A sequence of real numbers

$a_1, a_2, a_3, \ldots\ldots$

$\{a_n\}_{n=1}^{\infty}$ in short, or simply $\{a_n\}_n$

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$

with $f(n) = a_n$

<u>Remark.</u> A sequence must be distinguished from its range.

For instance,

- $1, 1, 1, \ldots\ldots = \{1\}_{n=1}^{\infty}$

  is a sequence, where as

  its range $= \{1\}$, a singleton.

- $1, 4, 3, 2, 5, 6, \ldots$ is a sequence

  its range is $\{1, 2, 3, 4, 5, 6, \ldots\}$.

Convergence of a sequence & its limit

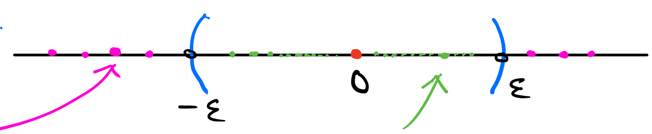
Null sequence.

A sequence $\{a_n\}_n$ is called a null sequence, or said to be converging to 0 if

(depending on ε)

for every $\varepsilon > 0$, $\exists$ some $n_0 \in \mathbb{N}$ s.t. $|a_n| < \varepsilon \;\; \forall \; n \geq n_0$.

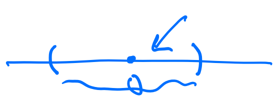
outside every nbhd. of 0 there can be at most finitly many terms of the sequence.



a_n where $n < n_0$ $\qquad$ a_n where $n \geq n_0$

only $a_1, a_2, \ldots, a_{n_0 - 1}$ can be outside $(-\varepsilon, \varepsilon)$

We write $\lim\limits_{n \to \infty} a_n = 0$



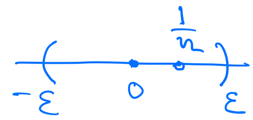
or simply $a_n \to 0$

$a_n \in I \;\; \forall \; n \geq 6.$

$1, 2, 3, 4, 5, 0, 0, 0, \ldots, 0$

$0 < \frac{1}{n} < \varepsilon$

$n\varepsilon > 1$



<u>Example</u>. • $\left\{\frac{1}{n}\right\}_n \longrightarrow 0$

Let $\varepsilon > 0$.

To show: $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n} < \varepsilon \quad \forall \quad n \geq n_0 .$$

By AP, $\exists$ a n s.t.

$$n\varepsilon > 1$$

By WOP, $\exists$ a <u>smallest</u> n with this property. Call it n_0.

Then
- $n_0 \varepsilon > 1$, and
- $n\varepsilon > 1 \quad \forall \quad n \geq n_0$

$$\Rightarrow \quad \frac{1}{n} < \varepsilon \quad \forall \quad n \geq n_0 .$$

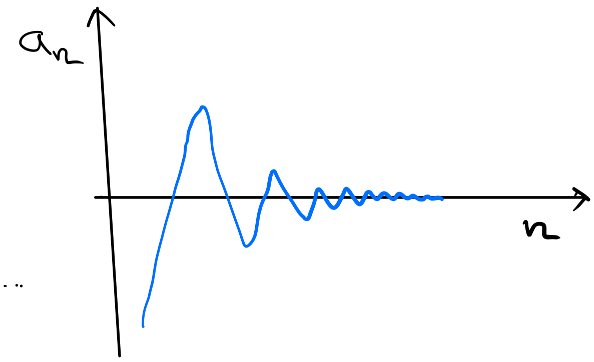
Thus, $\lim_{n\to\infty} \frac{1}{n} = 0$.

<u>Remark</u>: The choice of n_0 depends on ε.
e.g. here, $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$, the roof of $1/\varepsilon$

- $\left\{ \frac{(-1)^n}{n} \right\}_n \longrightarrow 0.$

  $\uparrow$

  $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$



Given every $\varepsilon > 0$, we have

$$\left| \frac{(-1)^n}{n} \right| = \frac{1}{n} < \varepsilon \quad \forall \; n \geq n_0$$

where n_0 is the same as in the previous example.

