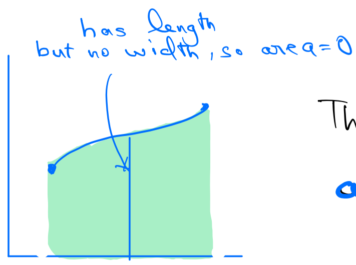
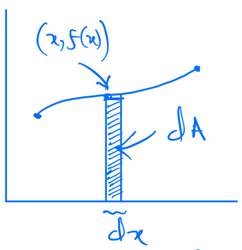
The Differential

A differential is an infinitesimal variable.
Meaning, given **any** $\varepsilon > 0$,

$$\varepsilon > dx > 0$$



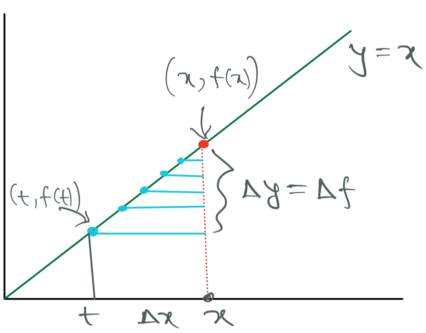
Thus, dx is a variable that is approaching 0 for eternity.

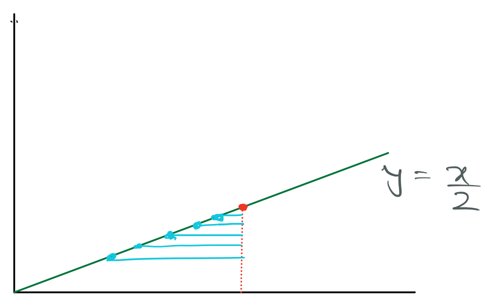


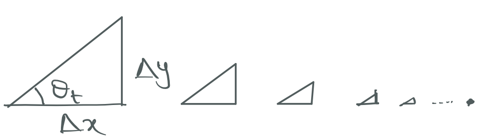
$dA = f(x)\,dx$

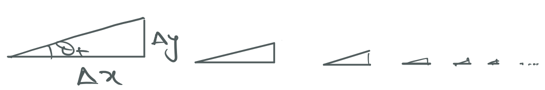
dx is an independent differential

If $y = f(x)$, a function of x,
then $dy = df$ is a dependent differential.









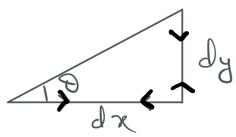
$$\frac{dy}{dx} = \lim_{t \to x} \tan\theta_t = 1$$

$$\frac{dy}{dx} = \lim_{t \to x} \tan\theta_t = \frac{1}{2}.$$

$$f'(x) = \lim_{t \to x} \tan\theta_t = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$\frac{\Delta y}{\Delta x}$ = average change in y over an iterval with respect to the change in x.

$\frac{dy}{dx}$ = instantaneous rate of change of $y = f(x)$ at x.



A forever shrinking triangle

Thus,
$$dy = f'(x)\,dx$$

or
$$df = f'(x)\,dx$$

The Difference Quotient

Suppose $f(x)$ is defined on a nbhd. I of a point c. For $t \in I \setminus \{c\}$, define

$$\varphi_f(t) = \frac{f(t) - f(c)}{t - c}.$$

$\varphi_f(t)$ is called the **Difference Quotient of $f(x)$ at $x = c$**

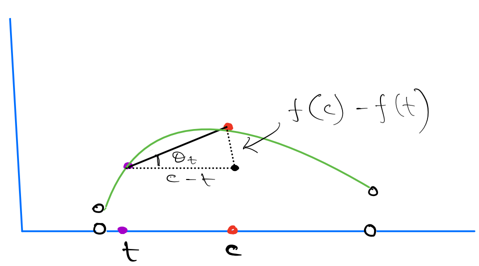
We say that $f(x)$ is differentiable at $x = c$ if

$$\lim_{t \to c} \varphi_f(t) \quad \text{exists (finitely).}$$

That is if

$$\lim_{t \to c} \frac{f(t) - f(c)}{t - c} \quad \text{exists.}$$

If the above limit exists, then it is called the derivative of $f(x)$ at $x=c$, and denoted by $f'(c)$.



$\varphi_f(t) = \tan \theta_t$

Thm. If f is diff. at c, then f is cont. at c.

