भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

7th Lecture on Differential Equation

(MA-1150)

What have we learnt?

- Equation of First Order but not First Degree
- Clairaut's Equation
- Singular Solution

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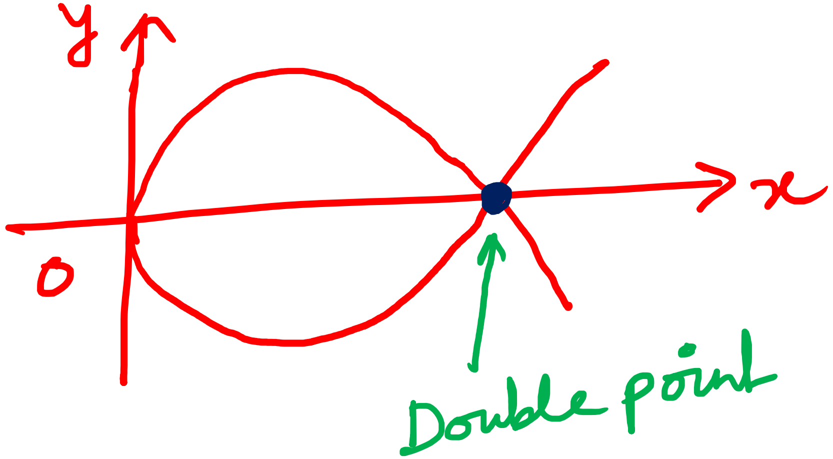
What will we learn today?

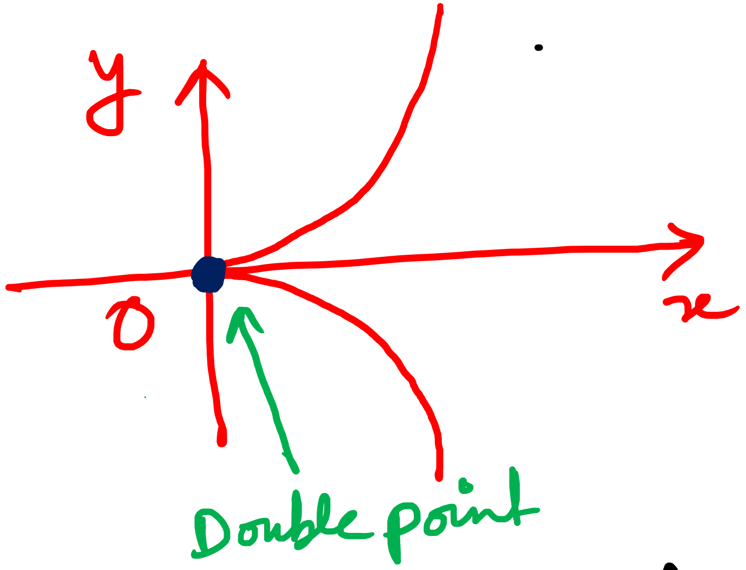
- Singular Solutions
- Initial Value Problems
- Second and Higher Order Linear ODE

Singular Solution

Double point: A point of a curve in a plane is called a double point if two branches of the curve pass through it.

Example:



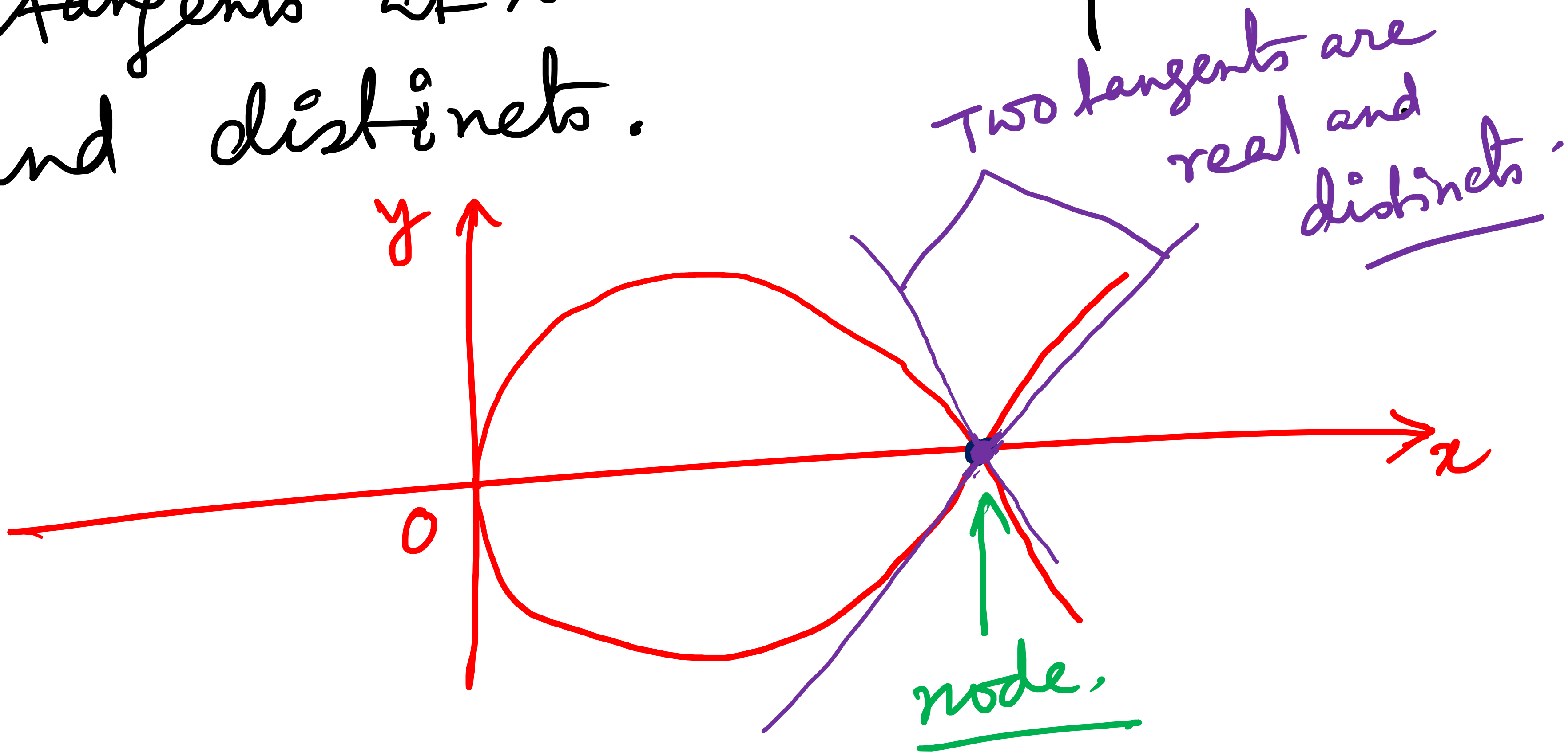


Note: In general, we expect two tangents (one for each branch) at a double point.

Singular Solution

① **Node:** The double point is called as node if two tangents at the double point are real and distinct.

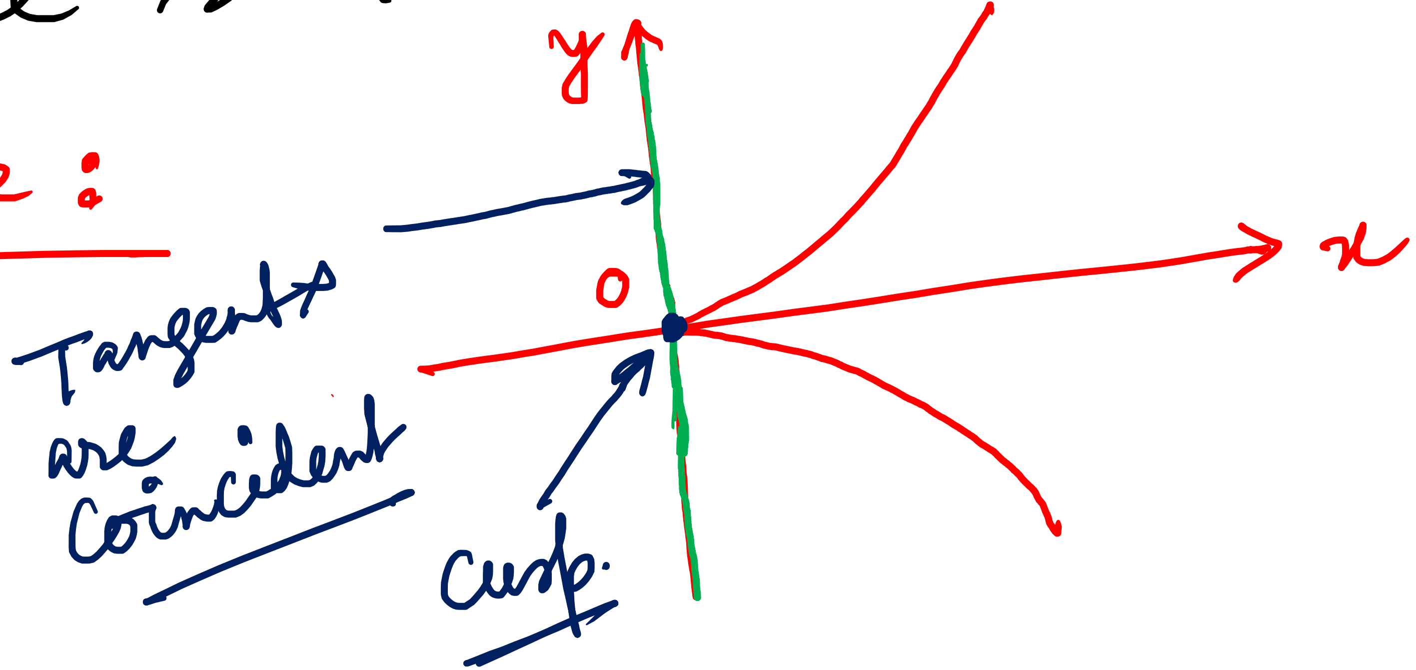
② **Example:**



Singular Solution

Cusp: The double point is called as cusp if two tangents at the double point are real and coincident.