- $\left\{ \frac{(-1)^{n+1}}{n^2} \right\}_n \longrightarrow 0$

$$\left| \frac{(-1)^{n+1}}{n^2} \right| = \frac{1}{n^2} < \frac{1}{n} < \varepsilon$$

whenever $n \geq n_0$.

Limit of a (convergent) sequence.

A sequence $\{a_n\}_n$ is said to converge to $\ell \in \mathbb{R}$ if the sequence $\{b_n\}_n$ is a null seq. where $b_n = a_n - \ell$.

"ℓ" above, is called the limit of $\{a_n\}$

We write $\lim\limits_{n\to\infty} a_n = \ell$

or, $a_n \longrightarrow \ell$.

A Direct Definition.

Let $\{a_n\}_n$ be a sequence.

A real number ℓ is said to be a limit of $\{a_n\}$ if given every $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t. $|a_n - \ell| < \varepsilon \quad \forall \; n \geq n_0$.

As a first example, we prove the following result.

<u>Proposition.</u> (Rational Approximation)

Let $a \in \mathbb{R}$. Then $\exists$ a sequence $\{a_n\}_n$ with $a_n \in \mathbb{Q}$ s.t.

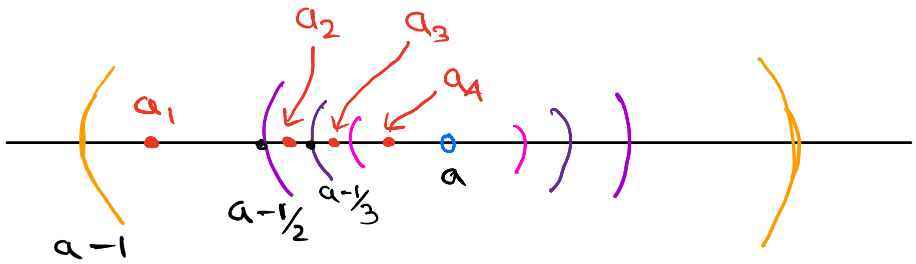
- $a_n \neq a$
- $a_n \longrightarrow a$.

<u>Proof.</u> Recall the system of nbhds. $I_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$

Since every interval contains ∞-ly many rational numbers,

choose $a_1 \in I_1$, $a_2 \in I_2 \setminus \{a_1\}$, $a_3 \in I_3 \setminus \{a_1, a_2\}$, ...

$$a_{n+1} \in I_{n+1} \setminus \{a_1, a_2, \dots, a_n\}$$



Note that $|a_n - a| < \frac{1}{n}$.

Claim: $a_n \longrightarrow a$

Let $\varepsilon > 0$ be given

To show: $\exists \; n_0 \in \mathbb{N}$ s.t.

$$|a_n - a| < \varepsilon \quad \forall \; n \geq n_0$$

By AP + WOP, $\exists$ a smallest $n_0 \in \mathbb{N}$ s.t. $n_0 \varepsilon > 1$

i.e., $n\varepsilon > 1 \quad \forall \; n \geq n_0$

$$\Rightarrow \frac{1}{n} < \varepsilon \quad \forall \; n \geq n_0.$$

Thus, for $n \geq n_0$, we have

$|a_n - a| < \frac{1}{n} < \varepsilon$, as required.

Examples.

- Constant sequence $\{\overset{a_n}{c}\}_n = c, c, \cdots\cdots$

  $b_n = a_n - c = 0$, so that $b_n \to 0$

  hence $a_n \to c$.

- $\underbrace{1 + \frac{1}{n}}_{a_n - 1 = \frac{1}{n}} \longrightarrow 1$

- $1 + \frac{(-1)^n}{n^3} \longrightarrow 1$

- $a_n = 2^{1/n}$?

Sandwich Theorem

Suppose $a_n \to \ell$ and $b_n \to \ell$

If $\{c_n\}_n$ is a sequence s.t.

$$a_n \le c_n \le b_n \quad \forall\ n \quad \text{---}(*)$$

Then $c_n \longrightarrow \ell$.

Proof. First suppose $\ell = 0$.

