Lecture 2

Applications of the Heisenberg Uncertainty Principle

Young's Double Slit Experiment

Wave packet; phase and group velocity

Dispersion Relation

The Uncertainty Principle

- One of the fundamental consequences of quantum mechanics is that it is IMPOSSIBLE to SIMULTANEOUSLY determine the POSITION and MOMENTUM of a particle with COMPLETE PRECISION

- Can be illustrated by a couple of “thought experiments”, for example the “photon picture” of single slit diffraction and the “Heisenberg Microscope”

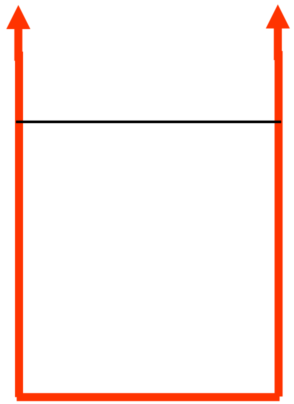
Some Applications of Heisenberg’s uncertainty principle

- Measurement of the position of a particle and its momentum in a Gamma ray Microscope is in accordance with the Heisenberg’s uncertainty principle
- It explain why free electrons cannot reside inside an atomic nucleus
- The diffraction of electrons through a narrow slit takes place in accordance with Heisenberg’s uncertainty principle.
- We can estimate the ground state energy of a linear harmonic oscillator
- Natural broadening of the spectral line

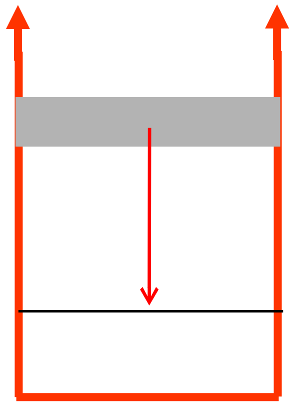
Energy-time Uncertainty

- Uncertainty principle also applies to simultaneous measurements of *energy* and *time*

$$\Delta E \Delta t \geq h$$



Stationary state
Zero energy spread



Decay to lower state with finite lifetime Δt: Energy broadening ΔE
(explains, for example “natural line-width”
In atomic spectra)

Heisenberg Microscope

Suppose we have a particle, whose momentum is, initially, precisely known. For convenience assume initial p = 0.

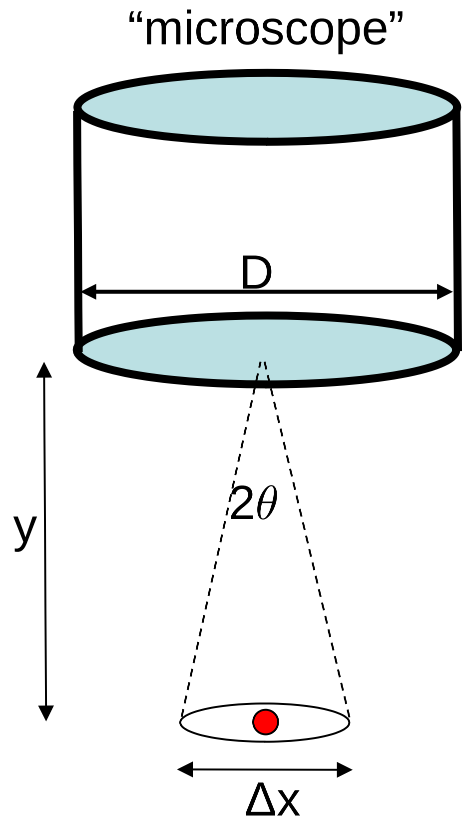
From wave optics (Rayleigh Criterion)

$$\sin\theta \approx \frac{\lambda}{D}$$

From our diagram:

$$\sin\theta \approx \frac{\Delta x}{2} \div y = \frac{\Delta x}{2y}$$

$$\Delta x \approx \frac{2y\lambda}{D}$$

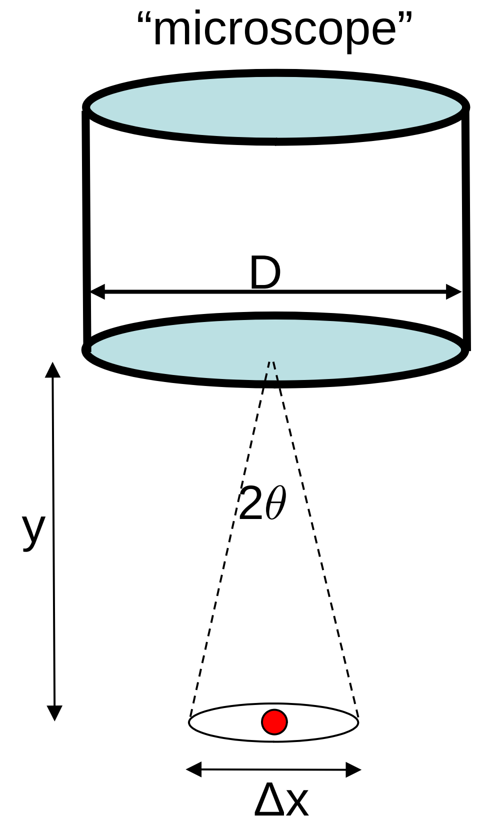


Heisenberg Microscope

$$\Delta x \approx \frac{2y\lambda}{D}$$

Since this is a "thought experiment" we are free from any practical constraints, and we can locate the particle as precisely as we like by using radiation of shorter and shorter wavelengths.

But what are the consequences of this?

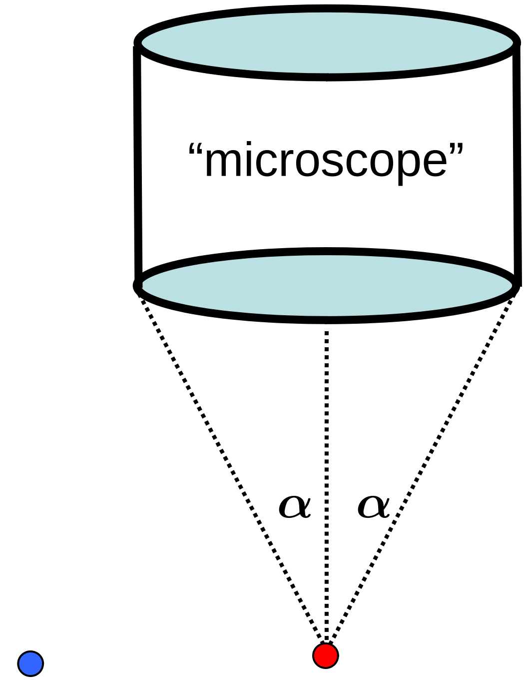


Heisenberg Microscope

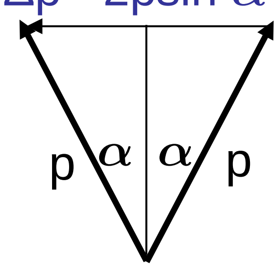
In order to see the particle, a photon must scatter off it and enter the microscope.

Thus process MUST involve some transfer of momentum to the particle…….

