WORKSHEET - 5 - KEY

1. Use the definition of continuity of a function at a point to prove the following.

(a) $f(x) = 2x + 1$ is continuous at $x = 1$.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$|x-1| < \delta \Rightarrow |2x+1-3| = 2|x-1| < \varepsilon$

A clear choice of δ is $\delta = \varepsilon/2$.

(b) $f(x) = x^2$ is continuous at $x = 2$.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|x-2| < \delta \implies |x^2 - 4| < \varepsilon$$

$$|x^2-4| = |x-2|\,|x+2|$$

Take $|x-2| < 1$

$$\implies |x+2| \le |x-2| + 4 < 5$$

Thus, if $|x-2| < 1$, then

$$|x-2|\,|x+2| < 5|x-2| < \varepsilon$$

provided $|x-2| < \varepsilon/5$

Finally, take $\delta < \min\{\varepsilon/5, 1\}$

(e.g., $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{6}\}$)

Then $|x-2| < \delta$

$$\begin{aligned} |x^2-4| &= |x-2|\,|x+2| \\ &< 5|x-2| \quad (\text{since } \delta < 1) \\ &< \varepsilon. \quad (\text{since } \delta < \varepsilon/5) \end{aligned}$$

(c) $f(x) = 1/x$ is continuous at $x = 1/2$.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|x - \tfrac{1}{2}| < \delta \Rightarrow |\tfrac{1}{x} - 2| < \varepsilon$$

Now, $|\frac{1}{x} - 2| = \frac{2}{|x|}|x - \frac{1}{2}|$

Take $|x - \frac{1}{2}| < \frac{1}{4}$ so that,

$$x > \frac{1}{4}$$

and then $\frac{1}{x} < 4$

Thus, $|\frac{1}{x} - 2| = \frac{2}{|x|}|x - \frac{1}{2}|$

$< 8|x - \frac{1}{2}| < \varepsilon$

provided $|x - \frac{1}{2}| < \varepsilon/8$

Take $\delta < \min\{\frac{1}{4}, \frac{\varepsilon}{8}\}$ (e.g. $\delta = \min\{\frac{1}{5}, \frac{\varepsilon}{9}\}$)

Then $|x - \frac{1}{2}| < \delta$

$$\Rightarrow |\tfrac{1}{x} - 2| = \tfrac{2}{|x|}|x - \tfrac{1}{2}|$$

$\leq 8|x - \frac{1}{2}|$ (since $\delta < \frac{1}{4}$)

$< \varepsilon$ (since $\delta < \varepsilon/8$)

2. Let
$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } \quad x \neq 0 \\ 0 & \text{if } \quad x = 0. \end{cases}$$
Prove that $f(x)$ is continuous everywhere on $\mathbb{R}$.

Supp. $c \neq 0$.

Then $\frac{1}{x}$ is cont. at c

$\Rightarrow \sin\frac{1}{x}$ is cont. at c

(sine is cont. everywhere & composition of two cont. fns. is cont.)

$\Rightarrow x\sin\frac{1}{x}$ is cont. at c

(since product of two cont. fns. is cont.).

Next supp. $c = 0$.

To show: given $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t.

$|x| < \delta \Rightarrow |f(x)| < \varepsilon$.

Now, $|f(x)| = 0 < \varepsilon$ if $x = 0$, and

$|f(x)| = |x\sin\frac{1}{x}| \leq |x|$ if $x \neq 0$.

Thus, $|f(x)| \leq |x| \;\forall x$

Hence, $|x| < \varepsilon \Rightarrow |f(x)| < \varepsilon$

Taking $\delta = \varepsilon$, we are done!

3. Let

$$f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Prove that $f(x)$ is not continuous at $x = 0$.

If f is cont. at 0, then we should have

$$f\left(\tfrac{1}{n}\right) \to f(0) = 0$$

But $f\left(\tfrac{1}{n}\right) = n \longrightarrow \infty$,

hence, f is not cont. at $x=0$

(by sequential continuity Thm.)

Remark: By sequential cont., $f(x)$ is cont. at every $x \neq 0$.

