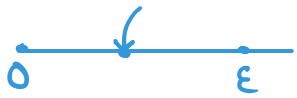
Quiz-6

need some element of E here



1. Question 1. 4 points

Determine the the infimum of the set $E = \left\{ \sqrt{n+1} - \sqrt{n} : n \in \mathbb{N} \right\}$.

0

- $\sqrt{n+1} - \sqrt{n} > 0$
- $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < 0+\varepsilon$ for $n > \frac{1}{4\varepsilon^2}$,

2. Question 2. 4 points

Which of the following functions is injective on $[-1, 1]$ and differentiable at $x = 0$?

(A) $f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0 \\ -x^2 & \text{if} \quad x < 0 \end{cases}$ diff at 0 with $f'(0)=0$, and injective on $[-1,1]$ by Horizontal line test

(B) $f(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ \sin x & \text{if} \quad x < 0 \end{cases}$ diff. at 0 with $f'(0)=1$, and clearly injective on $[-1,1]$

(C) $f(x) = \cos|x|$ ← diff. at $x=0$ with $f'(0)=0$, but not injective since $\cos|1| = \cos|-1|$.

(D) $f(x) = \begin{cases} 0 & \text{if} \quad x \geq 0 \\ x & \text{if} \quad x < 0 \end{cases}$ neither diff at 0 (LH limit=1, RH limit=0) nor injective ($f(0)=0=f(1)$).

Mark only one oval per row.

	Yes	No
A	✓	
B	✓	
C		✓
D		✓

3. Question 3.

4 points

Let $\{a_n\}_n$ be a sequence. Which of the following statements is/are always true?

(A) If $a_n \to 1$ and if $\{a_n^n\}_n$ is convergent, then $a_n^n \to 1$

(B) If $a_n \to 1/2$, then $\{a_n^n\}$ is a null sequence

(C) If $a_n > 0$, and $a_n \to 1$, then $\sqrt[n]{a_n} \to 1$

(D) If $(n!)a_n \to 1$, then $2^n a_n \to 1$

Mark only one oval per row.

	Yes	No
A		✓
B	✓	
C	✓	
D		✓

(A) $1+\frac{1}{n} \to 1$ but $\left(1+\frac{1}{n}\right)^n \to e > 1$

Or, $\frac{1}{n^{1/n}} \to 1$ but $\left(\frac{1}{n^{1/n}}\right)^n = \frac{1}{n} \to 0$.

(B) $a_n \to 1/2 \Rightarrow \exists\, n_0$ s.t. $\frac{1}{4} < a_n < \frac{3}{4} \ \forall n \geq n_0$

$\Rightarrow 0 < a_n^n < \left(\frac{3}{4}\right)^n \ \forall n \geq n_0$

$\Rightarrow a_n^n \to 0$.

(C) $a_n \to 1 \Rightarrow \exists\, n_0$ s.t. $\frac{1}{2} < a_n < 2 \ \forall n \geq n_0$

$\Rightarrow \frac{1}{2^{1/n}} < a_n^{1/n} < 2^{1/n} \ \forall n \geq n_0$

$\to 1$ $\quad$ $\to 1$

by Sandwich Thm. $a_n^{1/n} \to 1$.

(D) $\lim_{n\to\infty} 2^n a_n = \lim_{n\to\infty} \frac{2^n}{n!} \cdot \lim_{n\to\infty} (n!)a_n = 0 \cdot 1 = 0 \neq 1$

4. Question 4.

4 points

Which of the following inequalities hold for all sufficiently large n?
(That is, there is a threshold n_0 such that the left hand side is less than the right hand side for $n \geq n_0$.)

(A) $3^n + 4^n + 5^n < \frac{6^n}{\ln n}$

(B) $\sqrt{n} + \sqrt[3]{n+2} < \sqrt{n+1}$

(C) $n! < 10^{10n}$

(D) $\ln(n+1) + \ln(n+2) + \ln(n+3) < \ln(n^2+1)$

Mark only one oval per row.

	Yes	No
A	✓	
B		✓
C		✓
D		✓

(A) To show: $\frac{\ln n}{2^n} + \frac{\ln n}{1.5^n} + \frac{\ln n}{1.2^n} < 1$

Since $\frac{\ln n}{2^n} \to 0$, $\frac{\ln n}{1.5^n} \to 0$ & $\frac{\ln n}{1.2^n} \to 0$

$\exists\, n_0$ s.t $n \geq n_0 \Rightarrow \frac{\ln n}{2^n} < \frac{1}{3}$, $\frac{\ln n}{1.5^n} < \frac{1}{3}$ and $\frac{\ln n}{1.2^n} < \frac{1}{3}$, hence Done!

(B) $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2} < \sqrt[3]{n+2}$ $\forall n \geq 27$

(C) $\frac{10^{10n}}{n!} = \frac{(10^{10})^n}{n!} \to 0$, hence No.

(D) $\frac{n^2+1}{(n+1)(n+2)(n+3)} \to 0$, hence $\frac{n^2+1}{(n+1)(n+2)(n+3)} < 1$ for suff. large n.

5. Question 5. 4 points

Identify the convergent series.

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

sin x is increasing in [0,1], hence
$\sin\frac{1}{n} > \sin\frac{1}{n+1}$,
Also, $\sin\frac{1}{n} \to 0$
hence convg. by Leibniz Test (Alt. Series Test).