BUT there is an intrinsic uncertainty in the X-component of the momentum of the scattered photon, since we only know that the photon enters the microscope somewhere within a cone of half angle :



$\Delta p = 2p\sin\alpha$



By conservation of momentum, there must be the same uncertainty in the momentum of the observed particle……………

Heisenberg Microscope: Summary

Uncertainty in position of particle: $$\Delta x \approx \frac{2y\lambda}{D}$$

Can reduce as much as we like by making λ small……

Uncertainty in momentum of particle: $$\Delta p \approx 2\, p_{photon} \sin\alpha \approx \frac{2h}{\lambda}\frac{D}{2y}$$

So, if we attempt to reduce uncertainty in position by decreasing λ, we INCREASE the uncertainty in the momentum of the particle!!!!!!

Product of the uncertainties in position and momentum given by:

$$\Delta x \Delta p \approx \frac{2y\lambda}{D}\frac{Dh}{y\lambda} = 2h$$

The Uncertainty Principle

Our microscope thought experiments give us a rough estimate for the uncertainties in position and momentum:

$$\Delta x \Delta p \sim h$$

"Formal" statement of the Heisenberg uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Example: Diffraction

- Light emerging from a tiny hole or slit will diverge (diffract)
- We know its position very well (in at least one dimension)
  - so we give up knowledge of momentum in that dimension—thus the spread

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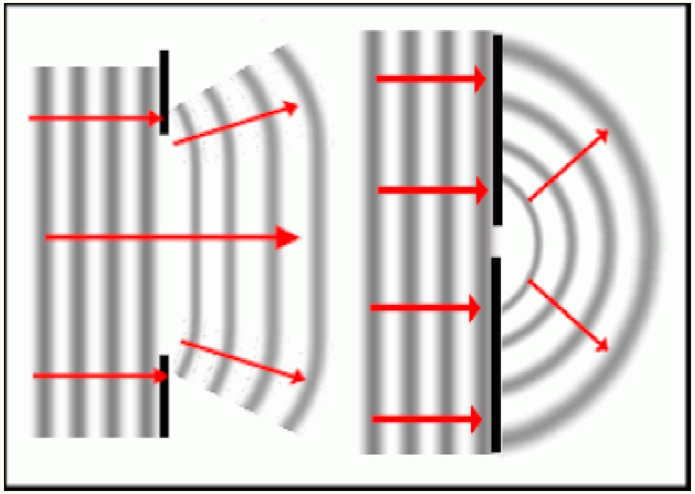

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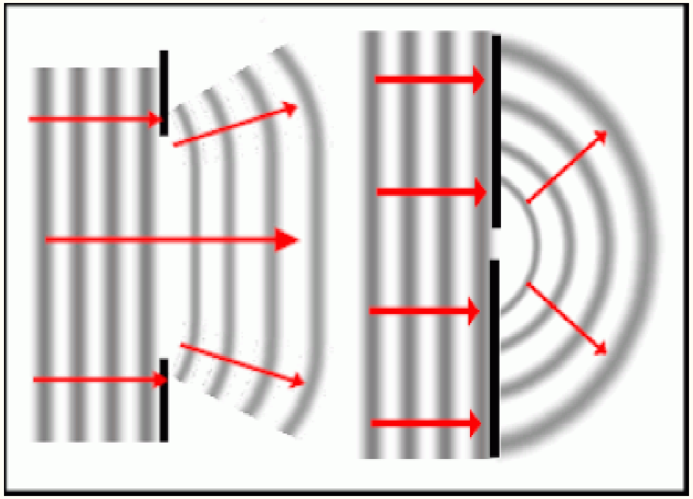
large opening: greater position uncertainty results in smaller momentum uncertainty, which translates to less of a spread angle



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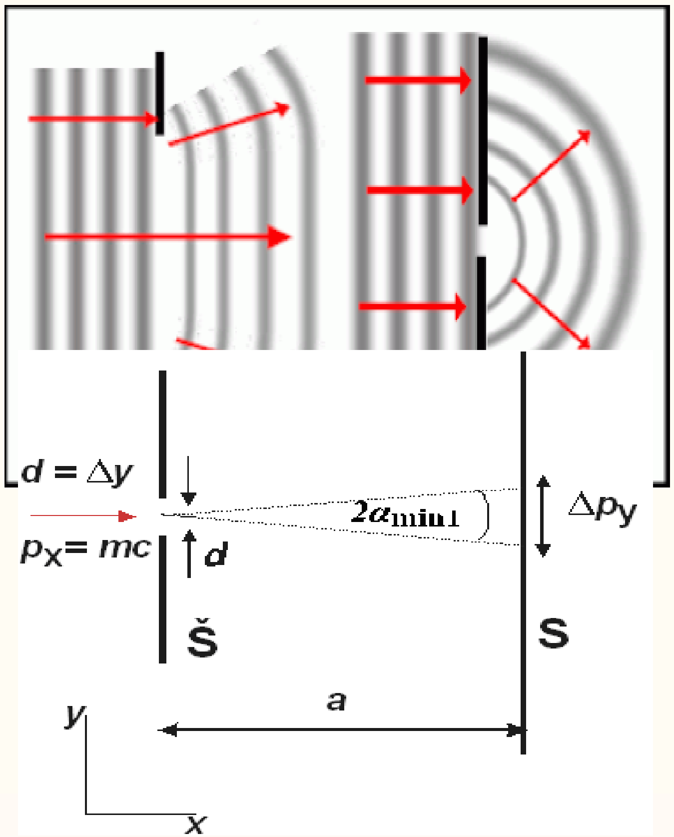
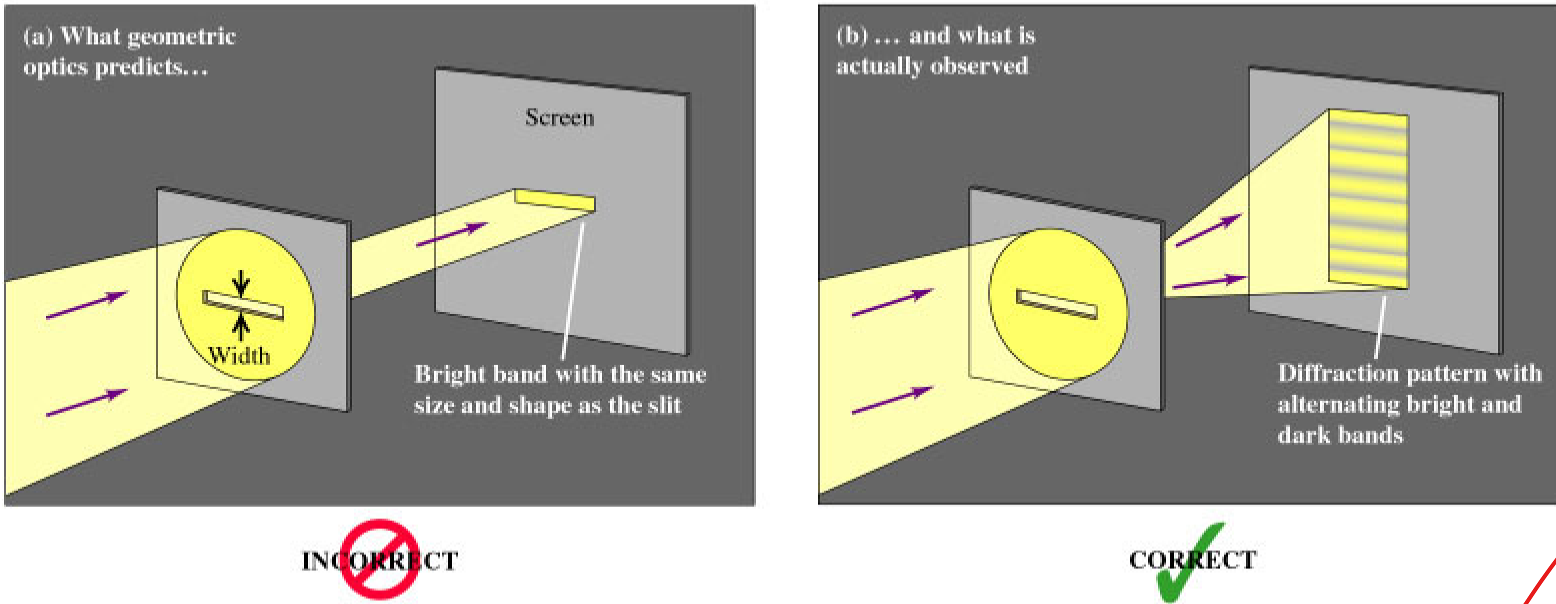


