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6th Lecture on Differential Equation

(MA-1150)

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What have we learnt?

- Method of solving non-exact first order and first degree ode:
  - ✓ Rule V
- Linear ODE
- Non-linear ODE: Bernoulli Equation

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What will we learn today?

- Equation of First Order but not First Degree
- Clairaut's Equation
- Singular Solution

Equations of First Order but not First Degree

- An eqn of first order and nth degree can be written as —

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \dots + P_{n-1}\frac{dy}{dx} + P_n = 0.$$

i.e., $\boxed{p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p_1 + P_n = 0}$

where $p = \frac{dy}{dx}$ and $P_1, P_2, \dots P_{n-1}, P_n$ are functions of x and y.

Equations of First Order but not First Degree

- We will discuss such eqⁿs in three special types —

(i) eqⁿs solvable for p
(ii) eqⁿs solvable for y
(iii) eqⁿs solvable for x.

Three Special types

Equations of First Order but not First Degree

Equation Solvable for p:

The given eqⁿ is

$$\boxed{p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0.} \quad -(1)$$

It is an algebraic eqⁿ in p.
So it has n roots $p_1, p_2, \ldots p_n$, which are functions of x and y.

Equations of First Order but not First Degree

So we can easily write the given ODE as

$$\boxed{(p-p_1)(p-p_2)\cdots(p-p_n)=0} \quad -(2)$$

Let $\phi_r(x,y,c_r)=0$ be the solution of $p-p_r=0$, for $r=1,2,\ldots n$.

Equations of First Order but not First Degree

All possible solutions of the ODE are written in the form

$$\boxed{\phi_1(x, y, c_1)\,\phi_2(x, y, c_2)\cdots\phi_n(x, y, c_n) = 0}$$

Since the given eqn is first order only, we must have only one constants but here we are having n constants.

Equations of First Order but not First Degree

So all constants must be same say c.

Hence the general solution is obtained as

$$\boxed{\phi_1(x,y,c)\,\phi_2(x,y,c)\cdots\phi_n(x,y,c)=0}$$

where c is any arbitrary constant.

Examples

Ex ①: $p^2 + p(x+y) + xy = 0$

$\Rightarrow (p+x)(p+y) = 0$

$(p+x) = 0$ gives $2y = -x^2 + c_1$

$(p+y) = 0$ gives $x = -\log y + c_2$

The general solution is

$$\boxed{(2y + x^2 - c)(x + \log y - c) = 0}$$

, where c is arbitrary constant.

Equations of First Order but not First Degree

Ex. 2 $xy\left\{\left(\frac{dy}{dx}\right)^2-1\right\} = (x^2-y^2)\frac{dy}{dx}$

Let $\frac{dy}{dx} = p$

Then the given eqn becomes —

$xy(p^2-1) = (x^2-y^2)p$

or, $(x-py)(y+px) = 0$

$x-py=0$ or, $y+px=0$

$\hookrightarrow x\,dx - y\,dy = 0 \Rightarrow \boxed{x^2-y^2=2c}$

$y+px=0$

or $x\,dy + y\,dx = 0$

$\Rightarrow \boxed{xy = c}$

Now the general solution is

$(x^2-y^2-2c)(xy-c) = 0$.

where c is an arbitrary constant.

Examples

Ex. 3 $\quad x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 = x^2y^2 + x^4$

Let $p = \frac{dy}{dx}$,

$$x^2p^2 - 2xyp + y^2 = x^2y^2 + x^4$$

or, $(xp-y)^2 = x^2(x^2+y^2)$

or, $(xp-y) = \pm x\sqrt{x^2+y^2}$

or, $p = \frac{y \pm x\sqrt{x^2+y^2}}{x}$

or, $\frac{dy}{dx} = \frac{y \pm x\sqrt{x^2+y^2}}{x}$

Examples

Put $y = vx$ or $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

we have,

$$v + x\frac{dv}{dx} = y/x \pm x\sqrt{1 + y^2/x^2}$$

$$= v \pm x\sqrt{1+v^2}$$

or $$x\frac{dv}{dx} = \pm x\sqrt{1+v^2}$$

or $$\frac{dv}{\sqrt{1+v^2}} = \pm\, dx$$

or $$\sinh^{-1} v = \pm x + c$$

or $$\sinh^{-1}(y/x) = \pm x + c$$

or $$\boxed{y = x\sinh(c \pm x)}$$

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Examples

we have, $y - x\sinh(x+c) = 0$
$y - x\sinh(c-x) = 0$

So, the general sol^n is

$$\boxed{\{y - x\sinh(x+c)\}\{y - x\sinh(c-x)\} = 0}$$

where c is an arbitrary constant.

