Improper integrals of first kind:

Area
Arc length
Volume
Surface area.

→ $\int_a^\infty f$ Convergent if f is integrable on $[a,x]$ $\forall x \geq a$ and $\lim_{x\to\infty} \int_a^x f$ exists.

→ $\int_{-\infty}^b f$ Convergent if f is integrable on $[x,b]$ $\forall x \leq b$ and $\lim_{x\to-\infty} \int_x^b f$ exists.

→ $\int_{-\infty}^\infty f$. convergent if $\int_0^\infty f$ and $\int_{-\infty}^0 f$ exist.

→ $\lim_{x\to\infty} \int_{-x}^x f$ exists and is equal to L, then it is called the Cauchy principal value of $\int_{-\infty}^\infty f$.

⇝ $\int_{-\infty}^\infty f$ converges $\Rightarrow$ CPV exists $\not\Rightarrow$ $\int_{-\infty}^\infty f$ converges.

$$\boxed{\int_{-\infty}^\infty \frac{2t}{t^2+1}\,dt}$$

$$\boxed{\log|1+a^2|} - \underbrace{\log|1|}_{0}$$

$$\int_0^a \frac{2t}{t^2+1}\,dt \underset{FTC (2)}{\overset{\text{Method of subst.}}{=\!=}} \log|1+t^2|\Big]_0^a = \log|1+a^2| \longrightarrow \infty \text{ as } a\to\infty$$

$\Rightarrow \int_0^\infty f$ does not exist.

$\Rightarrow \int_{-\infty}^\infty f$ does not exist.

$$\int_{-a}^a \frac{2t}{t^2+1}\,dt \underset{FTC (2)}{\overset{\text{Substitution}}{=\!=}} \log|1+t^2|\Big]_{-a}^a = 0$$

$\Rightarrow \lim_{a\to\infty} \int_{-a}^a \frac{2t}{1+t^2}\,dt = 0$ (CPV for $\int_{-\infty}^\infty \frac{2t}{1+t^2}\,dt$)

$\lim_{a\to\infty} (0) = 0$

$\int_1^\infty \frac{1}{t}\,dt$ does not converge.

$\int_1^\infty \frac{1}{t^p}\,dt$ converges if $p>1$.

$\int_a^b f$ $f: \boxed{[a,b]} \to \mathbb{R}$

f is unbounded.

$\int_{\mathbb{R}}$ unbounded.

$[a,\infty)$: $\int_a^\infty f$ $[a,x]$: $\lim_{x\to\infty} \int_a^x f$

$(-\infty, a]$ $\forall x \leq a$ $[x,a]$: $\lim_{x\to-\infty} \int_x^a f$.

$(-\infty, a]$ $[a,\infty)$ $(-\infty,\infty)$

$\lim_{x\to\infty} \int_{-x}^x f$ $\xrightarrow{CPV}$ $\int_{-\infty}^\infty f$ $\int_a^\infty f$ $\int_{-\infty}^a f$

$\int_{-\infty}^0 f$ $\int_0^\infty f$

$$\boxed{\int_0^{\infty} (1+2x)\cdot e^{-x}\, dx.}$$

$$= \boxed{\int_0^{\infty} e^{-x} dx} + 2\cdot\int_0^{\infty} x e^{-x} dx.$$

$$= I_1 + I_2$$

$$\lim_{t\to\infty} \boxed{\int_0^{t} (1+2x)e^{-x} dx} \quad \forall t \geq 0$$

$[0,t]$

$(1+2x)e^{-x}$

$$I_1 = \int_0^{\infty} e^{-x} dx. \qquad I_2 = 2\int_0^{\infty} x e^{-x} dx$$

$$\int_0^a e^{-x} dx \underset{=}{\text{FTC(2)}} -e^{-x}\Big]_0^a = -e^{-a} - (-1) = 1 - e^{-a} = I_1.$$

$$\lim_{a\to\infty} \int_0^a e^{-x} dx = \lim_{a\to\infty}\left(1 - \frac{1}{e^a}\right) = 1$$

$\frac{1}{e^a} \to 0$

$\boxed{f(x) = x}$, $\boxed{g(x) = e^{-x}}$ } all the conditions of integration by parts are satisfied.

