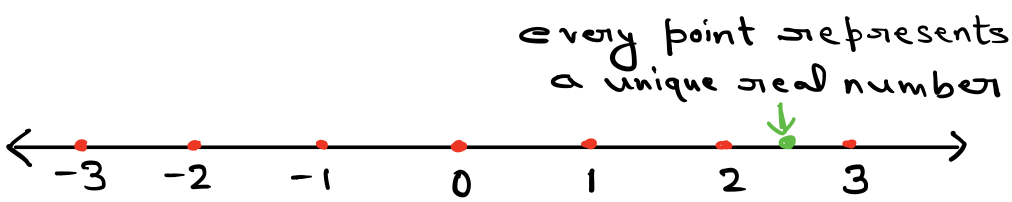
Subsets of $\mathbb{R}$

Geometrically $\mathbb{R}$ is represented by a line



$\mathbb{R}$ as a line

Intervals

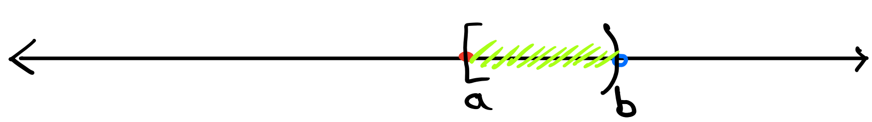
Bounded intervals

For real numbers $a < b < \infty$

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ – closed

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$ – open

similarly, define $[a,b)$ and $(a,b]$



$\mathbb{R}$ $[a,b)$

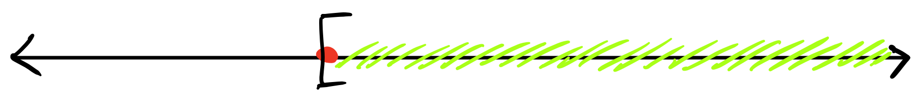
Typically, I will denote an interval

$\to$ I is one of $[a, b]$, (a, b), $[a, b)$ or $(a, b]$,

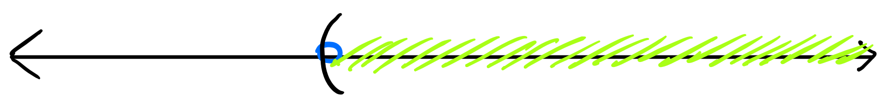
Unbounded intervals

Let $a \in \mathbb{R}$

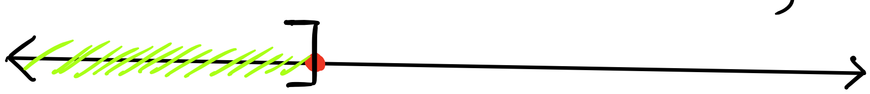
$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$



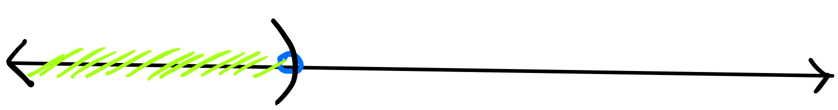
$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$



$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$



$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$



Some common examples

$\mathbb{R} = (-\infty, \infty)$ $\quad [-\infty \quad \infty]$

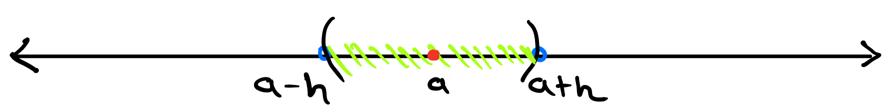
$\mathbb{R}_{>0} = (0, \infty)$

$\mathbb{R}_{\leq 0} = (-\infty, 0]$

Neighbourhood (nbhd.) of a point

Let $a \in \mathbb{R}$

A nbhd. of a is a symmetric open interval about a, i.e. of the form $(a-h, a+h)$ where $h > 0$