Proof. $$\lim_{t \to c} \left(f(t) - f(c) \right) = \lim_{t \to c} (t-c) \left(\frac{f(t)-f(c)}{t-c} \right)$$
$$= \lim_{t \to c} (t-c) \cdot \lim_{t \to c} \frac{f(t)-f(c)}{t-c}$$
$$= 0.$$
$$\Rightarrow \lim_{t \to c} f(t) = f(c).$$

Algebraic Properties of Derivatives

Thm. Let f, g be diff. at c. Then

(a) $f+g$ is diff. at c with

$$(f+g)'(c) = f'(c) + g'(c)$$

(b) $f \cdot g$ is diff. at c with

$$(f \cdot g)'(c) = f(c) \cdot g'(c) + f'(c) \cdot g(c)$$

(c) If $g(c) \neq 0$, then (f/g) is differentiable at c

with $$(f/g)'(c) = \frac{g(c) f'(c) - g'(c) f(c)}{g(c)^2}$$

Proof. (a) What is $\varphi_{f+g}(t)$?

$$\varphi_{f+g}(t) = \frac{(f+g)(t) - (f+g)(c)}{t-c}$$

$$= \frac{f(t) + g(t) - f(c) - g(c)}{t-c}$$

$$= \frac{f(t)-f(c)}{t-c} + \frac{g(t)-g(c)}{t-c}$$

Thus, $\lim\limits_{t \to c} \varphi_{f+g}(t) = f'(c) + g'(c)$.

(b) $$\varphi_{f\cdot g}(t) = \frac{(f\cdot g)(t) - (f\cdot g)(c)}{t-c}$$

$$= \frac{f(t)\,g(t) - f(c)\,g(c)}{t-c}$$

$$= \frac{(f(t)-f(c))\,g(t) + f(c)\,g(t) - f(c)\,g(c)}{t-c}$$

$$= g(t)\,\varphi_f(t) + f(c)\,\varphi_g(t)$$

$$\Rightarrow \lim_{t\to c} \varphi_{f\cdot g}(t) = \left(\lim_{t\to c} g(t)\right)\left(\lim_{t\to c} \varphi_f(t)\right) + f(c)\lim_{t\to c}\varphi_g(t)$$

$$= g(c)\,f'(c) + f(c)\,g'(c).$$

(c) $$\varphi_{f/g}(t) = \frac{(f/g)(t) - (f/g)\,c}{t-c}$$

$$= \frac{\frac{f(t)}{g(t)} - \frac{f(c)}{g(c)}}{t-c}$$

$$= \frac{f(t)\,g(c) - f(c)\,g(t)}{(t-c)\,g(t)\,g(c)}$$

Thus, $$\lim_{t\to c} \varphi_{f/g}(t) = \lim_{t\to c} \frac{f(t)g(t)-f(c)g(t)}{(t-c)\,g(t)\,g(c)}$$

$$= \lim_{t\to c} \frac{1}{g(t)\,g(c)} \cdot \lim_{t\to c} \frac{f(t)\,g(c) - f(c)g(t)}{t-c}$$

$$= \frac{1}{g(c)^2} \cdot \lim_{t\to c} \frac{(f(t)-f(c))\,g(c) - f(c)(g(t)-g(c))}{t-c}$$

$$= \frac{f'(c)\,g(c) - f(c)\,g'(c)}{g(c)^2}.$$

The "h" definition of Differentiability.

Continuity in terms of "h"

Recall that f is cont. at c if for every $\varepsilon>0$ $\exists$ $\delta>0$ s.t.

$$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$$

If we write $x=c+h$, then

f is cont. at c if for every $\varepsilon>0$, $\exists$ $\delta>0$ s.t.

$$|h|<\delta \Rightarrow |f(c+h)-f(c)|<\varepsilon \quad \text{---}(*)$$

Thus, if one sets

$$\Psi_f(h) = f(c+h)-f(c)$$

Then $(*)$ is equivalent to saying that

$$\lim_{h\to 0} \Psi_f(h) = 0.$$

Therefore, f is cont. at c if $\exists$ a function Ψ_f satisfying $\lim_{h\to 0} \Psi_f(h)=0$ such that

$$f(c+h) = f(c) + \Psi_f(h)$$

Thus, $f(c+h) \approx f(c)$

Examples.

