भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

6^{th} Lecture on Transform Techniques

(MA-2120)

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What did we learn in previous class?

- Laplace Transform of Periodic functions
- Initial Value Theorem
- Final Value Theorem
- Convolution Theorem
- Inverse Laplace Transform

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What will we learn today?

- Inverse Laplace Transform
- Properties of Inverse Laplace Transform

Properties of the Inverse Laplace Transform

1 Linearity:

If $F(s) = \mathcal{L}[f(t)]$, $s > \alpha_1$ and $G(s) = \mathcal{L}[g(t)]$, $s > \alpha_2$, then for constants α and β,

$$\boxed{\mathcal{L}^{-1}[\alpha F(s) + \beta G(s)] = \alpha f(t) + \beta g(t), \quad t \geq 0}$$

convergence region for $s > \max\{\alpha_1, \alpha_2\}$.

Inverse Transformation:

$$\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}, \quad s > a.$$

Example:

So, $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right]$. [using linearity property]

$$= \mathcal{L}^{-1}\left[\frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+a}\right]$$

$$= \frac{1}{2}e^{at} - \frac{1}{2}e^{-at}$$

$$= \sinh at.$$

So,

$$\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at.$$

Inverse Transform :

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at}$$

&

$$\boxed{\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at.}$$

Ex.

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}, \quad s>0.$$

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}, \quad s>0$$

Inverse Transform:

$$\boxed{\begin{aligned} &\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at \\ &\& \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at. \end{aligned}}$$

Example :

$$\mathcal{L}^{-1}\left[\frac{2s+5}{s^2+25}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2s}{s^2+25} + \frac{5}{s^2+25}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right] + \mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right]$$

[By using Linearity property]

$$= 2\cos 5t + \sin 5t$$

Ex. $\mathcal{L}^{-1}\left(\frac{1}{s^{7/2}}\right)$

Solⁿ: $\mathcal{L}^{-1}\left[\frac{1}{s^{7/2}}\right] = \frac{1}{\Gamma\frac{7}{2}}\mathcal{L}^{-1}\left[\frac{\Gamma\frac{7}{2}}{s^{7/2}}\right]$

$$= \frac{1}{\Gamma\frac{7}{2}} \cdot t^{5/2} = \frac{8t^2}{15}\left(t/\pi\right)^{1/2}.$$

Ex: $$\mathcal{L}^{-1}\left[\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right]$$

Sol^n: using the linearity property, we have

$$\mathcal{L}^{-1}\left[\frac{6}{2s-3}\right] - \mathcal{L}^{-1}\left[\frac{3}{9s^2-16}\right] - \mathcal{L}\left[\frac{4s}{9s^2-16}\right]$$

$$+\mathcal{L}^{-1}\left[\frac{8}{16s^2+9}\right]$$

$$-\mathcal{L}^{-1}\left[\frac{6s}{16s^2+9}\right]$$

$$= 6\mathcal{L}^{-1}\left[\frac{1}{2s-3}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{9s^2-16}\right] - 4\mathcal{L}^{-1}\left[\frac{s}{9s^2-16}\right]$$

$$+ 8\mathcal{L}^{-1}\left[\frac{1}{16s^2+9}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{16s^2+9}\right]$$

$$= 6/2\,\mathcal{L}^{-1}\left[\frac{1}{s-3/2}\right] - 3/9\,\mathcal{L}^{-1}\left[\frac{1}{s^2-(4/3)^2}\right]$$

$$- 4/9\,\mathcal{L}^{-1}\left[\frac{s}{s^2-(4/3)^2}\right]$$

$$+ 8/16\,\mathcal{L}^{-1}\left[\frac{1}{s^2+(3/4)^2}\right] - \frac{6}{16}\mathcal{L}^{-1}\left[\frac{s}{s^2+(3/4)^2}\right]$$

$$= 3e^{3t/2} - \frac{1}{4}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{2}{3}\sin\frac{3t}{4}$$

$$- \frac{3}{8}\cos\frac{3t}{4}.$$

Ex: $\mathcal{L}^{-1}\left[\frac{3s-2}{s^{3/2}} + \frac{(\sqrt{s}-1)^2}{s^2} + \frac{4s-18}{9-s^2}\right]$

$$= \frac{5}{\sqrt{\pi t}} - 4\left(\frac{t}{\pi}\right)^{1/2} + 1 + t - 4\cosh 3t + 6\sinh 3t.$$

(You try it)

Method of Partial Fractions:

Ex. $$\mathcal{L}^{-1}\left[\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right]$$

Soln

We use the partial fractions to write the given function $F(s)$ as the sum of three factors.

$$\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$$

$\Rightarrow$ $A=2$, $B=2$, $C=1$.