$$\boxed{\int_0^a x e^{-x} dx} = f(a)G(a) - f(0)G(0) - \int_0^a f'(x)\cdot G(x)\, dx$$

$$= a(-e^{-a}) - 0(-e^{0}) - \int_0^a -e^{-x} dx.$$

$$= a(-e^{-a}) + \boxed{\int_0^a e^{-x} dx}$$

$$= -ae^{-a} + I_1.$$

$$\Rightarrow I_2 = -2ae^{-a} + 2I_1$$

$$\Rightarrow I_1 + I_2 = -2ae^{-a} + 3I_1 = -2ae^{-a} + 3(1 - e^{-a})$$

$$= -e^{-a}(2a+3) + 3.$$

$$\int_0^a (1+2x)e^{-x} dx = -e^{-a}(2a+3) + 3$$

$$= 3 - e^{-a}(2a+3)$$

$$= 3 - \frac{2a+3}{e^a} \longrightarrow 3 \text{ as } a \to \infty.$$

$\frac{2a+3}{e^a} \to 0$

$\int_0^{\infty} (1+2x)e^{-x} dx$ exists and it is equal to 3.

$$\int_0^a e^{-x} dx$$

$$2\int_0^a x e^{-x} dx$$

$$\int_0^{\infty} (1+2x)e^{-x} dx \longrightarrow \text{converges.}$$

$$\frac{2a}{e^a} \to 0$$

$$\frac{3}{e^a} \to 0$$

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$$

$$\int_{b}^{0} \frac{1}{\sqrt{3-x}}\,dx = \int_{b}^{0} (3-x)^{-1/2}\,dx$$

$$\overset{FTC(2)}{=} -\frac{(3-x)^{1/2}}{1/2}\Bigg]_{b}^{0}$$

$$= -2\cdot\left(\sqrt{3} - \sqrt{3-b}\right)$$

$$\longrightarrow \infty \quad \text{as} \quad b \to -\infty$$

divergent.

$(-\infty, 0]$

$$\frac{1}{\sqrt{3-x}}$$

$x = 3$

$[b, 0]$

$$-2\sqrt{3} - 2\sqrt{3-b}$$

$b \to -\infty$

$3-b \to \infty$

$-2\sqrt{3-b} \to -\infty$

Convergent tests

→ **Direct comparison test:**

If $\int_a^b f$ exists for all $a \le b$ and $\boxed{0 \le f(x) \le g(x)}$ $\forall x \ge a$.

then $\int_a^\infty g$ converges $\Longrightarrow \int_a^\infty f$ converges.

$$0 \le \int_0^\infty f(x)\,dx \le \int_0^\infty g(x)\,dx$$

(Seen) $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges.

$\int_1^\infty \frac{1}{\sqrt{x^2 - 0.01}}\,dx$ diverges

$0 \le \frac{\cos x}{x^2} \le \frac{1}{x^2}$

$\boxed{\int_1^\infty \frac{1}{x}\,dx \quad \text{divergent}}$

$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \quad \text{converges if } p > 1}$

$\boxed{\int_2^\infty \frac{1}{x}\,dx}$

$\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$ converges.

$\frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$

$\int_2^\infty \frac{1}{x^2}\,dx$ converges

$\int_2^\infty \frac{\cos x}{x^2}\,dx$

$\int_a^\infty \frac{1}{x}\,dx$

$\int_1^\infty \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^\infty \frac{1}{x}\,dx$

$\int_3^\infty \frac{1}{x + e^x}\,dx$ converges. $\boxed{0 \le \frac{1}{x+e^x} \le \frac{1}{e^x}}$

$0 \le \frac{1}{x+e^x} \le \frac{1}{x}$

$0 \le f(x) \le g(x)$

$-\frac{1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2}$

$\int_3^\infty \frac{1}{x - e^{-x}}\,dx$ diverges. $\boxed{\frac{1}{x - e^{-x}} \ge \frac{1}{x} \ge 0}$ D.C.T.