Then given $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t.

$$|b_n| < \varepsilon \quad \forall\ n \ge n_0$$

Similarly, for the same $\varepsilon > 0$,
$\exists \; n_1 \in \mathbb{N}$ s.t.

$$|a_n| < \varepsilon \quad \forall \; n \geq n_1.$$

Thus, if $n_2 = \max\{n_0, n_1\}$, then

$$|b_n| < \varepsilon \;\; \& \;\; |a_n| < \varepsilon \quad \forall \; n \geq n_2.$$

Now, if $n \geq n_2$, then

$$-\varepsilon < a_n \leq c_n \leq b_n < \varepsilon$$

$$\Rightarrow \quad |c_n| < \varepsilon \quad \forall \; n \geq n_2$$

$$\Rightarrow \quad c_n \to 0.$$

Next suppose that $\ell \neq 0$.
Then $a_n - \ell \to 0$ and $b_n - \ell \to 0$

from $*$ we find that

$$a_n - \ell \leq c_n - \ell \leq b_n - \ell \quad \forall \; n \in \mathbb{N}.$$

by the previous part,

$$c_n - \ell \to 0$$

$$\Rightarrow \quad c_n \to \ell.$$

- Back to $a_n = 2^{1/n}$.

Clearly, $a_n > 1$.

Also, $a_n^n - 1 = 1$

i.e. $(a_n - 1)(a_n^{n-1} + a_n^{n-2} + \cdots + 1) = 1$

or, $a_n - 1 = \dfrac{1}{a_n^{n-1} + a_n^{n-2} + \cdots + 1} < \dfrac{1}{n}$

Thus,

$$0 < a_n - 1 < \frac{1}{n}$$

$$\Rightarrow a_n - 1 \longrightarrow 0$$

$$\Rightarrow a_n \longrightarrow 1.$$

Alternatively,

$$a_n = 2^{1/n} = e^{\frac{\log 2}{n}} \longrightarrow e^0 = 1.$$

- So, if $a_n \to \ell$, does $e^{a_n} \to e^{\ell}$?

Discussion.

Let $\{a_n\}_n$ be a sequence, and k be a fixed +ve integer. Then

Supp. $k=10$,

$$a_n \longrightarrow \ell \iff a_{n+k} \longrightarrow \ell$$

$\rightarrow a_1, a_2, \cdots, a_{10}, a_{11}, \cdots$

$\rightarrow a_{11}, a_{12}, \cdots\cdots,$

Remark: In view of the last result, Sandwich Thm. can generalized as

Sandwich Theorem (General version)

Suppose $a_n \to \ell$ and $b_n \to \ell$

If $\{c_n\}_n$ is a sequence s.t. there is a natural no. k for which

$$a_n \leq c_n \leq b_n \quad \forall \quad n \geq k \quad \text{—} (**)$$

Then $c_n \to \ell$.

Proof. Set $a_n' = a_{n+k-1}$ & $b_n' = b_{n+k-1}$ and $c_n' = c_{n+k-1}$

Then $a_n' \to \ell$ & $b_n' \to \ell$,

and $a_n' \leq c_n' \leq b_n' \quad \forall n \in \mathbb{N}$.

By the previous version of Sandwich Thm.,

$$c_n' \to \ell$$

$$\Rightarrow c_n \to \ell.$$

• Back to showing that $a_n \to \ell \Rightarrow e^{a_n} \to e^{\ell}$

First assume $\ell = 0$.

Since $a_n \to 0$, taking $\varepsilon = 1$, we see that $\exists$ a $n_0 \in \mathbb{N}$ s.t. $|a_n| < 1 \;\; \forall \;\; n \geq N_0$.

Let $n \geq n_0$

Now,
$$|e^{a_n} - 1| = \left|a_n + \frac{a_n^2}{2!} + \frac{a_n^3}{3!} + \cdots\right|$$
$$\leq |a_n| + \frac{|a_n|^2}{2!} + \frac{|a_n|^3}{3!} + \cdots$$
$$< |a_n|\left(1 + \frac{|a_n|}{2} + \frac{|a_n|^2}{2^2} + \cdots\right)$$
$$= |a_n| \cdot \frac{1}{1 - \frac{|a_n|}{2}} \leq 2|a_n|$$

(since $|a_n| < 1$)

$a_n \to \ell$

$\Rightarrow$

$e^{a_n} \to e^{\ell}$

$a_n - \ell \to 0$

$e^{a_n - \ell} \to 1$

e^{ℓ}

$e^{a_n} \to e^{\ell}$

Thus,

$$0 \leq |e^{a_n} - 1| < 2|a_n| \quad \forall \; n \geq n_0.$$

$$\Rightarrow |e^{a_n} - 1| \to 0 \; !$$

$|a_n| \to 0$

$\Rightarrow a_n \to 0$ ✓

$\Rightarrow e^{a_n} - 1 \to 0$

$\Rightarrow e^{a_n} \to 1$

- $a_n \to 0 \Rightarrow 2|a_n| \to 0.$

  Let $b_n = 2|a_n|$

  To show: given $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $|b_n| < \varepsilon \quad \forall\ n \geq n_0$.

