भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

First Lecture on Transform Techniques

(MA-2120)

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What will we learn in this class?

- Introduction to Integral Transforms
- Introduction and Motivation to Laplace Transforms
- Improper Integral
- Existence of Improper Integral

Laplace Transforms

Introduction:

Laplace transform is one of the integral transforms and it was introduced by a French mathematician P.S. Laplace in 1780.

It is one of the important and powerful tools for solving linear ordinary or partial differential equations, integral equations, integro-differential equations, etc in applied mathematics, mathematical physics, and engineering Science.

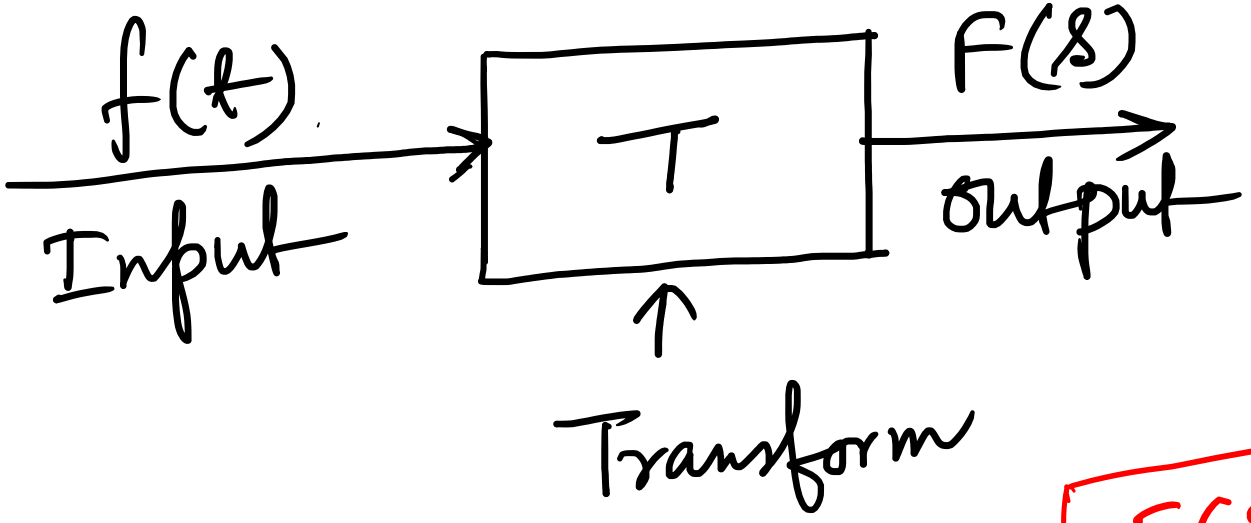
There exist so many differential equations or integral equations which are difficult to solve in its original form. To overcome this difficulty, this integral transform transforms the problem from its original domain to some other domain. That means the integral transform maps the problem from its original domain to some other domain. Therefore, the transformed problems in new domain are more easier to solve to get the solution. Thereafter, the solution is mapped back to the original domain by its inverse transformation.

By the Laplace transform, ODE is transformed to its equivalent algebraic equation which is very easy to solve.

In case of PDE, Laplace transform reduces two independent variables in the PDE to the only one independent variable (usually the variable t (time)). Therefore, the procedure of solving PDE much easier.

In this way, the use of Laplace transform provide a powerful technique of solving differential eq^{n}s and integral eq^{n}s.

Transform:



$$\boxed{T\left(f(t)\right) = F(s)}$$

F(s) is the image of f(t).

T is the transform operator.

Integral Transform:

$$T\big(f(t)\big) = F(s)$$

$$= \int_a^b K(s,t)\, f(t)\, dt$$

The above transform which is associated with integral form is known to be integral transform.

