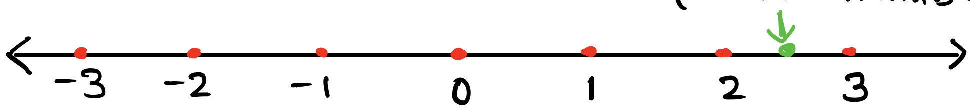
Subsets of $\mathbb{R}$

Geometrically $\mathbb{R}$ is represented by a line

every point represents a unique real number



$\mathbb{R}$ as a line

Intervals

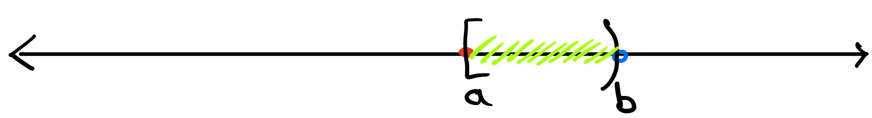
Bounded intervals

For real numbers $a < b < \infty$

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\} \text{ - closed}$$

$$(a,b) = \{x \in \mathbb{R} : a < x < b\} \text{ - open}$$

similarly, define $[a,b)$ and $(a,b]$



$\mathbb{R}$ $[a,b)$

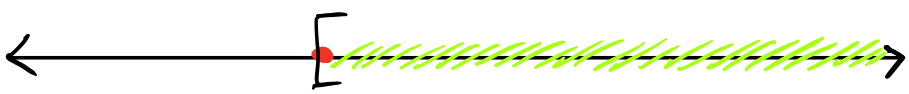
Typically, I will denote an interval

$\rightarrow$ I is one of $[a, b]$, (a, b), $[a, b)$ or $(a, b]$,

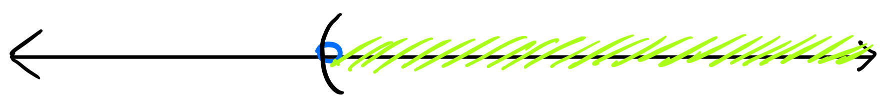
Unbounded intervals

Let $a \in \mathbb{R}$

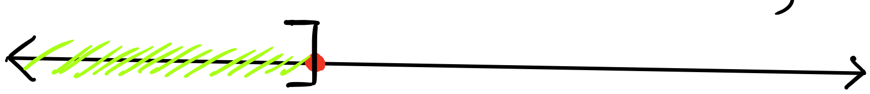
$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$



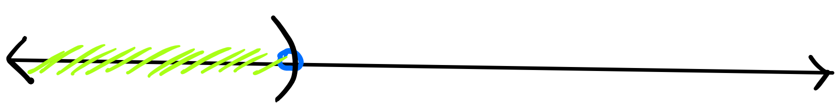
$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$



$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$



$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$



Some common examples

$\mathbb{R} = (-\infty, \infty)$ $\quad [-\infty \quad \infty]$

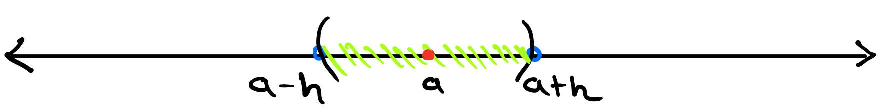
$\mathbb{R}_{>0} = (0, \infty)$

$\mathbb{R}_{\leq 0} = (-\infty, 0]$

Neighbourhood (nbhd.) of a point

Let $a \in \mathbb{R}$

A nbhd. of a is a symmetric open interval about a, i.e. of the form $(a-h, a+h)$ where $h > 0$



A nbhd. of a

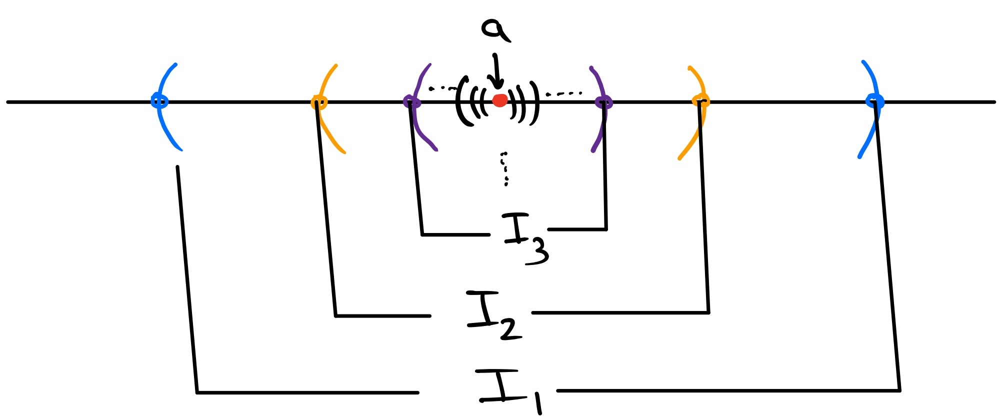
An important sequence of nbhds.