Example:



Singular Solution

c-discriminant:

The c-discriminant gives —

(i) Envelope locus (E)

(ii) Nodal locus (N)

(iii) Cuspidal locus (C)

Singular Solution

Note: The c-discriminant contains the envelope locus (E) as a factor once, the nodal locus (N) twice and the cusp locus (C) thrice.

So one can write the c-discriminant relation as

$$\boxed{\text{c-discriminant relation} = E N^2 C^3}$$

This comes from analytical theory.

Singular Solution

- **Envelope locus:** This locus will give the Singular Solution.
- **Nodal locus:** In general, this locus will not be a solution of the differential eqn.
- **Cuspidal locus:** In general, this locus will not also be a solution of the differential eqn.

Singular Solution

p- discriminant:

p-discriminant gives —

1. Envelope locus (E)
2. Tac locus (T)
3. Cuspidal locus (C)

one can write p-discriminant relation as —

$$\boxed{p\text{-discriminant relation} = ET^2C}$$

Here p-discriminant relation includes the Envelope locus as a factor once, cuspidal locus as a factor once and Tac locus as a factor twice.

Singular Solution

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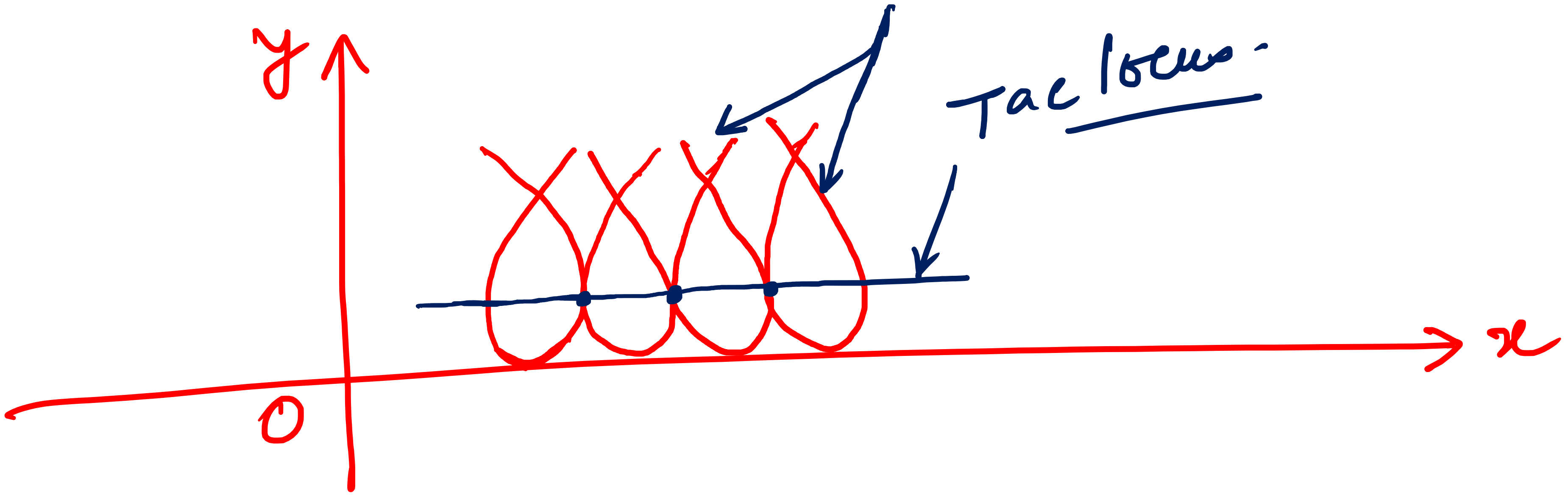
Tac Locus: This is the locus of all points at which two curves of the system touch each other.

Example:



Singular Solution

Example: Examine for Singular Solution of $8ap^3 = 27y$

Sol^n: The general Solution is obtained as

$$ay^2 = (x-c)^3$$ [Try] Home work.

p discriminant: we have $F(x,y,p) = 8ap^3 - 27y = 0$

To find the envelope: $\frac{\partial F}{\partial p} = 0$

Singular Solution

or $24ap^2 = 0$ or, $p^2 = 0 \Rightarrow p = 0$.

Now we have $8ap^3 = 27y \Rightarrow \boxed{y = 0.}$ [since $p=0$]

This is the p-discriminant relation

c-discriminant: We have from the general solution $f(x,y,c) = ay^2 - (x-c)^3 = 0$

To find the envelope: $\frac{\partial f}{\partial c} = 0 \Rightarrow (x-c) = 0 \Rightarrow c = x.$

Singular Solution

Now we have $y^2 = 0 \Rightarrow \boxed{y = 0}$

This is the c-discriminant relation.

Since $y=0$ occurs in both discriminants and also satisfies the differential eqn, $y=0$ is the singular solution.

Singular Solution

Example: Find the Singular Solution if any of $y = 2px + p^2$

Sol^n: Since the given eq^n is quadratic eq^n in p, the p-discriminant relation is

$p^2 + 2px - y = 0 \Rightarrow (2x)^2 + 4y = 0 \Rightarrow \boxed{x^2 + y = 0}$

It does not satisfy the differential equation.