small opening: less position uncertainty results in larger momentum uncertainty, which translates to more of a spread angle

Single Slit Diffraction

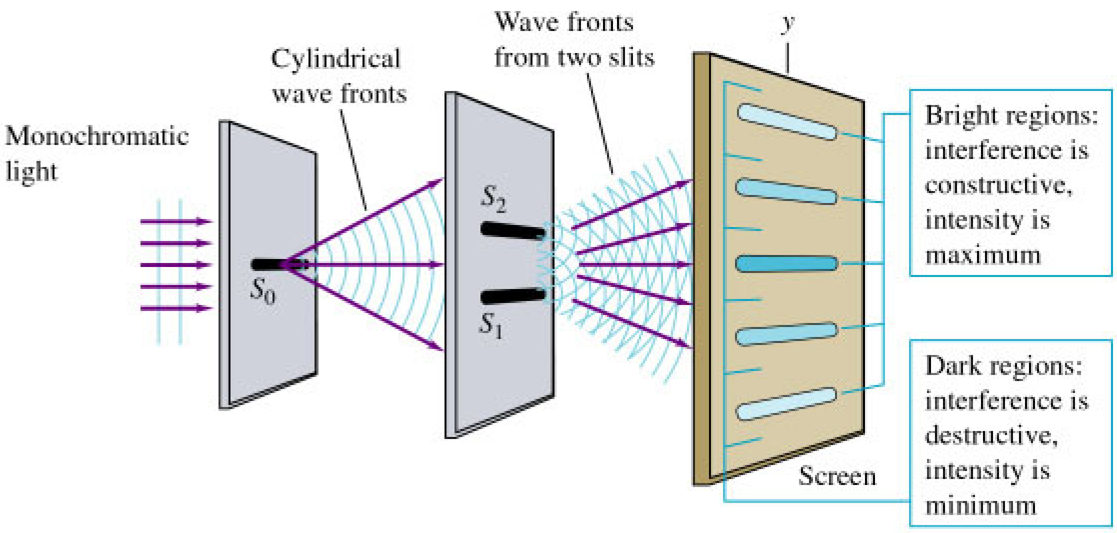


“geometrical” picture breaks down when slit width becomes comparable with wavelength

Young's Double Slit Experiment

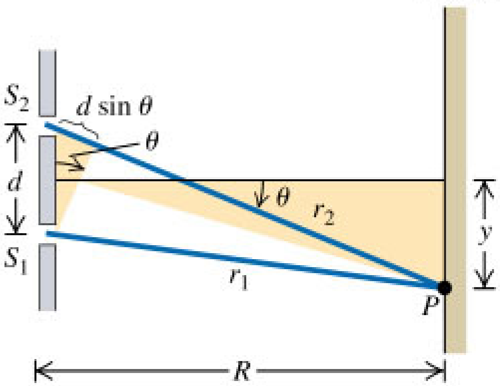
•Demonstrates wave nature of light

•Each slit S_1 and S_2 acts as a separate source of coherent light (like the loudspeakers for sound waves)

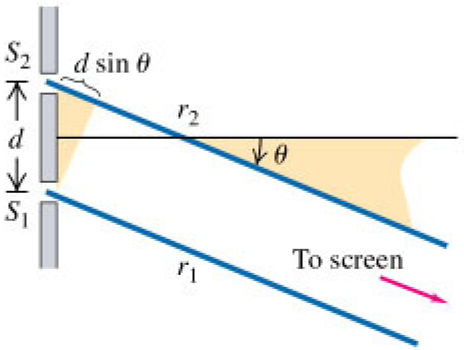


(a)

Young's Double Slit Experiment



(b) Slits S_1 and S_2 are horizontal and seen from the side in cross section



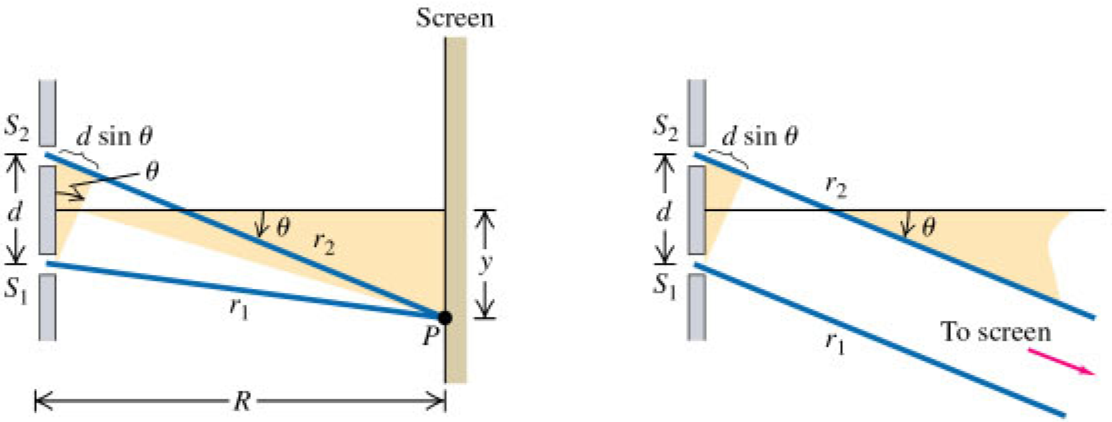
(c) Same as (b), but with screen very far from slits (R much greater than d)

Consider intensity distribution on screen as a function of θ (angle measured from central axis of apparatus)……………………….