Let $a \in \mathbb{R}$,

for every $n \in \mathbb{N}$, define

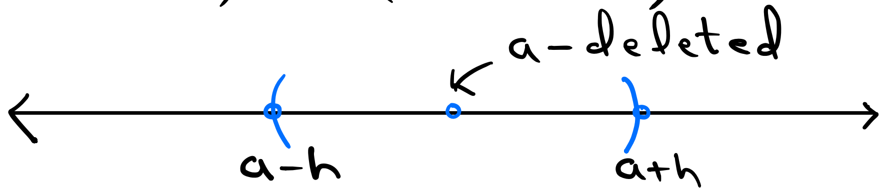
$$I_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$$

Then
- $I_1 \supset I_2 \supset I_3 \supset \cdots\cdots$
- $a \in I_n \;\forall\; n \in \mathbb{N}$.



Deleted Nbhd.

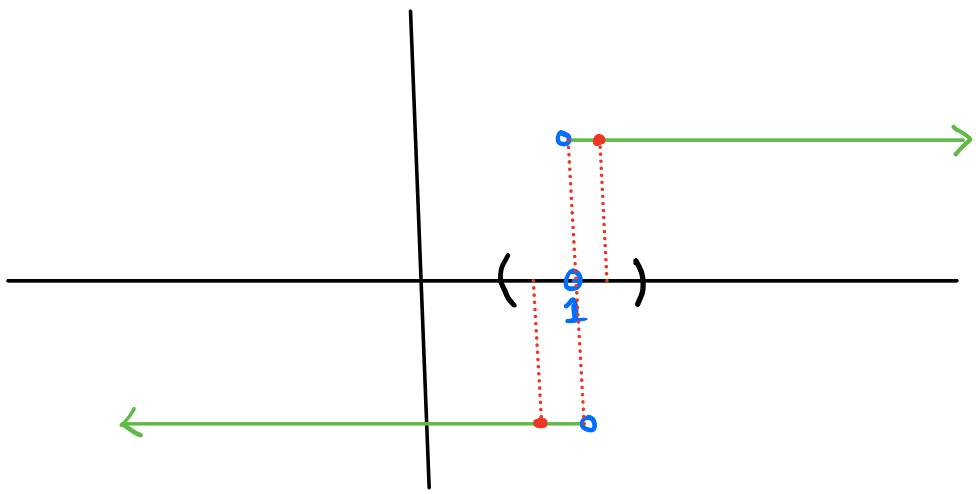
$$(a-h, a+h)' = (a-h, a+h) \setminus \{a\}$$



Example

Let $f(x) = \dfrac{|x-1|}{x-1}$

Then $f(x)$ is defined on any deleted nbhd. $(1-h, 1+h)'$



$f(x)$ defined on every deleted nbhd. of 1.

Distribution of Rationals (and irrationals)

- Between any two (distinct) rationals, there is another rational. (Ex.)

Thm. Let $x < y$ be real nos. Then $\exists$ a rational q satisfying
$$x < q < y.$$

Preparation

(1) (Archimedian Property (AP))

Let $x > 0$. Then $\exists$ a $n \in \mathbb{N}$ s.t. $nx > 1$.

(2) Let $x \in \mathbb{R}$. Then $\exists$ $m \in \mathbb{Z}$ s.t.
$$m-1 \leq x < m$$
$$x \in [m-1, m)$$

Proof of (1).

If $x > 1$, then done

(simply choose $n = 1$)

Now, suppose $0 < x < 1$

example. say $x = \frac{1}{3}$, then

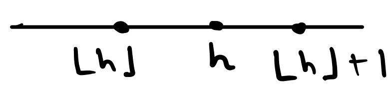
$$4x = \frac{4}{3} > 1$$

Since $x < 1$, $\exists$ a $h > 0$ s.t.

$$x = \underbrace{\frac{1}{1+\left(\frac{1}{x}-1\right)}} \qquad x = \frac{1}{1+h} \qquad h = \frac{1}{x} - 1 > 0$$

Let $n = 2 + \lfloor h \rfloor = 1 + (1 + \lfloor h \rfloor)$

$$> 1 + h$$

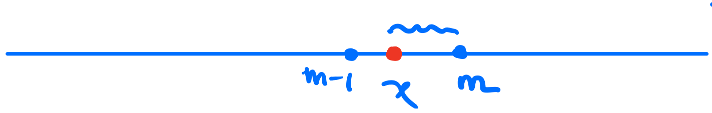


Thus, $nx > \dfrac{1+h}{1+h} = 1$.