So there can not be any Singular Sol^n.

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Singular Solution

Ex: Find the general and Singular Solution of $xp^2 - 2py + 4x = 0$

Soln: The given eqn is solvable for y.

$$y = \frac{p^2 x + 4x}{2p} = \frac{1}{2}px + 2\,x/p$$

Differentiating w.r.t x, we have

$$p = \frac{1}{2}p + \frac{1}{2}x\frac{dp}{dx} - \frac{2x}{p^2}\frac{dp}{dx} + 2/p$$

Singular Solution

$$\Rightarrow -\frac{p}{2}+\frac{y}{p}+x\frac{dp}{dx}\left(\frac{1}{2}-\frac{y}{p^2}\right)=0$$

$$\Rightarrow -p\left(\frac{1}{2}-\frac{y}{p^2}\right)+x\frac{dp}{dx}\left(\frac{1}{2}-\frac{y}{p^2}\right)=0$$

$$\Rightarrow \left(x\frac{dp}{dx}-p\right)\left(\frac{1}{2}-\frac{y}{p^2}\right)=0$$

This will give General Solution.

$$x\frac{dp}{dx}=p \quad \text{or} \quad \frac{1}{2}=\frac{y}{p^2}$$

This will give Singular Solution.

Singular Solution

$$x\frac{dp}{dx} = p$$

or $p = cx$.

$$xp^2 - 2py + 4x = 0$$

or $c^2x^2 - 2cy + 4 = 0$.

general sol^n.

we have

$$\frac{1}{2} = \frac{2}{p^2} \quad \text{or} \quad p^2 = 4 \Rightarrow xp^2 - 2py + 4x = 0$$

or $2py = 8x$

or $p^2y^2 = 16x^2$

or $y^2 = 4x^2 \Rightarrow y^2 - 4x^2 = 0$

Singular Solution

Here $y^2 = 4x^2$ or $\boxed{y = \pm 2x}$ are the singular solution, since it occurs in both p-discriminant and c-discriminant relation, and it also satisfies the differential equation.

$(-2y)^2 - 4 \cdot 4x \cdot x = 0$

$\Rightarrow y^2 - 4x^2 = 0$

$\Rightarrow y = \pm 2x$

$(-2y)^2 - 4 \cdot 4x^2 = 0$

or, $y^2 - 4x^2 = 0$

$y = \pm 2x$

Singular Solution

Example: $4xp^2 - (3x-1)^2 = 0$

Sol^n: $p^2 = \frac{(3x-1)^2}{4x} \Rightarrow p = \pm\left(\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}\right)$

Integrating we have

$$\boxed{(y+c)^2 = x(x-1)^2.}$$

General Sol^n

Now $x=0$ occurs in both the p-discriminant and the relation $\boxed{x(3x-1)^2=0}$

Singular Solution

c-discriminant relation $\boxed{x(x-1)^2=0}$

Also $x=0$ Satisfies the ODE (How?)

$x=0$ gives $\frac{dx}{dy}=0 \Rightarrow \frac{1}{p}=0$

Therefore we have $p^2 = \frac{(3x-1)^2}{4x}$

$\Rightarrow \frac{1}{p^2} = \frac{4x}{(3x-1)^2}$ $\left(x=0, \frac{1}{p}=0\right)$

$\Rightarrow \boxed{0=0} \Rightarrow$ Satisfies the ODE.

$x=0$

Note: $\frac{dy}{dx}$ does not exist here.

Singular Solution

Hence $x=0$ is a Singular Solution (Envelope locus)

p- discriminant relation: $x(3x-1)^2=0$

E T^2

Here $(3x-1)=0$ occurs twice in the p-discriminant, does not satisfy the differential equation and does not occur in the c-discriminant.