Equations of First Order but not First Degree

② Equations Solvable for y:

If the eq^n is solvable for y, then it is in the form of

$$\boxed{y = f(x, p)}$$

Differentiate w.r.t x, we have

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dx}$$

Examples

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dx} = g\left(x, p, \frac{dp}{dx}\right)$$

(Say).

Now we have to solve

$$p = g\left(x, p, \frac{dp}{dx}\right)$$ to get

$$\underline{h(x, p, c) = 0.}$$

Examples

Therefore the general solution is obtained by eliminating p between $y = f(x, p)$ and $h(x, p, c) = 0$.

Example $y + px = p^2 x^4$

Soln: differentiating w.r.t x, we get

$$p + p + x\frac{dp}{dx} = 2px^4\frac{dp}{dx} + p^2 \cdot 4x^3$$

Examples

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or $2p(1-2x^3p) + x\frac{dp}{dx}(1-2x^3p) = 0$

or $(1-2x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$

$\therefore 2p + x\frac{dp}{dx} = 0$ or $1-2x^3p = 0$

$\Rightarrow p = c/x^2$ (after integration)

Examples

Now we have

$$p = c/x^2 \quad \text{--- } (1)$$

and $$y + px = p^2 x^4 \quad \text{--- } (2)$$

Now eliminating p from (1) and (2)

we have

$$y = -\frac{c}{x^2}\cdot x + \frac{c^2}{x^4}\cdot x^4$$

$$= -c/x + c^2.$$

Examples

Hence $xy + c = c^2 x$ is the general sol^n.

From $(1 - 2x^3 p) = 0$, we have

$$\Rightarrow p = \frac{1}{2x^3}.$$

we have from the given eq^n

$$y = -px + p^2 x^4 = -\frac{1}{4x^2}$$

Examples

$\therefore\ 4x^2y+1=0$ is the singular

Solution.

Examples

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② Equation Solvable for x:

Similarly we can find out the general solution.

Ex.: $p^2 y + 2px = y$

Sol^n: we can easily write the given eq^n in the form

Examples

$$x = \frac{y}{2p}(1-p^2) = \frac{y}{2}\left\{\frac{1}{p} - p\right\}$$

Differentiating w.r.t y

$$\frac{dx}{dy} = \frac{1-p^2}{2p} + y\left\{-\frac{1}{2}\frac{(1+p^2)}{p^2}\cdot\frac{dp}{dy}\right\}$$

$$\Rightarrow \frac{1}{p} = \frac{1-p^2}{2p} - \frac{y}{2}\frac{(1+p^2)}{p^2}\frac{dp}{dy}.$$

Examples

$$\text{or}\quad (1+p^2)\left\{\frac{1}{2p} + \frac{1}{2p^2}\, y\, \frac{dp}{dy}\right\} = 0$$

$$\text{or,}\quad 1 + \frac{y}{p}\frac{dp}{dy} = 0 \qquad \left[\because (1+p^2) \neq 0\right]$$

$$\text{or}\quad \boxed{py = c} \Rightarrow p = \frac{c}{y}.$$

Examples

From the given eqⁿ, we have

$$x = \frac{y}{2p}(1-p^2)$$

or, $\boxed{y^2 = c^2 + 2cx}$, This is the general solⁿ.

Examples

* Here we can't get the singular solution.

Examples

Ex (3)

$$x = py - p^2$$

Diff. w.r.t y

or $\dfrac{dx}{dy} = p + y\dfrac{dp}{dy} - 2p\dfrac{dp}{dy}$

or, $\dfrac{1}{p} = p + y\dfrac{dp}{dy} - 2p\dfrac{dp}{dy}$

or, $\left(\dfrac{1}{p} - p\right) = (y - 2p)\dfrac{dp}{dy}$

or, $\dfrac{dy}{dp}\left(\dfrac{1}{p} - p\right) = y - 2p$

or, $\dfrac{dy}{dp} + \dfrac{y}{\left(p - \frac{1}{p}\right)} = \dfrac{2p}{\left(p - \frac{1}{p}\right)}$, which is linear in y with p as its independent variable.