$\square$

Proof of (2) (obvious)

m - nearest integer to x that is $> x$



Thus, every real number lies in some $[m-1, m)$ where $m \in \mathbb{Z}$.

$\square$

Proof of Thm.

We have to show that $\exists$ $m, n \in \mathbb{Z}$ with $n > 0$ s.t.

~~integer~~
natural no.

$$x < \frac{m}{n} < y \quad \text{---} (*)$$

$\frac{3}{5}$, $-\frac{2}{9}$ $\frac{-2}{9}$

$(*)$ can be proved by showing $\exists$ $m \in \mathbb{Z}$, $n \in \mathbb{N}$ s.t.

$$0 < m - nx < n(y-x) \quad \text{---} (**)$$

$(**)$ can be proved by showing $\exists$ $m \in \mathbb{Z}$, $n \in \mathbb{N}$ s.t.

$$\left.\begin{array}{l} 0 < m - nx \leq 1 \\ \text{and} \quad 1 < n(y-x) \end{array}\right\} \Rightarrow (**)$$

Since $y - x > 0$, by AP, $\exists$ a $n \in \mathbb{N}$ s.t. $n(y-x) > 1$

by (2), $\exists$ a $m \in \mathbb{Z}$ s.t.

$$m-1 \leq nx < m$$

$$\Rightarrow \quad 0 < m - nx \leq 1 .$$

$\square$

Ex: Prove the <u>Thm.</u> with rationals replaced by irrationals.

<u>Corollary</u>. Between any two real nos. $\exists$ infinitely many rationals. (or irrationals)

<u>Corollary</u>. Let I be an interval, then I contains infinitely many rationals (or irrationals).

Bounded (bdd.) subsets of $\mathbb{R}$

$[-2,2] = \{x \in \mathbb{R} : -2 \leq x \leq 2\}$

bdd. from below

bdd. from above

$|x| \leq 2$

$(-2,2) = \{x \in \mathbb{R} : -2 < x < 2\}$

$|x| < 2$ bdd. from both sides

$(2,\infty) = \{x \in \mathbb{R} : x > 2\}$

$\cancel{|x| < *}$

bdd. from below

not bdd. from above

$(-\infty,-2) = \{x \in \mathbb{R} : x < -2\}$

bdd. from above

not bdd. from below

$(-\infty,-2) \cup (2,\infty)$

$= \{x \in \mathbb{R} : x < -2 \text{ or } x > 2\}$

not bdd. from either side

if some M_1 exists

E is bdd. from above →

- A real number M_1 is said to be an upper bound of E if $x \leq M_1 \ \forall \ x \in E$

if Some M_2 exists

E is bdd. from below →

- Similarly, M_2 is a lower bound of E if

$$x \geq M_2 \ \forall \ x \in E$$

Remark: If a set is bounded from above (resp. below), then $\exists$ infinitely many upper (resp. lower) bounds.

Defn. • Let $E \subseteq \mathbb{R}$.

$E \subseteq [-M, M]$

Then E is said to be bdd.

if $\exists$ a (Some) $M > 0$ s.t.

$$|x| \leq M \ \forall \ x \in E$$

Proposition: If $E \subseteq \mathbb{R}$ is not bounded (unbounded), then given any real $M > 0$, there is a (some) $x \in E$ s.t. $|x| > M$.

Defn. The **Least Upper Bound** (lub) or the **Supremum** of a set.

$E = \{1, 2, 3, 4\}$

$\text{Max } E = 4$

It is the <u>smallest</u> upper bound.

- $E = [-2, 2]$

  Set of upper bounds of E is

  $$\{x \in \mathbb{R} : x \geq 2\} = [2, \infty)$$

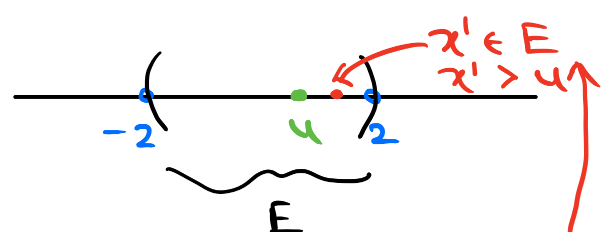
  So, $\sup E = 2$

- $E = (-2, 2)$

  Set of upper bounds is $[2, \infty)$

  $\sup E = 2$

If $u < 2$, then u is <u>not</u> an upper bound.



so u fails to be an upper bound.

