Some questions:

(1) Let $y(x)$ be a continuous solution of IVP $\frac{dy}{dx} + 2y = f(x)$, $y(0)=0$, where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1. \end{cases}$$

Then $y(3/2)$ is equal to -

(A) $\frac{\sinh(1)}{e^3}$ (B) $\frac{\cosh(1)}{e^3}$

(C) $\frac{\sinh(1)}{e^2}$ (d) $\frac{\cosh(1)}{e^2}$

Ans: (C)

Solⁿ:

$$\frac{dy}{dx} + 2y = f(x)$$

with $y(0) = 0$.

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Case 1: Let $0 \le x \le 1$,

then $f(x) = 1$.

$$\frac{dy}{dx} + 2y = 1$$

linear eqⁿ in y

$$IF = e^{\int 2dx} = e^{2x}.$$

Now the general solⁿ is

$$\boxed{y = \frac{1}{2} + Ce^{-2x}}$$ where C is an arbitrary constant.

$y(0)=0$ gives $c=-\frac{1}{2}$.

$$\therefore\ y=\frac{1}{2}\left(1-e^{-2x}\right),\quad \underline{0\le x\le 1}$$

<u>Case 2:</u> Let $x>1$.

$$f(x)=0$$

$$\therefore\ \frac{dy}{dx}+2y=0$$

$$\Rightarrow\ y(x)=c_1 e^{-2x},\quad x>1$$

$$\therefore\ y(x)=\begin{cases}\frac{1}{2}\left(1-e^{-2x}\right), & 0\le x\le 1\\ c_1 e^{-2x}, & x>1\end{cases}$$

Since $y(x)$ is continuous at $x=1$, we have

$$\lim_{x\to 1+} y(x) = y(1)$$

$$\Rightarrow c_1 e^{-2} = \frac{1}{2}(1-e^{-2})$$

$$\Rightarrow \boxed{c_1 = \frac{1}{2}(e^{2}-1)}$$

$$\therefore y(x) = \begin{cases} \frac{1}{2}(1-e^{-2x}), & 0\le x\le 1 \\ \frac{1}{2}(e^{2}-1)\,e^{-2x}, & x>1 \end{cases}$$

$$y(3/2) = \frac{1}{2}(e^{2}-1)\times e^{-3}$$

$$= \frac{e-e^{-1}}{2e^{-2}}$$

$$= \frac{1}{e^{2}}\cdot \sinh(1)$$

Here $\sinh x = \dfrac{e^x - e^{-x}}{2}$

and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

EX 2 $\dfrac{dy}{dx} + y = |x|$, $x \in R$, $y(-1) = 0$. Then $y(1)$ is equal to —

(a) $\left(\frac{2}{e} - \frac{2}{e^2}\right)$ (b) $\left(\frac{2}{e} - 2e^2\right)$

(c) $\left(2 - \frac{2}{e}\right)$ (d) $2 - 2e$

Ans: (a) Try similarly as EX. (1).

EX 3 Consider a body of unit mass falling from the rest under gravity with velocity v. If the air resistance retards the acceleration by cv where c is a constant, then —

(A) $v = g/c\,(1+e^{ct})$

(B) $v = g/c\,(1+e^{-ct})$

(C) $v = g/c\,(1-e^{-ct})$

(D) $v = g/c\,(1-e^{ct})$

Ans: (C).

Solⁿ: Here we have

$$m\frac{dv}{dt} = m(g - cv)$$

or, $$\frac{dv}{g-cv} = dt$$

Integrating we have

$$-\frac{1}{c}\log|g-cv| = t + c_1$$

c_1 is arbitrary constant.

$v=0,\ t=0.$

$$\Rightarrow\quad C_1 = -\frac{1}{c}\log|g|$$

$$-\frac{1}{c}\log|g-cv| = t-\frac{1}{c}\log|g|$$

$$\text{or}\quad \frac{1}{c}\log\left|\frac{g}{g-cv}\right| = t$$

$$\text{or}\quad \left|\frac{g-cv}{g}\right| = e^{-ct} \Rightarrow \left|\frac{g-cv}{g}e^{ct}\right| = 1$$

$$\Rightarrow\quad \frac{g-cv}{g}\cdot e^{ct} = 1$$ [since $\frac{g-cv}{g}e^{ct}$ is continuous]

Theorem:

If $f:\mathbb{R}\to\mathbb{R}$ be continuous and $|f(x)|$ is constant, then $f(x)$ is constant

See my lecture note 2 to get this theorem.

$$\text{or}\quad \boxed{v = g/c\left(1-e^{-ct}\right)}$$

Ex 4 If x^3y^2 is an integrating factor of $(6y^2+axy)dx$

$+ (6xy + bx^2)\,dy = 0$, where $a, b \in \mathbb{R}$, then

(A) $3a - 5b = 0$ (B) $2a - b = 0$

(C) $3a + 5b = 0$ (D) $2a + b = 0$

Ans: (A)

Solⁿ: Since x^3y^2 is an integrating factor of the given ODE, multiplying the given ODE by IF, we will get the exact ODE.