Examples

IF is $e^{\int \frac{p}{p^2-1}\,dp} = e^{\frac{1}{2}\log(p^2-1)} = \sqrt{p^2-1}$

multiplying by IF and then integrating, we find

$$y\sqrt{p^2-1} = \int \frac{2p^2}{\sqrt{p^2-1}}\,dp + c$$

$$= 2\int \frac{p^2-1+1}{\sqrt{p^2-1}}\,dp + c$$

$$= 2\int \sqrt{p^2-1}\,dp + 2\int \frac{dp}{\sqrt{p^2-1}} + c$$

$$= p\sqrt{p^2-1} + \cosh^{-1} p + c$$

Examples

a) $y = p + (c + \cosh^{-1} p)(p^2-1)^{-1/2}$

This relation and the given eqn

$$x = py - p^2 = p(c + \cosh^{-1} p)(p^2-1)^{-1/2}$$

Constitute the parametric solution of the given solution.

Clairaut's Equation

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- A non-linear ODE of the form

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

or, $\boxed{y = xp + f(p)}$, where $p = \frac{dy}{dx}$.

is called the Clairaut's equation. This equation is an interesting equation as it always has a singular solution. The general solution is

$\boxed{y = cx + f(c)}$, replacing p by constant c in the given equation.

representing the family of straight lines.

Clairaut's Equation

Clairaut's eqn in the form —

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

or $y = xp + f(p)$, where $p = \frac{dy}{dx}$.

Differentiate w.r.t x,

$$p = p + \{x + f'(p)\}\frac{dp}{dx}$$

or $\{x + f'(p)\}\frac{dp}{dx} = 0$

Clairaut's Equation

$\therefore \frac{dp}{dx} = 0$ or $x + f'(p) = 0$

this gives general sol^n

this gives singular Sol^n

integrating

$p = c$, Now from the given eq^n, we have $y = cx + f(c)$, which is general solution.

Clairaut's Equation

This general solution represents the family of straight lines.

Now we have

$$\left.\begin{array}{l} x + f'(p) = 0 \Rightarrow x = -f'(p) \\ \text{and } y = -p f'(p) + f(p) \end{array}\right\} \quad -(1)$$

This gives Singular Solution

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Clairaut's Equation

* Note: The eqn ① represent a curve which is known as the envelope of the family of lines $y = cx + f(c)$.

This envelope of the family of lines is known as Singular Solution

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Definition of Envelope:

Consider the equation of one parameter family of curves in the form — $\boxed{f(x,y,c)=0}$, where c is an arbitrary constant. For different values of c, we obtain different members of the family of curves. If each member of an infinite family (one parameter family) of curves is tangent to a certain curve C_1 and if at each point on the curve C_1, atleast one member of the family is tangent, then the curve C_1 is called the envelope of the given one parameter family of curves.

Envelope

- $f(x,y,c) = 0$ — (1)

  and $\dfrac{\partial f}{\partial c} = 0$ — (2)

  $\Rightarrow$ provides the equation of envelope.

The envelope of the family of curves is the curve obtained after eliminating constant c from eqⁿs (1) and (2)

- Note: Here we are eliminating constant c from the given family of curves $f(x,y,c)=0$ to get the envelope. So this process is known as c-discriminant.

Envelope

p-discriminant to find envelope:

Here we will consider the given ODE in the form $F(x, y, p) = 0$, where $p = \frac{dy}{dx}$ is treated as parameter.

To find envelope:

$$\boxed{\begin{aligned} &F(x,y,p) = 0 \\ &\text{and } \frac{\partial F}{\partial p} = 0 \end{aligned}}$$

after removing p, it give the envelope eq^n.

Envelope

In this way, to get the envelope of the given eqn is known to be p-discriminant of the differential eqn.

Remark: The envelope of the family of curves is the singular solution of the differential equation.