Therefore $$\mathcal{L}^{-1}\left[\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2}{s-1} + \frac{2}{s-2} + \frac{1}{s+3}\right]$$

$$= 2e^{t} + 2e^{2t} + e^{-3t}.$$

Ex. $\mathcal{L}^{-1}\left[\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = e^{-t}+2e^{-2t}+3e^{-3t}.$

Ex. $\mathcal{L}^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \dfrac{a\sin at - b\sin bt}{a^2-b^2}.$

Hints: $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)} = \dfrac{1}{(a^2-b^2)}\left[\dfrac{a^2}{s^2+a^2} - \dfrac{b^2}{s^2+b^2}\right]$

② Inverse Laplace Transform on first shifting theorem:

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then $\mathcal{L}^{-1}[F(s-a)]$

$= e^{at}\mathcal{L}^{-1}[F(s)] = e^{at} f(t).$

$$\Rightarrow \boxed{\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)}$$

Ex. $\mathcal{L}^{-1}\left[\frac{1}{s^2-4s+8}\right]$

Solⁿ :

$$\frac{1}{s^2-4s+8} = \frac{1}{(s-2)^2+2^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2+2^2}\right] = \frac{1}{2}e^{2t}\sin 2t.$$

Ex. $\mathcal{L}^{-1}\left[\frac{4}{s^2-s+2}\right] = \mathcal{L}^{-1}\left[\frac{4}{(s-1/2)^2+(\sqrt{7}/2)^2}\right]$

$$= \frac{4}{\sqrt{7}/2}e^{t/2}\sin\left(\sqrt{7}/2\, t\right) = \frac{8}{\sqrt{7}}e^{t/2}\sin\left(\frac{\sqrt{7}t}{2}\right)$$

Ex. $\mathcal{L}^{-1}\left[\frac{3s-1}{(s-2)^2}\right] = 3e^{2t} + 5t\,e^{2t}$ (Try).

Ex. $\mathcal{L}^{-1}\left[\frac{s+8}{s^2+6s+13}\right] = e^{-3t}\cos 2t + \frac{3}{2}e^{-3t}\sin 2t$.

(Try).

Ex. $\mathcal{L}^{-1}\left[\frac{s}{(s+1)^5}\right] = \mathcal{L}^{-1}\left[\frac{(s+1)-1}{(s+1)^5}\right]$

$= e^{-t}\mathcal{L}^{-1}\left[\frac{s-1}{s^5}\right] = e^{-t}\mathcal{L}^{-1}\left[\frac{1}{s^4} - \frac{1}{s^5}\right]$

$= e^{-t}\left(\frac{t^3}{3!} - \frac{t^4}{4!}\right).$

Ex. $\mathcal{L}^{-1}\left[\dfrac{s}{(s+1)^{5/2}}\right] = 2e^{-t}\left(\dfrac{t}{\pi}\right)^{1/2}\left(1-\dfrac{2t}{3}\right)$ (Try).

Ex. $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4s+1}\right]$

$$= e^{-2t}\cosh\sqrt{3}\,t - \frac{2}{\sqrt{3}}\, e^{-2t}\sinh\sqrt{3}\,t$$

(3) Inverse Laplace Transform on Second Shifting theorem:

If $\mathcal{L}^{-1}[f(t)] = F(s)$, then

$$\boxed{\mathcal{L}^{-1}\left[e^{-as} F(s)\right] = H(t-a)\, f(t-a)}$$

$a \geq 0$.

Ex: $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s+2}\right]$

$= \mathcal{L}^{-1}\left[e^{-3s}\cdot\dfrac{1}{s+2}\right]$

$= H(t-3)\,f(t-3)$

$= H(t-3)\,e^{-2(t-3)}$

Here $a=3$.

$F(s) = \dfrac{1}{s+2}$

and

$f(t) = \mathcal{L}^{-1}[F(s)]$

$= \mathcal{L}^{-1}\left[\dfrac{1}{s+2}\right]$

$= e^{-2t}$

Ex.

$$\mathcal{L}^{-1}\left[\frac{4e^{-(s\pi/2)}}{s^2+16}\right]$$

$$= H(t-\pi/2)\sin 4t.$$

Ex: $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = (t-2)H(t-2)$

Try!

