भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

4th Lecture on Transform Techniques

(MA-2120)

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What did we learn in previous class?

- Linearity Property of Laplace Transform
- Some Examples
- First Shifting or Translation Theorem
- Second Translation Theorem
- Some Examples

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What will we learn today?

- Scaling Property
- Laplace Transform of Derivatives
- Differentiation of Laplace Transform
- Division by t
- Laplace Transform of Integral

④ Change of Scale (Scaling Property):

Let $\mathcal{L}[f(t)] = F(s)$, $Re(s) > \alpha$.

Let $a > 0$ be a constant. Then

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a} F(s/a)}, \quad Re(s) > a\alpha.$$

and for a constant b,

$$\boxed{\mathcal{L}[f(at+b)] = \frac{1}{a} e^{sb/a} F(s/a)}, \quad a > 0$$

Proof: $\mathcal{L}[f(at)]$

$$= \int_0^\infty e^{-st} f(at)\, dt$$

Change of variables:

Let $u = at \Rightarrow du = a\,dt$

$t = 0 \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$ since $\underline{a > 0}$

$$\int_0^\infty f(at)\, e^{-st}\, dt$$

$$= \int_0^\infty f(u)\, e^{-s(u/a)} \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty f(u)\, e^{-s/a \cdot u}\, du$$

$$= \frac{1}{a} F(s/a), \qquad Re(s) > a\alpha.$$

Ex, $\mathcal{L}[\sin 5t]$

Sol: $\mathcal{L}[\sin t] = \dfrac{1}{s^2+1} = F(s)$, $s>0$

$\mathcal{L}[\sin 5t] = \frac{1}{5} F(s/5)$

$= \dfrac{1}{5} \cdot \dfrac{1}{(s/5)^2+1}$

$= \dfrac{5}{s^2+25}$, $s>0$

Ex: $\mathcal{L}[e^{at}]$, $a>0$

$$\mathcal{L}[e^{t}] = \frac{1}{s-1}, \quad s>1$$

$$\mathcal{L}[e^{at}] = \frac{1}{a}\,\frac{1}{s/a-1}$$

$$= \frac{1}{s-a}, \quad s>a\cdot 1 = a.$$

Ex: If $\mathcal{L}[f(t)] = \dfrac{s^2 - s + 1}{(2s+1)^2 (s-1)}$, prove that

$$\mathcal{L}[f(2t)] = \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}.$$

Soln:

$$\mathcal{L}[f(2t)] = \frac{1}{2} F\left(\frac{s}{2}\right).$$

$$= \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}.$$

⑤ Laplace transform of Derivatives: (Derivative Theorem)

Let $f(t)$ be a continuous function on $[0,\infty)$ and of exponential order α and $f'(t)$ is piecewise continuous on $[0,\infty)$. Then

$$\boxed{\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)}$$

$$\Rightarrow \mathcal{L}\left[\frac{d}{dt}\big(f(t)\big)\right] = s\,\mathcal{L}[f(t)] - f(0),$$

$Re(s) > \alpha$

Remark: If $f(t)$ is continuous on $(0,\infty)$ and of exponential order α and $f'(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}\left[\frac{d}{dt}(f(t))\right] = s\,\mathcal{L}[f(t)] - f(0^+), \quad \mathrm{Re}(s) > \alpha.$$

$$\boxed{f(0^+) = \lim_{t \to 0^+} f(t)}$$

Proof:

$$\mathcal{L}\left[f'(t)\right]$$

$$= \int_0^\infty e^{-st} f'(t)\,dt = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)\,dt$$

Integrating by parts —

$$= \lim_{R\to\infty} \left[e^{-st} f(t) \Big|_0^R + s\int_0^R e^{-st} f(t)\,dt \right]$$

$$= \lim_{R\to\infty} \left[e^{-st} f(t) \Big|_0^R \right] + \lim_{R\to\infty} s\int_0^R e^{-st} f(t)\,dt$$

[Since f is of exponential order α, $\lim_{R\to\infty} e^{-sR} f(R) = 0$, $\forall s > \alpha$]

$$= -f(0) + s\mathcal{L}[f(t)]$$

$$\Rightarrow \boxed{\mathcal{L}\left[\frac{d}{dt}\left(f(t)\right)\right] = s\mathcal{L}[f(t)] - f(0)}$$

(Proved).

Remark: An interesting feature of the derivative theorem is that we do not need to consider that $f'(t)$ is an exponential order.