- $f(x) = x$

  $\varphi_f(h) = c+h-c = h \to 0$ as $h \to 0$

- $f(x) = x^2$

  $\varphi_f(h) = (c+h)^2 - c^2 = h(2c+h) \to 0$ as $h \to 0$

- $f(x) = \sin x$

  $\varphi_f(h) = \sin(c+h) - \sin c$

  $= 2\sin\frac{h}{2} \cdot \cos\left(c+\frac{h}{2}\right) \to 0$

  as $h \to 0$

Differentiability in terms of "h"

Recall: f is diff. at c if there is a constant $(f'(c))$ s.t. for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$\boxed{\lim_{t\to c} \frac{f(t)-f(c)}{t-c} = f'(c)}$$

$$|t-c| < \delta \Rightarrow |\varphi_f(t) - f'(c)| < \varepsilon \quad -(*)$$

Writing $t = c+h$, we see that $(*)$ is same as saying that $\exists$ a constant $(f'(c))$ s.t.

$$|h| < \delta \Rightarrow |\varphi_f(c+h) - f'(c)| < \varepsilon$$

that is $\displaystyle\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$

In other words, $\exists$ a function

$$\Psi_f(h) = \frac{f(c+h) - f(c)}{h} - f'(c) \quad \text{—} (**)$$

satisfying $\lim\limits_{h\to 0} \Psi_f(h) = 0$

Observe that $(**)$ can be expressed as

$$f(c+h) = f(c) + h f'(c) + h\Psi_f(h)$$

i.e. $$f(c+h) \approx f(c) + h f'(c)$$

Examples.

- $f(x) = x^2$

$$\Psi_f(h) = \frac{(c+h)^2 - c^2}{h} - 2c$$

$$= 2c + h - 2c = h.$$

- $f(x) = \sin x$

$$\Psi_f(h) = \frac{\sin(c+h) - \sin c}{h} - \cos c$$

$$= \frac{2\sin\frac{h}{2}\cos\left(c+\frac{h}{2}\right)}{h} - \cos c$$

$$\to 0 \quad \text{as } h \to 0.$$

The Chain Rule.

Thm. If f is diff. at c, and g is diff. at $f(c)$, then

$$(g\circ f)'(c) = \underbrace{g'(f(c))\cdot f'(c)}.$$

Proof.

$$(g\circ f)(c+h) = g(f(c+h))$$

$$= g\big(f(c) + h(f'(c)+\psi_f(h))\big)$$

$$= g(f(c)) + h(f'(c)+\psi_f(h))\,g'(f(c))$$

$$+ h(f'(c)+\psi_f(h))\,\psi_g\big(h(f'(c)+\psi_f(h))\big)$$

$\lim_{h\to 0}\psi_{g\circ f}(h) = 0$

$(g\circ f)(c+h) = (g\circ f)(c) + h(\cdots) + h\,\psi_{g\circ f}(h)$.

Thus, $(g\circ f)(c+h) = (g\circ f)(c) + h\,f'(c)\,g'(f(c))$

$$+ \underbrace{h\,\psi_f(h)\,g'(f(c)) + h(f'(c)+\psi_f(h))\,\psi_g\big(h(f'(c)+\psi_f(h))\big)}$$

$$= h\Big\{\underbrace{\psi_f(h)\,g'(f(c)) + (f'(c)+\psi_f(h))\,\psi_g\big(h(f'(c)+\psi_f(h))\big)}_{\psi_{g\circ f}}\Big\}$$

suff. to show $\lim_{h\to 0}\psi_g\big(h(f'(c)+\psi_f(h))\big) = 0$.

To show: given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$|h| < \delta \Rightarrow \left| \psi_g \left(h \left(f'(c) + \psi_f(h) \right) \right) \right| < \varepsilon$$

Since $\lim_{k \to 0} \psi_g(k) = 0$.

given $\varepsilon > 0$ $\exists$ a $\delta' > 0$ s.t.

$|k| < \delta' \Rightarrow |\psi_g(k)| < \varepsilon$.

Thus, need to figure out what $\delta > 0$ ensures that

$|h| < \delta \Rightarrow |h(f'(c) + \psi_f(h)| < \delta'$

Now, $\lim_{h \to 0} \psi_f(h) = 0$

$\Rightarrow$ $\exists$ a δ_1 s.t.

$|h| < \delta_1 \Rightarrow |\psi_f(h)| < 1$

Then, $|f'(c) + \psi_f(h)| < 1 + |f'(c)|$

so that,

$$|h| < \frac{\delta'}{1+|f'(c)|} \Rightarrow |h(f'(c) + \psi_f(h))| < \delta'$$

Thus $\delta = \dfrac{\delta'}{1+|f'(c)|}$, and $|h| < \delta \Rightarrow |\psi_g(h(f'(c) + \psi_f(h))) < \varepsilon$