④ Inverse Laplace Transform on change of Scale property:

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{a}F(s/a)\right] = f(at). \quad a>0}$$

$$\mathcal{L}^{-1}[aF(as)] = f(t/a). \quad a>0$$

$$\Rightarrow \mathcal{L}^{-1}[aF(as)] = f(t/a)$$

$$\Rightarrow \mathcal{L}^{-1}[F(s.a)] = \frac{1}{a}f(t/a).$$

Ex: If $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\sin t$, find $\mathcal{L}^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\}$.

Soln:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2}\times\frac{1}{a}\times t/a \sin t/a$$

For the present problem, we put $a=4$ and obtain

$$\mathcal{L}^{-1}\left\{\frac{8.4s}{(16s^2+1)^2}\right\} = 8.t/_{32} \sin t/4.$$

$$= t/4 \sin t/4.$$

(5) Inverse Laplace Transform of Derivatives :

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\boxed{\mathcal{L}^{-1}\left[\frac{d^n F(s)}{ds^n}\right] = (-1)^n t^n f(t).}$$

$$= (-1)^n t^n \mathcal{L}^{-1}[F(s)]$$

Ex:

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

Solⁿ:

Let $F(s) = \dfrac{1}{s^2+a^2}$.

$$F'(s) = \frac{dF}{ds} = -\frac{2s}{(s^2+a^2)^2}$$

$$\Rightarrow \frac{s}{(s^2+a^2)^2} = -\frac{1}{2}F'(s)$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[-\frac{1}{2}F'(s)\right] = -\frac{1}{2}\mathcal{L}^{-1}\left[F'(s)\right]$$

$$= -\frac{1}{2}\cdot(-1)\cdot t\cdot\mathcal{L}^{-1}\left[F(s)\right]$$

$$= -\frac{1}{2} \cdot -1 \cdot t \cdot \frac{1}{a} \cdot \sin at$$

$$= \frac{t}{2a} \sin at.$$

$$\therefore \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a} \sin at$$

Ex $\mathcal{L}^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right]$

$$\frac{2(s+1)}{(s^2+2s+2)^2} = \frac{2(s+1)}{\left[(s+1)^2+1\right]^2} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -F'(s)$$

$$\mathcal{L}^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right] = \mathcal{L}^{-1}\left[-F'(s)\right]$$

$$= t f(t) = t e^{-t} \sin t.$$

where $f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$$= e^{-t} \sin t.$$

Ex: $\mathcal{L}^{-1}\left[\frac{1}{(s+5)^4}\right]$

$$\frac{1}{(s+5)^4} = -\frac{1}{6}\frac{d^3}{ds^3}\left[\frac{1}{s+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{d^3}{ds^3}\left(\frac{1}{s+5}\right)\right] = (-1)^3 t^3 e^{-5t}.$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+5)^4}\right] = \frac{t^3}{6} e^{-5t}.$$

Ex.

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+9)^2}\right] = \frac{1}{18}\left[\frac{1}{3}\sin 3t - t\cos 3t\right].$$ (Try)

Ex: $\mathcal{L}^{-1}\left[\log\left(1+a/s\right)\right]$

Solⁿ:

$$F(s) = \log\left(1+a/s\right)$$
$$= \log(s+a) - \log s.$$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s}.$$

we know $\mathcal{L}^{-1}\left[F'(s)\right] = (-1).\,t\,f(t)$
$$= -t\,\mathcal{L}^{-1}\left[F(s)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[F(s)\right] = -\frac{1}{t}\,\mathcal{L}^{-1}\left[F'(s)\right].$$

$$\mathcal{L}^{-1}\left[F(s)\right] = -\frac{1}{k}\mathcal{L}^{-1}\left[\frac{1}{s+a} - \frac{1}{s}\right]$$

$$= -\frac{1}{k}\left(e^{-at} - 1\right).$$

(6) Inverse Laplace Transform of integral:

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t},$$ provided

$\int_s^\infty F(s)\,ds$ exists.

Ex: $\mathcal{L}^{-1}\left[\int_s^{\infty} \frac{1}{s(s+1)}\,ds\right]$

Solⁿ: Let $F(s) = \frac{1}{s(s+1)}$, $\mathcal{L}^{-1}[F(s)] = f(t)$.