A nbhd. of a

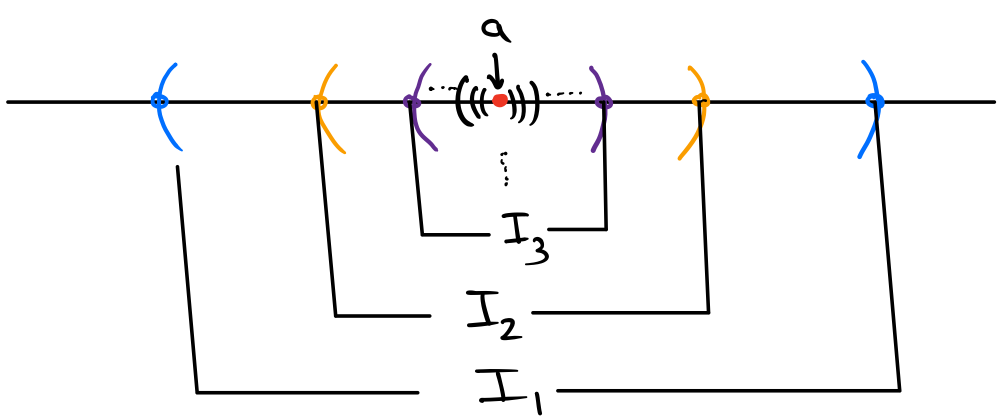
An important sequence of nbhds.

Let $a \in \mathbb{R}$,

for every $n \in \mathbb{N}$, define

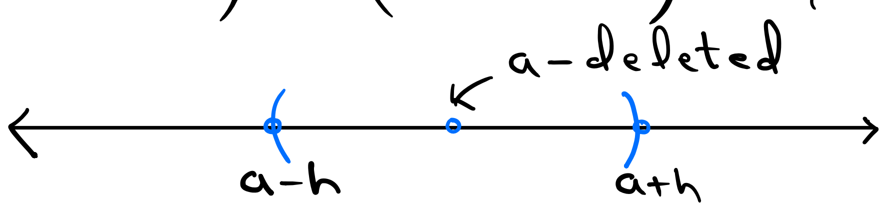
$$I_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$$

Then
- $I_1 \supset I_2 \supset I_3 \supset \cdots\cdots$
- $a \in I_n \quad \forall\ n \in \mathbb{N}$.



Deleted Nbhd.

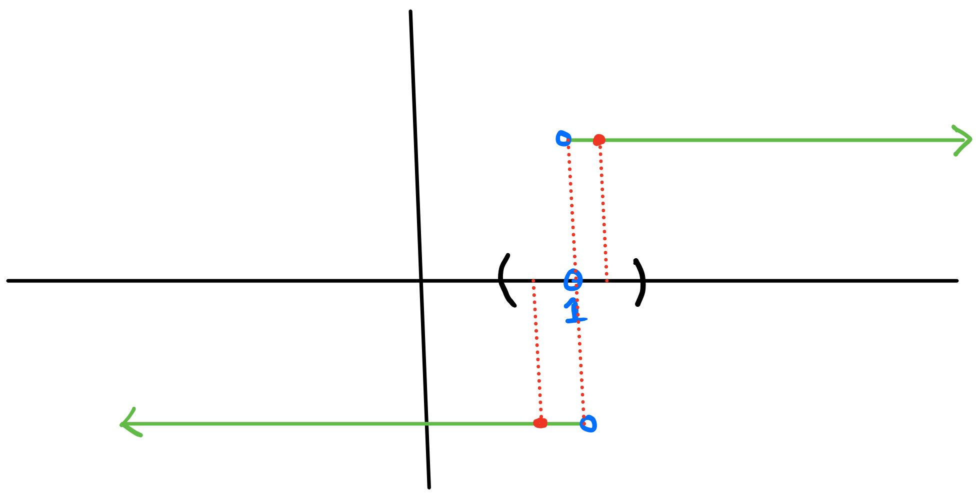
$$(a-h, a+h)' = (a-h, a+h) \setminus \{a\}$$



Example

Let $f(x) = \dfrac{|x-1|}{x-1}$

Then $f(x)$ is defined on <u>any</u> deleted nbhd. $(1-h, 1+h)'$



$f(x)$ defined on every deleted nbhd. of 1.

Distribution of Rationals
(and irrationals)

- Between any two (distinct) rationals, there is another rational. (Ex.)

Thm. Let $x < y$ be real nos. Then $\exists$ a rational q satisfying
$$x < q < y.$$

Preparation

(1) (Archimedian Property (AP))

Let $x > 0$. Then $\exists$ a $n \in \mathbb{N}$ s.t. $nx > 1$.