  Since $a_n \to 0$, for the given ε $\exists$ a $n_0 \in \mathbb{N}$ s.t.

  $$|a_n| < \frac{\varepsilon}{2} \quad \forall\ n \geq n_0.$$

  Now, for $n \geq n_0$, we have

  $$|b_n| = 2|a_n| < \varepsilon.$$

  $\square$

- $|e^{a_n} - 1| \to 0$

  $\Rightarrow e^{a_n} - 1 \to 0$

  $\Rightarrow e^{a_n} \to 1$

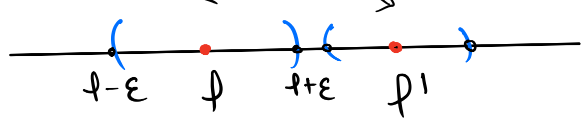
  To prove: if $|b_n| \to 0$, then $b_n \to 0$ (Ex.).

Main Results on Convergent Sequences

Thm. If $\{a_n\}_n$ is a convergent sequence, then $\lim_{n\to\infty} a_n$ is <u>unique</u>.

Proof. Suppose $a_n \to \ell$ and $a_n \to \ell'$.

one of these intervals can contain only finitely many terms of a sequence.



Given $\varepsilon > 0$, $\exists$ n_0 & n_0' in $\mathbb{N}$ s.t.

$$|a_n - \ell| < \varepsilon/2 \quad \forall\ n \geq n_0, \text{ and}$$

$$|a_n - \ell'| < \varepsilon/2 \quad \forall\ n \geq n_0'$$

$0 \leq x < \varepsilon$
$\Rightarrow x = 0$

Thus, if $n \geq \max\{n_0, n_0'\}$, then

$$|a_n - \ell| < \varepsilon/2 \quad \& \quad |a_n - \ell'| < \varepsilon/2$$

so that

if $x < \varepsilon$ (≤ 0) $\forall \varepsilon > 0$
$\Rightarrow x = 0$

$$0 \leq |\ell - \ell'| \leq |\ell - a_n| + |a_n - \ell'| < \varepsilon$$

$$\Rightarrow \ell = \ell' \quad (\text{since } \varepsilon > 0 \text{ is arbitrary}).$$

Thm. A convergent sequence is bounded.

That is, if $a_n \to \ell$, then $\exists$ a $M > 0$ s.t.

$$|a_n| \leq M \quad \forall\, n \in \mathbb{N}.$$

Proof. Suppose $a_n \to \ell$

Then taking $\varepsilon = 1$, we see that $\exists\, n_0 \in \mathbb{N}$ s.t.

$$|a_n - \ell| < 1 \quad \forall\, n \geq n_0$$

$$\Rightarrow \quad |a_n| \leq |a_n - \ell| + |\ell|$$

$$< 1 + |\ell| \quad \forall\, n \geq n_0$$

Let $M_0 = \max\{|a_1|, |a_2|, \cdots, |a_{n_0-1}|\}$

Set $M = \max\{M_0, 1 + |\ell|\}$

Then $|a_n| \leq M \quad \forall\, n \in \mathbb{N}$.

$a_n \to \ell$

given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $|a_n - \ell| < \varepsilon \quad \forall\, n \geq n_0$

As $\varepsilon \to 0$

$|a_n - \ell| \to 0$

every nbhd. of ℓ contains all but finitely many terms of $\{a_n\}_n$.

Algebra of convergent sequences

Let $a_n \to \ell$ & $b_n \to s$.