So we have after multiplying by IF —

$$(6x^3y^4 + ax^4y^3)\,dx + (6x^4y^3 + bx^5y^2)\,dy = 0$$

Here $M = 6x^3y^4 + ax^4y^3$

$N = 6x^4y^3 + bx^5y^2$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow 3a = 5b \Rightarrow \boxed{3a - 5b = 0}$

(5) Consider an ODE

$y'(t) = -y^3 + y^2 + 2y$,

$y(0) = y_0 \in (0,2)$. Then

$\lim_{t \to \infty} y(t)$ belongs to -

(A) $[-1,0]$ (B) $[-1,2]$ (C) $[0,2]$

(D) $[0,\infty)$

Ans: (B)

Sol^n:

$$\frac{dy}{dt} = y(2+y-y^2)$$
$$= y(1+y)(2-y)$$

or $$\frac{dy}{y(1+y)(2-y)} = dt$$

or, $$\frac{dy}{2y} - \frac{dy}{3(1+y)} + \frac{dy}{6(2-y)} = dt$$

or $$\frac{1}{2}\log|y| - \frac{1}{3}\log|1+y| - \frac{1}{6}\log|2-y| + \frac{1}{6}\log|C| = 6t$$

or $$\log|1+y|^2 - \log|y|^3 + \log|2-y| - \log|C| = -6t$$

or $$(1+y)^2\left|\frac{(2-y)}{Cy^3}\right| = e^{-6t}$$

or, $$(1+y)^2|2-y| = |Cy^3|e^{-6t}$$

Using the initial condition $y(0) = y_0$, we have

$$(1+y_0)^2 |2-y_0| = |C y_0^3|$$

$$\Rightarrow C = \frac{(1+y_0)^2(2-y_0)}{y_0^3}, \text{ since } y_0 \in (0,2)$$

Here C is nonzero positive real number.

Now from the solution

$$(1+y(t))^2 |2-y(t)| = C |y(t)|^3 e^{-6t}$$

Taking $t \to \infty$,

$$\Rightarrow \left\{1 + \lim_{t\to\infty} y(t)\right\}^2 \left|2 - \lim_{t\to\infty} y(t)\right| = C \left|\lim_{t\to\infty} y(t)\right|^3 \lim_{t\to\infty} e^{-6t}$$

$$\Rightarrow \left\{1 + \lim_{t\to\infty} y(t)\right\}^2 \left|2 - \lim_{t\to\infty} y(t)\right| = 0$$

$$\Rightarrow \lim_{t\to\infty} y(t) = -1 \text{ or } \lim_{t\to\infty} y(t) = 2.$$

$\Rightarrow \lim_{t\to\infty} y(t)$ belongs to $[-1, 2]$.

Ex 6 The general solution of the differential eqⁿ $xy' = y + \sqrt{x^2+y^2}$ for $x > 0$ is given by (with an arbitrary constant K) —

(a) $Ky^2 = x + \sqrt{x^2+y^2}$

(b) $Kx^2 = x + \sqrt{x^2+y^2}$

(c) $Kx^2 = y + \sqrt{x^2+y^2}$

(d) $Ky^2 = y + \sqrt{x^2+y^2}$

Ans: (c)

Solⁿ: $\frac{dy}{dx} = y/x + \sqrt{1 + (y/x)^2}$

Let $y/x = v$, Therefore try it.

Ex 7 The general solution of

$$\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$$

is

(a) $2\sin y = (x + c - \sin x \cos x)\sec x$

(b) $\sin y = (x+c)\cos x$

(c) $\cos y = (x+c)\sin x$

(d) $\sec y = (x+c)\cos x$

Ans: (b)

Sol^n:

$$\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$$

$$\Rightarrow \cos y \frac{dy}{dx} + \sin y \tan x = \cos x$$

Let $\sin y = v$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} + \tan x \, v = \cos x.$$

Linear in v.