⑨ **Example:** $f(t) = \cos(e^{t^2})$ $\left(|f(t)| \le 1\right)$

bounded function and it is of exponential order 0.

$$\boxed{f'(t) = -2t e^{t^2} \sin(e^{t^2})}$$

This $f'(t)$ is not of exponential order.

But $\mathcal{L}[f'(t)]$ exists.

$\Rightarrow$ This example implies that for existence of $\mathcal{L}[f'(t)]$, we do not need to consider that $f(t)$ is an exponential order.

$$\mathcal{L}[f''(t)] = \mathcal{L}\left[\frac{d}{dt}(f'(t))\right]$$

$$= s\,\mathcal{L}[f'(t)] - f'(0)$$

$$= s\left[s\,\mathcal{L}[f(t)] - f(0)\right] - f'(0)$$

$$= s^2\,\mathcal{L}[f(t)] - s f(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s^3\,\mathcal{L}[f(t)] - s^2 f(0) - s f'(0) - f''(0)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $f(t), f'(t), f''(t), \ldots f^{(n-1)}(t)$ are continuous on $[0,\infty)$ and are of exponential order, and $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$.

Theorem: If f is continuous on $[0,\infty)$ except for a jump discontinuity at $t = t_1 > 0$, and f has exponential order α with f' piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) - e^{-t_1 s}\left(f(t_1^+) - f(t_1^-)\right)$$

$Re(s) > \alpha$.

Proof: $\mathcal{L}[f'(t)]$

$$= \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \lim_{R\to\infty} \left[e^{-st} f(t) \Big|_0^{t_1^-} + e^{-st} f(t) \Big|_{t_1^+}^{R} + s \int_0^R e^{-st} f(t)\, dt \right]$$

$$= s\,\mathcal{L}[f(t)] - f(0) - e^{-st_1}\left(f(t_1^+) - f(t_1^-)\right)$$

Note: If $0 = t_0 < t_1 < t_2 \cdots < t_n$ are a finite number of jump discontinuities, then we have

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) - \sum_{k=1}^{n} e^{-st_k}\left(f(t_k^+) - f(t_k^-)\right)$$

Example: $\mathcal{L}[\sin^2 \omega t]$

$f(t) = \sin^2 \omega t$. $\quad f'(t) = 2\omega \sin \omega t \cos \omega t$

$= \omega \sin 2\omega t$

$$\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)$$

or $\mathcal{L}[\omega \sin 2\omega t] = s\,\mathcal{L}[f(t)]$

or $\mathcal{L}[f(t)] = \frac{1}{s}\mathcal{L}[\omega \sin 2\omega t]$

$$= \frac{\omega}{s}\mathcal{L}[\sin 2\omega t] = \frac{\omega}{s}\cdot\frac{2\omega}{s^2+4\omega^2}$$

$$= \frac{2\omega^2}{s(s^2+4\omega^2)}$$

Ex

$$\mathcal{L}\left[\cos^2 \omega t\right]$$

$$= \frac{s^2 + 2\omega^2}{s(s^2 + 4\omega^2)} \quad \text{(Try!)}$$

⑥ Differentiation of Laplace Transform:

Theorem: Let $f(t)$ be piecewise continuous on $[0,\infty)$ of exponential order α and $\mathcal{L}[f(t)] = F(s)$. Then

$$\boxed{\mathcal{L}[t\,f(t)] = -F'(s)} = -\frac{d}{ds}\left[\mathcal{L}[f(t)]\right]$$

and

$$\boxed{\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n} = (-1)^n \cdot F^{(n)}(s)}$$

$$= (-1)^n \frac{d^n}{ds^n}\left[\mathcal{L}[f(t)]\right], \quad (s>\alpha)$$

Note: If $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order, then $t f(t)$ is also piecewise continuous and of exponential order.

$\Rightarrow \mathcal{L}[t f(t)]$ exists.

Proof: $\mathcal{L}[t f(t)] = \int_0^\infty t f(t) e^{-st}\, dt$

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\, dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds}\left(\int_0^\infty f(t) e^{-st}\, dt\right)$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(f(t) e^{-st}\right) dt.$$

$$= -\int_0^\infty t f(t) e^{-st}\, dt = -\mathcal{L}[t f(t)]$$

$$\Rightarrow \mathcal{L}[t f(t)] = -\frac{d}{ds} F(s).$$

Counter examples where interchange of interchange of integral and derivative does not exist :

$$f(y) = \int_0^\infty \frac{\sin(xy)}{x}\,dx$$

$$f'(y) = \int_0^\infty \frac{\partial}{\partial y}\left(\frac{\sin(xy)}{x}\right)dx$$

$$= \int_0^\infty \cos(xy)\,dx,$$ does not exist for any y.