Envelope

If the family of curves $f(x,y,c)=0$ forms quadratic eqn in constant c or the given ODE $F(x,y,p)=0$ forms quadratic eqn in p, then we can find a simple approach to get the envelope or singular solution. See this at next page.

Envelope

Alternative approach for finding envelope :

If the one parameter family of curves are in the form —

$$f(x,y,c) = \alpha(x,y)c^2 + \beta(x,y)c + \gamma(x,y) = 0$$

c-discriminant :

$$\therefore \boxed{\alpha(x,y)c^2 + \beta(x,y)c + \gamma(x,y) = 0}$$

quadratic eq^n in c.

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Envelope

then the equation of envelope is given by

$$\boxed{\beta^2 - 4\alpha\gamma = 0}.$$

This is singular solution.

This is also known as c-discriminant relation.

This is only valid for quadratic form of $f(x,y,c)=0$.

Envelope

Alternative approach to find an envelope:

If $f(x,y,p) = A(x,y)p^2 + B(x,y)p + C(x,y)$ be the ODE in quadratic form in p, then the envelope is given by the p-discriminant relation

p-discriminant: $\boxed{B^2 - 4AC = 0}$

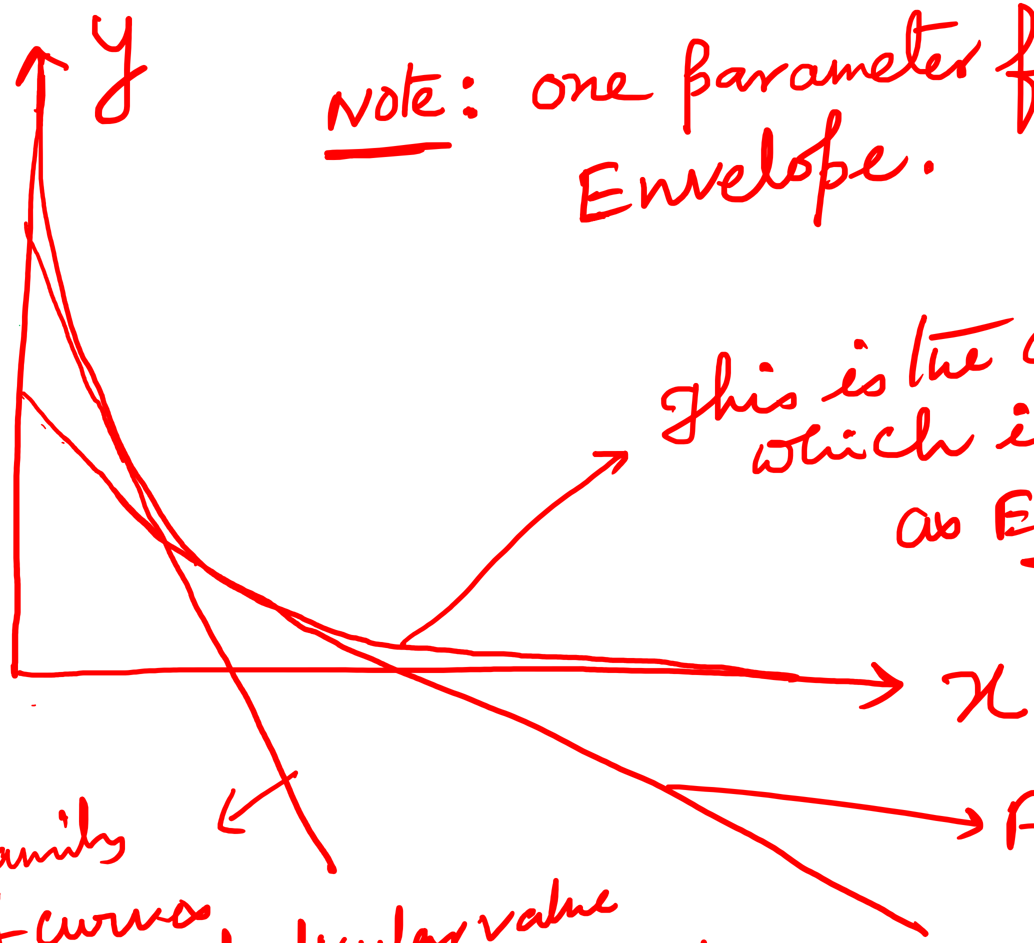
This will provide the envelope.