Now try it.

(8) An integrating factor for

$$(\cos y \sin 2x)\,dx + (\cos^2 y - \cos^2 x)\,dy = 0$$

is

(a) $\sec^2 y + \sec y \tan y$

(b) $\tan^2 y + \sec y \tan y$

(c) $\dfrac{1}{\sec^2 y + \sec y + \tan y}$

(d) $\dfrac{1}{\tan^2 y + \sec y \tan y}$

Ans: (a)

Solⁿ: Here $M(x,y) = \cos y \sin 2x$

$N(x,y) = \cos^2 y - \cos^2 x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$ not exact.

Find the IF. Apply Rule III

$$\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = -(\sec y + \tan y)$$
$$= \varphi(y)$$

IF = ? (Try).

(9) Let $y_1(x)$ and $y_2(x)$ be the solⁿs of the differential eqⁿ $\frac{dy}{dx} = y + 7$ with initial condition $y_1(0) = 0$, $y_2(0) = 1$,

then

(A) y_1 and y_2 will never intersect

(B) y_1 and y_2 will intersect at $x=17$

(C) y_1 and y_2 will intersect at $x=e$

(D) y_1 and y_2 will intersect at $x=1$

Ans: (A)

Sol^n: $\dfrac{dy}{dx} = y+7$

Here $y_1(x) = 17(e^x - 1)$

and $y_2(x) = 18e^x - 17$

At the point of intersection of y_1 and y_2, we must have

$y_1(x) = y_2(x)$

$\Rightarrow 17(e^x - 1) = 18e^x - 17$

$\Rightarrow \boxed{e^x = 0}$ Shows that x does not exist.

Hence, y_1 and y_2 will never intersect.

10. Let $y : R \to R$ satisfy the IVP $y'(t) = 1 - y^2(t)$, $t \in R$, $y(0) = 0$. Then

(A) $y(t_1) = 1$ for some $t_1 \in R$

(B) $y(t) > -1 \ \forall \ t \in R$

(C) $y(t)$ is strictly decreasing in R.

(D) $y(t)$ is increasing in $(0,1)$ and decreasing in $(1,\infty)$

<u>Ans:</u> (B)

<u>Soln:</u> $\frac{dy}{dt} = 1 - y^2, \ t \in \mathbb{R}$

Integrating we have

$$\frac{1}{2}\log\left|\frac{1+y}{1-y}\right| = t + c$$

$$y(0) = 0 \Rightarrow c = 0$$

$$\log\left|\frac{1+y}{1-y}\right| = 2t$$

$$\Rightarrow \left|\frac{1+y}{1-y}\right| = e^{2t}$$

$$\Rightarrow y(t) = \frac{e^{2t}-1}{e^{2t}+1} \quad (\text{try!})$$

$$y(t) = \frac{e^{2t}-1}{e^{2t}+1}$$

Let $y(t_1) = 1$, for $t_1 \in \mathbb{R}$

$$\Rightarrow \frac{e^{2t_1}-1}{e^{2t_1}+1} = 1$$

$$\Rightarrow \boxed{-1 = 1}$$ which is not possible

So option (A) is wrong.

$$y(t) = \frac{e^{2t}-1}{e^{2t}+1} = -1 + \frac{2e^{2t}}{e^{2t}+1}$$

$\Rightarrow$ $y(t) > -1 \;\; \forall \cancel{t \in R}$, since

$$\frac{2e^{2t}}{1+e^{2t}} > 0 \;\; \forall t \in R.$$

Hence option (B) is correct.

we have

$$\frac{dy}{dt} = \frac{4e^{2t}}{(e^{2t}+1)^2} > 0 \;\; \forall \cancel{t \in R}$$

Shows that $y(t)$ is strictly increasing in R

$\Rightarrow$ options (C) and (D) are incorrect.

(11) The singular integral/Solution of the ODE

$(xy'-y)^2 = x^2(x^2-y^2)$ is

(A) $y = x\sin x$

(B) $y = x\sin(x+\pi/4)$

(C) $y = x$

(D) $y = x+\pi/4$.