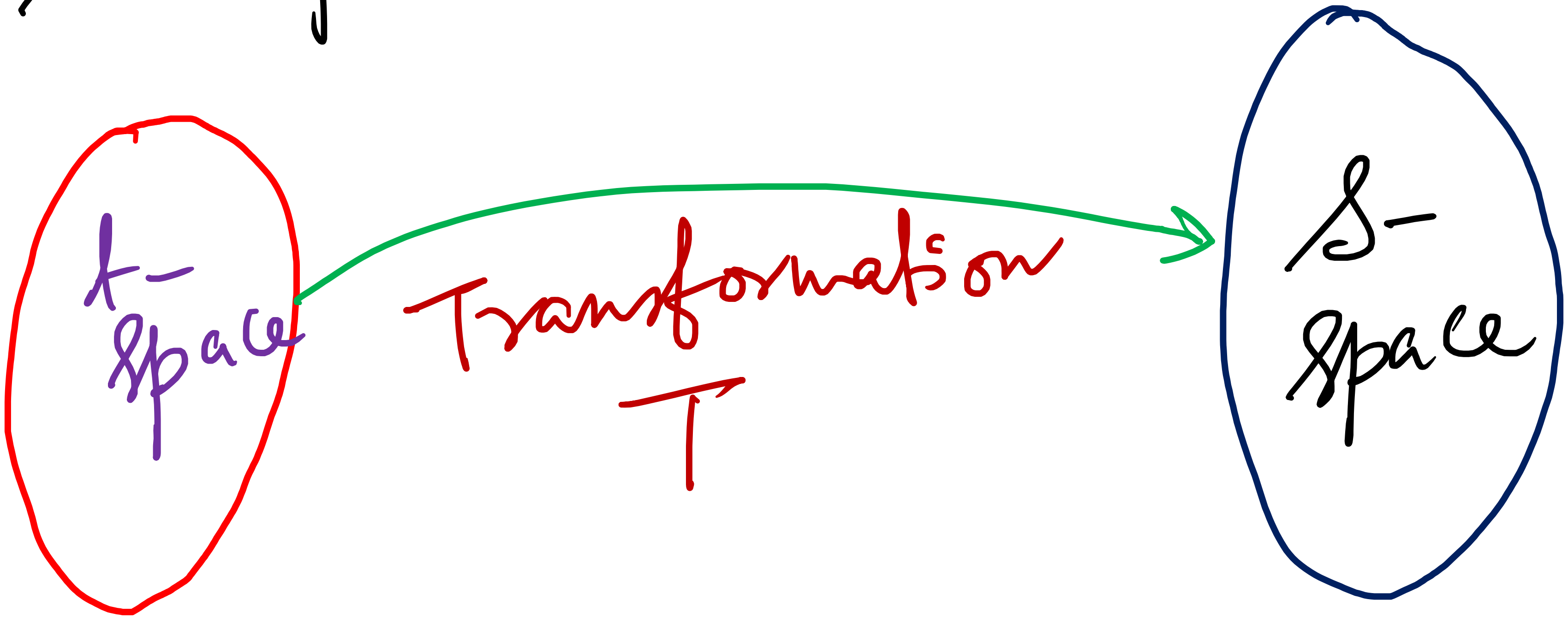
This is the integral transform of a function $f(t)$ defined in $a \le t \le b$.

Here $K(s,t)$ which is given function of two variables x and t, is called the kernel of the transform.

The operator T is known to be an integral transform operator or an integral transformation.

The transform function $F(s)$ is referred to as the image of the given function $f(t)$ and s is called the transform variable. The variable s belongs to some domain on the real line or in the complex plane.

One can say that this is the transformation from t-space to s-space.



Similarly, the integral transform of a function of several variables is defined by

$$T\{f(t)\} = F(s) = \int_S K(s,t)\, f(t)\, dt$$

where $t = (t_1, t_2, \dots, t_n)$ and $s = (s_1, s_2, \dots, s_n)$ and $S \subset \mathbb{R}^n$.

The idea of the integral transform operator T is somewhat similar to that of the well known linear differential operator $D \equiv \frac{d}{dt}$, which acts on a function $f(t)$ to produce another function $f'(t)$, i.e.,

$$\boxed{D f(t) = f'(t)}$$

Here $f'(t)$ is called the derivative or the image of $f(t)$ under the linear transformation D.

Note : choosing different kernels and different values of a and b, we get the different integral transforms.

Examples: The important integral transforms are

These are defined by choosing different kernels $K(s,t)$ and different values of a and b.

(i) Laplace transform

(ii) Fourier transform

(iii) Hankel transform

(iv) Mellin transform

(i) Laplace transform :

For $K(s,t) = e^{-st}$, $a=0$, $b=\infty$,

the improper integral

$$\int_0^{\infty} e^{-st} f(t)\, dt$$ is called

Laplace transform of $f(t)$.

i.e., $T\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$

Here $f(t)$ is a function defined for $t \geq 0$.