If light behaves as a conventional wave, then we expect high intensity (bright line) at a position on the screen for which $r_2 - r_1 = n\lambda$

Expect zero intensity (dark line) at a position on the screen for which $r_2 - r_1 = (n+1/2)\lambda$

Young’s Double Slit Experiment



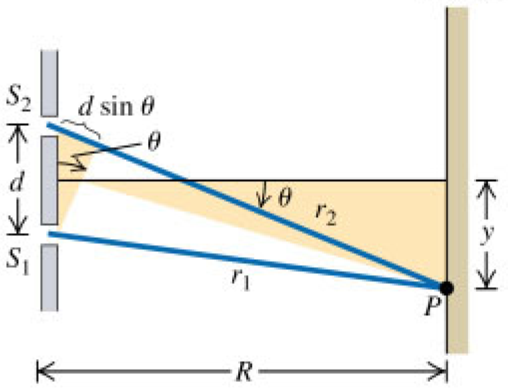
(b) Slits S_1 and S_2 are horizontal and seen from the side in cross section

(c) Same as (b), but with screen very far from slits (R much greater than d)

Assuming (justifiably) that R>>d, then lines r_2 and r_1 are approximately parallel, and path difference for the light from the 2 slits given by:

$$r_2 - r_1 = d \sin\theta$$

Young's Double Slit Experiment



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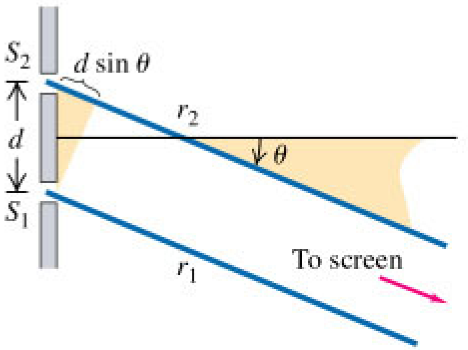
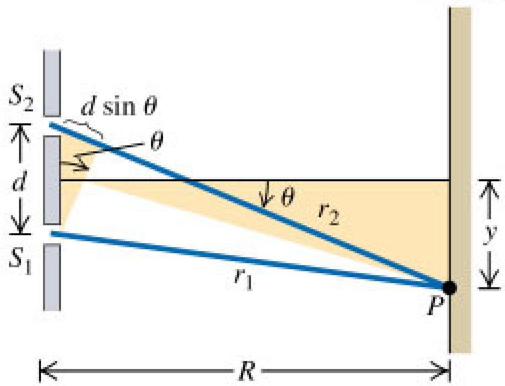


(c) Same as (b), but with screen very far from slits (R much greater than d)

Constructive interference: $d \sin\theta = n\lambda$

Destructive interference: $d \sin\theta = \left(n + \frac{1}{2}\right)\lambda$

Young's Double Slit Experiment



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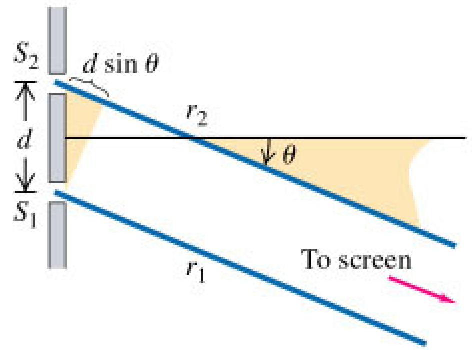


(c) Same as (b), but with screen very far from slits (R much greater than d)

Y-position of bright fringe on screen: $y_m = R\tan\theta_m$

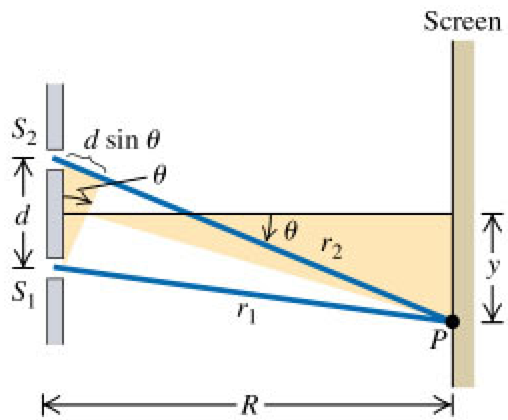
Small θ, ie r_1, $r_2 \approx R$, so <u>$\tan\theta \approx \sin\theta$</u>

<u>So, get bright fringe when:</u>

$$y_m = R\frac{n\lambda}{d}$$

(small θ only)

Young's Double Slit Experiment: Intensity Distribution



(b) Slits S_1 and S_2 are horizontal and seen from the side in cross section

For some general point P, the 2 arriving waves will have a path difference which is some fraction of a wavelength.

This corresponds to a difference ϕ in the phases of the electric field oscillations arriving at P:

$$E_1 = E_0 \sin(\omega t)$$

$$E_2 = E_0 \sin(\omega t + \phi)$$

Young's Double Slit Experiment: Intensity Distribution

Total Electric field at point P:

$$E_{TOT} = E_1 + E_2 = E_0 \sin(\omega t) + E_0 \sin(\omega t + \phi)$$

Trig. Identity:

$$\sin\alpha + \sin\beta = 2\cos\left[\frac{1}{2}(\alpha - \beta)\right]\sin\left[\frac{1}{2}(\alpha + \beta)\right]$$

With α = (ωt + φ), β= ωt, get:

$$E_{TOT} = \left[2E_0 \cos\frac{\phi}{2}\right]\sin(\omega t + \phi)$$

$$E_{TOT} = \left[2E_0 \cos\frac{\phi}{2}\right] \sin(\omega t + \phi)$$

So, E_{TOT} has an "oscillating" amplitude: $\left[2E_0 \cos\frac{\phi}{2}\right]$

Since intensity is proportional to amplitude squared:

$$I_{TOT} \propto 4E_0^{\ 2} \cos^2\left(\frac{\phi}{2}\right)$$

Or, since $I_0 \propto E_0^2$, and proportionality constant the same in both cases:

$$I_{TOT} = 4I_0 \cos^2\left(\frac{\phi}{2}\right)$$

$$\frac{\text{phase difference}}{2\pi} = \frac{\text{path difference}}{\lambda}$$

$$\frac{\phi}{2\pi} = \frac{d \sin \vartheta}{\lambda}$$

$$I_{TOT} = 4I_0 \cos^2\left(\frac{\phi}{2}\right)$$

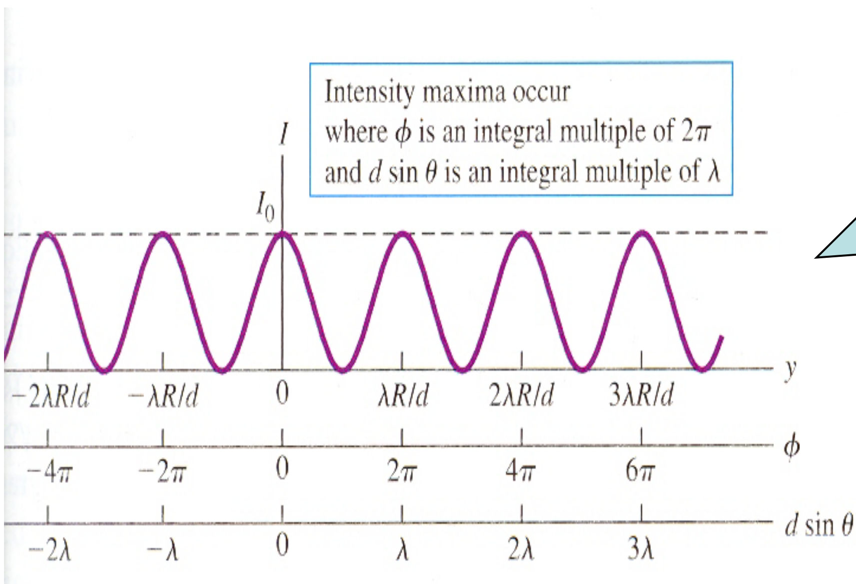
$$I_{TOT} = 4I_0 \cos^2\left(\frac{\pi d \sin \theta}{\lambda}\right)$$