4. Define the function $f(x)$ as

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

Show that $f(x)$ is continuous at exactly one point, namely at $x = 0$, and discontinuous everywhere else.

Let $c \neq 0$. If c is rational, let $\{x_n\}_n$ be an irrational sequence with $x_n \to c$

I $f(x)$ is cont at c, then

$$f(x_n) \longrightarrow f(c)$$

But $f(x_n) = 0 \ \forall n$ since x_n is irrational and $f(c) = c \neq 0$, hence f is not cont. at c.

Next, let c be an irrational.

Then $\exists$ a rational sequence $\{x_n\}_n$ with $x_n \to c$ —— $(*)$

If f is cont. at c, then

$$f(x_n) \to f(c) \quad \text{——} \ (**)$$

But $f(x_n) = x_n \ \forall n$ since $x_n \in \mathbb{Q}$. and $f(c) = 0$.

Thus, $(**)$ would imply that

$$x_n \to 0 \quad \text{——} \ (***)$$

But now $(*)$ and $(***)$ contradict each other since $c \neq 0$.

Next, we show that f is cont. at 0.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|x| < \delta \implies |f(x)| < \varepsilon.$$

Now, $$|f(x)| = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ |x| & \text{if } x \in \mathbb{Q} \end{cases}$$

Thus, $|f(x)| \leq |x| \quad \forall\ x \in \mathbb{R}$.

That is $|x| < \varepsilon \implies |f(x)| \leq |x| < \varepsilon$

$\implies f(x)$ is cont. at 0 (with $\delta = \varepsilon$).

5. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Lipchitz* if there is a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \quad \forall \quad x, y \in \mathbb{R}.$$

Show that a Lipchitz function is continuous everywhere on $\mathbb{R}$.

Let $c \in \mathbb{R}$.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

But the Lipschitz condition

$$\Rightarrow |f(x) - f(c)| \leq M|x - c| < \varepsilon$$

whenever $|x - c| < \dfrac{\varepsilon}{M}$.

Thus, we are done by taking $\delta = \dfrac{\varepsilon}{M}$

Remark: Examples of Lipschitz functions.

constant fn.

$f(x) = x, |x|, \sin x, \cos x$

In fact, by Mean Value Thm., any function with bdd. derivative is Lipschitz

6. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere. Define the function $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = \max\{f(x), 0\}$. Show that $g(x)$ is continuous everywhere.

To show: given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|x - c| < \delta \implies |g(x) - g(c)| < \varepsilon.$$

The key observation here is that for a real no. a,

$$\max\{a, 0\} = \frac{a + |a|}{2}$$

This is easy to check! (Do it yourself)

Thus, $g(x) = \dfrac{f(x) + |f(x)|}{2}$

Now, $f(x)$ - cont. $\implies |f(x)|$ - cont.

$\implies g(x)$ is cont.

7. Suppose f is continuous on $[0,1]$. Show that if $\{x_n\}_n$ is a Cauchy sequence in $[0,1]$, then $\{f(x_n)\}_n$ is also a Cauchy sequence.

$\{x_n\}_n$ – Cauchy $\Rightarrow x_n \to c$ for some $c \in \mathbb{R}$.

Claim: $c \in [0,1]$.

Let E = Range of $\{x_n\}_n \subseteq [0,1]$

If E is finite, then $c \in E$ (Why?) in which case $c \in [0,1]$.

If E is infinite, then $c \in E'$

$\Rightarrow c$ is a limit point of $[0,1]$

$\Rightarrow c \in [0,1]$ (since $[0,1]$ is closed).

This proves the claim.

Thus, $x_n \to c$ with $c \in [0,1]$

Now, f – cont. on $[0,1]$

$\Rightarrow f(x_n) \to f(c)$

$\Rightarrow \{f(x_n)\}_n$ is Cauchy (conv. seq. is Cauchy).

8. Does the conclusion of Problem 7. hold if $[0, 1]$ is replaced by $(0, 1)$?

Answer is a No!