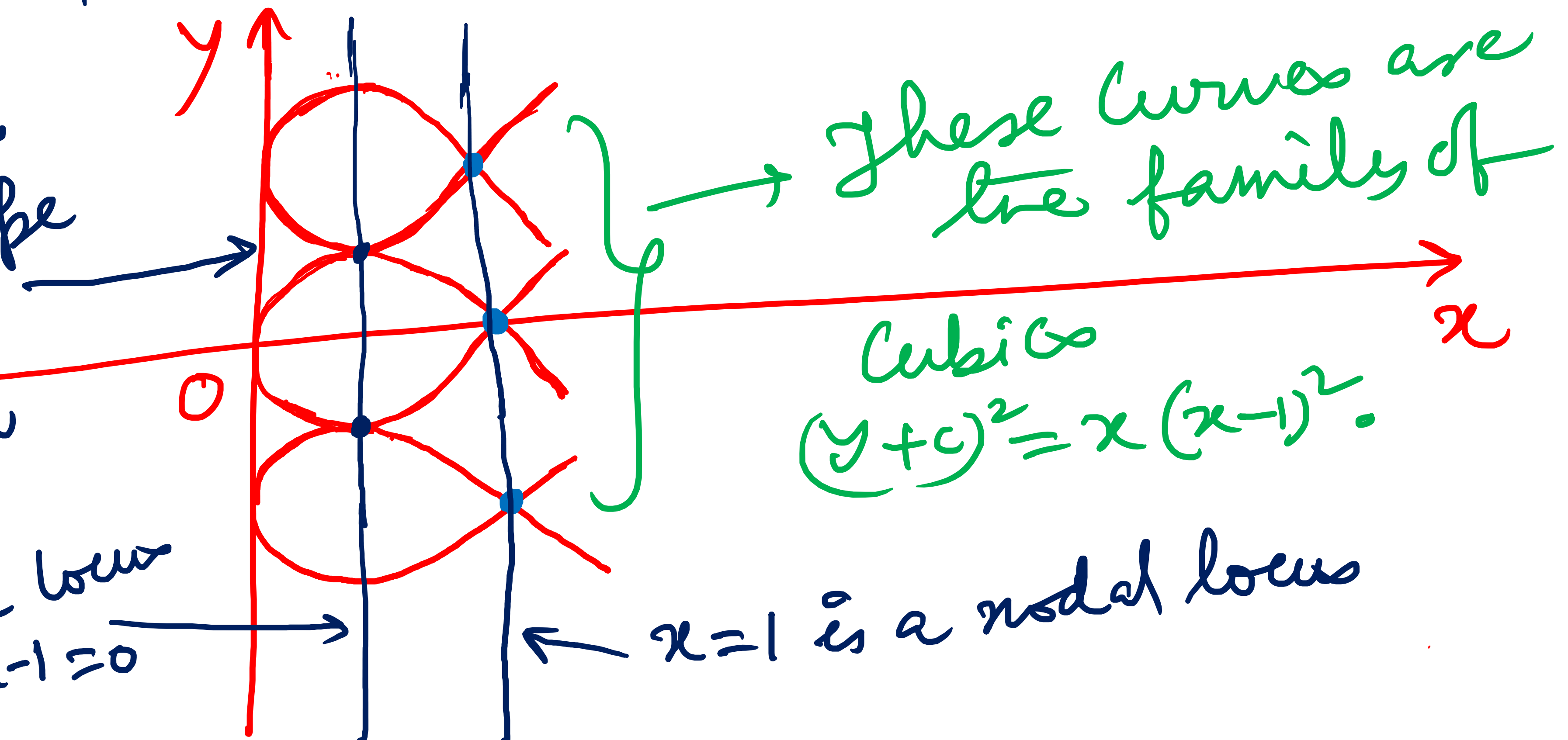
Therefore, $3x-1=0$ is a Tac-Locus.

c- discriminant relation: $x(x-1)^2=0$

E N^2

Singular Solution

$x-1=0$ is a nodal locus since it occur twice in the c-discriminant, does not satisfy the differential equation and does not occur in the p-discriminant relation.



Singular Solution

Example: Solve $9yp^2+4=0$ and examine for singular solutions.

Sol^n: $9yp^2+4=0$

or, $y=-\dfrac{4}{9p^2}$

Differentiating w.r.t x, we have.

Singular Solution

or, $dx = \frac{8}{9}\frac{1}{p^4}\,dp$

or, $x + c = -\frac{8}{27p^3}$

Eliminating p between this relation and the given eqn, we have

$$\boxed{y^3 + (x+c)^2 = 0}$$ general soln.

Singular Solution

p-discriminant : $y = 0$

c-discriminant : $y^3 = 0$

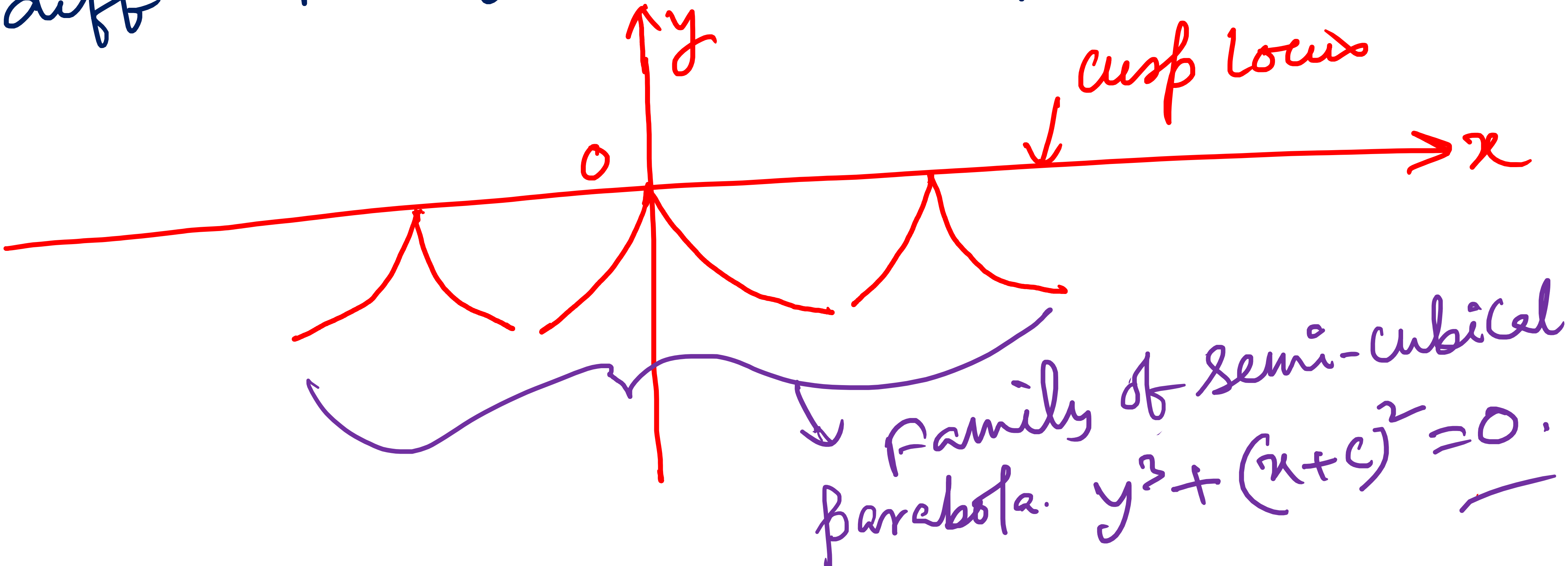
but $y=0$ does not satisfy the eqⁿ.