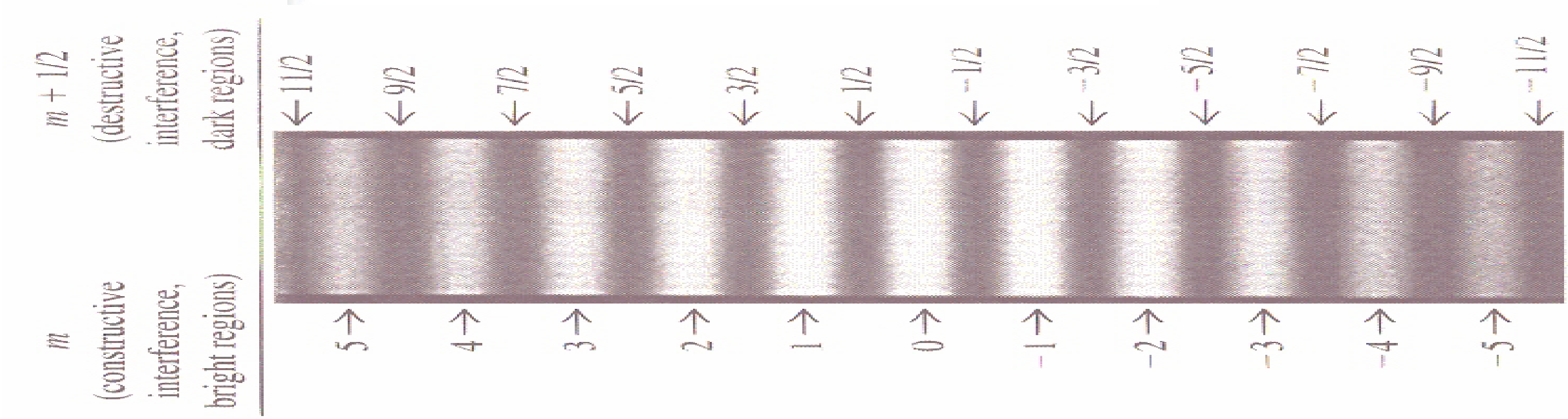
For the case where y<<R, sinθ ≈ y/R:

$$\boxed{I_{TOT} = 4I_0 \cos^2\left(\frac{\pi d y}{R\lambda}\right)}$$

Young's Double Slit Experiment: Intensity Distribution

$$I_{TOT} = I_0' \cos^2\left(\frac{\pi d y}{R\lambda}\right)$$

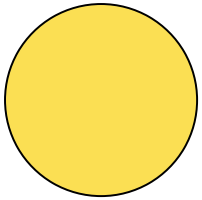




Phase Velocity and Group Velocity

Particle

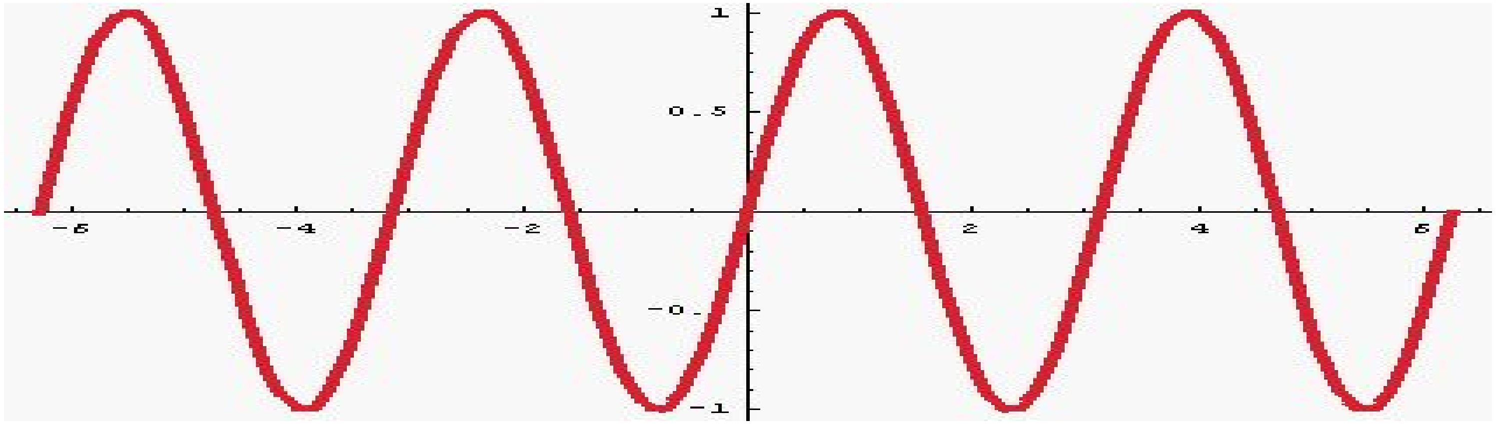
Our traditional understanding of a particle…



“Localized” - definite position, momentum, confined in space

Wave

Our traditional understanding of a wave….



