first degree ODE is given by

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{—} \ (1).$$

If $M(x,y)$ and $N(x,y)$ are both homogeneous functions of x and y with same degree n, then $M(x,y)$ and $N(x,y)$ can be written as

$$M(x,y) = x^n \phi(y/x)$$

and $N(x,y) = x^n \psi(y/x)$

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So we have

$$M\,dx - N\,dy = 0$$

or $x^n \phi(y/x)\,dx - x^n \psi(y/x)\,dy = 0$ $(\because x^n \neq 0)$

or $$\boxed{\frac{dy}{dx} = \frac{\phi(y/x)}{\psi(y/x)} = f(y/x)}$$

This is the form of homogeneous equation. This equation can be reduced to separable eqn by a change of variables.

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Theorem: If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous eqⁿ, then the change of variables $\boxed{y = vx}$ transforms the given ODE into a separable eqⁿ in the variables x and v.

Proof: Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be homogeneous eqⁿ.

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Then we have

$$\frac{dy}{dx} = f(y/x)$$

Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$\Rightarrow v + x\frac{dv}{dx} = f(v)$$

or $x\frac{dv}{dx} = f(v) - v$

or, $\frac{dv}{f(v)-v} = \frac{dx}{x}$

Transferred into Separable eqn in the variable v and x.

Now use the Separation of variables method to find out the general solⁿ —

$$\int \frac{dv}{f(v)-v} = \int \frac{dx}{x} + c$$

Let $F(v) = \int \frac{dv}{f(v)-v}$

or $F(v) = \ln|x| + c$

$$\Rightarrow \boxed{F(y/x) = \ln|x| + c}$$

general solⁿ

Examples

Ex. 1 $\quad 2xy\dfrac{dy}{dx} = y^2 - x^2$

Homogeneous function of degree 2 since $y^2 - x^2 = x^2\left(\left(\frac{y}{x}\right)^2 - 1\right) = x^2\varphi(y/x)$

and $2xy = x^2\left(\frac{2y}{x}\right) = x^2\psi(y/x)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy} = \dfrac{(y/x)^2 - 1}{2(y/x)}$, Let $y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

or $\quad v + x\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v}$

or, $\quad x\dfrac{dv}{dx} = -\dfrac{v^2 + 1}{2v} \Rightarrow \dfrac{2v\,dv}{1 + v^2} + \dfrac{dx}{x} = 0$

or $\quad \log|(1 + v^2)| + \log|x| = \log c$

$\Rightarrow |x(v^2 + 1)| = c$,

$\Rightarrow \boxed{y^2 + x^2 = cx}$ general sol^n, $x > 0$

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Examples

Ex. 2 $\quad x^2 dy + y(x+y)dx = 0, \quad (x,y) \in D$ rectangular domain

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{y(x+y)}{x^2}$, Let $y = vx$.

$\Rightarrow x\dfrac{dv}{dx} = -v(2+v) \Rightarrow \dfrac{dx}{x} + \dfrac{dv}{v(v+2)} = 0$

$\Rightarrow \log|x| + \frac{1}{2}\log\left|\dfrac{v}{v+2}\right| = \log c_1$

$\Rightarrow \log|x^2| + \log\left|\dfrac{v}{v+2}\right| = \log c, \quad \boxed{c = c_1^2}$

$\Rightarrow \left|\dfrac{x^2 y}{y+2x}\right| = c \Rightarrow \boxed{x^2 y = c(2x+y)}, \; x>0, \; y>0$

General solⁿ

Equation Reducible to Homogeneous Form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C} \quad \text{where } \frac{a}{A} \neq \frac{b}{B}$$

$$\left.\begin{aligned} ax+by+c &= 0 \\ Ax+By+C &= 0 \end{aligned}\right\}$$ Two intersecting straight lines.

The point of intersection is (h, k) which is obtained by $h = \dfrac{bC - cB}{aB - bA}$, $k = \dfrac{cA - aC}{aB - bA}$

we now transfer the origin by putting, $x = h + X$, $y = k + Y$

$\Rightarrow dx = dX$, $dy = dY$

Now we have, $\boxed{\dfrac{dY}{dX} = \dfrac{aX + bY}{AX + BY}} \rightarrow$ This is homogeneous

we can solve it.

Examples

Ex. 1 $\quad \dfrac{dy}{dx} = \dfrac{y-x+1}{y+x+5}$, $\quad \left.\begin{array}{l} y-x+1=0 \\ y+x+5=0 \end{array}\right\}$ Solve.

The two lines $y-x+1=0$ and $y+x+5=0$ intersect at the point $(-2,-3)$.
The substitution $x = X-2$, $y = Y-3$ gives

$$\frac{dY}{dX} = \frac{Y-X}{Y+X} = \frac{Y/X - 1}{Y/X + 1}$$

Put $Y = vX$, we will get,

$$X\frac{dv}{dX} = -\frac{v^2+1}{v+1}$$

$$\Rightarrow \frac{(v+1)\,dv}{v^2+1} + \frac{dX}{X} = 0$$

$$\Rightarrow \frac{1}{2}\log|(v^2+1)| + \tan^{-1} v + \log|X| = \log c$$

Examples

$$\text{or, } \log\left[\left(\frac{y+3}{x+2}\right)^2+1\right]+2\tan^{-1}\frac{y+3}{x+2}+2\log(x+2)=K \text{ (say)}$$

$$\text{or, } \boxed{\log\left[(y+3)^2+(x+2)^2\right]+2\tan^{-1}\frac{y+3}{x+2}=K}$$

General Sol^n.

Examples

Ex 2 $$\frac{dy}{dx} = \frac{3x-4y-2}{6x-8y-5}$$

Here the two lines $3x-4y-2=0$ and $6x-8y-5=0$ are parallel. We observe, here that the eqn may be written as

$$\frac{dy}{dx} = \frac{(3x-4y)-2}{2(3x-4y)-5},$$ Put $3x-4y = v$

$$\Rightarrow \frac{dv}{dx} = 3-4\frac{dy}{dx}.$$

Hence we obtain,

$$\frac{3}{4} - \frac{1}{4}\frac{dv}{dx} = \frac{v-2}{2v-5},$$

This reduces to, $$\frac{2v-7}{2v-5} = \frac{dv}{dx} \Rightarrow \frac{(2v-5)\,dv}{2v-7} = dx$$

Integrating, $v + \log|(2v-7)| = x + c$

Examples

$$\boxed{\log|(6x-8y-7)| = c - 2x + 4y}$$

This is the general solution.