Then (i) $a_n \pm b_n \to \ell \pm s$

(ii) If $c \in \mathbb{R}$, then $ca_n \to c\ell$

(iii) $a_n \cdot b_n \to \ell \cdot s$

(iv) If $b_n \neq 0 \ \forall n$ and $s \neq 0$, then

$$a_n/b_n \to \ell/s.$$

$|-3a_n + 3\ell| < \varepsilon$

$|a_n - \ell| < \varepsilon/3$

n_0

if $n \geq n_0$

Proof. Outline of (ii)

Say $c = \boxed{-3}$.

To show: $a_n \to \ell \Rightarrow -3a_n \to -3\ell$

Set $b_n = -3a_n$

Given $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$|a_n - \ell| < \varepsilon/3 \quad \forall\ n \geq n_0$$

Then

$$|b_n + 3\ell| = |-3a_n + 3\ell| = 3|a_n - \ell| < \varepsilon$$

$$\forall\ n \geq n_0.$$

Outline of (iii)

To show: $a_n \to \ell$ & $b_n \to s$

$$\Rightarrow a_n b_n \to \ell s$$

$$a_n b_n - \ell s = a_n b_n - b_n \ell + b_n \ell - \ell s$$
$$= b_n (a_n - \ell) + \ell (b_n - s)$$

Thus,

$$|a_n b_n - \ell s| \leq |b_n| |a_n - \ell| + |\ell| |b_n - s|$$

Now, $\{b_n\}_n$ – convergent $\Rightarrow \{b_n\}_n$ – bdd.

$\Rightarrow \exists \; M_0 > 0$ s.t

$$|b_n| \leq M_0 \;\forall\; n \in \mathbb{N}$$

Set $M = M_0 + |\ell|$.

Then

$$|a_n b_n - \ell s| \leq M \left(|a_n - \ell| + |b_n - s| \right)$$

Now, given $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$

s.t. $|a_n - \ell| < \varepsilon/2M$ & $|b_n - s| < \varepsilon/2M$ $\forall\; n \geq n_0$

$\Rightarrow$ for $n \geq n_0$,

$$|a_n b_n - \ell s| < \varepsilon$$

Limit points of a set and limit of a sequence.

Thm. Let a be a limit point of a set E.
Then there is a sequence $\{a_n\}_n$ with $a_n \in E \ \forall n$, and $a_n \to a$.

Remark. Not every limit is a limit point.
Let $\{a_n\}_n$ with $a_n \in E$ and $a_n \to a$, then a is not necessarily a limit pt. of E.

e.g. Let $E = (0,1) \cup \{2\}$ and define $a_n = 2 \ \forall n \in \mathbb{N}$.
Then $a_n \to 2$, $a_n \in E$ but 2 is not a limit pt.

However, if the Range of $\{a_n\}_n$ is infinite & $a_n \to a$, then a is a limit pt. of E.

Bounded & Unbounded sequences

- A sequence $\{a_n\}_n$ is bdd. if $\exists$ some $M>0$

$$|a_n| \le M \quad \forall\ n \in \mathbb{N}.$$

- A sequence is unbounded if for every $M>0$, $\exists$ some $n \in \mathbb{N}$ s.t. $|a_n| > M$.

$\left.\begin{array}{l}\lim_{n\to\infty} a_n = \infty \\ a_n \to \infty\end{array}\right\}$

- A sequence $\{a_n\}_n$ is divergent if for every $M>0$ $\exists$ some $n_0 \in \mathbb{N}$ s.t. $|a_n| > M \ \forall\ n \ge n_0$.

Examples.

- $1, 2, 3, \ldots$

  i.e $a_n = n$ is divergent

$|a_n| > 1000$

$\forall\ n \ge n_0$

$M = 1000$

$a_{2n} = \frac{1}{2n} < 1 < 1000$

- $1, \tfrac{1}{2}, 3, \tfrac{1}{4}, \ldots$

$$a_n = \begin{cases} n & \text{if } n \text{ is odd} \\ 1/n & \text{if } n \text{ is even} \end{cases}$$

Then a_n is unbounded but not divergent.

Montone Sequences

A sequence $\{a_n\}_n$ is monotone increasing (resp. decreasing) if

$$a_1 \leq a_2 \leq a_3 \leq \cdots\cdots$$

or $\quad a_n \leq a_{n+1} \quad \forall \; n \in \mathbb{N}$

(resp. $a_1 \geq a_2 \geq a_3 \geq \cdots\cdots$

i.e $\; a_{n+1} \geq a_n \quad \forall \; n \in \mathbb{N}$).