(2) Let $x \in \mathbb{R}$. Then $\exists$ $m \in \mathbb{Z}$ s.t.
$$m-1 \leq x < m$$

Proof of (1).

If $x > 1$, then done

$x = 1$ (simply choose $n = 1$)

Now, suppose $0 < x < 1$

example. say $x = \frac{1}{3}$, then

$$4x = \frac{4}{3} > 1$$

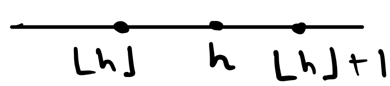
Since $x < 1$, $\exists$ a $h > 0$ s.t.

$$x = \frac{1}{1+h} \leftarrow h = \frac{1}{x} - 1$$

Let $n = 2 + \lfloor h \rfloor = 1 + (1 + \lfloor h \rfloor)$

$$> 1 + h$$

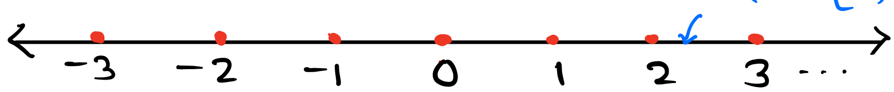
$h \in [\lfloor h \rfloor, \lfloor h \rfloor + 1)$



Thus, $nx > \frac{1+h}{1+h} = 1$. $\square$

Proof of (2) (obvious)

$2.2 \in [2, 3)$



Thus, every real number lies in some $[m-1, m)$ where $m \in \mathbb{Z}$.

$\square$

<u>Proof of Thm.</u>

We have to show that $\exists\ m, n \in \mathbb{Z}$ with $n > 0$ s.t.

$\frac{m}{n}$

$$x < \frac{m}{n} < y \quad \text{—} (*)$$

$(*)$ can be proved by showing $\exists\ m \in \mathbb{Z},\ n \in \mathbb{N}$ s.t.

$$0 < m - nx < n(y-x) \quad \text{—} (**)$$

$(**)$ can be proved by showing $\exists\ m \in \mathbb{Z},\ n \in \mathbb{N}$ s.t.

$$\left.\begin{aligned} &0 < m - nx \leq 1 \\ \text{and} \quad &1 < n(y-x) \end{aligned}\right\} \Rightarrow (**)$$

Since $y - x > 0$, by AP, $\exists$ a $n \in \mathbb{N}$ s.t. $n(y-x) > 1$

by (2), $\exists$ a $m \in \mathbb{Z}$ s.t.

$$m-1 \leq nx < m$$

$$\Rightarrow \quad 0 < m - nx \leq 1 .$$

$\square$

Ex: Prove the Thm. with rationals replaced by irrationals.

Corollary. Between any two real nos. $\exists$ infinitely many rationals. (or irrationals)

Corollary. Let I be an interval, then I contains infinitely many rationals (or irrationals).

Bounded (bdd.) subsets of $\mathbb{R}$

$$[-2,2] = \{x \in \mathbb{R} : -2 \le x \le 2\}$$

bdd. from below

bdd. from above

$$(-2,2) = \{x \in \mathbb{R} : -2 < x < 2\}$$

bdd. from both sides

$$(2,\infty) = \{x \in \mathbb{R} : x > -2\}$$

bdd. from below

not bdd. from above

$$(-\infty,-2) = \{x \in \mathbb{R} : x < 2\}$$

bdd. from above

not bdd. from below

$$(-\infty,-2) \cup (2,\infty)$$

$$= \{x \in \mathbb{R} : x < -2 \text{ or } x > 2\}$$

not bdd. from either side

Defn. • Let $E \subseteq \mathbb{R}$ and $E \neq \phi$.

Then E is said to be bdd.
if $\exists$ a $M > 0$ s.t.

$$|x| < M \quad \forall\, x \in E$$

E is bdd. from above

• A real number M_1 is said to be an upper bound of E if

$$x \leq M_1 \quad \forall\, x \in E$$

E is bdd. from below

• Similarly, M_2 is a lower bound of E if

$$x \geq M_2 \quad \forall\, x \in E$$

Remark: If a set is bounded from above (resp. lower), then $\exists$ infinitely many upper (resp. lower) bounds.