So $y=0$ is not singular solution.

we can't get any singular solution.

Singular Solution

Here $y=0$ occurs once in the p-discriminant, twice in c-discriminant and does not satisfy the differential eqⁿ, it is a cuspidal locus.



Singular Solution

Example: $p^3 - 4xyp + 8y^2 = 0$

Sol^n: General Sol^n: $y = c(x-c)^2$

p discriminant: $y = 0, y = \frac{4}{27}x^3$

c- discriminant: $y = 0, y = \frac{4}{27}x^3$

Both discriminants have same factors.

Singular Solution

Both $y=0$ and $y=\frac{4}{27}x^3$ Satisfy the differential eqn.

However $y=0$ is a particular solution, it can be obtained from the general solution by putting $c=0$.

So $y=\frac{4}{27}x^3$ is only a singular solution.

Singular Solution

Example: $p^2 y + p(x-y) - x = 0$

Sol^n: p discriminant:

$$(x-y)^2 + 4xy = 0$$

$$\Rightarrow (x+y)^2 = 0 \Rightarrow y = -x.$$

But $y = -x$ is not a sol^n of differential eq^n. Hence it cannot be

Singular Solution

a singular solution.

The eq^n can be written as

$$(p-1)(py+x)=0$$

$$\Rightarrow \quad y-x=c \quad \text{and} \quad y^2+x^2=c$$

The general sol^n is $(y-x-c)(y^2+x^2-c)=0$

It shows that if a first order ODE can be reducible to linear and rational factors, it has no singular solution.

Initial value problem (IVP):

IVP (Initial value Problem):

Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function of two variables x and y. This function may be linear or non-linear.

The IVP is defined by First order ODE with initial condition

IVP
$$\begin{cases} \dfrac{dy}{dx} = f(x,y) & \text{— (1)} \\ y(x_0) = y_0 \, , \quad x \in [x_0, b] & \text{— (2)} \end{cases}$$

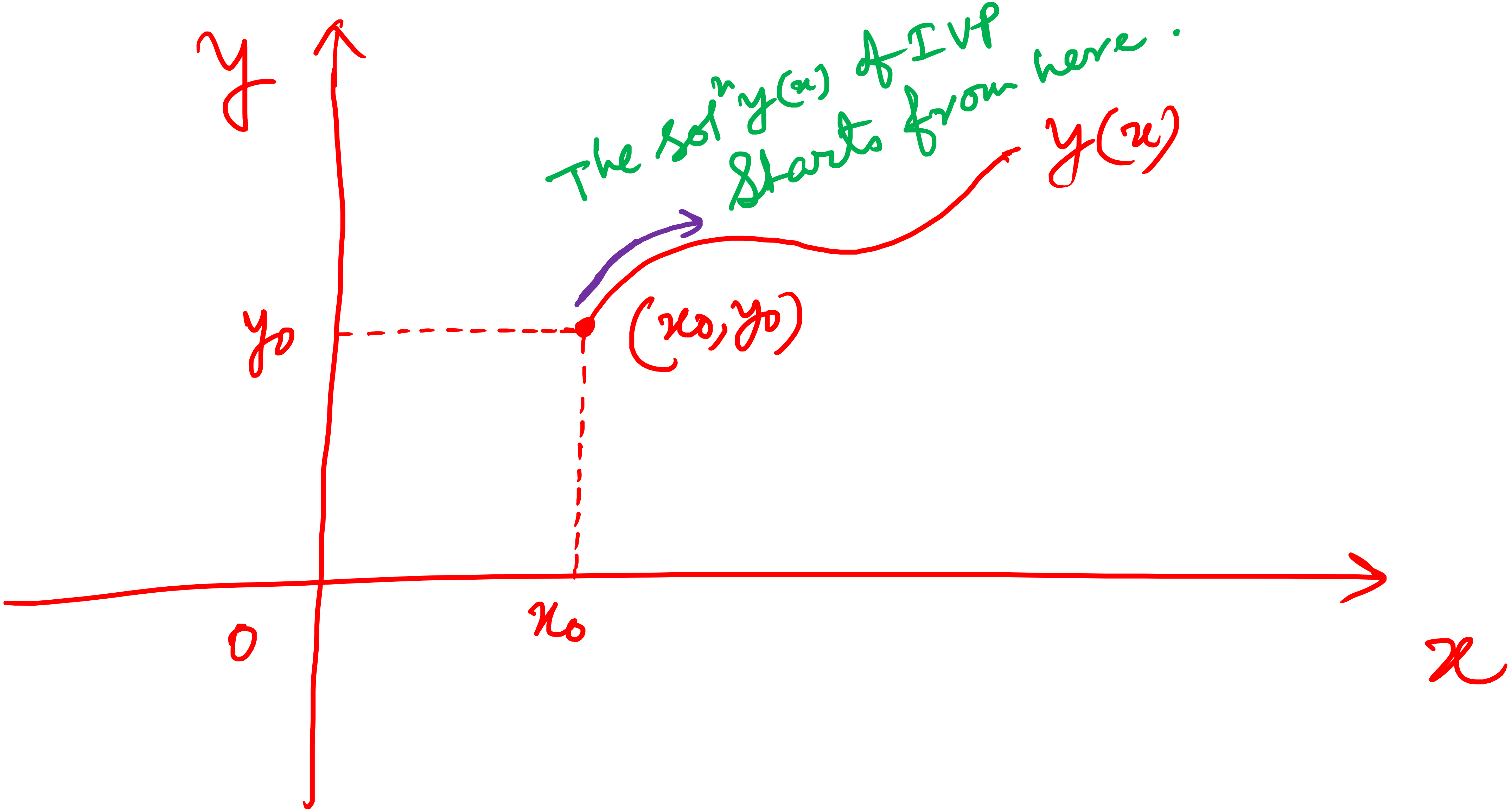
Solⁿ of IVP and Physical Significance of IVP :