"de-localized" – spread out in space and time

General Wave properties

Osillations at a particlar point

$$\Psi = A\cos 2\pi\nu\, t$$

travelling waves $\left(\text{1-d}\right)$

$$\Psi = A\cos\left(\omega t - kx\right)$$

$$\omega = 2\pi\nu \;\left(\text{angular frequency}\right)$$

$$k = \frac{2\pi}{\lambda}\;\left(\text{wave number}\right)$$

$$\text{in } 3-d : \Psi = A\cos\left(\omega t - \vec{k}\cdot\vec{r}\right)$$

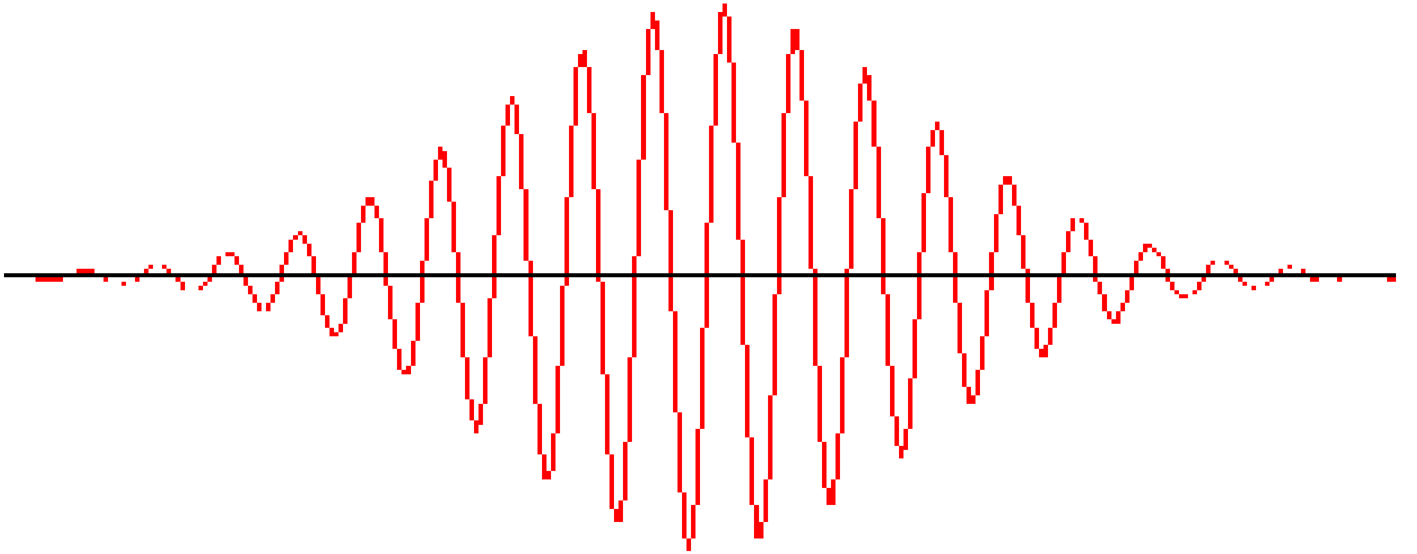
How do we associate a wave nature to a particle?

What could represent both wave and particle?

Find a description of a particle which is consistent with our notion of both particles and waves……

- Fits the “wave” description
- “Localized” in space

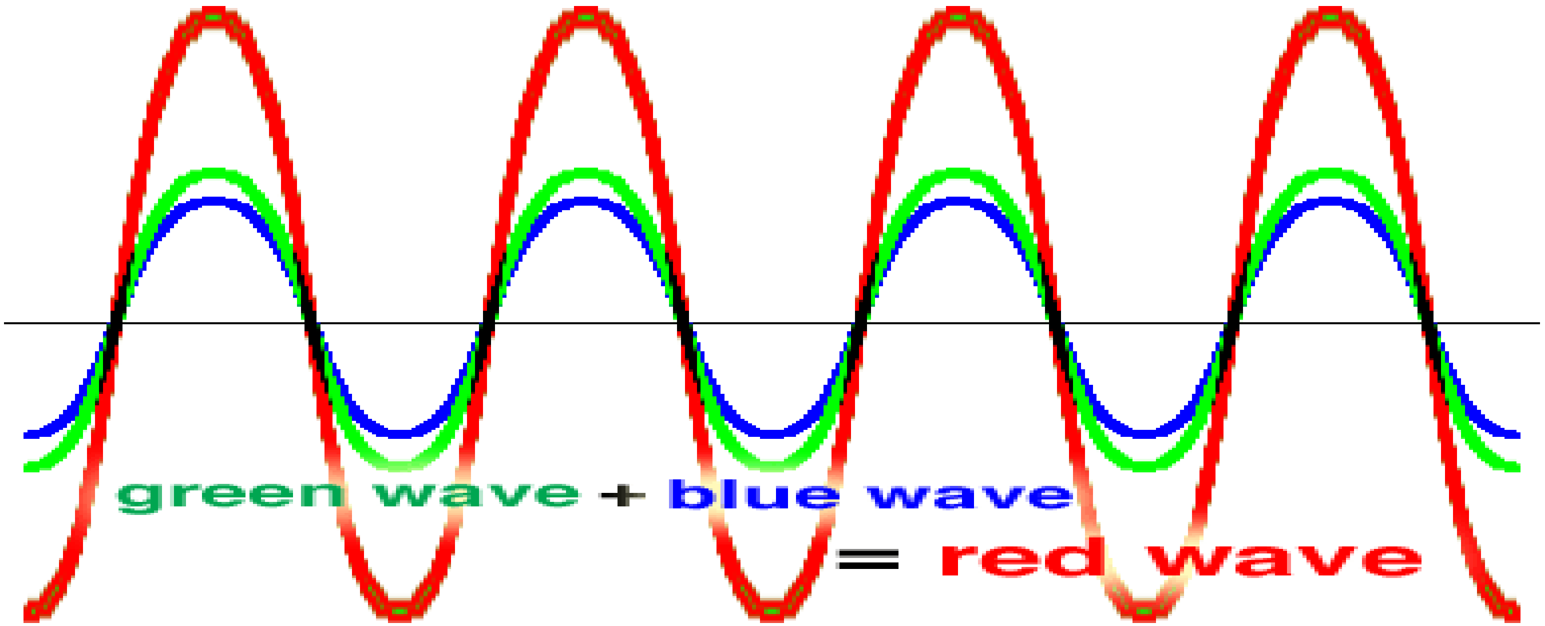
A particle thus can be described by a“Wave Packet”