Consider $f(x) = \frac{1}{x}$ cont. on $(0,1)$

The sequence $\{\frac{1}{n}\}_n$ is Cauchy in $(0,1)$

but $\{f(\frac{1}{n})\}_n = \{n\}_n$ is not Cauchy.

9. (Removable Discontinuity) Let $c \in \mathbb{R}$. Suppose that $f(x)$ is a function that is defined on a neighbourhood I of c but not defined at $x = c$, and that $\lim_{x \to c} f(x)$ exists finitely. Define $g(x)$ on I as

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \neq c \\ \lim_{x \to c} f(x) & \text{if} \quad x = c. \end{cases}$$

Prove that $g(x)$ is continuous at $x = c$. The function $g(x)$ is called the *continuous extension* of the function $f(x)$ at $x = c$. Is it possible to extend $f(x) = \dfrac{1-\sqrt{x}}{1-x}$ continuously at $x = 1$?

To show: given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$|x - c| < \delta \implies |g(x) - g(c)| < \varepsilon.$$

Let $\lim_{x \to c} f(x) = \ell$ (which is given to be existent)

Thus, given $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t.

$$|x - c| < \delta \implies |f(x) - \ell| < \varepsilon.$$

Take δ small enough so that $(c - \delta, c + \delta) \subseteq I$.

Now, if $|x - c| < \delta$, then

$$g(x) = \begin{cases} f(x) & \text{if } x \neq c \\ \ell & \text{if } x = c \end{cases}$$

Thus, $|x - c| < \delta$

$$\implies |g(x) - g(c)| = \begin{cases} |f(x) - \ell| & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$$

$\implies |g(x) - g(c)| < \varepsilon$, as required.

Answer to the second part is a yes since

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2} \text{ exists!}$$

10. Suppose $f(x)$ is continuous at $x = c$ with $f(c) \neq 0$. Show that there is a neighbourhood I of c such that $f(x) \neq 0$ for every $x \in I$.

It suffices to consider the case that $f(c) > 0$

(If $f(c) < 0$, then consider the fn. $g(x) = -f(x)$)

Since f is cont. at c, and $f(c) > 0$

for $\varepsilon = \frac{f(c)}{2}$, $\exists$ a $\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2}$$

$$\Rightarrow f(x) > \frac{f(c)}{2} > 0,$$

as required.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that the range of f is contained in $\mathbb{Z}$. Show that $f(x)$ must be a constant function.

A classic application of IVP.

Suffices to show that

$$f(x) = f(0) \quad \forall \; x \in \mathbb{R}.$$

First assume $c > 0$.

If $f(c) \neq f(0)$ for some $c \in \mathbb{R}$.

Applying IVP to f which is cont. on $[0, c]$,

we deduce that if r is in between $f(0)$ and $f(c)$, then $\exists$ a $c' \in (0, c)$ s.t. $f(c') = r$.

But we can choose r to be irrational (since there are ∞-ly many irrationals between $f(c)$ and $f(0)$),

so that, $f(c') = r$ is impossible, since f takes only integer values.

12. Discuss the continuity (or the lack of it) of $\sin(1/x)$ at $x = 0$.

It is discontinuous, and the continuity is <u>not</u> removable.

Clearly, $\sin \frac{1}{x}$ is not defined at 0.

Furthermore, $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

else, $\lim_{n \to \infty} \sin n$ exists (why?)

which is a contradiction!

13. (Piecewise Continuous Functions) Let $f(x)$ be continuous on $[a, c]$ and $g(x)$ continuous on $[c, b]$ with $f(c) = g(c)$. Define

$$h(x) = \begin{cases} f(x) & \text{if} \quad x < c \\ g(x) & \text{if} \quad x \geq c. \end{cases}$$

Prove that $h(x)$ is continuous on $[a, b]$.

Let $u \in [a,b]$

If $u \neq c$, then $h(x)$ coincides with either $f(x)$ or $g(x)$ in a nbhd. of u, and hence cont. at u.

Only thing to show is that

$$\lim_{x \to c} h(x) = h(c)$$

Note that $f(x)$ is cont. from right at c.