The function $y(x)$ which is differentiable and satisfies ① and ②, is known to be a solution of IVP.

Here we are interested to obtain a solution $y(x)$ which is not only differentiable but also whose slope at any point (x,y) is $\left.\frac{dy}{dx}\right|_{(x,y)} = f(x,y)$

In addition, it is important to say that the solution $y(x)$ starts from the initial point (x_0, y_0).

See the figure —



The solⁿ y(x) of IVP
Starts from here.
y(x)
(x0, y0)
y
y0
0
x0
x

Well posedness of IVP in the Sense of Hadamard:

⇒ At first, a french mathematician Hadamard proposed wellposedness of IVP. See the details of it.

The IVP is well posed if the following three properties are satisfied

(i) There exists a solution to the IVP. (Existence Problem)

(ii) The solution of IVP is unique. (Uniqueness Problem)

(iii) The behaviour of the solution to the IVP changes continuously w.r.t. initial condition. (Stability problem)

Ill-posed IVP:

The IVP is said to be ill-posed if it does not satisfy the Hadamard well-posedness conditions.

Solutions of IVP

Unique Solution

Infinitely many solutions

No Solution

See the examples:

Example: Consider IVP —

$$\frac{dy}{dx} = \frac{2}{x} y, \quad y(0) = 0.$$

Sol^n: By using the method of separation of variables, we have the general sol^n

$$\boxed{y = cx^2},$$

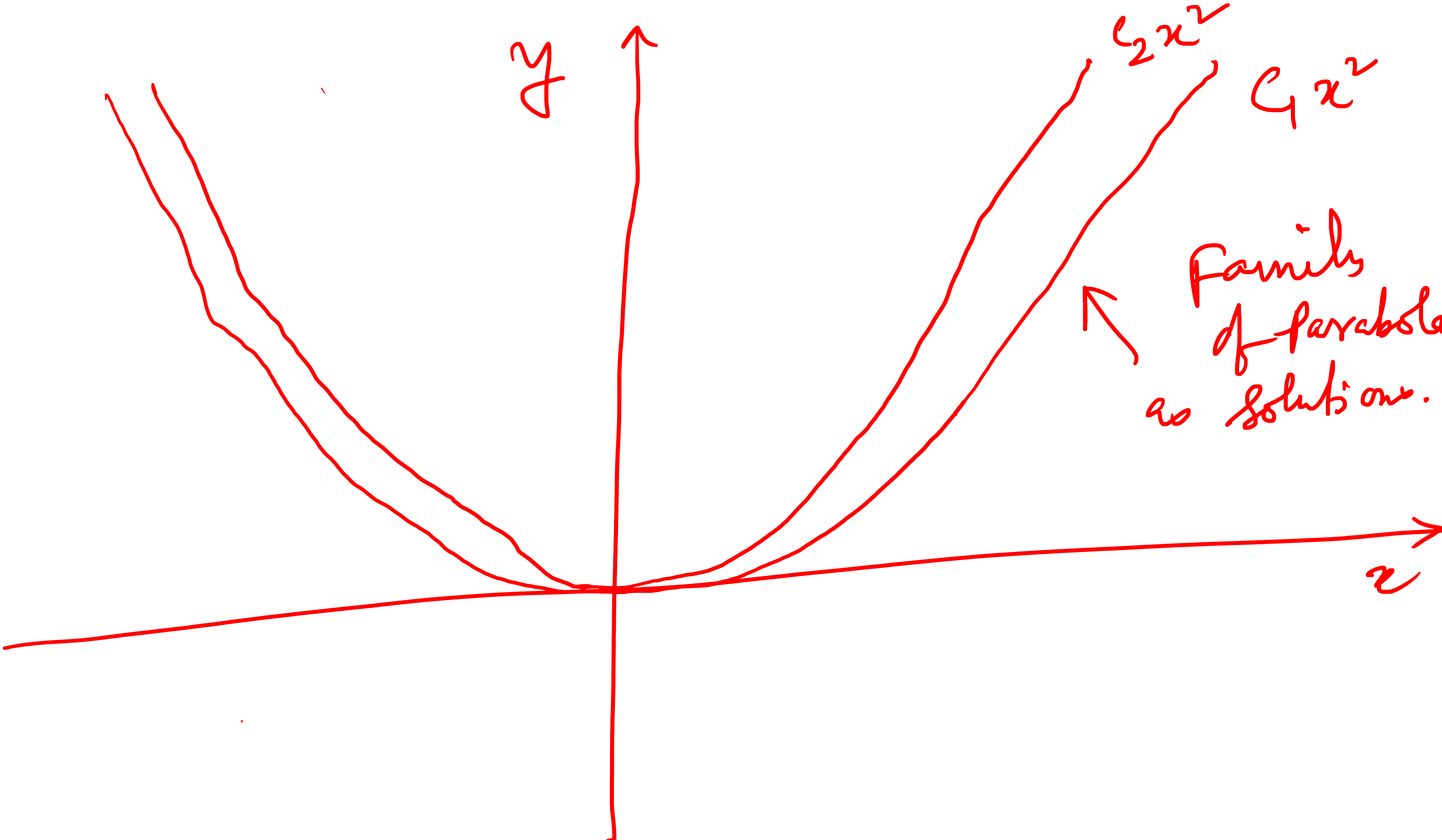
where c is an arbitrary constant.

Now by applying the initial condition, we have $y(0) = 0$

$$\Rightarrow \boxed{c \cdot 0^2 = 0} \Rightarrow c \text{ is arbitrary constant}$$

for which the initial condition is satisfied.

So the given IVP has infinitely many solutions as $y = cx^2$.


y
$c_2 x^2$
$c_1 x^2$
Family of parabolas as solutions.
x

Example: Consider IVP

$$\frac{dy}{dx} = \frac{2}{x} y, \quad y(1) = 3.$$

Sol^n: The general sol^n to the ODE $\frac{dy}{dx} = \frac{2}{x} y$ is

$$\boxed{y = cx^2}$$, where c is an arbitrary constant.

Initial condition gives —

$y(1) = 3 \Rightarrow c.1 = 3$

$\Rightarrow \underline{c = 3}.$

Therefore the IVP has <u>unique solution.</u>

Example : Consider IVP -

$$\frac{dy}{dx} = \frac{2}{x} y, \quad y(0) = 1.$$