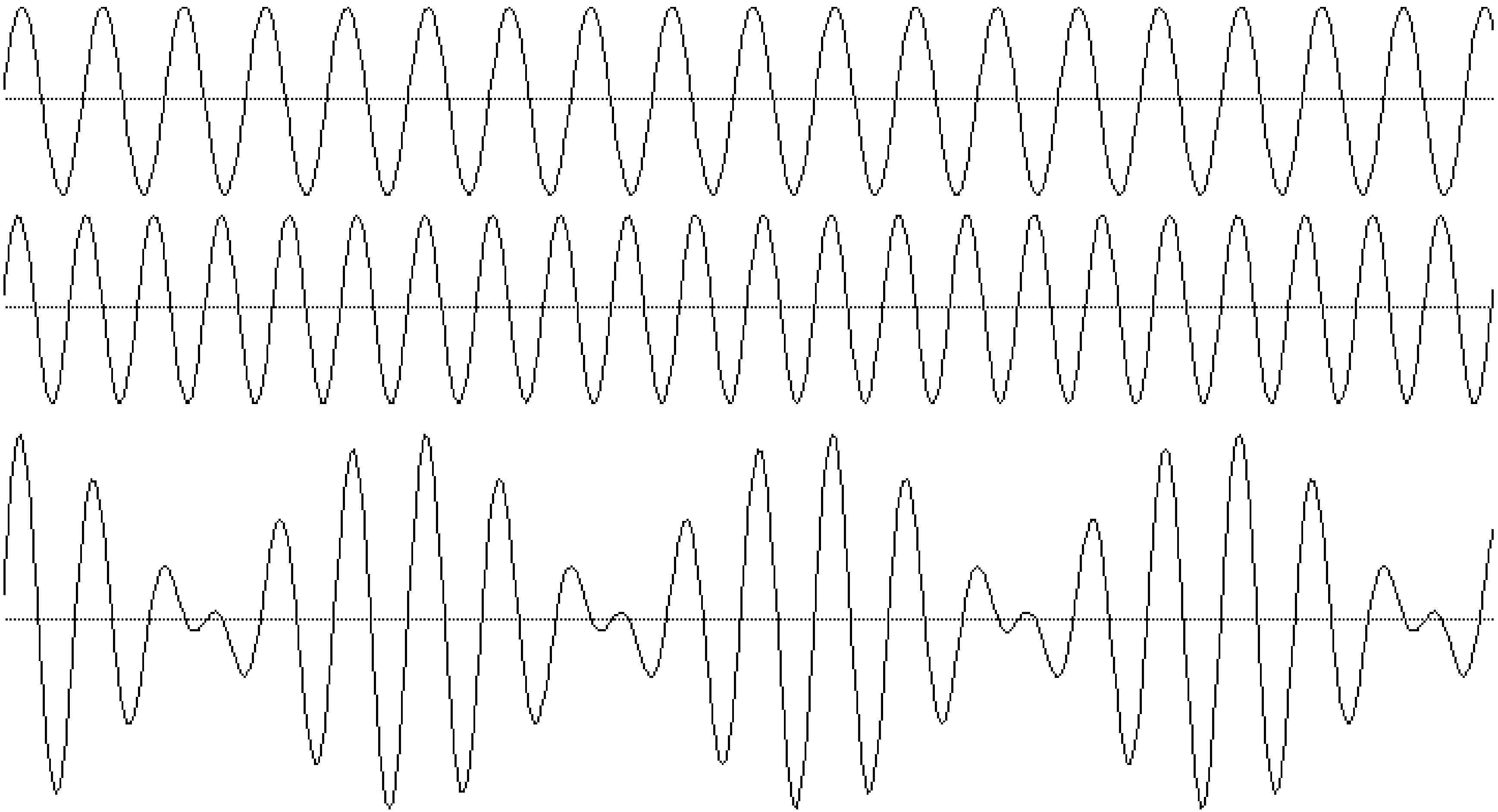
How do you construct a wave packet?

What happens when you add up waves of same frequencies?

The Superposition principle

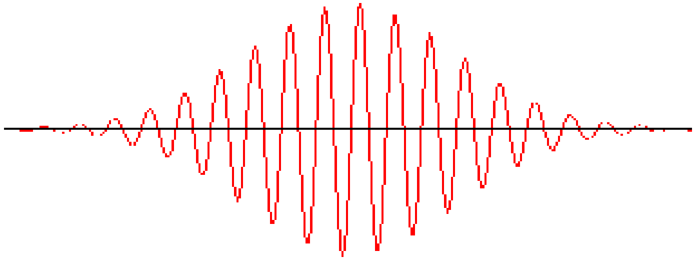


Adding up waves of different frequencies…..



Constructing a wave packet by adding up several waves …………

If several waves of different wavelengths (frequencies) and phases are superposed together, one would get a resultant which is a localized wave packet



A wave packet describes a particle

- A wave packet is a group of waves with slightly different wavelengths interfering with one another in a way that the amplitude of the group (envelope) is non-zero only in the neighbourhood of the particle

- A wave packet is localized – a good representation for a particle!

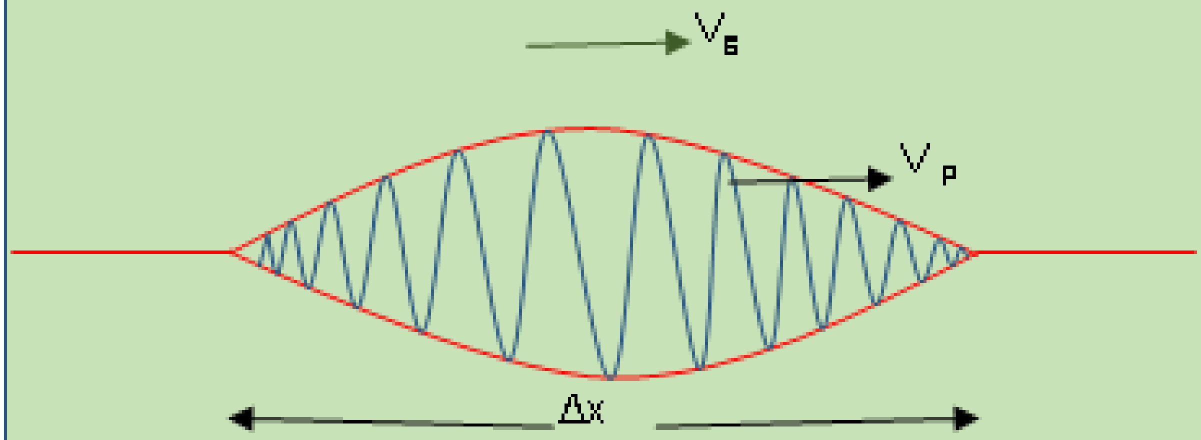
- The spread of wave packet in wavelength depends on the required degree of localization in space – the central wavelength is given by

$$\boxed{\lambda = \frac{h}{p}}$$

- What is the velocity of the wave packet?

Wave packet, phase velocity and group velocity

- The velocities of the individual waves which superpose to produce the wave packet representing the particle are different - the wave packet as a whole has a different velocity from the waves that comprise it
- Phase velocity: The rate at which the phase of the wave propagates in space
- Group velocity: The rate at which the envelope of the wave packet propagates



An identical wave packet

Figure: 01

Phase velocity

- Phase velocity is the rate at which the phase of the wave propagates in space.
- This is the velocity at which the phase of any one frequency component of the wave will propagate.
- You could pick one particular phase of the wave and it would appear to travel at the phase velocity.
- The phase velocity is given in terms of the wave's angular frequency ω and wave vector *k* by

$$v_{\mathrm{p}} = \frac{\omega}{k}$$

Phase Velocity of de-Broglie Waves

- de Broglie said a wave is associated with the wave packet which travel with the same velocity as that of the wave.

- Let the de-Broglie wave velocity is V_p, so

$$V_p = \upsilon\lambda$$

Phase Velocity - continued

- we know that

$$p=mv;\ \lambda=h/mv$$

- From Planck's Law

$$E = h\nu$$

- From mass-energy relation

$$E = mc^2$$

Phase Velocity for a massive particle and massless Particle

phase velocity

$$v_p = \lambda \nu$$

for a massive particle

$$v_p = \frac{h}{mv}\frac{mc^2}{h} = c\frac{c}{v} > c$$

for a massless particle

$$v_p = \frac{h}{p}\frac{E}{h} = \frac{1}{p}\frac{pc}{1} = c$$

phase velocity does not describe particle motion

Group Velocity=Particle velocity

$$y_1 = A\cos(\omega t - kx)$$

$$y_2 = A\cos[(\omega + d\omega)t - (k + dk)x]$$

$$y = y_1 + y_2$$

$$y = A\cos(\omega t - kx) + A\cos[(\omega + d\omega)t - (k + dk)x]$$

$$y = 2A\cos\left[\frac{(2\omega + d\omega)t}{2} + \frac{(2k + dk)x}{2}\right]\cos\left[\frac{(d\omega)t}{2} - \frac{(dk)x}{2}\right]$$

$$with\, d\omega << \omega, dk << k$$

$$y \cong 2A\cos\left[\frac{d\omega}{2}t - \frac{dk}{2}x\right]\cos[\omega t - kx] ------(1)$$