Ans: (C)

Soln: $(xp-y)^2 = x^2(x^2-y^2)$

or $x^2p^2 - 2xyp + (y^2-x^4+x^2y^2) = 0$

[$Ap^2+Bp+C=0$ in this form]

which is quadratic in p.

we p-discriminant relation

$$B^2-4AC=0$$

$$\Rightarrow (-2xy)^2 - 4x^2(y^2-x^4+x^2y^2)=0$$

$$\Rightarrow \boxed{x^2(y-x)(y+x)=0}$$

Now we have to check the c-discriminant relation from the general sol^n.

Here to get the general sol^n will be little more complicated.

Now we will see that without finding the c-discriminant relation, still we can get the singular solution.

See it.

we got the p-discriminant relation

$$x^2 (y-x)(y+x) = 0$$

We know that p-discriminant relation gives the envelope locus (E), Tac locus (T), Cuspidal Locus (C) in the form ET^2C. (See the p-discriminant relation).

Note: Tac and cuspidal locus will never give the solution but Envelope locus will give the singular solⁿ

Aim: To check that which is the envelope locus i.e, Solution in the p-discriminant relation $\boxed{x^2 (y-x)(y+x) = 0}$

Clearly $x=0$ and $y+x=0$ will not satisfy the differential eqn.

$(y-x)$ is the solution to the given ODE, which is envelope locus and it is singular solution.

So the singular solution is

$\boxed{y = x}$ (Ans).

Remarks: If you want to solve the ODE to get general soln try in this way —

$$(xp-y)^2 = x^2(x^2-y^2)$$

$$xp-y = \pm\sqrt{x^2(x^2-y^2)}$$

$$= \pm x\sqrt{(x^2-y^2)}$$

$$\Rightarrow xp = y \pm x\sqrt{x^2-y^2}$$

$$\frac{dy}{dx} = p = y/x \pm \sqrt{x^2-y^2}$$

$$= y/x \pm x\sqrt{1-(y/x)^2}$$

Let $y/x = v$. (Try)

(12) The family of straight lines passing through the origin is represented

by

(a) $y\,dx + x\,dy = 0$

(b) $x\,dx + y\,dy = 0$

(c) $x\,dy - y\,dx = 0$

(d) $x\,dx - y\,dy = 0$

Ans: (c)

Solⁿ: The family of straight lines passing through the origin is $y = mx$, where m is constant.

$$\boxed{\frac{dy}{dx} = m}$$

[Aim: to remove m]

$$y = mx = \frac{dy}{dx} \cdot x$$

or $\boxed{x\,dy - y\,dx = 0}$

(13) The singular solution of $(xp-y)^2 = p^2 - 1$, where $p = \frac{dy}{dx}$ is

(A) $x^2+y^2=1$ (B) $x^2-y^2=1$

(C) $x^2+y^2=2$ (D) $x^2-y^2=2$

Ans: (B) (Try it)

(14) which of the following transformation reduce the differential eqn

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^{x} \sec y$$

into the form —

$$\frac{dv}{dx} + P(x)\,v = Q(x)\,?$$

(A) $v = \cos y$ (B) $v = \sec y$

(C) $v = \sin y$ (D) $v = \tan y$

Ans: (C) (Try)

(15) Which of the following is an integrating factor of the eq $(y^2 - 3xy)\,dx + (x^2 - xy)\,dy = 0$ with an arbitrary constant k?

(A) $\frac{k}{x^2 y}$ (B) $\frac{k}{xy}$ (C) $\frac{k}{xy^2}$ (D) $\frac{k}{x^2 y^2}$

Ans: (A) (Try)

Hint: The eqn is homogeneous but not exact.

Find I.F.

(16) The initial value problem

$y' = 2\sqrt{y}$, $y(0) = a$, has

(A) a unique solution if $a < 0$

(B) no solution if $a < 0$

(C) infinitely many solution if $a = 0$

(D) a unique solution if $a \geq 0$

Ans: (B)

Solⁿ: $y' = 2\sqrt{y}$

$\Rightarrow 2\sqrt{y} = 2x + c$

$y(0) = a \Rightarrow c = 2\sqrt{a}$,

for $a < 0$, c does not exist.

Thus the problem has no solution for $a < 0$.

Note: For most of the problem, we can't get the solution directly to check the answers.

For this case, we will check the continuity of the function $f(x,y)$ in the given domain D, and the continuity of $\frac{\partial f}{\partial y}$ on D or the Lipschitz continuity of $f(x,y)$. Here the given ODE is $\frac{dy}{dx} = f(x,y)$.

This shows the existence of the solⁿ on D.

This shows the uniqueness of the solⁿ on D

We will study this later in high level course on ODE.