Defn. The Least Upper Bound (lub) or the Supremum of a set.

It is the <u>smallest</u> upper bound.

- $E = [-2, 2]$

  Set of upper bounds of E is

  $$\{x \in \mathbb{R} : x \geq 2\} = [2, \infty)$$

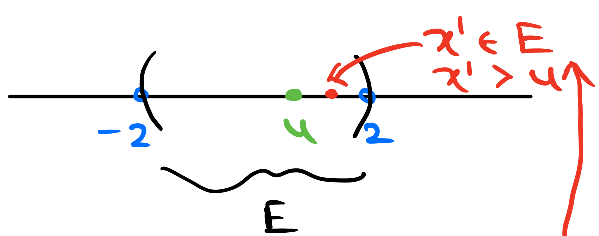
  So, $\sup E = 2$

- $E = (-2, 2)$

  Set of upper bounds is $[2, \infty)$

  $\sup E = 2$

  If $u < 2$, then u is <u>not</u> an upper bound.



so u fails to be an upper bound.

Mathematical Description of Sup

$L = \text{Sup } E$ if both of the following hold

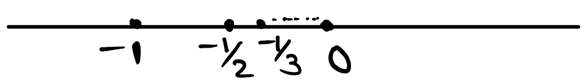
- $x \leq L \ \forall \ x \in E$
- For every $\varepsilon > 0$, $\exists$ a $x \in E$ s.t.

$$L - \varepsilon < x \leq L$$

Examples.

- Find Sup E where

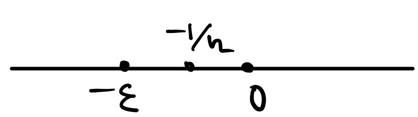
$$E = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$$



$\rightarrow$ E is bdd.

Claim: $\text{Sup } E = 0$

Clearly $-1/n < 0 \ \forall n \in \mathbb{N}$



Let $\varepsilon > 0$ (be arbitrary)

by AP, $\exists\ n \in \mathbb{N}$ s.t.

$$n\varepsilon > 1$$

$$\Rightarrow \quad \varepsilon > \frac{1}{n}$$

$$\Rightarrow \quad -\varepsilon < -\frac{1}{n}$$

hence $\sup E = 0$.

- Let $E = \mathbb{Q} \cap (-\infty, 2)$

Determine $\sup E$.

$$E = \{x \in \mathbb{Q} : x < 2\}.$$

<u>Claim:</u> $\sup E = 2$

It is clear that 2 is an upper bound

Let $\varepsilon > 0$. By density property $\exists$ a rational q satisfying

$$2-\varepsilon < q < 2$$

Since, this $q \in E$, it follows that

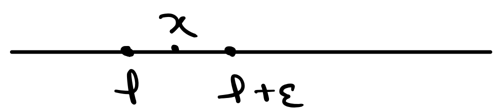
$$\sup E = 2.$$

Defn. The Greatest Lower Bound (glb) or the infimum of a set.

$\ell = \inf E$ if both of the following hold.

- $x \geq \ell \quad \forall\, x \in E$
- For every $\varepsilon > 0$, $\exists$ a $x \in E$ s.t.

$$\ell \leq x < \ell + \varepsilon$$



Examples

- Let $E = \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\}$

  Then $\inf E = 1$ (Ex.)

- Let $E = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : x > 0 \right\}$

  Then $\inf E = 0$ (Ex.).

The Diameter of a set E.

Let $E \neq \emptyset$. Then

$$\operatorname{diam} E = \sup\{|x-y| : x, y \in E\}.$$

Remark. $\operatorname{diam} E \geq 0$

Examples.

- $\operatorname{diam} \{1\} = 0$
- $\operatorname{diam} \{1,2,3\} = 2$
- $\operatorname{diam} \mathbb{N} = \infty$

- $\operatorname{diam} (a,b) = b-a$

  if $x, y \in (a,b)$, then

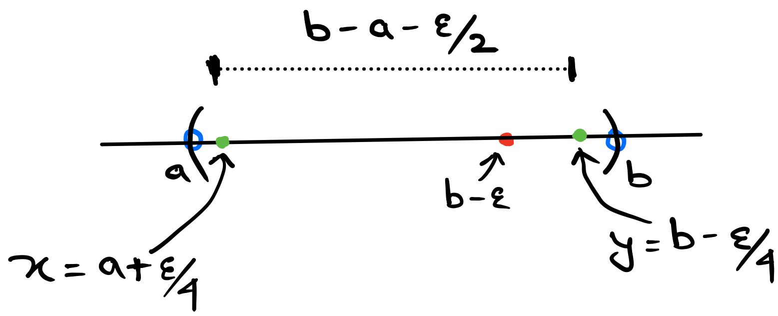
  $$|x-y| < b-a$$

  so, $b-a$ is an upper bd.

