Fundamental Theorem of Calculus

Lecture 2

FTC 1: If f is continuous on $[a,b]$, then

$$\left(\int_a^x f(t)\,dt\right)' = f(x). \qquad \forall x \in [a,b]$$

FTC 2: If f' exists and f' is integrable in $[a,b]$, then

$$\int_a^b f' = f(b) - f(a).$$

Examples:

① Compute the derivative of $g(x) = \int_1^x \sqrt{t^3+4t}\, dt$ at $x=2$.

$$\boxed{g'(x) = \sqrt{x^3+4x}}$$

Solution: $\int_a^x f(t)\, dt$

$$f(t) = \sqrt{t^3+4t}$$

FTC 1

$f(t)$ continuous.

$$\left(\int_a^x f(t)dt\right)' = f(x)$$

$f : [1,3] \to \mathbb{R}$ is continuous on $[1,3]$.

Thus FTC ① applies and we obtain

$$g'(2) = f(2) = \sqrt{2^3+4\cdot 2} = 4.$$

$2=1$

$$x^2 = x \cdot x$$
$$= \underbrace{x+x+\cdots+x}_{x \text{ times}}$$

$$2x = \underbrace{1+1+\cdots+1}_{x \text{ times}}$$

$\Rightarrow 2x = x$

$\Rightarrow 2 = 1$

$\lim\limits_{h\to 0}$

② Determine the derivative of $g(x) = \int_{-\frac{\pi}{2}}^{x} \sqrt{\sin^2 t + 2}\, dt$.

at $\boxed{x = \frac{\pi}{6}}$

Define $f(t) = \sqrt{\sin^2 t + 2}$

$x \in [a, b]$

f is continuous on $[-\frac{\pi}{3}, \pi]$.

Thus FTC ① applies and $g'(\frac{\pi}{6}) = f(\frac{\pi}{6}) = \sqrt{\sin^2(\frac{\pi}{6}) + 2}$.

$$= \sqrt{\frac{1}{4} + 2}$$

$$= \frac{3}{2}.$$

3. Find the derivative of: $g(x) = \int_1^{x^3} t^2 \, dt$.

We define $f(t) = t^2$.

Since f is continuous on $[1, a]$ for some sufficiently large value of a, FTC applies, and we obtain that $F(x) = \int_1^x t^2 \, dt$ is differentiable and $F'(x) = x^2$.

Now $g(x) = F(x^3)$.

(chain rule) $\Rightarrow$ $g'(x) = \frac{d}{dx^3} F(x^3) \cdot \frac{dx^3}{dx} = 3x^2 \, F'(x^3)$

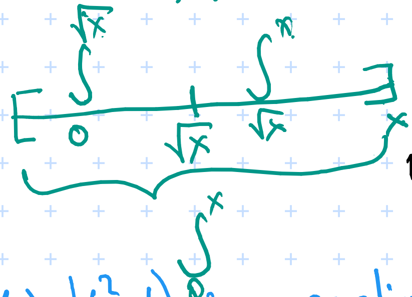
$= 3x^2 \cdot x^6$

$= 3x^8$.

4. Compute the derivative of $g(x)=\int_{\sqrt{x}}^{x}(t^2-t)\,dt$ at $x=2$

$\sqrt{x}<x$

$\sqrt{x}>x$



(domain additivity) $= \int_0^x (t^2-t)\,dt - \int_0^{\sqrt{x}}(t^2-t)\,dt.$

$= F(x) - F(\sqrt{x}).$

where $F(x)=\int_0^x (t^2-t)\,dt.$

$f(t)=(t^2-t)$ is continuous on $[0,a]$ for a sufficiently large. Thus FTC ① applies and yields $F'(x)=x^2-x$.

Thus $g'(x)=F'(x)-F'(\sqrt{x})$

$=(x^2-x)-\frac{1}{2\sqrt{x}}(x-\sqrt{x})$

⑤ Find the derivative of $g(x) = \int_{x^2}^{x^3} t\,dt$

$g(x) = F(x^3) - F(x^2)$.

$F(x) = \int_0^x t\,dt$

$\xRightarrow{FTC(1)} F'(x) = x$.

$\Rightarrow g'(x) = 3x^2 F'(x^3)$
$- 2x F'(x^2)$.

$= 3x^2 \cdot x^3 - 2x \cdot x^2$.

$= 3x^5 - 2x^3$.

⑥ Compute $\int_0^1 \left(\sqrt[3]{t} - \sqrt{t}\right) dt$.

$$t^{1/3} = \frac{t^{4/3}}{4/3}$$

Let $f(t) = \frac{t^{4/3}}{4/3} - \frac{t^{3/2}}{3/2}$.