(ii) Fourier transform:

If we set $K(s,t) = e^{-ist}$, $a = -\infty$, $b = \infty$, then $\int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$ is called Fourier transform of $f(t)$.

$$T\{f(t)\} = F(s)$$

$$= \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt.$$

Applications: (1) Solve linear differential eqns and system of differential eqns.

(2) Solve integral eqn or integro-differential equations.

(3) Application in Probability and Statistics.

(4) Applications in Signal processing.

Note: It is worth noting that the Laplace transform is a special case of the Fourier transform for a class of functions defined on the positive real axis, but it is more simple than the Fourier transform.

Motivation for Laplace Transform :

Power Series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad x > 0$$

Example : (i) $a_n = 1 \quad , \quad n \geq 0$

$$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad , \quad |x| < 1$$

Since $x > 0$, the region of the convergence for power series $\sum_{n=0}^{\infty} x^n$ is

$$0 < x < 1.$$

(ii) $a_n = \frac{1}{n!}$, $n \geq 1$

$$A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \quad \forall\, x > 0$$

So the region of convergence is $x > 0$.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x > 0$$

$$= \sum_{n=0}^{\infty} a(n)\, x^n. \rightarrow \underline{\text{Discrete}}$$

Now change from discrete to continuous case.

<u>Continuous Analog:</u>

Discrete variable: $n = 0, 1, 2, \ldots$

Continuous variable (t): $\underline{0 \le t < \infty}$.

Continuous :

$$A(x) = \int_0^{\infty} a(t)\, x^t \, dt$$

$x = e^{\log x}$

$x^t = (e^{\log x})^t$

choose $0 < x < 1$.

$\log x < 0$.

If we choose $s = -\log x$.

then $s > 0$.

$$x^t = (e^{\log x})^t = e^{-st}.$$

$$A(e^{-s}) = F(s) = \int_0^{\infty} a(t)\, e^{-st} \, dt$$

$= \mathcal{L}[a(t)]$

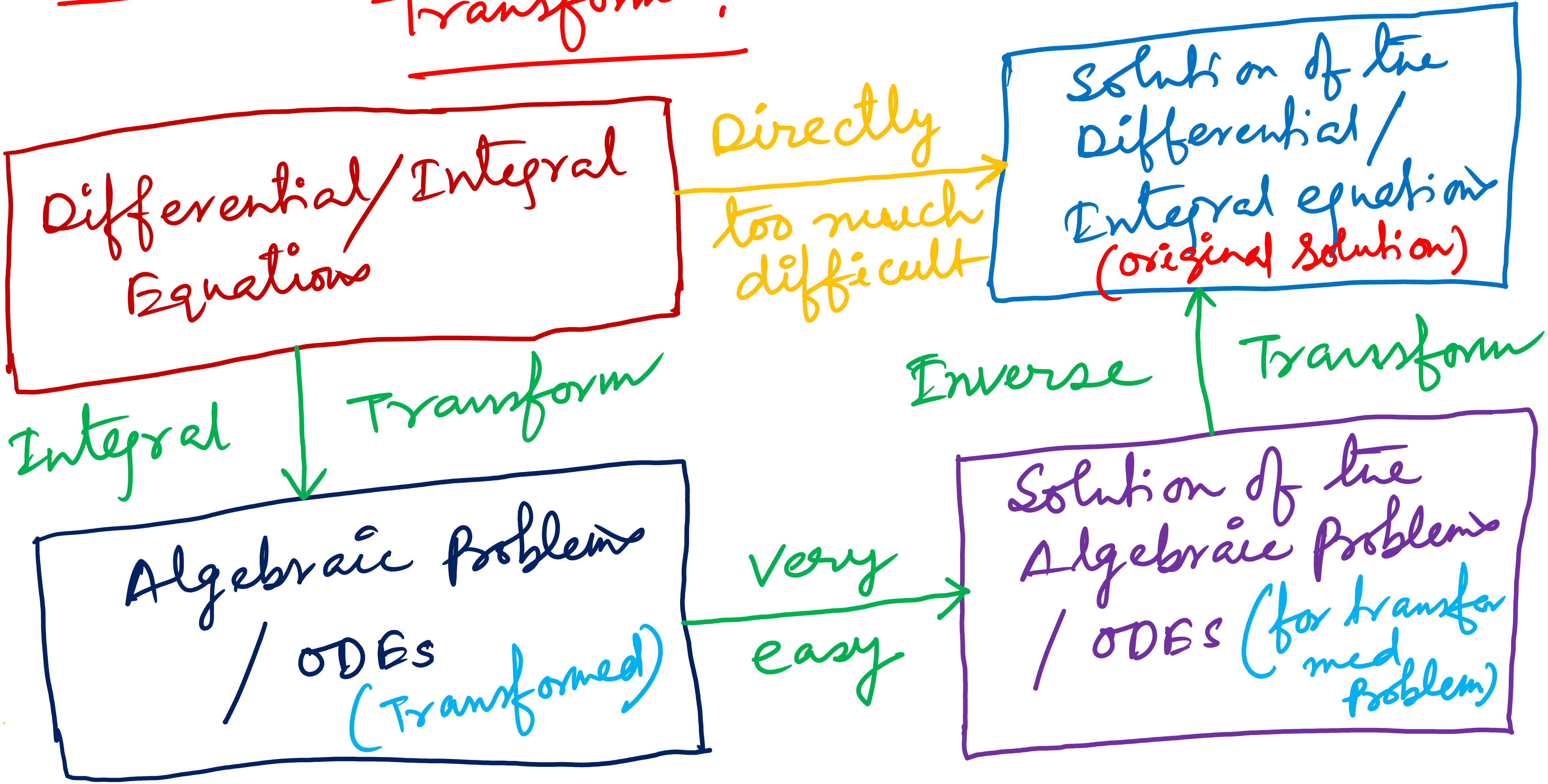
Laplace Transform.