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$= 1 - e^{-t}$$

$$\mathcal{L}^{-1}\left[\int_s^{\infty} F(s)\,ds\right] = \frac{f(t)}{t}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\int_s^{\infty} \frac{1}{s(s+1)}\,ds\right] = \frac{(1-e^{-t})}{t}.$$

(7) Multiplication by power of s: (Derivative of Inverse Laplace Transform)

Theorem: If $\mathcal{L}^{-1}\left[F(s)\right] = f(t)$ and $f(0) = 0$,

then $\mathcal{L}^{-1}\left[sF(s)\right] = f'(t)$.

Theorem: If $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, then

$$\mathcal{L}^{-1}\left[s^n F(s)\right] = f^{(n)}(t).$$

Ex: $\mathcal{L}^{-1}\left[\dfrac{s^2}{(s^2+4)^2}\right]$

Solⁿ: Let $F(s) = \dfrac{1}{(s^2+4)}$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{\sin 2t}{2}$$

$$\frac{d\,F(s)}{ds} = -2\cdot\frac{s}{(s^2+4)^2}$$

$$\mathcal{L}^{-1}\left[F'(s)\right] = -2\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{t}{4}\sin 2t.$$

Let $g(t) = \frac{t}{4}\sin 2t$

and $G(s) = \frac{s}{(s^2+4)^2}$

Here $g(0) = 0$. So we have

$$\mathcal{L}^{-1}\left[sG(s)\right] = g'(t)$$

$$= \frac{\sin 2t + 2t\cos 2t}{4}.$$

(8) Division by powers of s: (Integral of Inverse Laplace Transform):

Let $\mathcal{L}^{-1}[F(s)] = f(t)$. If $f(t)$ is piecewise continuous and of exponential order α such that $\lim_{t \to 0} \left\{ \frac{f(t)}{t} \right\}$ exists, then for $s > \alpha$,

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(\tau)\, d\tau.$$

Theorem: If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}\left[\frac{f(s)}{s^n}\right] = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n \text{ times}} f(\tau)(d\tau)^n$$

Ex:

$$\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$$

Sol^n:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+1}\right] = \int_0^t \sin\tau \, d\tau$$

$$= 1 - \cos t.$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right] = \int_0^t (1-\cos\tau)\, d\tau$$

$$= t - \sin t.$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{s^2+1}\right]$$

$$= \int_0^t \left(\tau - \sin\tau\right) d\tau$$

$$= \frac{t^2}{2} + \cos t - 1 .$$

Ex: $\mathcal{L}^{-1}\left[\frac{1}{s}\log\frac{s+2}{s+1}\right]$

$$= \int_0^t \frac{e^{-x} - e^{-2x}}{x}\,dx.$$

Try it

Ex: $\mathcal{L}^{-1}\left[\frac{1}{s^4(s^2+1)}\right] = \frac{t^3}{6} + \sin t - t.$

9 Convolution Theorem :

If $\mathcal{L}^{-1}[F(s)] = f(t)$ and $\mathcal{L}^{-1}[G(s)] = g(t)$, then $\mathcal{L}^{-1}[F(s)\,G(s)]$

$$= \int_0^t f(u)\,g(t-u)\,du = f*g(t).$$

Here $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$.

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left[F(s)\,G(s)\right] = (f*g)(t).}$$

Ex:

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right]$$

$$\frac{1}{(s^2+\omega^2)^2} = \frac{1}{(s^2+\omega^2)} \cdot \frac{1}{(s^2+\omega^2)}$$

$$= F(s) \cdot G(s).$$

Where $F(s) = \frac{1}{s^2+\omega^2} = G(s)$

We have $f(t) = g(t) = \sin \omega t/\omega$.