Examples.

- $1, 1, 1, 1, \cdots\cdots$

  is a monotone sequence

- $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots\cdots$

  is a monotone decreasing

$1-\frac{1}{n}$

- $1-1 < 1-\frac{1}{2} < 1-\frac{1}{3} < 1-\frac{1}{4} < \cdots\cdots$

  is monotone increasing

- $1 < 2 < 3 < \cdots\cdots$ is monotone increasing

Thm. A bounded monotone sequence is convergent.

Meaning: If $a_1 \le a_2 \le a_3 \le \cdots$ and $\exists$ a M s.t. $a_n \le M \ \forall\ n \in \mathbb{N}$, then $\{a_n\}_n$ is convergent.

$\{a_n\}_n \uparrow$

E = Range of $\{a_n\}_n$.

Sup E exists

$L = \sup E$.

$\varepsilon > 0$

a_{n_0}

$L-\varepsilon$ $\quad$ L

$L-\varepsilon < a_n \le L \quad \forall\ n \ge n_0$

$a_n \in (L-\varepsilon, L+\varepsilon)$

OR

If $a_1 \ge a_2 \ge a_3 \ge \cdots$ and $\exists$ a m s.t. $a_n \ge m \ \forall\ n \in \mathbb{N}$, then $\{a_n\}_n$ is convergent.

Proof. Suppose $a_1 \le a_2 \le a_3 \le \cdots$ and $\exists$ a M s.t. $a_n \le M \ \forall\ n \in \mathbb{N}$.

Let E = Range of $\{a_n\}_n$.

Then E is bounded above.

Also, $E \ne \emptyset$

By the lub property of $\mathbb{R}$,

$L = \sup E$ exists.

<u>Claim:</u> $a_n \longrightarrow L$.

Given $\varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$L - \varepsilon < a_{n_0} \leq L$$

(from the defn. of sup)

Since, $a_{n_0} \leq a_n \quad \forall\ n \geq n_0$,

$-1/n \longrightarrow 0$

$1/n \longrightarrow 0$

$$L - \varepsilon < a_{n_0} \leq a_n \leq L \quad \forall\ n \geq n_0$$

$\Rightarrow a_n \in (L-\varepsilon, L+\varepsilon) \ \forall\ n \geq n_0$

$\Rightarrow a_n \longrightarrow L$.

<u>Proposition</u>. If $\{a_n\}_n$ is an unbounded monotone seq., then $a_n \longrightarrow \infty$

Subsequences

Let $\{a_n\}_n$ be a sequence, and

$$n_1 < n_2 < n_3 < \cdots\cdots$$

be an infinite chain of natural numbers.

$\{\frac{1}{n}\}_n$

$2 < 4 < 6 < \cdots$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$

$\{\frac{1}{2n}\}$.

Example.

- $2 < 4 < 6 < \cdots\cdots$
- $1 < 3 < 5 < \cdots\cdots$
- $1 < 4 < 9 < 16 < \cdots\cdots$

Then $a_{n_1}, a_{n_2}, a_{n_3}, \cdots\cdots$

is a subsequence of $\{a_n\}_n$.

Denoted by $\{a_{n_k}\}_k$

If $b_k = a_{n_k}$, then the subsequence is $\{b_k\}_k$

Examples.

- $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots\cdots$

  is a subsequence of $\{\frac{1}{n}\}_n$

- $1, 1, 1, \ldots\ldots$

  every subsequence is also

  $1, 1, 1, 1, \ldots\ldots$

- $1, -1, 1, -1, \ldots$

  $$a_n = (-1)^{n-1}$$

  $\{a_{2n}\}_n = -1, -1, -1, \ldots\ldots$

  is a subsequence

  $\{a_{2n+1}\}_n = 1, 1, 1, \ldots$

  is a subsequence

- $1, 1/2, 3, 1/4, \ldots\ldots$

  $$a_{2n+1} = 2n+1$$

  $$a_{2n} = \frac{1}{2n}$$

  Subsequences

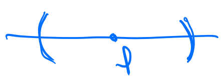
  $\{a_{2n+1}\}_n = 1, 3, 5, \ldots$

  $\{a_{2n}\}_n = 1/2, 1/4, \ldots$

Main results on subsequences

Thm. If $a_n \longrightarrow \ell$, then every subsequence of $\{a_n\}_n$ converges to ℓ.