$\int_3^\infty \frac{1}{x}\,dx$ divergent

$0 \le \frac{\sin x}{x^2} + \frac{1}{x^2} \le \frac{1}{2x^2}$

$\boxed{\int_0^\infty \frac{\sin x + 1}{x^2}\,dx}$ ← Nakul converges

$\frac{|\sin x|}{x^2} \le \frac{1}{x^2}$

$\frac{\sin x}{x^2}$

$\lim_{a \to \infty} \int_{[0,a]}$

$\int_1^\infty \frac{1 + 3\sin^4(2x)}{\sqrt{x}}$ diverges

Check!!

$\int_{1/2}^\infty e^{-x^2}\,dx$ converges.

$= \int_{1/2}^1 e^{-x^2}\,dx + \underbrace{\int_1^\infty e^{-x^2}\,dx}$ ← converges.

Check!!

$\frac{1}{e^{x^2}} \le \frac{1}{e^x}$

$e^{x^2} \ge x^2$

$x \le x^2$

$e^x \le e^{x^2}$

$\frac{1}{e^{x^2}} \le \frac{1}{e^x}$

Limit Comparison Test:

Suppose $\int_a^b f$ and $\int_a^b g$ exist for all $b \geq a$ where $f(x) \geq 0$ $\forall x \geq a$
and $g(x) > 0$

If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = c$, $c \neq 0$. then

$\int_a^\infty f$ and $\int_a^\infty g$ converge or diverge simultaneously.

(Seen) $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ converges.

$\displaystyle\int_1^\infty \frac{3}{e^x+5}\,dx$ converges

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

$f(x) = \dfrac{1-e^{-x}}{x} \geq 0$ for all $x \geq 1$.

$g(x) = \dfrac{1}{x} > 0$. for all $x \geq 1$.

$\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

$$\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\Big(1-\underbrace{\frac{1}{e^x}}_{\downarrow\,0}\Big) = 1 \neq 0$$

Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges, by limit comparison test

$\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ also diverges.

Improper Integrals of Second kind

$\int_a^b f$ ← unbounded.

bounded intervals
unbounded functions

① $f:(a,b]\to\mathbb{R}$ f is unbounded but integrable on $[x,b]$ for $a<x\le b$

$\boxed{f:(0,1]\to\mathbb{R},\ f(x)=\frac{1}{x}}$

$(0,1]$ $\frac{1}{x}$

$\int_{a<t\le b} f$ is convergent iff $\lim_{x\to a^+}\int_x^b f$ exists

$$\int_{a<t\le b} f = \boxed{\lim_{x\to a^+}\int_x^b f}$$

$[x,b]$

② $f:[a,b)\to\mathbb{R}$, f unbounded but integrable on $[a,x]$ for all $x\in[a,b)$

$\int_{a\le t<b} f$ is convergent iff $\lim_{x\to b^-}\int_a^x f$ exists

$[0,1)$ $\frac{1}{1-x}$ $[0,a]\subset[0,1)$

$$\int_{a\le t<b} f = \lim_{t\to b^-}\int_a^x f$$

③ $f:(a,b)\to\mathbb{R}$ f is unbounded on (a,b) but integrable on $[x,y]$ $\forall x,y\in(a,b)$

$\int_{a<t<b} f$ converges $\Longleftrightarrow$ for $c\in(a,b)$, $\int_{a<t\le c} f$ and $\int_{c\le t<b} f$ converge.

$$\lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{b-\varepsilon} f$$

④ $a<c<b$.
f is integrable on $[a,x]$ $\forall x\in[a,c)$ but unbounded on $[a,c)$

And

f is integrable on $[x,b]$ $\forall x\in(c,b]$ but unbounded on $(c,b]$.