  Now, let $\varepsilon > 0$.

  we have to show that $\exists$ $x, y \in (a,b)$ s.t.

  $$b-a-\varepsilon < |x-y|$$



$$|x-y| = b-a-\varepsilon/2$$
$$> b-a-\varepsilon.$$

- $\operatorname{diam}\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 1$

  Suppose, $m > n$, then

  $$\frac{1}{n} - \frac{1}{m} < 1 - 0 = 1$$

  Now, let $\varepsilon > 0$,
  we have to show that
  $1-\varepsilon$ is not a upper bd. of E

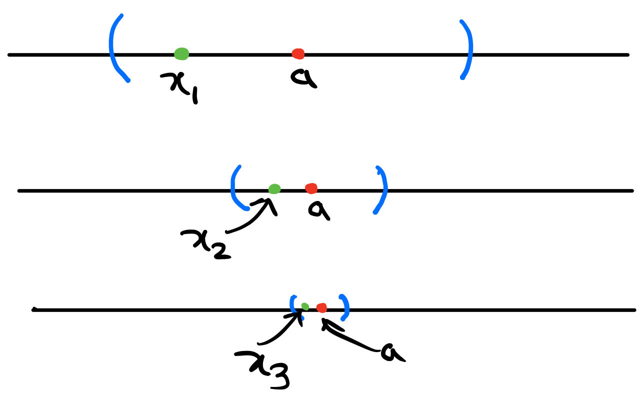
  Apply, A.P.
  $\exists\ n \in \mathbb{N}$ s.t. $n\varepsilon > 1$
  can assume $n > 1$
  then $\frac{1}{n} < \varepsilon$

  $$\Rightarrow 1 - \frac{1}{n} > 1 - \varepsilon.$$

Limit Point of a set

Let $E \neq \phi$.

Then "a" is a limit point of E if every nbhd. of "a" contains a point of E.



$x_1, x_2, x_3 \in E$.

Alternatively,

"a" is a limit point of E if for every $\varepsilon > 0$,

$\exists \; x \in E$ s.t

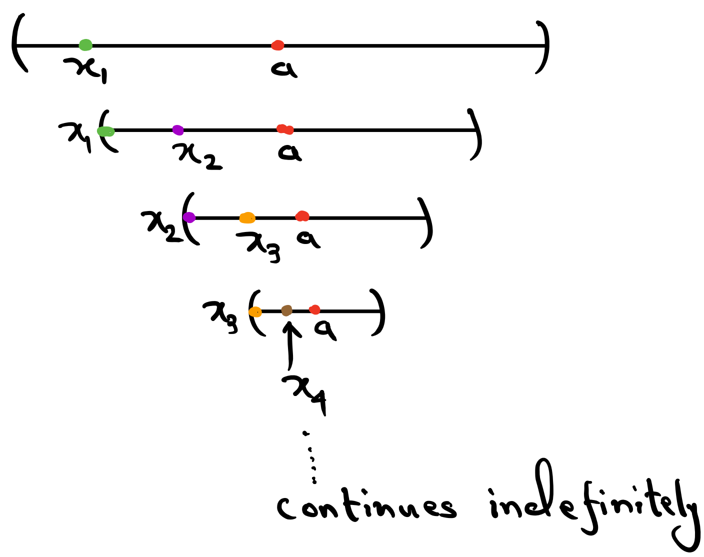
$$|x - a| < \varepsilon$$

or

$$x \in (a - \varepsilon, a + \varepsilon)$$

Thm. If "a" is a limit point of E, then every nbhd. of "a" contains infinitely many elements of E.

A geometric proof



Corollary. If E has a limit pt., then E is infinite.