(B) $\sum_{n=1}^{\infty}\sqrt{\frac{1}{n^2}+\frac{1}{(n+1)^2}}$

$\sqrt{\frac{1}{n^2}+\frac{1}{(n+1)^2}} > \sqrt{\frac{2}{(n+1)^2}} = \frac{\sqrt{2}}{n+1}$

since $\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{n+1}$ diverges, the given series divgs.

(C) $\sum_{n=1}^{\infty}\frac{1}{n^{\ln n}}$

$\frac{1}{n^{\ln n}} \le \frac{1}{n^2}$ $\forall\, n \ge e^2$.
and $\sum_n \frac{1}{n^2}$ convgs.

(D) $\sum_{n=1}^{\infty}\frac{1}{n^{\sin n}}$

$\frac{1}{n} \le \frac{1}{n^{\sin n}}$ $\forall\, n \in \mathbb{N}$.

$\Rightarrow \sum_n \frac{1}{n^{\sin n}} \ge \sum_n \frac{1}{n} \to \infty$

Mark only one oval per row.

	Yes	No
A	✓	
B		✓
C	✓	
D		✓

6. Question 6. 4 points

If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, then $\sum_{n=1}^{\infty} a_n \sin n$ is absolutely convergent.

Mark only one oval.

- [x] True
- [ ] False

$|a_n \sin n| \le |a_n|$ $\forall\, n$.

$\Rightarrow \sum_n |a_n \sin n| \le \sum_n |a_n| < \infty$.

7. Question 7. 4 points

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Which of the following sets is always closed?

(A) $\{x \in \mathbb{R} : f(x) > 0\}$

(B) $\{x \in \mathbb{R} : f(x) \geq 0\}$

(C) $\{x \in \mathbb{R} : f(x) \neq 0\}$

(D) $\{x \in \mathbb{R} : f(x) = 0\}$

Mark only one oval per row.

	Yes	No
A		✓
B	✓	
C		✓
D	✓	

(A) Supp. $f(c) > 0$, then f-cont $\Rightarrow \exists$ a nbhd. I of c s.t. $f(x) > 0 \; \forall x \in I$ $\Rightarrow I \subseteq \{x \in \mathbb{R} : f(x) > 0\}$ $\Rightarrow \{x \in \mathbb{R} : f(x) > 0\}$ is open.

(B) Let $E = \{x \in \mathbb{R} : f(x) \geq 0\}$. If $E' = \phi$, then done, else if $p \in E$, $\exists \{x_n\}_n$ with $x_n \in E$ s.t. $x_n \to p$

f-cont $\Rightarrow f(x_n) \to f(p)$

$x_n \in E \Rightarrow f(x_n) \geq 0 \Rightarrow \lim_{n\to\infty} f(x_n) \geq 0 \Rightarrow f(p) \geq 0 \Rightarrow p \in E$.

(C) Same as (A)

(D) Same as (B) – $\underset{\substack{\| \\ 0 \; \forall n}}{f(x_n)} \to f(p)$ hence $f(p) = 0$ and $p \in E$.

8. Question 8. 4 points

Consider the function $f : [-1, 1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \dfrac{1}{x\lfloor \frac{1}{x} \rfloor} & \text{if } \quad x \neq 0 \\ a & \text{if } \quad x = 0. \end{cases}$$

What value of a would make $f(x)$ continuous at $x = 0$?

1

for any real y, $y-1 < \lfloor y \rfloor \leq y$

setting $y = 1/x$, $x \neq 0$, $\frac{1-x}{x} < \lfloor \frac{1}{x} \rfloor \leq \frac{1}{x}$

$\sim \quad 1 \leq \frac{1}{x\lfloor \frac{1}{x} \rfloor} < \frac{1}{1-x}$

By Sandwich Thm., $\lim_{x\to 0} \frac{1}{x\lfloor \frac{1}{x} \rfloor} = 1$.

9. Question 9. 4 points

Suppose $f(x)$ is differentiable everywhere on $\mathbb{R}$ with $f\left(\frac{1}{n}\right) = \frac{n^2}{1+n^2}$.

Then $f'(0) = ?$

0

f-diff $\Rightarrow$ f-cont. $\Rightarrow f(0) = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$

since $f'(0)$ exists, $f'(0) = \lim_{n\to\infty} \dfrac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{-n}{n^2+1} = 0$.

10. Question 10. 4 points

Determine the number of real solutions of $e^x + x^3 + x + 1 = 0$.

1

Let $f(x) = e^x + x^3 + x + 1$

For $n \in \mathbb{N}$

$$\lim_{n \to -\infty} (e^n + n^3 + n + 1) = \lim_{n \to \infty} (e^{-n} - n^3 - n + 1)$$

$$= \lim_{n \to \infty} (-n^3)\left(\frac{-1}{n^3 e^n} + 1 + \frac{1}{n^2} - \frac{1}{n^3}\right) = -\infty$$

$\Rightarrow \exists\ m \in \mathbb{N}$ s.t. $f(-m) < 0$

Sim. $\exists\ m' \in \mathbb{N}$ s.t. $f(m') > 0$

Since, f cont on $[-m, m']$, by IVP,

$\exists$ a $c \in (-m, m')$ s.t. $f(c) = 0$

i.e. $e^c + c^3 + c + 1 = 0$.

If there is another solution c', then $f(c') = 0$, and by MVT on (c, c') (or (c', c))

$\exists\ d \in (c, c')$ s.t

$$0 = \frac{f(c) - f(c')}{c - c'} = f'(d).$$

But $f'(d) = e^d + 3d^2 + 1$

$e^d > 0$

and $3d^2 + 1 > 0$

so that $f'(d) > 0$.

hence, unique solution.