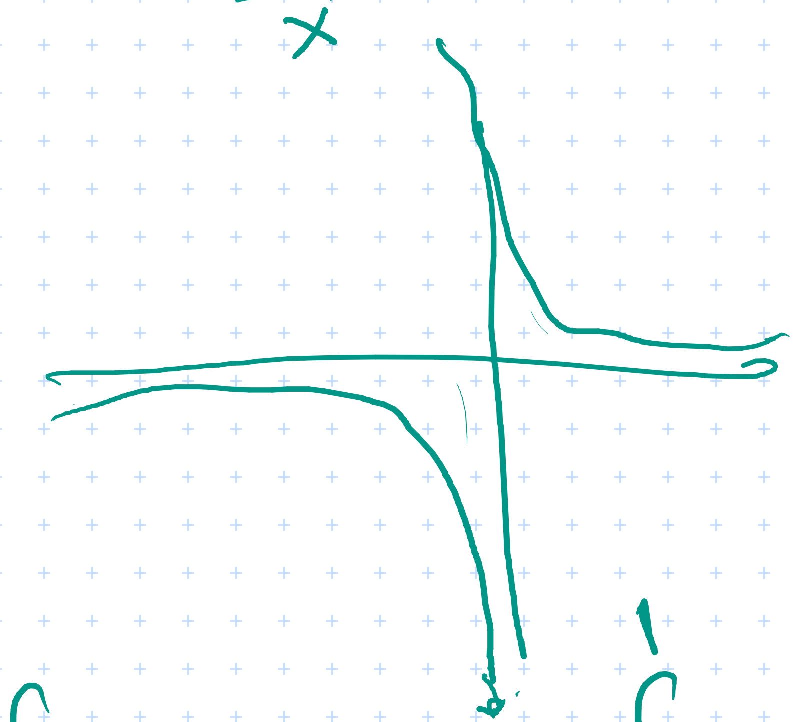
Then $\int_a^b f$ converges iff $\int_{a\le t<c} f$ and $\int_{c<t\le b} f$ converge.

$[a,x]$ $a\le x<c$

$[x,b]$ $c<x\le b$

$$\int_{a<t\le c} f + \int_{c\le t<b} f = \int_{a<t<b} f$$

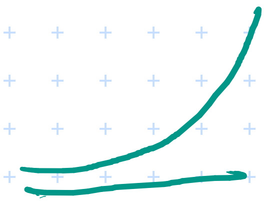
$[1,-1]$ $\frac{1}{x}$



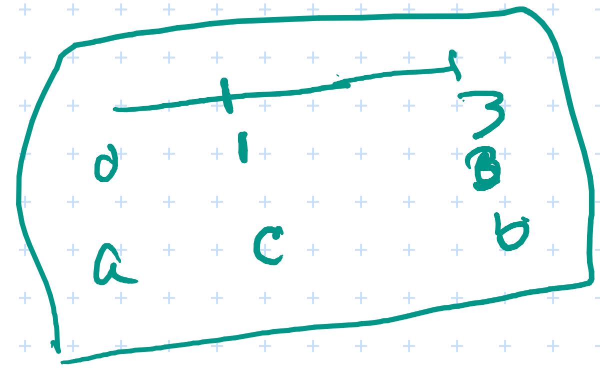
$$\int_{-1\le x<0} + \int_{0<x\le 1} = \int_{-1}^{1}$$

(Seen). $\int_0^1 \frac{1}{1-x}\,dx$ diverges

$\int_0^3 \frac{1}{(t-1)^{2/3}}\,dt$ converges.



$$\int_0^4 \frac{x}{x^2-9}\,dx.$$



$a=0, \quad c=3, \quad b=4.$

$\forall x \in [0,3)$, $\int_0^x \frac{t}{t^2-9}\,dt$ exists

$x=3$

$$\underset{\text{Subst.}}{\overset{\text{FTC(2)}}{=}} \frac{1}{2}\log|t^2-9|\Big]_0^x$$

$[0,3)$.

$(3,4]$

$$= \frac{1}{2}\log(9-x^2) - \frac{1}{2}\log 9$$

$\xrightarrow{x\to 3^-} -\infty.$

$$\lim_{x\to 3^-}\int_0^x \frac{t}{t^2-9}\,dt$$

$$\lim_{x\to 3^+}\int_x^4 \frac{t}{t^2-9}\,dt$$

Not convergent.

X

$$\int_1^4 \frac{1}{x^2+x-6}\,dx$$

x^2+x-6
$=(x+3)(x-2)$.