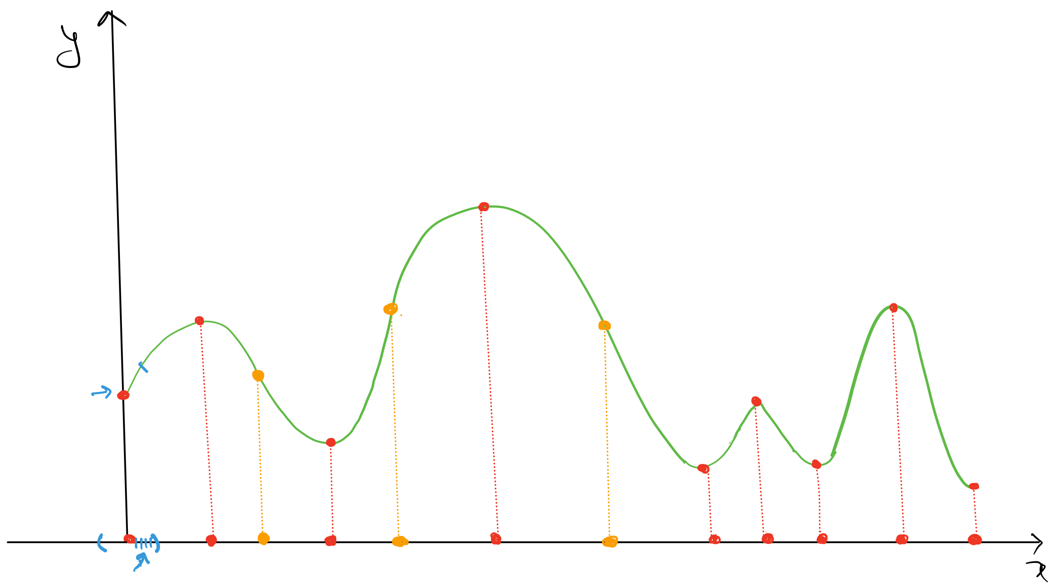
$$\Rightarrow \lim_{h \to 0} \psi_g(h(f'(c) + \psi_f(h))) = 0.$$

Local Extremum of a diff. function

Suppose f is defined on a set D containing c. Then c is said to be a local maxima (minima) of f on D if $\exists$ a nbhd. I of c such that

$$f(x) \leq f(c) \quad \forall\, x \in I \cap D$$

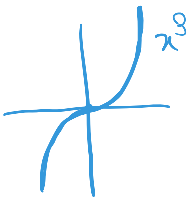
$$\left(\text{resp. } f(x) \geq f(c) \quad \forall\, x \in I \cap D\right)$$



Red dots on the x-axis are local extremum while, orange dots are not.

$[a,b]$ ✓

(a,b)



Thm. Let f be **defined on** D, and $c \in D$ be an **interior point** of D (meaning, $\exists$ a nbhd. of c that is contained inside D). **If** c **is a local extremum** of f on D, **and** f **is diff. at** c, then

$$f'(c) = 0.$$

Proof. We prove when c is a local max.

Consider $$\varphi_f(t) = \frac{f(t) - f(c)}{t - c}.$$

Since, c is a local max. of f, as well as an interior pt. of D, $\exists$ a nbhd. $(c-\delta, c+\delta)$ of c s.t.

- $(c-\delta, c+\delta) \subseteq D$
- $f(t) \leq f(c) \ \forall\, t \in (c-\delta, c+\delta)$

If $c - \delta < t < c$, then

$$\varphi_f(t) = \frac{f(t) - f(c)}{t - c} \geq 0$$

$$\Rightarrow \lim_{t \to c^-} \varphi_f(t) \geq 0$$

similarly, if $c < t < c + \delta$,

$$\varphi_f(t) = \frac{f(t) - f(c)}{t - c} \leq 0$$

$$\Rightarrow \lim_{t \to c^+} \varphi_f(t) \leq 0.$$

Since f is diff. at c,

$\lim_{t \to c} \varphi_f(t)$ exists,

hence

$$0 \leq \lim_{t \to c^-} \varphi_f(t) = \lim_{t \to c} \varphi_f(t) = \lim_{t \to c^+} \varphi_f(t) \leq 0$$

$$\Rightarrow \lim_{t \to c} \varphi_f(t) = 0.$$

Differentiable Functions on Intervals

Let • f be cont. on $[a,b]$, and
• f be diff. on (a,b).

Absolute Maxima/Minima on $[a,b]$

Let $E = \{ f(x) : x \in [a,b] \}$

Since f is cont. on $[a,b]$, it attains its **max** and **min** on $[a,b]$.

That is, $\exists$ x_M and x_m in $[a,b]$ s.t.

$$f(x_M) = M = \text{Max } E \quad (\sup E)$$
$$f(x_m) = m = \text{Min } E \quad (\inf E).$$

Defn: The point x_M is called an **Absolute Maxima** of $f(x)$ on $[a,b]$, and x_m is called an **Absolute Minima** of f on $[a,b]$.