So Laplace transform is the Continuous analog to the Power Series

How does the Laplace Transform work?

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{y''(t) - y(t) = t} \arrow[r, "\text{Laplace}", "\text{Transform}"'] \arrow[d, "\text{Solution}"] & \boxed{\text{Algebraic eq}^{n} \text{ in } s} \arrow[d, "\text{Solution}"] \\
\boxed{y(t)} & \boxed{F(s)} \arrow[l, "\text{Inverse}"', "\text{Laplace Transform}"]
\end{tikzcd}
$$

Basic Idea of the procedure in the Laplace Transform :



Laplace Transform:

Let $f(t)$ be a function defined for $t \geq 0$. Then the Laplace transform of a function $f(t)$ is defined as -

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

provided the improper integral converges for some s.

Recall: The integral $\int_0^\infty e^{-st} f(t)\,dt$ is said to be convergent (absolutely convergent) if

$$\lim_{R\to\infty} \int_0^R e^{-st} f(t)\,dt \quad \left(\lim_{R\to\infty} \int_0^R \left|e^{-st} f(t)\right| dt\right)$$

exists (as a finite number).

1. Find the Laplace transform of $f(t)=1,\ t\geq 0$

<u>Sol^n:</u> $\mathcal{L}(1) = \int_0^\infty e^{-st}\,dt$

$$= \lim_{R\to\infty} \int_0^R e^{-st}\,dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^R$$

$$= \frac{1}{s}, \text{ provided } \underline{s>0.}$$

Question: Do Laplace Transform exist for any kind of functions?

Ans: No.

Example: Consider the function $f(t) = e^{t^2}$. Then $F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{t^2} \cdot e^{-st} \, dt$

$$= \lim_{R \to \infty} \int_0^R e^{t^2 - st} \, dt.$$

This improper integral does not converge.

So $\mathcal{L}[e^{t^2}]$ does not exist.

Question: Then a common question comes out in the mind that

For which class of functions, the Laplace improper integral converges i.e., Laplace transform exist?

This question can be answered by the "Sufficient Conditions for existence of Laplace Transform".

- Before we will discuss about the Conditions for existence of Laplace Transform, we should know the conditions for existence of improper integral. So let's know about 'improper integral.'

Improper Integral :

$$\int_0^\infty f(t)\,dt = \lim_{R\to\infty} \int_0^R f(t)\,dt$$

The integral exists if the limit on the RHS is finite.