$a < c < b$, $a = 1$, $c = 2$, $b = 4$.

$$\frac{1}{x^2+x-6} = \frac{1}{(x+3)(x-2)} = \frac{1}{5}\left(\frac{1}{x-2} - \frac{1}{x+3}\right)$$

$$\int_1^2 \frac{1}{x-2}\,dx$$

For $t \in [1, 2)$, $\int_1^t \frac{1}{x-2}\,dx$

$= \log|x-2|\Big]_1^t$

$= \log(2-t) \xrightarrow{t \to 2^-} -\infty$

Not convergent.

$$\int_1^\infty = \int_1^2 + \int_2^\infty$$

$$\int_1^4 \frac{1}{x+3}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x}\,dx$$

$x=0$

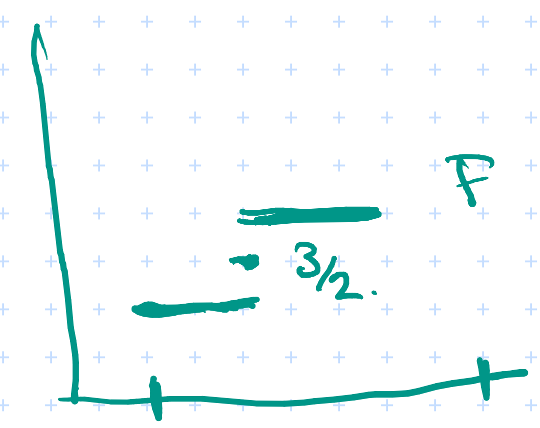
Darboux's Theorem

$F : [a,b] \to \mathbb{R}$ differerentiable

F' must have IVP.

$[x]$

$[1, 3]$



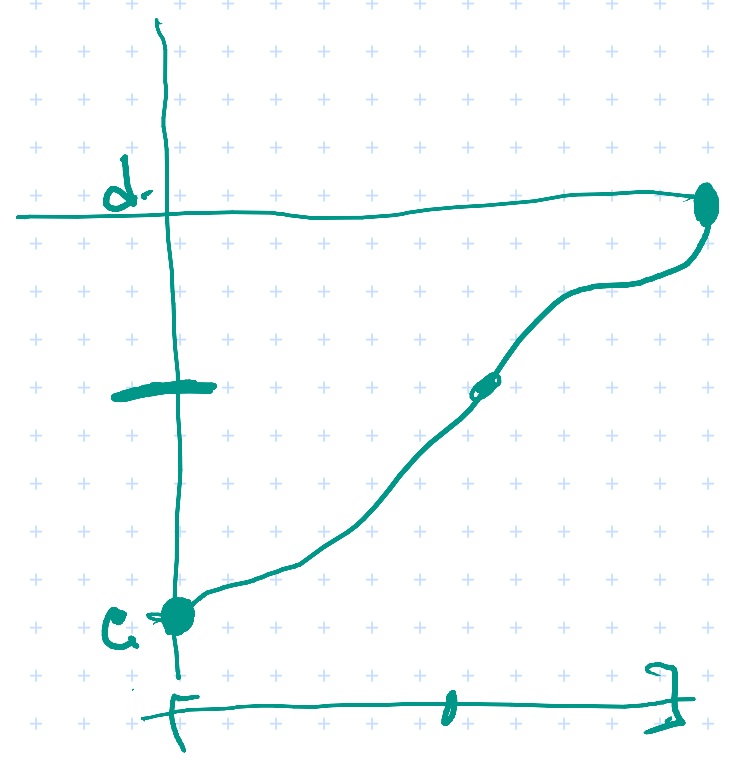
3/2.

$[1,3]$

$G' = F$.

$G' = F$

G



f satisfies IVP on $[a,b]$

if $f(a)=c$. $f(b)=d$.

$c<d$.

then for every $c<y<d$.

$\exists\, x \in [a,b]$ s.t.

$f(x)=y$

$$\lim_{n\to\infty} \frac{1}{n^{17}} \sum_{i=1}^{n} i^{16}$$

$$\lim_{n\to\infty} \frac{1}{n^{17}} + \frac{2^{16}}{n^{17}} + \cdots + \frac{n^{16}}{n^{17}}$$

$$\sum \frac{1}{n}$$

$n=1$ $\quad \frac{1}{1^{17}}$

$n=2$ $\quad \frac{1}{2^{17}} + \frac{2^{16}}{2^{17}}$

$n=3$ $\quad \frac{1}{3^{17}} + \frac{2^{16}}{3^{17}} + \frac{3^{16}}{3^{17}}$