Mathematical Description of Sup

$L = \sup E$ if both of the following hold

$L' < L \longleftrightarrow L-\varepsilon$

- $x \le L \;\; \forall\, x \in E$
- For every $\varepsilon > 0$, $\exists$ a $x \in E$ s.t.

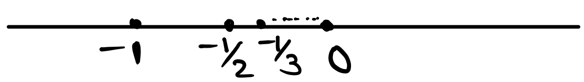
$$L - \varepsilon < x \le L$$

if $L' < L$ can always find an $x \in E$ s.t. $L' < x \le L$

Examples.

- Find $\sup E$ where

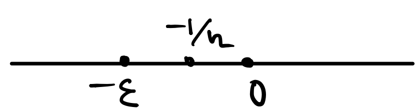
$$E = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$$



$\rightarrow E$ is bdd.

Claim: $\sup E = 0$

Clearly $-1/n < 0 \;\; \forall\, n \in \mathbb{N}$



Let $\varepsilon > 0$ (be arbitrary)

by AP, $\exists\ n \in \mathbb{N}$ s.t.

$$n\varepsilon > 1$$

$$\Rightarrow \quad \varepsilon > \frac{1}{n}$$

$$\Rightarrow \quad -\varepsilon < -\frac{1}{n}$$

hence $\sup E = 0$.

- Let $E = \mathbb{Q} \cap (-\infty, \sqrt{2})$

Determine $\sup E$.

$$E = \{x \in \mathbb{Q} : x < \sqrt{2}\}.$$

Claim: $\sup E = \sqrt{2}$

It is clear that $\sqrt{2}$ is an upper bound

Let $\varepsilon > 0$. By density property $\exists$ a rational q satisfying

$$\sqrt{2} - \varepsilon < q < \sqrt{2}$$

Since, this $q \in E$, it follows that

$$\sup E = \sqrt{2}.$$

Defn. The Greatest Lower Bound (glb) or the infimum of a set.

$\ell = \inf E$ if both of the following hold.

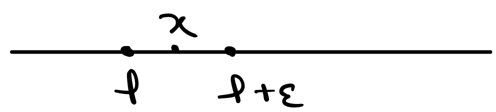
$\inf(-2,2) = -2$

$(-\infty, -2]$

- $x \geq \ell \quad \forall\, x \in E$

- For every $\varepsilon > 0$, $\exists$ a $x \in E$ s.t.

$$\ell \leq x < \ell + \varepsilon$$



Examples

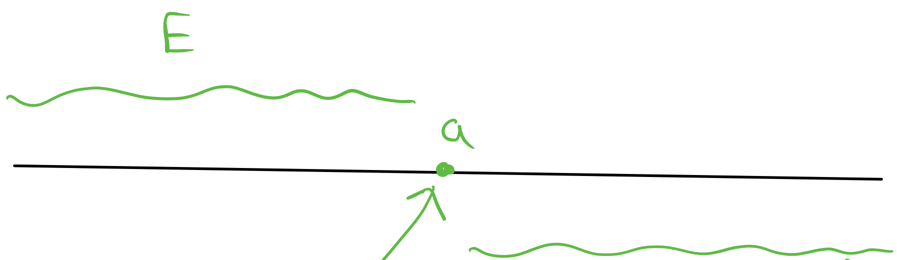
- Let $E = \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\}$

  Then $\inf E = 1$ (Ex.)

- Let $E = \{ x \in \mathbb{R} \setminus \mathbb{Q} : x > 0 \}$

  Then $\inf E = 0$ (Ex.).

Thm. Let $\emptyset \neq E \subseteq \mathbb{R}$, and E is bdd. above, then $\underline{\sup E}$ exists (in $\mathbb{R}$).



upper bounds of E.

$\exists$ a pt. a on $\mathbb{R}$ that separates the two sets!

Skip!

Corollary. If $\emptyset \neq E \subseteq \mathbb{R}$ is bdd. below, then $\underline{\inf E}$ exists.

Proof. Let $F = \left\{ x \in \mathbb{R} : \begin{array}{l} x \text{ is a} \\ \text{lower bound} \\ \text{of } E \end{array} \right\}$

Then $F \neq \emptyset$, and F is bdd. above

By the Thm. above $\sup F$ exists.

Let $L = \sup F$.