So $\lim_{x \to c^-} h(x) = \lim_{x \to c^-} f(x) = f(c)$

similarly,

$$\lim_{x \to c^+} h(x) = \lim_{x \to c^+} g(x) = g(c)$$

Since $f(c) = g(c)$, we have our result.

14. [redacted] Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(x+y) = f(x) + f(y)$ for all x and y in $\mathbb{R}$. Show that $f(x) = ax$. (Hint: [redacted] show that $f(x)$ is continuous at every $x \in \mathbb{R}$. Then show that $f(x) = f(1) \cdot x$ for all $x \in \mathbb{Q}$. Deduce that $f(x) = f(1) \cdot x$ for every $x \in \mathbb{R} - \mathbb{Q}$)

Note that $f(0) = f(0+0) = f(0) + f(0)$

$$\Rightarrow f(0) = 0$$

Thus $0 = f(0) = f(c + (-c)) = f(c) + f(-c)$

$$\Rightarrow f(-c) = -f(c) \quad \forall c \in \mathbb{R}.$$

Let $c \in \mathbb{R}$ and $x_n \to c$ be arbitrary.

$\Rightarrow x_n - c$ is a null sequence.

Since f is cont. at 0, hence

$$f(x_n - c) \to 0$$

$$\Rightarrow f(x_n) - f(c) \to 0$$

$$\Rightarrow f(x_n) \to f(c)$$

Thus, f is cont. at every c.

Now, if $p, q \in \mathbb{Z}$ with $q \neq 0$, then

$$q \cdot f\left(\frac{p}{q}\right) = f\left(q \cdot \frac{p}{q}\right) \qquad \left(n \cdot f(x) = f(n \cdot x) \text{ by induction}\right)$$

$$= f(p)$$

$$= p \cdot f(1)$$

$$\Rightarrow f\left(\frac{p}{q}\right) = f(1) \cdot \frac{p}{q}.$$

Now, let c be an irrational no.

Then $\exists$ a rational $\{x_n\}_n$

s.t. $x_n \longrightarrow c$

since f - cont. at c,

$$f(x_n) \rightarrow f(c)$$

on the other hand, since $x_n \in \mathbb{Q}$,

$$f(x_n) = f(1) \cdot x_n \longrightarrow f(1) \cdot c$$

(since $x_n \rightarrow c$)

$$\Rightarrow f(c) = f(1) \cdot c.$$

Thus, if $f(1) = a$, then

$$f(x) = ax \quad \forall \ x.$$

15. (Homeomorphism) Let I and J be two intervals. A function $f : I \to J$ is called a *homeomorphism* if it satisfies the following conditions.

- $f : I \to J$ is continuous.
- $f(x)$ is bijective.
- $f^{-1} : J \to I$ is continuous.

We say that I and J are homeomorphic if there is a homeomorphism $I \to J$, and denote this relationship by $I \sim J$. Prove the following.

(a) $(0, 1) \sim (a, b)$.

(b) $(0, 1) \sim \mathbb{R}$ (i.e., $(0, 1) \sim (-\infty, \infty)$).

(c) $(0, 1) \not\sim [0, 1]$.

(a) Linear maps $Ax+B$ are continuous, invertible with the inverse $\frac{x-B}{A}$ cont.