Therefore, $\mathcal{L}^{-1}\left[F(s)\,G(s)\right]$

$$= (f*g)(t) = \int_0^t f(x)\,g(t-x)\,dx$$

$$= \frac{1}{\omega^2}\int_0^t \sin \omega x \sin(t-x)\,dx$$

$$= \frac{1}{2\omega^3}\left[\sin \omega t - \omega t \cos \omega t\right].$$

Ex. $\mathcal{L}^{-1}\left[\dfrac{1}{(s-2)(s+3)}\right]$

$$\frac{1}{(s-2)(s+3)} = \frac{1}{(s-2)} \cdot \frac{1}{(s+3)} = F(s)\,G(s)$$

where, $F(s) = \dfrac{1}{(s-2)}$ and $G(s) = \dfrac{1}{(s+3)}$

$\mathcal{L}^{-1}[F(s)] = e^{2t} = f(t)$ and $\mathcal{L}^{-1}[G(s)] = e^{-3t} = g(t)$

Therefore, $\mathcal{L}^{-1}[F(s)\,G(s)] = (f*g)(t)$

$$= \int_0^t e^{2x} e^{-3(t-x)}\,dx = \frac{1}{5}\left[e^{2t} - e^{-3t}\right].$$

Ex: $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \frac{1}{2}(\sin t - \cos t + e^{-t})$.

(try),

Ex: $\mathcal{L}^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right]$

$F(s) = \frac{1}{s-2}$ and $G(s) = \frac{1}{(s+2)^2}$

$f(t) = \mathcal{L}^{-1}[F(s)] = e^{2t} = f(t)$

$g(t) = \mathcal{L}^{-1}[G(s)] = e^{-2t}\,\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$

$= e^{-2t} \cdot t. = g(t)$

Now. $\mathcal{L}^{-1}\left[F(s)\,G(s)\right] = (f*g)(t)$

$$= \int_0^t f(x)\, g(t-x)\, dx$$

$$= \int_0^t e^{2x} \cdot (t-x)\, e^{-2(t-x)}\, dx.$$

$$= \frac{1}{16}\left\{ e^{2t} - (4t+1)\, e^{-2t} \right\}.$$

Heaviside's Expansion Theorem :

If $F(s) = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in s and the degree of $Q(s)$ is higher than that of $P(s)$, then

$$\boxed{\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{t\alpha_k}}$$

where $Q(s) = (s-\alpha_1)\cdots(s-\alpha_n)$

$\alpha_1, \alpha_2 \ldots \alpha_n$ are distinct constants.

Proof: Let $Q(s) = (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_k)$

Now we have by partial fractions

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \cdots + \frac{A_n}{s-\alpha_n}$$

$$= \sum_{k=1}^{n} \frac{A_k}{s-\alpha_k}$$

To compute A_k, multiplying by $(s-\alpha_k)$ and taking limit as $s \to \alpha_k$, we have

$$A_k = \lim_{s \to \alpha_k} \frac{P(s) \times (s - \alpha_k)}{Q(s)}$$

$$= P(\alpha_k) \lim_{s \to \alpha_k} \frac{s - \alpha_k}{Q(s)}$$

$$= P(\alpha_k) \lim_{s \to \alpha_k} \frac{1}{Q'(s)}$$ [by L'Hospital's Rule]

$$= P(\alpha_k) \cdot \frac{1}{Q'(\alpha_k)}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right]$$

$$= \sum_{k=1}^{n} \mathcal{L}^{-1}\left[\frac{A_k}{s-\alpha_k}\right]$$

$$= \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} \cdot e^{\alpha_k t}.$$

Ex:

$$\mathcal{L}^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s}\right]$$

Solⁿ:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{2s^2+5s-4}{s^3+s^2-2s}$$

$$P(s) = 2s^2+5s-4$$

$$Q(s) = s^3+s^2-2s = \underbrace{s(s-1)(s+2)}_{\Downarrow}$$

$$Q'(s) = 3s^2+2s-2.$$

$\alpha_1 = 0,\ \alpha_2 = 1$ and $\alpha_3 = 2.$

$$\mathcal{L}^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s}\right]$$

$$= \sum_{K=1}^{3} \frac{P(\alpha_K)}{Q'(\alpha_K)} e^{\alpha_K t}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t}$$

$$= 2 + e^{t} - e^{-2t}.$$

Ex :

$$\mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$$

$$= 2e^t + \sin t - 2\cos t.$$

Ex :

$$\mathcal{L}^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = \frac{e^t}{2} + e^{2t} + \frac{5e^{3t}}{2}$$

Initial value Theorem :

(i) $\lim_{s \to \infty} [sF(s)] = f(0)$

(ii) $\lim_{s \to \infty} [s^2F(s) - sf(0)] = f'(0)$

(iii) $\lim_{s \to \infty} [s^{n+1}F(s) - s^nF(s) - \cdots sf^{(n-1)}(0)]$

$= f^{(n)}(0)$