Observe that if x_M (or x_m) is an interior pt. of $[a,b]$ (that is a point of (a,b)), then

$$f'(x_M) = 0 \quad \left(\text{resp. } f'(x_m) = 0\right).$$

Algorithm to compute the absolute Max/Min on $[a,b]$.

- Let $F = \{ x \in (a,b) : f'(x) = 0 \}$
  $= \{ x_1, x_2, \cdots, x_k \}$, say

- Compute $f(x_i)$ for $i = 1, 2, \cdots, k$.
  $f(a)$ and $f(b)$.

Now pick the points in $\{x_1, \cdots, x_k, a, b\}$ which give the maximum & and minimum values of f.

Mean Value Theorems

Thm. Let f and g be cont. on $[a,b]$, and both differentiable on (a,b). Then $\exists\ c \in (a,b)$ s.t.

$$(f(b)-f(a))\,g'(c) = (g(b)-g(a))\,f'(c).$$

Proof. Consider the function

$$h(t) = (f(b)-f(a))\,g(t) - (g(b)-g(a))\,f(t)$$

Thus, h is cont. on $[a,b]$ and diff. on (a,b).

To show: $h'(c) = 0$ for some $c \in (a,b)$.

Observe that

$$h(a) = f(b)g(a) - g(b)f(a) = h(b).$$

Now, h being cont. on $[a,b]$ attains both its max and min on $[a,b]$. Let x_M and x_m be the points at which h attains its max & min.

Claim: If h is not constant, then at least one of x_M and x_m belongs to (a,b).

(If h happens to be constant on $[a,b]$, then $h'(c)=0$ for every $c \in (a,b)$ whence, we are done.)

If x_M and $x_m \in \{a,b\}$, then

$$h(x_M) = h(x_m)$$

$\Rightarrow$ h is constant, proving our claim.

Thus, x_M or $x_m \in (a,b)$, say x_M.

Then x_M is an interior point of $[a,b]$, and by a Thm.

$$h'(x_M) = 0,$$

as required.

<u>Corollary</u>: If f is cont. on $[a,b]$, and diff. on (a,b), then $\exists$ a $c \in (a,b)$ s.t. $f(b)-f(a) = f'(c)(b-a)$

<u>Proof</u>. Take $g(x) = x$. ✓

<u>Corollary</u>. If f is cont. on $[a,b]$, and diff. on (a,b). Then

$x < y$ $[x,y]$

$f(y)-f(x) = f'(c)(y-x)$

- $f'(x) \geq 0 \;\; \forall\, x \in (a,b) \Rightarrow f$ is monotone increasing on $[a,b]$
- $f'(x) = 0 \;\; \forall\, x \in (a,b) \Rightarrow f$ is constant on $[a,b]$
- $f'(x) \leq 0 \;\; \forall\, x \in (a,b) \Rightarrow f$ is monotone decreasing on $[a,b]$

Darboux Thm. Suppose f is differentiable on $(a-h, b+k)$ for some $h, k > 0$. Further suppose that $f'(a) < f'(b)$ and $f'(a) < r < f'(b)$. Then $\exists\ c \in (a, b)$ s.t. $f'(c) = r$.

cont. on $[a,b]$
diff on (a,b)

Proof. Consider the function

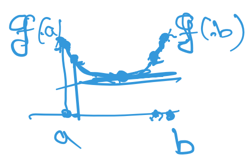
$$g(x) = f(x) - rx.$$

Then g is diff. on $(a-h, b+k)$.

Note that

$$g'(a) = f'(a) - r < 0$$

hence a is not a local maximum for g.



g on $[a,b]$

Claim: $\exists\ t$ satisfying $a < t < b$ s.t. $g(t) < g(a)$.

Else, if $g(t) - g(a) \geq 0 \quad \forall\ a<t<b$

then $\varphi_g(t) = \dfrac{g(t) - g(a)}{t-a} \geq 0 \quad \forall\ a<t<b$

$$\Rightarrow \lim_{t \to a^+} \varphi_g(t) \geq 0$$

But $\lim_{t \to a^+} \varphi_g(t) = g'(a) < 0$.

Thus $\exists$ a t satisfying $a < t < b$
s.t. $g(t) < g(a)$

Similarly, $\exists$ $s \in (a, b)$ s.t.

$$g(s) < g(b).$$

Back to the proof.

Since g is cont. on $[a, b]$, it attains its **minimum** on $[a, b]$.

The preceding argument shows that g certainly attains its **minimum** in (a, b).

$\Rightarrow$ $g'(x) = 0$ for some $x \in (a, b)$.