Group Velocity = Particle velocity

- Equation represent a wave of angular velocity w and wave number k which has superimposed upon it a wave $\left(\text{the process is called modulation}\right)$ of angular velocity $\frac{d\omega}{2}$ and wave number $\frac{\mathrm{dk}}{2}$

$$\text{phase velocity = wave velocity of carrier} : v_p = \frac{\omega}{k}$$

$$\text{group velocity = wave velocity of envelope} : v_g = \frac{\Delta\omega}{\Delta k}$$

$$\text{for more than two wave contiributions} : v_g = \frac{d\omega}{dk}$$

Group velocity of De Broglie waves

$$\omega = 2\pi\nu$$

$$\omega = 2\pi \frac{E}{h}$$

NR: $E = 1/2\ mv^2$

Relativistic treatment:

$$\omega = 2\pi \frac{mc^2}{h}$$

$$\boxed{\omega = 2\pi \frac{m_0 c^2}{h\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$k = \frac{2\pi}{\lambda}$$

$$k = \frac{2\pi}{h} mv$$

$$\boxed{k = \frac{2\pi}{h} \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$\boxed{\omega = \frac{2\pi}{h}\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$d\omega/dv = \frac{2\pi m_0 c^2}{h}\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[-2v/c^2\right]$$

$$\boxed{d\omega/dv = \frac{2\pi}{h}\frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}$$

$$\boxed{k = \frac{2\pi}{h}\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$dk/dv = \frac{2\pi m_0}{h}\left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} + (-\frac{1}{2})\left(1 - (v/c)^2\right)^{-3/2}\left[-2v/c^2\right]\right]$$

$$dk/dv = \frac{2\pi m_0}{h}\left(1 - (v/c)^2\right)^{-3/2}\left[1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right]$$

$$\boxed{dk/dv = \frac{2\pi m_0}{h}\frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

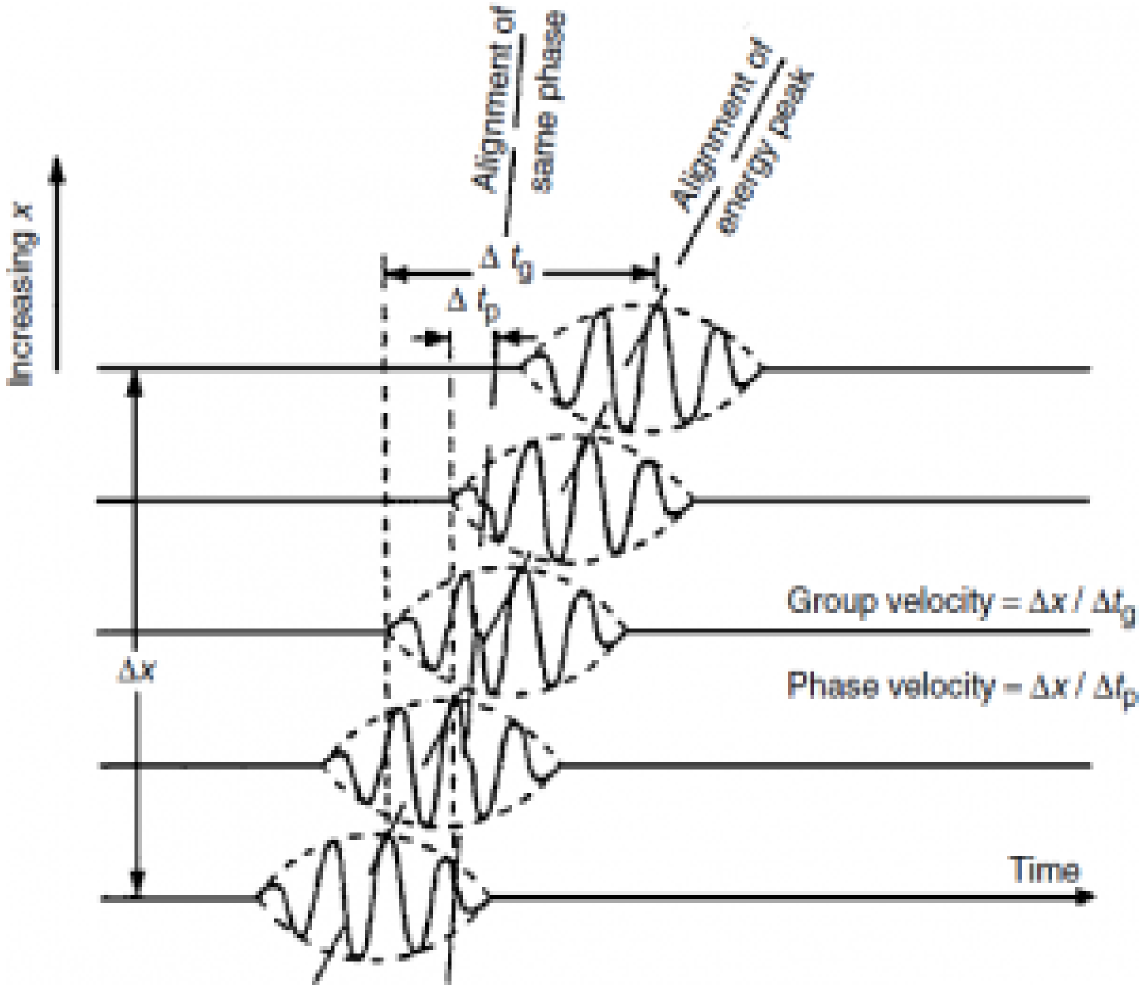
$$\boxed{d\omega/dv = \frac{2\pi}{h}\frac{m_0 v}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}}$$

$$\boxed{dk/dv = \frac{2\pi m_0}{h}\frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}}$$

$$\Rightarrow \; v_g = v$$

This is true for the non-relativistic case as well

Scheme



Increasing x
Alignment of same phase
Alignment of energy peak
Δ t_g
Δ t_p
Δx
Group velocity = Δx / Δt_g
Phase velocity = Δx / Δt_p
Time

Relation Between Group Velocity and Phase Velocity

$$\omega = v_p k$$

$$v_g = \frac{d\omega}{dk}$$

$$v_g = \frac{d(v_p k)}{dk}$$

$$v_g = v_p + k\frac{dv_p}{dk}$$

$$v_g = v_p + \frac{k}{dk}dv_p$$

$$k = 2\pi / \lambda$$

$$dk = -\frac{2\pi}{\lambda^2}d\lambda$$

$$\frac{k}{dk} = \frac{2\pi / \lambda}{-\frac{2\pi}{\lambda^2} d\lambda}$$

$$\frac{k}{dk} = -\frac{\lambda}{d\lambda}$$

$$\boxed{v_g = v_p - \lambda \frac{dv_p}{d\lambda}}$$

This gives us the relation between Phase velocity and Group velocity, known as Dispersion Relation