So, enough to find a linear map

$$f : (0,1) \to (a,b).$$

say, $f(x) = (b-a)x + a$

Need to make sure that

$$f(x) \in (a,b) \quad \forall \ x \in (0,1).$$

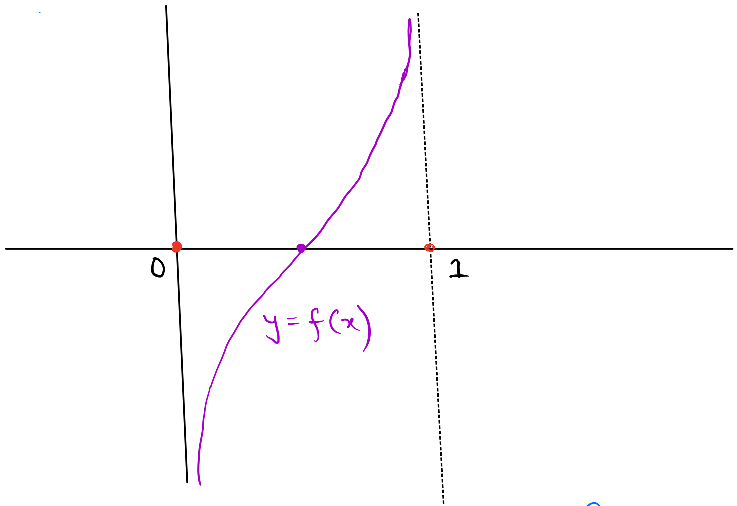
But since $(b-a) > 0$, hence

$$0 < x < 1$$

$$\Rightarrow \quad (b-a)\cdot 0 + a < (b-a)\cdot x + a < (b-a)\cdot 1 + a$$

i.e $\quad a < f(x) < b.$

(b)



want something like this.

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (-\infty, \infty)$$

is cont. bijective with

$$\tan^{-1} : (-\infty, \infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ cont.}$$

Thus $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim (-\infty, \infty)$

Since $(0,1) \sim \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, deduce that

$(0,1) \sim (-\infty, \infty)$ (Think about this!)

(c) Two ways to think about this

First Method: If $\exists$ a homeomorphism

$$f : [a,b] \to (a,b),$$

then (a,b) satisfies the fixed point Thm. which would be contradiction.(*)

(*) e.g. the constant fn. $g(x) = b$ has no fixed pt. on (a,b).

For, if $g : (a,b) \to (a,b)$ is cont., then

consider $f^{-1} \circ g \circ f : [a,b] \to [a,b]$

$$\begin{tikzcd}
(a,b) \arrow[r, "g"] & (a,b) \arrow[d, "f^{-1}"] \\
{[a,b]} \arrow[u, "f"] \arrow[r, "f^{-1} \circ g \circ f"'] & {[a,b]}
\end{tikzcd}$$

being a composition of cont. fns.,

$f^{-1} \circ g \circ f$ is cont. $[a,b] \to [a,b]$

We proved this for $[0,1]$. But the proof goes through, word for word, for $[a,b]$.

$\Rightarrow$ $f^{-1} \circ g \circ f$ has a fixed pt.

$\Rightarrow$ $\exists\, c \in [a,b]$ s.t

$$\left(f^{-1} \circ g \circ f\right)(c) = c$$

that is, $f^{-1}\left(g \circ f(c)\right) = c$

$$\Rightarrow \quad g \circ f(c) = f(c)$$

$$\Rightarrow \quad g\left(f(c)\right) = f(c)$$

Since $f(c) \in (a,b)$

$\Rightarrow$ $f(c)$ is a fixed pt. of g.

Second Method:

Supp. $f : (a,b) \longrightarrow [a,b]$ is a homeomorphism.

$\exists$ a sequence $\{x_n\}_n$ in (a,b)

s.t. $x_n \to a$

$\Rightarrow \{x_n\}_n$ is Cauchy.

Now, $\{f(x_n)\}_n$ is a sequence in $[a,b]$,

and hence bdd.

$\Rightarrow \exists$ a subsequence $\{f(x_{n_k})\}_k$ that

is convergent

let $f(x_{n_k}) \to l$

As explained in an earlier prob.

$l \in [a,b]$

$\Rightarrow l = f(a')$ for some $a' \in (a,b)$

Thus, $f(x_{n_k}) \longrightarrow f(a')$ in $[a,b]$

Now, $f^{-1} : [a,b] \longrightarrow (a,b)$ is cont.

$\Rightarrow f^{-1}(f(x_{n_k})) \longrightarrow f^{-1}(f(a'))$

$\Rightarrow x_{n_k} \to a'$ —— $(*)$

but $\{x_{n_k}\}_k$ is a subsequence of $\{x_n\}_n$

with $x_n \to a$

$\Rightarrow x_{n_k} \to a$ —— $(**)$

But then $(*)$ and $(**)$ $\Rightarrow a = a'$,

a contradiction since $a' \in (a,b)$.