Proof. Let $a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$ be a subsequence of $\{a_n\}_n$.



For convenience, set

$$b_k = a_{n_k}$$

Now, $a_n \longrightarrow \ell$

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$

s.t. $|a_n - \ell| < \varepsilon \quad \forall\ n \geq N$

$\exists$ a k_0 s.t.

$$N \leq n_{k_0}$$

But then, for $k \geq k_0$

$$|a_{n_k} - \ell| < \varepsilon$$

i.e. $|b_k - \ell| < \varepsilon \quad \forall\ k \geq k_0$

$\Rightarrow$ $b_k \longrightarrow \ell$.

Thm. (Bolzano-Weierstrass Theorem)

Let $\{a_n\}_n$ be a bounded sequence.

$1, -1, 1, -1, \ldots$

Then $\{a_n\}_n$ has a convergent subsequence.

Proof. Let E = The range of $\{a_n\}_n$

Consider two cases, namely

(i) E is finite

(ii) E is infinite

(i) If E is finite, then $\exists$ some $a \in E$ s.t.

$\left.\begin{matrix}1\\2\\3\end{matrix}\right\}$ $a_n = a$ for infinitely many n.

$\Rightarrow$ $A = \{n \in \mathbb{N} : a_n = a\}$ is an infinite subset of $\mathbb{N}$

so that $A = \{n_1, n_2, \ldots, n_k, \ldots\}$

$$n_1 < n_2 < \cdots < n_k < \cdots$$

& $a_{n_k} = a \quad \forall \; k \in \mathbb{N}$

$\Rightarrow$ $a_{n_k} \longrightarrow a$ (constant sequence).

Case (ii) requires some preparation.

Digression.

$[a, b]$

Thm. (Nested Interval Property (NIP)).

Suppose $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of closed intervals with $\operatorname{diam} I_n \longrightarrow 0$.

Then

$$\bigcap_{n=1}^{\infty} I_n = \{a\} \text{ for some } a.$$

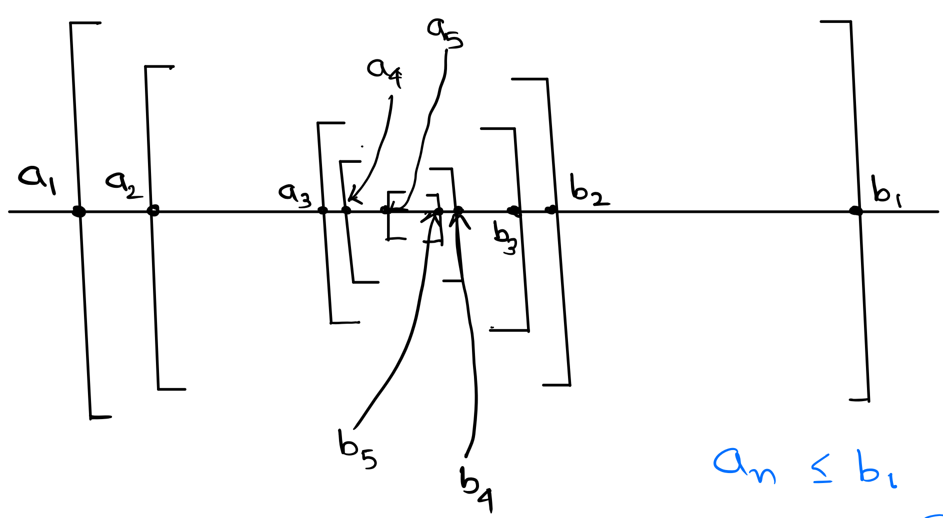
$J_n = (0, \frac{1}{n})$

$J_1 \supset J_2 \supset J_3 \supset \cdots$

$\bigcap_{n=1}^{\infty} J_n = \emptyset.$

$J_n = (-\frac{1}{n}, \frac{1}{n})$

$\bigcap_{n=1}^{\infty} J_n = \{0\}.$



$a_n \leq b_1$

$a_n < b_n < b_1$

If $I_n = [a_n, b_n]$, then

- $a_1 \leq a_2 \leq a_3 \leq \cdots$
- $b_1 \geq b_2 \geq b_3 \geq \cdots$
- $\text{diam } I_n = b_n - a_n$