This is the singular solution.

This alternative process is only valid for quadratic form in p of given ODE.

Graphical Presentation:

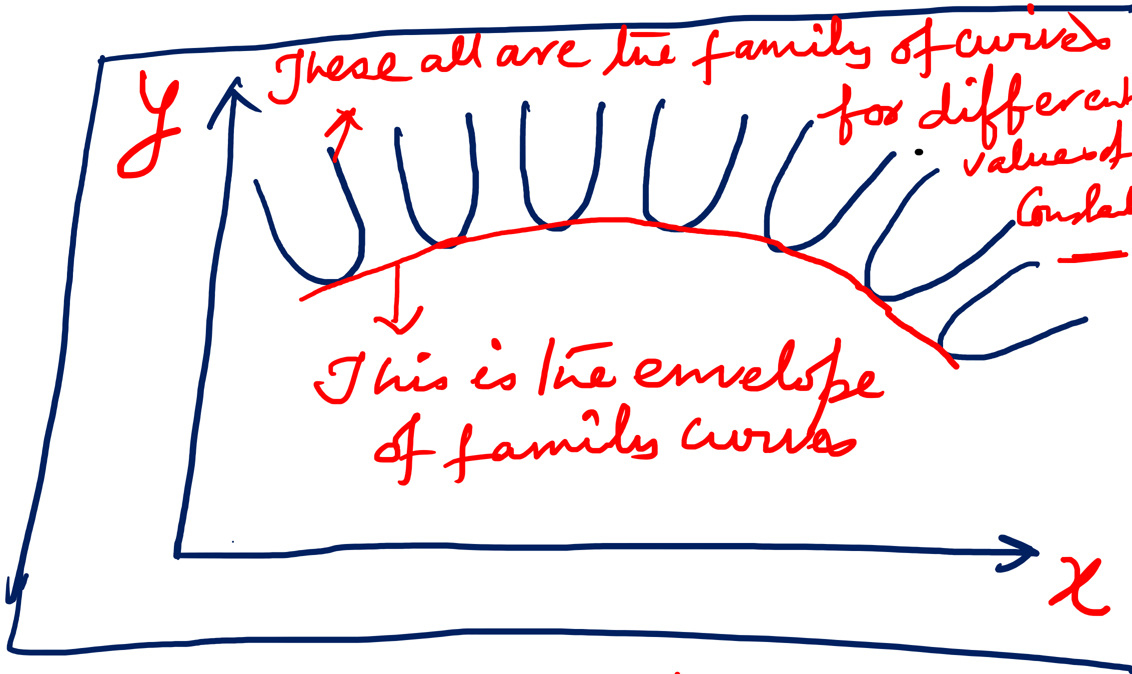
Note: one parameter family of curves are tangent to the Envelope.



This is the curve which is called as Envelope.

Family of curves for a particular value of c say 1

Family of curves for a particular value of c say 2



Envelope

The concept of envelope is that the curve given by singular solution i.e, the envelope touches every member of the family of lines given by the general solution.

Envelope

Note: Remember that if p-discriminant and c-discriminant of the differential eqn will give the same equation which is also satisfying the given differential equation, then the obtained eqn is known as **Singular Solution**.

Examples

Ex ①:

$y = xp + a/p \rightarrow$ This is Clairaut's eqⁿ in y.

Differentiating w.r.t x we have

$$\frac{dy}{dx} = x\frac{dp}{dx} + p - \frac{a}{p^2}\frac{dp}{dx}$$

$$\Rightarrow p = x\frac{dp}{dx} + p - \frac{a}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

$$\therefore \frac{dp}{dx} = 0 \quad \text{or} \quad x - \frac{a}{p^2} = 0$$

$\rightarrow p = c$, where c is an arbitrary constant.

Examples

$$y = xp + a/p$$

$$\Rightarrow \boxed{y = cx + a/c}$$ [putting $p = c$]

This is general solⁿ and it is family of straight line.

we have $x - a/p^2 = 0$

$\Rightarrow x = a/p^2$, and $y = xp + a/p$

gives $y = a/p^2 \cdot p + a/p = 2a/p$.