FTC 2

Then $f'(t) = \sqrt[3]{t} - \sqrt{t}$ exists and it is continuous on $[0,1]$. Thus FTC 2 applies and we have

$$\int_0^1 \left(\sqrt[3]{t} - \sqrt{t}\right) dt = \left.\frac{t^{4/3}}{4/3} - \frac{t^{3/2}}{3/2}\right]_0^1$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}.$$

7. Compute $\int_{-2}^{1} |x^2-1|\,dx$.

$$|x^2-1| = \begin{cases} x^2-1 & \text{if} \quad x>1 \text{ or } x<-1 \\ 1-x^2 & \text{if} \quad -1 \le x \le 1 \end{cases}$$

$$\int_{-2}^{1} |x^2-1|\,dx = \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^{1} (1-x^2)\,dx.$$

(Domain additivity)

$$= \frac{8}{3}$$

Integration by parts:

Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ be continuous functions such that

(a) f is differentiable on $[a,b]$ and f' is integrable on $[a,b]$.

(b) g is integrable on $[a,b]$ with antiderivative G on $[a,b]$.

Then $$\int_a^b f(x)\,g(x)\,dx = f(b)\,G(b) - f(a)\,G(a) - \int_a^b f'(x)\,G(x)\,dx$$

e^{-x}.

Example:

$$\int_0^4 x e^{-x}\,dx$$

$f(x) = x$, $g(x) = e^{-x}$ ← continuous

(1) f differentiable; $f'(x) = 1$ integrable on $[0,4]$.

(2) $g = e^{-x}$ is continuous and hence integrable with an antiderivative $G(x) = -e^{-x}$ on $[0,4]$

Let $f:[a,b]\to\mathbb{R}$ be a bounded function.

$$\int_0^4 f(x)\,g(x)\,dx = f(4)\,G(4) - f(0)\,G(0) - \int_0^4 f'(x)\,G(x)\,dx$$
$$= (4)\cdot(-e^{-4}) - 0\cdot(-e^{0}) - \int_0^4 1\cdot(-e^{-x})\,dx$$
$$= -4e^{-4} + \int_0^4 e^{-x}\,dx$$
$$= -4e^{-4} - e^{-4} + 1$$
$$= 1 - 5e^{-4}.$$

$$\int_0^4 e^{-x}\,dx = -e^{-x}\Big]_0^4 = -e^{-4} + 1$$

$f(x) = e^{-x}$ integrable.
$F(x) = -e^{-x}$ is an antiderivative of e^{-x} on $[0,4]$
FTC (2) applies

Next Session: Method of substitution and Riemann sum (Lectures 3 and 4).

Integration by substitution:

$$\begin{tikzcd} {[\alpha,\beta]} \arrow[r, "\phi"'] \arrow[rr, bend left, "\phi'"] \arrow[rr, bend right] & {[a,b]} \arrow[r, "f"'] & \mathbb{R}. \end{tikzcd}$$

$\rightarrow$ f is continuous on $[a,b]$.

$\rightarrow$ ϕ is differentiable on $[\alpha,\beta]$

$\rightarrow$ ϕ' is integrable on $[\alpha,\beta]$.

$$\boxed{\phi([\alpha,\beta]) = [a,b]}$$

$\Longrightarrow$ $(f\circ\phi)$ integrable on $[\alpha,\beta]$

$\Longrightarrow$ $\boxed{(f\circ\phi)\cdot\phi'}$ integrable on $[\alpha,\beta]$

$$\boxed{\int_{\alpha}^{\beta} (f\circ\phi)\,\phi' = \int_{\phi(\alpha)}^{\phi(\beta)} f}$$

(2) $\displaystyle\int_{\boxed{0}}^{\boxed{1}} \frac{x^3}{\sqrt{x^4+1}}\,dx$

α, β $\quad$ $\varphi(\alpha), \varphi(\beta)$

a, b $\quad$ $\varphi?$ $\quad$ $f?$

- Identify the intervals $[\alpha, \beta]$, $[a, b]$
- Identify f, φ
- Check whether f is continuous
  φ is differentiable
  φ' is integrable.
- Apply method of substitution.

$\displaystyle\int_\alpha^\beta (f\circ\varphi)\cdot\varphi'$

$\alpha = 0$
$\beta = 1$.

Let $\boxed{f = \frac{1}{\sqrt{x}}}$

$\phi = \boxed{x^4+1}$

$\phi'(x) = 4x^3$

$f\circ\varphi(x) = f(x^4+1) = \frac{1}{\sqrt{x^4+1}}$

$[\alpha, \beta] = \boxed{[0,1]}$

$\phi[\alpha, \beta] = [1, 2]$

$\boxed{[0,1]} \xrightarrow{\varphi} \underset{[a,b]}{[1,2]} \xrightarrow{f} \mathbb{R}$

continuous

φ differentiable on $[0,1]$
and ϕ' is integrable on $[0,1]$.