Example: (i) $f(t) = e^{-t}$.

$$\int_0^\infty f(t)\,dt = \lim_{R\to\infty} \int_0^R e^{-t}\,dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{-R} - 1}{-1} \right]$$

$$= 1.$$

(ii) $f(t) = t^{-3/2}, \quad 1 \le t < \infty$

$$\int_1^\infty f(t)\,dt = \lim_{R\to\infty} \int_1^R f(t)\,dt$$

$$= \lim_{R\to\infty} \left[-t^{-1/2} \right]_1^R$$

$$= 1.$$

(iii) $f(t) = t, \quad t \geq 0$

$$\int_0^\infty f(t)\,dt = \lim_{R\to\infty} \int_0^R f(t)\,dt$$

$$= \lim_{R\to\infty} \left[\frac{t^2}{2} \right]_0^R$$

$$\longrightarrow \infty \text{ as } R \to \infty.$$

This improper integral does not exist.

(iv) $f(t) = \sin t, \ t \geq 0$

$$\int_0^\infty f(t)\,dt = \lim_{R\to\infty} \int_0^R \sin t\,dt$$

$$= \lim_{R\to\infty} \left[1 - \cos R\right]$$

This limit does not exist.

(iv) $f(t) = \sin t, \ t \geq 0$

$$\int_0^{\infty} f(t)\, dt = \lim_{R \to \infty} \int_0^R \sin t \, dt$$

$$= \lim_{R \to \infty} \left[1 - \cos R \right]$$

This limit does not exist.

Sufficient Condition for existence of definite integral:

Theorem: If $f(t)$ is a continuous function on $[a, b]$ then $\int_a^b f(t)\,dt$ exists.

Here $0 < a < b < \infty$ and a and b are finite value.

② Sufficient condition for existence of improper integral:

Theorem: The improper integral $\int_a^{\infty} f(t)\,dt$ exists

if

(i) $f(t)$ is piecewise continuous on $[a, R]$ for all $R > a$

(ii) $f(t) \geq 0$ on $[a, \infty)$

(iii) $\int_a^R f(t)\,dt \le M$ for some $M > 0$, for every $R > a$

Theorem: Assume (i) and (ii) in the above theorem. Let there be a function $g(t)$ s.t. $f(t) \le g(t)$, $\forall\, t \in [a, \infty)$ and $\int_a^\infty g(t)\,dt$ exists. Then $\int_a^\infty f(t)\,dt$ exists.

Example :

(i) $f(t) = e^{-t}$. $\int_0^{\infty} f(t)\,dt$ exists.

Apply the first theorem

$$\int_0^R e^{-t}\,dt \le M \quad \forall R > 0$$
$$M > 0$$

or $1 - e^{-R} \le 1$,

(ii) $f(t) = e^{-t^2}, \quad 1 \le t < \infty$

$$\int_1^{\infty} f(t)\,dt = \lim_{R\to\infty} \int_1^{R} e^{-t^2}\,dt$$

for $t \ge 1$, $\quad t^2 \ge t$

$$e^{t^2} \ge e^{t}$$

$$\Rightarrow e^{-t} \ge e^{-t^2}, \quad \forall\, t \ge 1$$

Apply the second theorem with $g(t) = e^{-t}$

$$\int_1^\infty e^{-t}\, dt \quad \text{exists.}$$

Now we will discuss the Piecewise continuous function.