However $\displaystyle\int_0^\infty \frac{\sin xy}{x}\, dx$

$$= \int_0^\infty \frac{\sin u}{u}\, du, \quad \text{Let } \underline{u = xy.}$$

exist

$$= \underline{\pi/2.}$$

Ex: $\mathcal{L}[t\cos\omega t]$

$$= -\frac{d}{ds}\mathcal{L}[\cos\omega t]$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \quad \underline{s>0}$$

Ex: $\mathcal{L}[t^2 \cos 3t]$

$\mathcal{L}[\cos 3t] = \dfrac{s}{s^2+9}$.

$$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+9}\right].$$

$$= \frac{2s(s^2-27)}{(s^2+9)^3} \quad \text{(Try)}.$$

Ex,

$$\mathcal{L}[t^2 e^{-2t}] = \frac{2}{(s+2)^3} .$$ (Try).

Homework!

(7) Division by t: (Integration of Laplace Transform):

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order α with $\mathcal{L}[f(t)] = F(s)$ and $\lim_{t\to 0^+} f(t)/t$ exists, then

$$\boxed{\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(x)\,dx}\ , \quad s > \alpha$$

Proof:

$\dfrac{f(t)}{t}$ is piecewise continuous

and is also of exponential order

[Since $f(t)$ is of exponential order

$\left|\dfrac{f(t)}{t}\right| \leq |f(t)| \leq Me^{\alpha t}$, for $t \geq t_0$]

if $t \geq 1$

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$$

Proof:

$$= \int_0^\infty f(t) \cdot \frac{e^{-st}}{t}\, dt$$

$$= \int_0^\infty f(t) \cdot \left(\int_s^\infty e^{-ut}\, du \right) dt$$

$$\left[\text{since } \int_s^\infty e^{-ut}\, du = e^{-st}/t \right]$$

$$= \int_s^\infty \left(\int_0^\infty f(t)\, e^{-ut}\, dt \right) du$$

[The interchange of order of integration is possible if $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists and $f(t)$ is of exponential order.]

$$= \int_s^\infty F(u)\, du . \quad \text{(Proved)}$$

Example: $\mathcal{L}\left[\frac{\sin t}{t}\right]$

$$= \int_s^\infty \mathcal{L}[\sin t]\, dt$$

$$= \int_s^\infty \frac{du}{u^2+1} = \left[\tan^{-1} u\right]_s^\infty$$

$$= \pi/2 - \tan^{-1} s, \quad s>0$$

$$= \tan^{-1}(1/s), \quad s>0$$

Ex. $\mathcal{L}\left[\frac{\sin 2t}{t}\right]$ (Try)!

$= \pi/2 - \tan^{-1} s/2 = \cot^{-1} s/2$

Ex. $\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

Ex: $\mathcal{L}\left[\frac{\sinh \omega t}{t}\right] = \frac{1}{2}\log\frac{s+\omega}{s-\omega}$, $s > |\omega|$.

(8) Laplace Transform of Integral:

Theorem: If f is piecewise continuous on $[0,\infty)$ of exponential order α and

$$g(t) = \int_0^t f(u)\,du,$$

then $\mathcal{L}[g(t)] = \frac{1}{s}\mathcal{L}[f(t)] = \frac{F(s)}{s}$, $s>\alpha$

where $\mathcal{L}[f(t)] = F(s)$.

Proof:

$$g(t) = \int_0^t f(u)\, du.$$

$$\Rightarrow g'(t) = f(t).$$

Now we have ——

$$\mathcal{L}[g'(t)] = s\,\mathcal{L}[g(t)] - g(0), \quad s > \alpha$$

$$\text{or } \mathcal{L}[f(t)] = s\,\mathcal{L}[g(t)]$$

$$\Rightarrow \mathcal{L}[g(t)] = \frac{1}{s}\,\mathcal{L}[f(t)], \quad s > \alpha.$$

(Proved).

Ex: $\mathcal{L}\left[\int_0^t \frac{\sin x}{x}\,dx\right]$

Sol^n: $\mathcal{L}\left[\int_0^t \frac{\sin x}{x}\,dx\right] = \frac{1}{s}\,\mathcal{L}\left[\frac{\sin t}{t}\right]$

$$= \frac{1}{s}\int_s^\infty \frac{1}{u^2+1}\,du$$

$$= \frac{1}{s}\cdot \tan^{-1}u\Big|_s^\infty = \frac{1}{s}\left(\pi/2 - \tan^{-1}s\right)$$

$$= \frac{\cot^{-1}s}{s}.$$