<u>Claim:</u> $\inf E = \sup F$.

To show:

(i) $x \geq L \quad \forall \quad x \in E$.

(ii) If $L' > L$, then $\exists \; x \in E$ satisfying $L \leq x < L'$.

<u>Pf. of (i)</u>. Supp. $x \in E$ & $x < L$

Since $\sup F = L$, by defn. $\exists \; y \in F$ s.t.

$y \leq x$

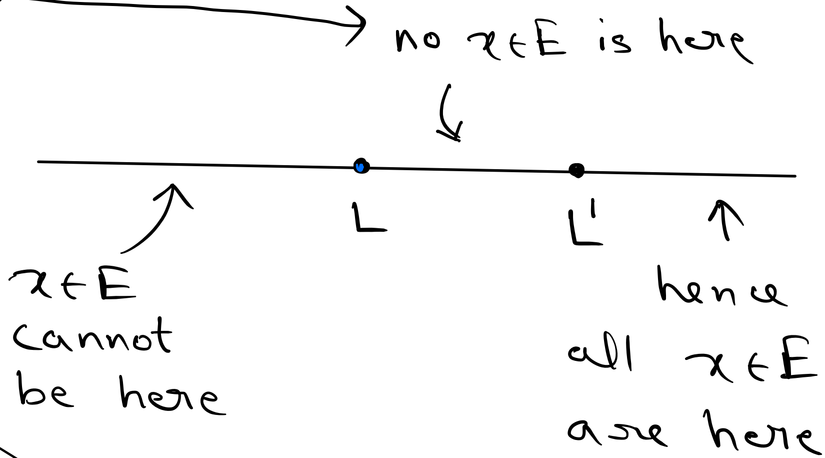
$$x < y < L$$

But this contradicts that $y \leq x$ (since $y \in F$ & $x \in E$).

Pf. of (ii). If $L' > L$ and suppose that $\exists$ no $x \in E$ satisfies

$$L < x < L',$$

then $x \geq L'$ $\forall\, x \in E$



But now, implies that

$$L' \in F$$

Since $L' > L$, L cannot then be the greatest lower bound of E, a contradiction.

Hence $\exists\, x \in E$ s.t. $L < x < L'$.

The Diameter of a set E.

Let $E \neq \emptyset$. Then

if E contains only nonnegative elements and E has a sup, then $\sup E \geq 0$

$$\operatorname{diam} E = \sup\{|x-y| : x, y \in E\}.$$

Remark. $\operatorname{diam} E \geq 0$

Examples.

- $\operatorname{diam} \{1\} = 0$
- $\operatorname{diam} \{1,2,3\} = 2$
- $\operatorname{diam} \mathbb{N} = \infty$

$\operatorname{diam} [a,b] = b-a$

- $\operatorname{diam} (a,b) = b-a$

if $x, y \in (a,b)$, then

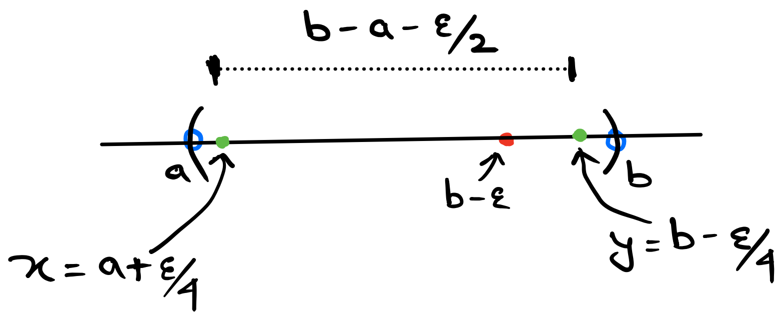
$$|x-y| < b-a$$

so, $b-a$ is an upper bd.

Now, let $\varepsilon > 0$.

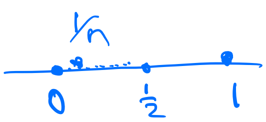
We have to show that $\exists$ $x, y \in (a,b)$ s.t.

$$b-a-\varepsilon < |x-y|$$



$$|x-y| = b-a-\varepsilon/2$$
$$> b-a-\varepsilon.$$

- $\text{diam}\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 1$



$1-\frac{1}{n} \to 1$

Suppose, $m > n$, then

$$\frac{1}{n} - \frac{1}{m} < 1 - 0 = 1$$

Now, let $\varepsilon > 0$,

we have to show that

$1-\varepsilon$ is not an upper bd. of E

Apply, AP.