$f\circ\phi(x) = f(x^4+1) = \frac{1}{\sqrt{x^4+1}}$.

$\phi'(x) = 4x^3$.

By method of substitution,

$$\int_0^1 f\circ\phi(t)\,\phi'(t)\,dt = \int_1^2 f(x)\,dx.$$

$$\Rightarrow \int_0^1 \frac{4t^3}{\sqrt{t^4+1}}\,dt = \boxed{\int_1^2 x^{-1/2}\,dx.}$$

$$\Rightarrow \int_0^1 \frac{t^3}{\sqrt{t^4+1}}\,dt = \frac{1}{4}\int_1^2 x^{-1/2}\,dx.$$

$h(x) = \frac{x^{1/2}}{1/2}$ is an antiderivative of $x^{-1/2}$ on $[1,2]$

and $h'(x) = x^{-1/2}$ is continuous on $[1,2]$

and hence integrable.

Thus, by FTC (2), $\displaystyle\int_1^2 x^{-1/2}\,dx = \left.\frac{x^{1/2}}{1/2}\right]_1^2 = 2(\sqrt{2}-1)$.

$$\Rightarrow \int_0^1 \frac{t^3}{\sqrt{t^4+1}}\,dt = \frac{1}{2}(\sqrt{2}-1).$$

$$\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt.$$

$[\alpha, \beta] = [0, 1]$.

$f(x) = \sqrt{x}$.

$\phi(t) = t^5+2t$ $\qquad \phi'(t) = 5t^4+2$

$\phi([0,1]) = [0,3]$.

$\|$ Check all the conditions of method of substitution.

$$\int_0^3 \sqrt{x}\,dx.$$

$\|$ Check all the conditions of FTC 2

$$\left.\frac{x^{3/2}}{3/2}\right]_0^3 = \frac{2}{3}\left(3\sqrt{3}\right) = 2\sqrt{3}$$

Riemann Sum:

Given a partition $P = \{x_0, x_1, \dots, x_n\}$.

define mesh. $\mu(P) = \max\{x_i - x_{i-1} \mid i = 1, \dots, n\}$.

(*) Let $f : [a,b] \to \mathbb{R}$ be integrable and let $\{P_n\}$ be a sequence of partitions satisfying $\mu(P_n) \to 0$ as $n \to \infty$.

Then $U(P_n, f) - L(P_n, f) \to 0$ as $n \to \infty$.

If $S(P_n, f)$ is the Riemann sum corresponding to P_n,

then $S(P_n, f) \to \int_a^b f$

$$S(P_n, f) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$t_i \in [x_{i-1}, x_i]$.

A natural way of computing $\int_a^b f$.

Define $P_n = [a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \ldots, a+\frac{n(b-a)}{n}]$

$\mu(P_n) = \frac{1}{n} \to 0$ as $n \to \infty$.

Choose the tag set $T = \{a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \ldots, a+\frac{n(b-a)}{n}\}$

$S(P_n, f)$ $S_n = \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\cdot(x_k - x_{k-1})$.

0

$= \frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) \xrightarrow[n\to\infty]{} \int_a^b f$.

In particular, if $[a,b] = [0,1]$, then $\int_0^1 f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{\frac{i}{n}+1}}$$

$$\xrightarrow[n\to\infty]{} \int_0^1 \frac{dx}{\sqrt{x+1}}$$

by means of: FTC (2)

$$\max\left\{\frac{i}{n}-\frac{i-1}{n}\right\}$$

$$= \max\left\{\frac{1}{n}\right\}$$

$$= \frac{1}{n}.$$

Must mention:

$[a,b] = [0,1]$

$P_n = \left\{0, \frac{1}{n}, \ldots, \frac{n}{n}\right\}$

$\mu(P_n) = \frac{1}{n} \to 0.$

$\mu(P_n) \to 0$

$f = \boxed{\frac{1}{\sqrt{x+1}}}$ is integrable on $[0,1]$.

$$S_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \frac{1}{n}\sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \frac{1}{n}\sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}$$

$$\frac{1}{n}\left(\left(\frac{n+1}{n}\right)^{3/2} + \left(\frac{n+2}{n}\right)^{3/2} + \cdots + \left(\frac{n+n}{n}\right)^{3/2}\right)$$

$$= \frac{1}{n}\left(\left(1+\frac{1}{n}\right)^{3/2} + \left(1+\frac{2}{n}\right)^{3/2} + \cdots + \left(1+\frac{n}{n}\right)^{3/2}\right)$$

$$\xrightarrow{n\to\infty} \int_0^1 (1+x)^{3/2}\,dx$$

$$= \left.\frac{(1+x)^{5/2}}{5/2}\right]_0^1 = \frac{2}{5}\left(2^{5/2}-1\right)$$

Next day: Improper integrals.

Area.