Sol^n: The general solution is

$y = cx^2$, where c is arbitrary.

Initial condition gives $y(0) = 1$

$\Rightarrow c \cdot 0^2 = 1$

$\Rightarrow 0 = 1$ which is not possible

Therefore the given IVP has no solution.

Note: It is worth mentioning that the same problem has been introduced with different initial conditions. This implies that the initial conditions play a crucial role for determining the solutions to the IVP.

Note: The continuity of $f(x,y)$ is so much important for existence, uniqueness and stability of the solution to the ODE.

We will study the conditions of $f(x,y)$ on existence, uniqueness and stability of solution to the IVP later.

I am skiping that.

Linear Differential Equation

- A linear ordinary differential eqⁿ of order n is written as —

$$\boxed{a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x)}$$

or,

$$\boxed{a_0(x)y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_{n-1}(x)\, y' + a_n(x)y = r(x)}$$

where y is dependent variable and x is independent variable and $a_0(x) \neq 0$

- If $r(x) = 0$, then it is called a homogeneous linear equation.

when $r(x) \neq 0$, it is called a non-homogeneous linear eqⁿ.

- $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are functions of x.

Second Order Differential Equation

- Second order linear differential eq^n is written as —

$$\boxed{a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = r(x)}$$

This is non-homogeneous eq^n. When $r(x)=0$, this eq^n reduces to homogeneous eq^n.

- If $a_0(x), a_1(x), \ldots, a_n(x)$ are only constants, then the above eq^n is known as linear second order constant coefficient equation.

Ex. (i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x^2e^x$

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$

Linear Differential Equation

IVP: The nth order linear non-homogeneous ODE is given by

$$\boxed{a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = r(x), \quad \forall x \in I}$$

with the initial conditions

$y(x_0) = c_1,\ y'(x_0) = c_2,\ \ldots,\ y^{(n-1)}(x_0) = c_n$, where $x_0 \in I$ and $c_1, c_2, \ldots, c_n$ are n known constants. Here I is an interval.