$\therefore$ $\boxed{x = a/p^2 \text{ and } y = 2a/p}$.

Now eliminate p from these relations.

Examples

$y = 2a/p \Rightarrow p = \frac{2a}{y}$ and $x = a/p^2 = \frac{a}{\frac{4a^2}{y^2}}$

$\boxed{y^2 = 4ax}$ This is singular Solution.

Simple way to get the singular solution:

Clairaut's form: $y = xp + a/p \Rightarrow \boxed{xp^2 - py + a = 0}$

Now p-discriminant relation:

$y^2 - 4ax = 0 \Rightarrow \boxed{y^2 = 4ax}$

This is singular sol^n.

Examples

OR we can get from the general solution by using c discriminant relation.

we have the general solution

$$y = cx + a/c \Rightarrow c^2x - cy + a = 0$$

now c-discriminant relation is

$$y^2 - 4ax = 0 \Rightarrow \boxed{y^2 = 4ax}$$

This is the Singular Solⁿ.

Examples

Note: Here p-discriminant and c-discriminant have same solution which is also satisfying the different eq^n.

This is always possible for getting the singular solution from clairaut's eq^n.

Examples

Note: However for other differential eqns (first order but not first degree), we may not get always singular solution. For this ODEs, the p-discriminant and c-discriminant give different eqns. We will discuss this later.

Examples

Ex 2:

$y = px + \sqrt{1+p^2}$

This is in clairaut's form.

Diff. w.r.t x, we have

$$p = p + x\frac{dp}{dx} + \frac{p}{\sqrt{p^2+1}}\frac{dp}{dx}$$

or, $$\frac{dp}{dx}\left(x + \frac{p}{\sqrt{p^2+1}}\right) = 0$$

$\frac{dp}{dx} = 0$ gives $p = c$, therefore general solution is $y = cx + \sqrt{1+c^2}$ family of straight lines

$x + \frac{p}{\sqrt{p^2+1}} = 0$ gives $y = \frac{1}{\sqrt{1+p^2}}$ (From the given eqn)

Note: For this eqn, the p-discriminant and c-discriminant will give the singular solution as

$x^2+y^2=1$ (You try)

Examples

Hence the ~~general~~ Singular solution is

$$\boxed{x^2+y^2=\frac{p^2}{1+p^2}+\frac{1}{1+p^2}=1} \Rightarrow \text{Singular Sol}^n$$

Note: Here you can apply alternative approaches of p-discriminant and c-discriminant to get the envelope (singular solution).

Examples

Now see the another problem of clairaut's eqⁿ —

Ex. 2 $\sin\left(x\frac{dy}{dx}\right)\cos y = \cos\left(x\frac{dy}{dx}\right)\sin y + \frac{dy}{dx}$.

Solⁿ: putting $\frac{dy}{dx} = p$, the given equation reduces to

$\sin px \cos y - \cos px \sin y = p$

or, $\sin(px - y) = p$ or $\boxed{y = px - \sin^{-1} p}$ clairaut's form.

Examples

Diff. w.r.t. x, we have

$$p = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or $$\frac{dp}{dx}\left(x - \frac{1}{\sqrt{1-p^2}}\right) = 0$$

$\frac{dp}{dx} = 0 \Rightarrow$ gives $p = c$, where c is an arbitrary constant.

Now the general solⁿ is

$$\boxed{y = cx - \sin^{-1}c}$$

Examples

Also we have

$$\left(x - \frac{1}{\sqrt{1-p^2}}\right) = 0 \Rightarrow p^2 = \frac{x^2-1}{x^2}$$

or $p = \dfrac{\sqrt{x^2-1}}{x}$ (Considering only positive values)

From, $y = px - \sin^{-1} p$, we have

$$\boxed{y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}}$$

is the Singular Solution.

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Ex. $(px-y)(x-py) = 2p$

Let $x^2 = u$, $y^2 = v$,

So we have $2x\,dx = du$ and $2y\,dy = dv$

or $dx = \frac{1}{2x}du$ and $dy = \frac{1}{2y}dv$.