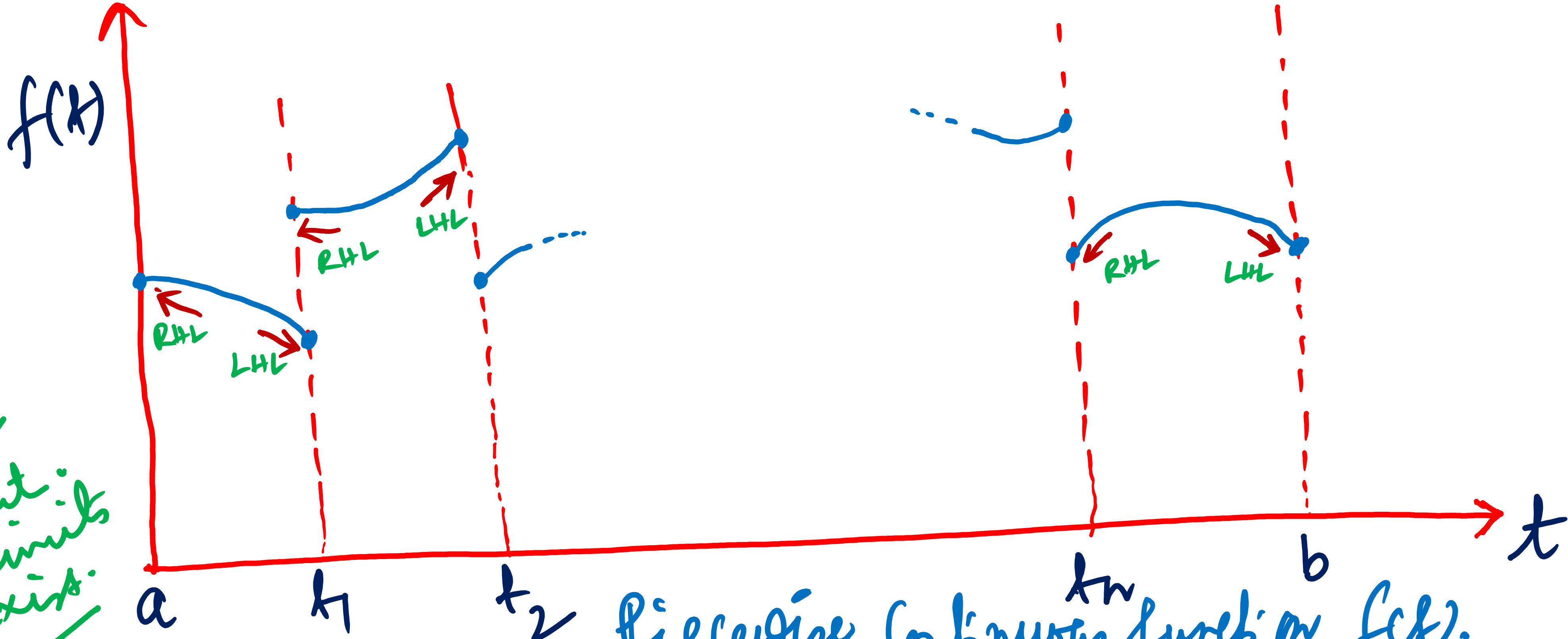
Piecewise Continuous function :

A function $f(t)$ is called piecewise continuous on $[a, b]$ if there are finite number of points $a < t_1 < t_2 \cdots < t_n < b$ such that $f(t)$ is continuous on each open subinterval (a, t_1), $(t_1, t_2), \ldots, (t_n, b)$ and all the following limits exist $\lim_{t \to a^+} f(t)$, $\lim_{t \to b^-} f(t)$, $\lim_{t \to t_j^-} f(t)$, $\lim_{t \to t_j^+} f(t)$, $\forall j$.

Note: A function $f(t)$ is said to be piecewise continuous on $[0,\infty)$ if it is piecewise continuous on every finite interval $[0,b]$, $b \in \mathbb{R}^+$

Example:

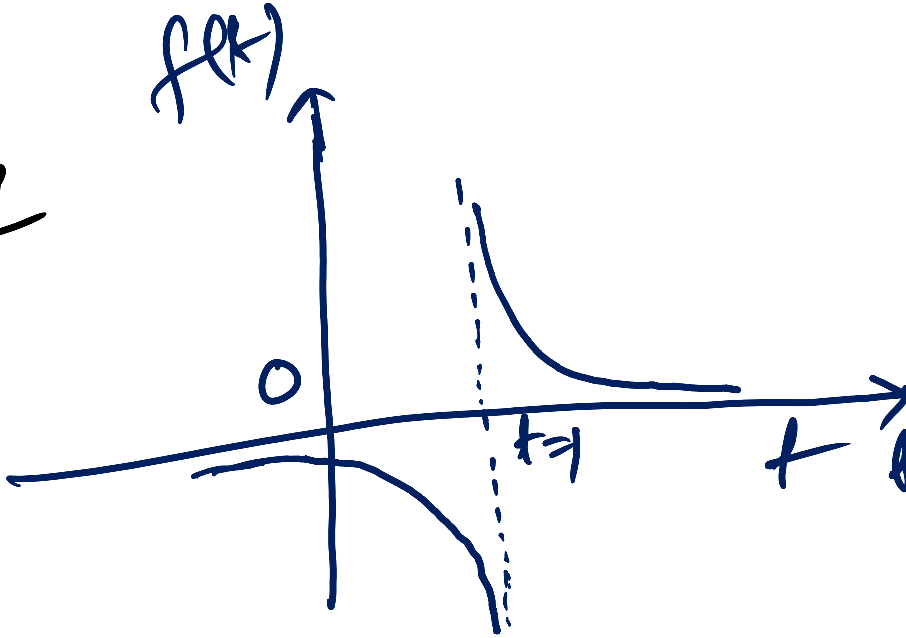
Here RHL means right hand limit and LHL means left hand limit. These limits exist.