$\exists\ n \in \mathbb{N}$ s.t. $n\varepsilon > 1$

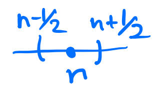
then $\frac{1}{n} < \varepsilon$

$$\Rightarrow 1 - \frac{1}{n} > 1 - \varepsilon.$$

Limit Point of a set

Let $E \neq \phi$.

Then "a" is a limit point of E if every deleted nbhd. of a contains an element of E.

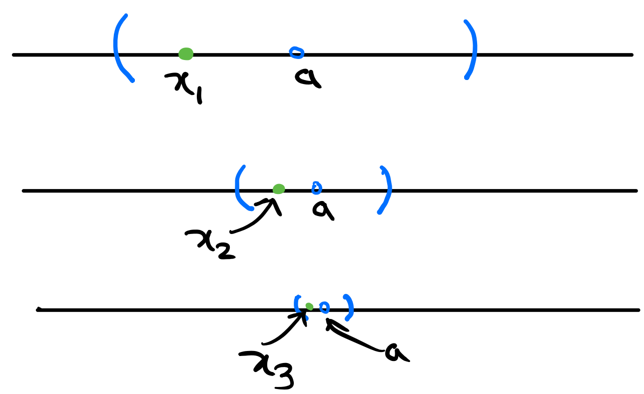


Let $E \subseteq \mathbb{R}$.

A pt. $c \in E$ is called an isolated pt. of E if c is not a limit pt. of E.

Equivalently, there is some deleted nbhd. I of c s.t

$I \cap E = \phi$



$x_1, x_2, x_3 \in E$.

Alternatively,

"a" is a limit point of E if for every $\varepsilon > 0$,

$\exists$ $x \in E$ s.t

$$0 < |x - a| < \varepsilon$$

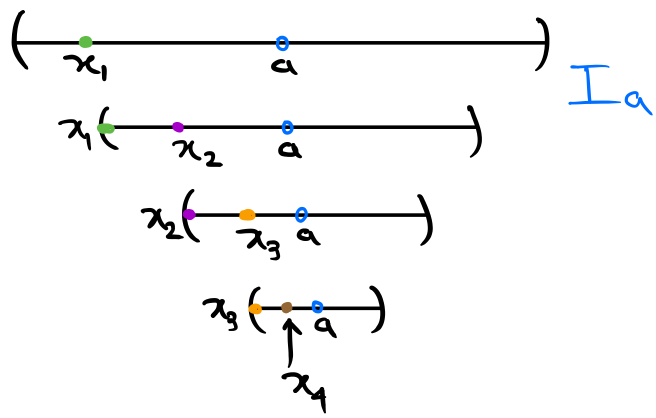
or

$$x \in (a-\varepsilon, a+\varepsilon)'$$

Thm. If "a" is a limit point of E, then every deleted nbhd. of "a" contains infinitely many elements of E.

A geometric proof

$x_1, x_2, \ldots$ of E contained in I.



continues indefinitely

Corollary. If E has a limit pt., then E is infinite.

Derived set and the Closure

If $E \neq \emptyset$ and $E \subseteq \mathbb{R}$, then

$E' = \{\text{limit points of } E\}$ ← Derived set of E

$\bar{E} = E \cup E'$ ← Closure of E.

Examples.

- $E = [a,b]$

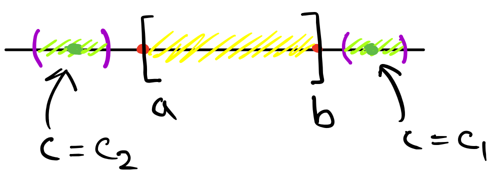
Claim! $E' = E$ (so that $\bar{E} = E$)

Two things to show

- $E' \subseteq E$

  i.e. every limit pt. of E is contained in E

suffices to show that if $c \notin [a,b]$, then $c \notin E'$. But this is clear as



- $E \subseteq E'$

  i.e. every pt. of E is a limit pt. of E

A mathematical proof.

If $c = c_1$, then

let $h = \frac{c-b}{2}$

and $I_c = (c-h, c+h)'$

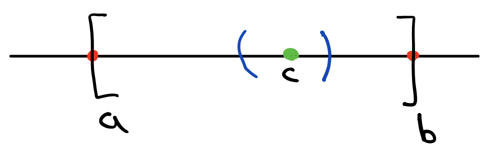
Claim: $I_c \cap [a,b] = \emptyset$

If $x \in I_c \cap [a,b]$

Then,

$$\begin{aligned} x - b &= x - c + c - b \\ &> -h + c - b \\ &= \frac{c-b}{2} > 0 \end{aligned}$$

This is a contradiction since $x \in [a,b] \Rightarrow x - b \leq 0$.