$p = \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} = \frac{x}{y}q$, Since $q = \frac{dv}{du}$.

we are transforming $x \to u$ and $y \to v$. | $x \to$ independent variable, $y \to$ dependent variable.

The given eqn transforms into —

$(x \cdot \frac{x}{y}q - y)(x - \frac{x}{y}q \cdot y) = 2\frac{x}{y}q$

Singular Solution

or, $\left(\frac{x^2 q}{y} - y\right)(x - xq) = 2\frac{x}{y}q$

or, $(x^2 q - y^2)(1-q) = 2q$

or, $(uq - v)(1-q) = 2q$

or, $\boxed{v = uq - \frac{2q}{1-q}}$ $\Rightarrow$ which is clairaut's form.

Now it can be rewritten as

$uq^2 + (2-u-v)q + v = 0$. —— (1)

. This is the transformed eqn in u and v

Singular Solution

Now differentiating w.r.t q, we have

$$2q \cdot \frac{dq}{du} \cdot u + q^2 + (2-u-v) \cdot \frac{dq}{du} + q\left(-1 - \frac{dv}{du}\right) + \frac{dv}{du} = 0$$

or, $2uq \frac{dq}{du} + q^2 + (2-u-v)\frac{dq}{du} - q(1+q) + q = 0.$

or, $(2uq + 2 - u - v)\frac{dq}{du} = 0$

$\Rightarrow \frac{dq}{du} = 0$ or, $2uq + 2 - u - v = 0$

$\boxed{q = c} \Rightarrow$ From transformed eq^n (1), we have $c^2u + (2-u-v)c + v = 0$

$\Rightarrow \boxed{c^2x^2 + c(2 - x^2 - y^2) + y^2 = 0}$ This is general solution.

Singular Solution

Also we have, $2uq + 2 - u - v = 0$

$$\Rightarrow q = -\frac{2-u-v}{2u}.$$

So, by putting the value of q in the transformed eqn (1), we have,

$$\frac{(2-u-v)^2}{4u} = v$$

$$\Rightarrow (2-u-v)^2 = 4uv$$

$$\Rightarrow \boxed{(2-x^2-y^2)^2 = 4x^2y^2}$$ Singular Solution

Singular Solution

Alternative Process using Envelope:

$$(px-y)(x-py) = 2p$$

So, using transformation $x^2 = u$, and $y^2 = v$, the transformed eqn is obtained as

$$\boxed{v = uq - \frac{2q}{1-q}}$$

, this is the clairaut's eqn.

So, the general solution is

$$v = cu - \frac{2c}{1-c}$$

Singular Solution

Since the eqn is quadratic in c, we have

$\beta = 2 - x^2 - y^2$

$\alpha = x^2$

$\gamma = y^2$

$c^2 u + (2-u-v)c + v = 0$

or

$$\boxed{c^2 x^2 + c(2 - x^2 - y^2) + y^2 = 0}$$

This is general solⁿ

This is one parameter family of curves

Let $f(x,y) = c^2 x^2 + c(2 - x^2 - y^2) + y^2 = 0$.

Now for finding envelope, we have

$\frac{\partial f}{\partial c} = 0 \Rightarrow 2cx^2 + (2 - x^2 - y^2) = 0$

$\Rightarrow c = -\frac{2 - x^2 - y^2}{2x^2}$.

This is c-discriminant

Singular Solution

Now substituting the value of c in the general solution, we have

$$\frac{(2-x^2-y^2)^2}{4x^4}\cdot x^2 - \frac{(2-x^2-y^2)}{2x^2}\cdot(2-x^2-y^2) + y^2 = 0$$

or, $$\frac{(2-x^2-y^2)^2}{4x^2} - \frac{(2-x^2-y^2)^2}{2x^2} + y^2 = 0$$

or, $$\boxed{(2-x^2-y^2)^2 = 4x^2y^2}$$

This is singular solution

Also one can directly get this sol^n by using $\beta^2-4\alpha\gamma=0$.

This process is very simple and we can get the singular solution quickly.

Singular Solution

Remember: p-discriminant will also give the same envelope as singular solution.

If the eqn is clairaut's eqn, there is no need of checking two discriminant. One will be sufficient to give the singular solution.