Piecewise continuous function $f(t)$.

Example: Discuss the piecewise continuity of the function $f(t) = \dfrac{1}{t-1}$.

Solⁿ: $f(t)$ is not continuous in any interval containing 1 since $\lim_{t \to 1^{\pm}} f(t)$ does not exist.



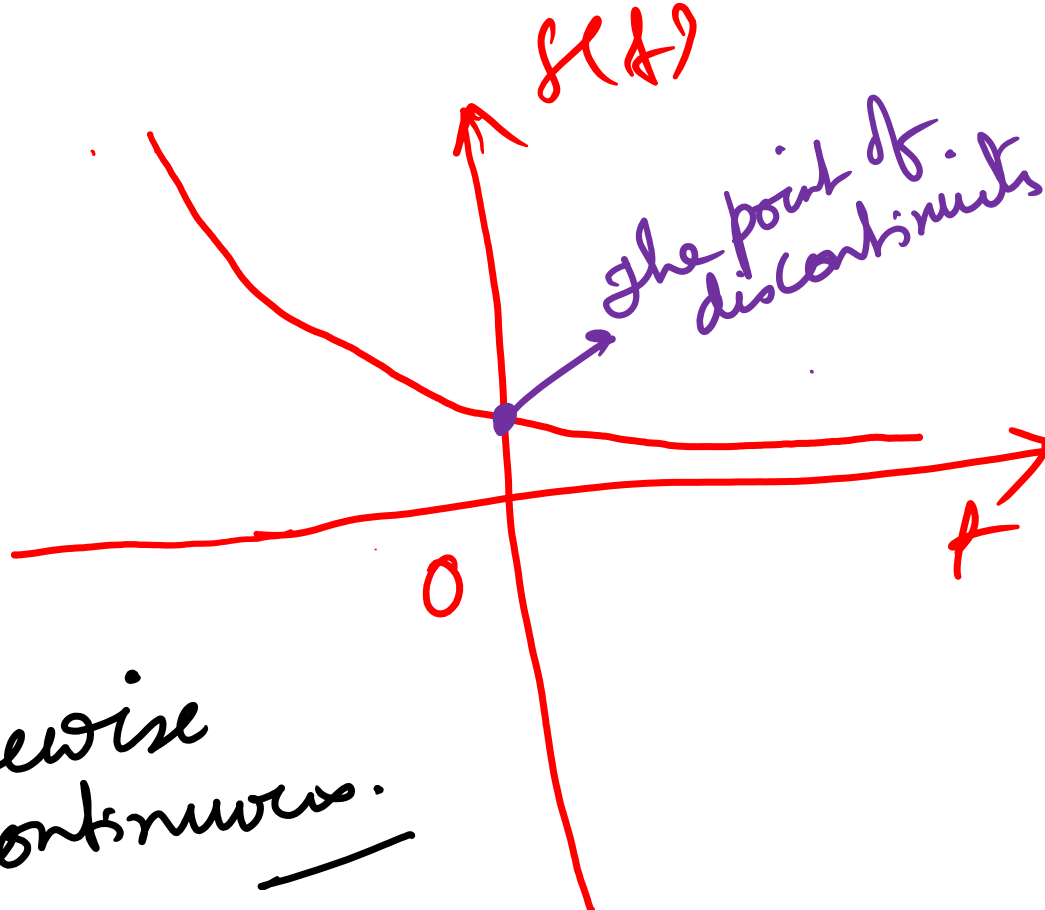
Example: check whether the function

$$f(t) = \begin{cases} \dfrac{1-e^{-t}}{t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

is piecewise continuous or not.

Sol^n: $\displaystyle\lim_{t \to 0^+} \frac{1-e^{-t}}{t} = 1$

and $\displaystyle\lim_{t \to 0^-} \frac{1-e^{-t}}{t} = 1$



$f(t)$ is piecewise continuous.