Supp. $c \in [a,b]$ and $I = (c-h, c+h)'$ be any nbhd. of c.

If $I \cap [a,b] = \emptyset$, then either

$c - h > b$

$\Rightarrow c > b$

or $c + h < a$

$\Rightarrow c < a$

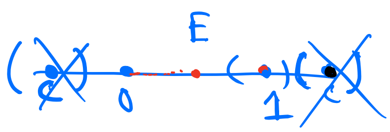
both are absurd.

- Similarly, if $E = (a,b)$, then
  $E' = E \cup \{a,b\}$
  So that $\bar{E} = [a,b]$

- $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

  Then, $E' = \{0\}$ so that $\bar{E} = E \cup \{0\}$.

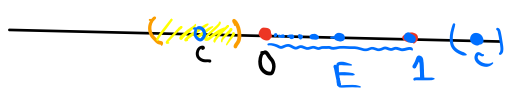
  It is easy to see that $0 \in E'$ (Ex.)



  Now, suppose $c \neq 0$.

  Claim: $c \notin E'$.

  If $c < 0$, then done.



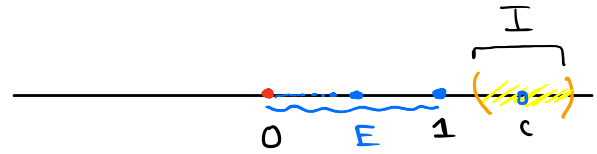
  Now, let $c > 0$

  Subcases - $c \geq 1$, $c < 1$.

  If $c > 1$, then

  set $h = \frac{c-1}{2}$

  and $I = (c-h, c+h)$



$$E \cap I = \emptyset$$

$$\Rightarrow c \notin E'$$

Next, $c = 1$ is not a limit pt. of E (Ex.)

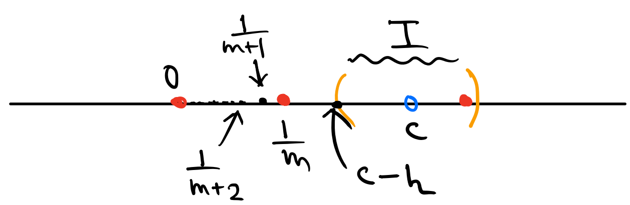
Now, let $0 < c < 1$.

By a homework problem,

$\exists$ a $m \in \mathbb{N}$ with $m \geq 2$

s.t. $$\frac{1}{m} < c \leq \frac{1}{m-1}$$

Let $h = \dfrac{c - \frac{1}{m}}{2}$, and $I = (c-h, c+h)$



Now, $\frac{1}{m} < c-h \Rightarrow \frac{1}{n} < c-h \;\forall\; n \geq m$.

$\Rightarrow I \cap E$ cannot be an infinite set

$$\Rightarrow c \notin E'.$$

Closed sets. Let $E \subseteq \mathbb{R}$. Then E is said to be closed if E contains all its limit points.

That is, $E' \subseteq E$

Equivalently, $E = \overline{E}$

$\overline{E} = E \cup E'$

Examples.

- Any finite set is closed.

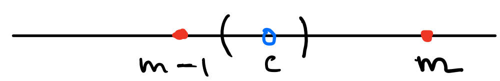
  Let $E \subseteq \mathbb{R}$ and $|E| < \infty$.

  Then $E' = \emptyset$

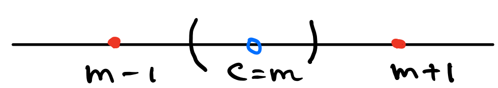
  $\Rightarrow E = \overline{E}$.

- $\mathbb{N}$, $\mathbb{Z}$ are closed.

  $\mathbb{N}' = \emptyset$ and $\mathbb{Z}' = \emptyset$



if c is not an integer



if c is an integer

- $\mathbb{Q}$ is <u>not</u> closed.

<u>Claim</u>: $\mathbb{Q}' = \mathbb{R}$, hence $\overline{\mathbb{Q}} = \mathbb{R}$.

Let $r \in \mathbb{R}$, then any nbhd. of r contains ∞-ly many rationals. Thus, any deleted nbhd. of r contains ∞-ly many rationals.

<u>Dense subsets of $\mathbb{R}$</u>.

Let $E \subseteq \mathbb{R}$. Then E is called dense in $\mathbb{R}$ if $\overline{E} = \mathbb{R}$ (equivalently, $E' = \mathbb{R}$).