Thus, $(b_n - a_n) \longrightarrow 0$

<u>Proof.</u> Since $\{a_n\}_n$ is a monotone increasing sequence and bdd. above (by, say b_1), hence $a_n \longrightarrow \sup_{n \in \mathbb{N}} a_n = a$ (say)

Similarly $b_n \longrightarrow \inf_{n \in \mathbb{N}} b_n = b$ (say)

Thus, $b_n - a_n \longrightarrow b - a$

$\Rightarrow a = b$ (since $a_n - b_n \longrightarrow 0$ and limit is unique)

$a \in \bigcap_{n=1}^{\infty} I_n$

<u>Claim</u>: $a \in \bigcap_{n=1}^{\infty} I_n$

This is clear since

$a_n \leq a \leq b_n \quad \forall\, n \in \mathbb{N}$.

Claim: $\bigcap_{n=1}^{\infty} I_n = \{a\}$.

Suppose $x \in \bigcap_{n=1}^{\infty} I_n$, then

$$a_n \le x \le b_n \quad \forall\, n \in \mathbb{N}$$

Taking limits $n \to \infty$,

$$a \le x \le a$$

$$\Rightarrow \quad x = a.$$

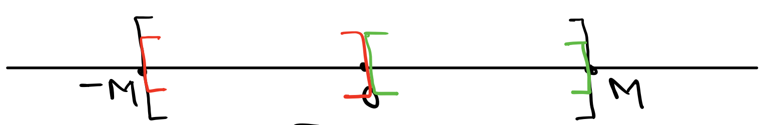
$\square$

Back to the proof of B-W Thm.

$\{a_n\}_n$, $\quad |a_n| \le M \quad \forall\, n \in \mathbb{N}$

and $E = \text{Range}\,\{a_n\}_n$ is infinite.

Note that $E \subseteq [-M, M]$



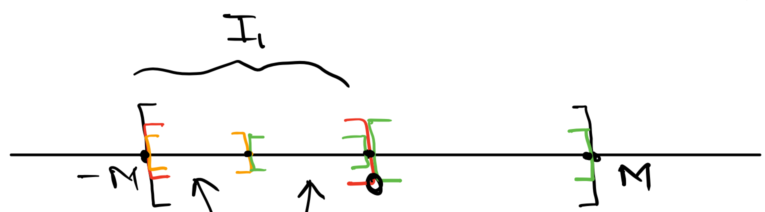
one of these intervals contains ∞-ly elements of E

One of $[-M, 0]$ and $[0, M]$ contains ∞-ly many elements of E.

Call that interval I_1

Pick $a_{n_1} \in I_1 \cap E$

Divide I_1 into two equal parts



one of these intervals contains ∞-ly elements of E

Call that I_2

Since $I_2 \cap E$ is infinite,

can pick $a_{n_2} \in I_2 \cap E$ s.t. $a_{n_2} \neq a_{n_1}$

Note that $I_2 \subsetneq I_1$

$\operatorname{diam} I_1 = M$, $\operatorname{diam} I_2 = M/2$

Continue this ....

After k-steps, we have

- picked k distinct elements of E $a_{n_1}, a_{n_2}, \dots, a_{n_k}$

with $a_{n_j} \in I_j \quad \forall \ j = 1, 2, \dots, k.$

$\{a_{n_k}\}_{k=1}^{\infty}$ • $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$, and

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

diam $I_n \to 0$ • $\text{diam } I_n = M/2^{n-1}$.

By NIP, $\bigcap_{n=1}^{\infty} I_n = \{a\}$.

Claim: $a_{n_k} \to a$

Since a and a_{n_k} are in I_k,

$$|a - a_{n_k}| \leq \text{diam } I_k = M/2^k$$

$$\Rightarrow a_{n_k} \to a.$$

Corollary: Every infinite bounded subset of $\mathbb{R}$ has a limit point.

Proof. Think about it!