- $[a,b]$ is closed but, (a,b), $[a,b)$ and $(a,b]$ are not closed.

- $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ is <u>not</u> closed but $E \cup \{0\}$ is.

<u>Proposition:</u> Let $E \subseteq \mathbb{R}$. Then $\overline{\overline{E}} = \overline{E}$.
(That is $\overline{E}$ is closed).

<u>Proof.</u> Set $F = \overline{E}$

We have to show that $F' \subseteq F$

Let $p \in F'$, and I_p be a an arbitrary deleted nbhd. of p.

To show: $I_p \cap E \neq \phi$.
(because then $p \in \overline{E} = F$)

$\boxed{F = E \cup E'}$

By defn. of a limit pt.,

$I_p \cap F \neq \emptyset$

Let $u \in I_p \cap F$

If $u \in E$, then done!

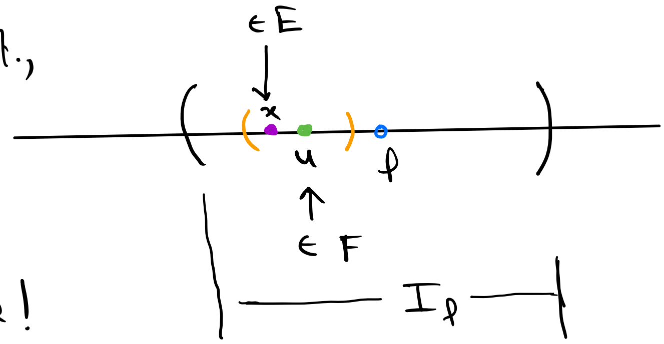
else, $u \in E'$

Let I_u be a deleted nbhd. of u s.t. $I_u \subseteq I_p$

By defn. of a limit pt.,

$$I_u \cap E \neq \emptyset$$

$$\Rightarrow I_p \cap E \neq \emptyset.$$



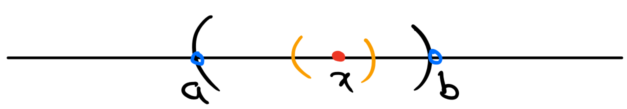
$F = E \cup E'$

Open set.

Let $E \subseteq \mathbb{R}$. Then E is open if for any $x \in E$, there is some nbhd. I_x of x s.t. $I_x \subseteq E$.

Examples.

- (a, b) is open.



$h = \min\left\{\frac{x-a}{2}, \frac{b-x}{2}\right\}$

Then $(x-h, x+h) \subseteq (a,b)$.

- $[a, b)$ is not open

  No nbhd. of $a \in [a,b)$ is $\subseteq [a,b)$.

  ($[a,b)$ is also not closed since $b \notin [a,b)$)

- No nonempty countable subset of $\mathbb{R}$ can be open

  If E is countable, then E cannot contain a nbhd. which is uncountable.

  So, $\mathbb{N}$, $\mathbb{Q}$ are not open.

Notation: Let $E \subseteq \mathbb{R}$. The symbol E^c denotes the complement of E in $\mathbb{R}$. That is $E^c = \mathbb{R} \setminus E$.

Thm. Let $E \subseteq \mathbb{R}$. Then

$$E \text{ is open} \iff E^c \text{ is closed.}$$

$\underbrace{\mathbb{R}\setminus\{1,2\}}_{=\{1,2\}^c}$

Proof. "$\Rightarrow$" part

Suppose E is open.

If $(E^c)' = \phi$, then done.

(since then $\overline{E^c} = E^c$ hence, closed.)

So assume that $(E^c)' \neq \phi$, and let p be a lim. pt. of E^c.

If $p \in E$, then $\exists$ a nbhd. I_p of p s.t. $I_p \subseteq E$

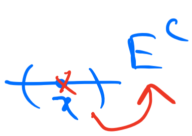
But then $I_p \cap E^c \neq \phi$, which is impossible since p is a lim. pt. of E^c.

Thus, $p \in E^c$.

"$\Leftarrow$" part

Now suppose E^c is closed.

Let $x \in E$.



If every deleted nbhd. of x intersects E^c, then

$x \in (E^c)' \Rightarrow x \in \overline{E^c} = E^c$ since E^c is closed.

But this is impossible

Thus, $\exists$ some a deleted nbhd.

$(x-h, x+h)'$ of x s.t.

$(x-h, x+h)' \subseteq E$.

But then since $x \in E$,

$$(x-h, x+h) = (x-h, x+h)' \cup \{x\} \subseteq E$$

$\Rightarrow$ E is open.