# Series

Let $\{a_n\}_n$ be a sequence — defining sequence.

Define: $S := a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \sum_n a_n$

is called a <u>series</u> (even if the sum does not exist)

$$S = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 2$$

$$S = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\left.\begin{aligned} S &= 1 + 2 + 3 + \cdots = \sum_{n=1}^{\infty} n \\ S &= 1 + 1 + 1 + \cdots = \sum_{n=1}^{\infty} 1 \end{aligned}\right\} S = \infty$$

$$S = 1 - 1 + 1 - 1 + 1 - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}.$$

$$(1-1) + (1-1) + (1-1) + \cdots = 0$$

$$1 + (-1+1) + (-1+1) + (-1+1) - \cdots = 1$$

Let $S = \sum_{n=1}^{\infty} a_n$

Let $n \in \mathbb{N}$, then

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$

Remark: If $a_n \geq 0 \ \forall n$, then $0 \leq S_1 \leq S_2 \leq S_3 \leq \cdots$ i.e. monotone increasing seq.

$$S = \underbrace{a_1 + a_2 + \cdots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{E_n}$$

$S_n$: n-th Partial Sum

$E_n$: n-th Error sum.

The sequence of partial sums for $S$ is

$\{S_n\}_{n=1}^{\infty}$ ← underlying sequence.

Defn. The series $S$ is said to converge (or the sum exists) if

$\lim_{n\to\infty} S_n$ exists.

$\left( S_n = a_1 + a_2 + \cdots + a_n \longrightarrow a_1 + a_2 + \cdots \text{ as } n \to \infty \right)$

The defn. makes sense!

If $\lim_{n\to\infty} S_n$ exists, then we write $S = \lim_{n\to\infty} S_n$ (the value of the limit)

or, $\boxed{S < \infty}$

## Examples.

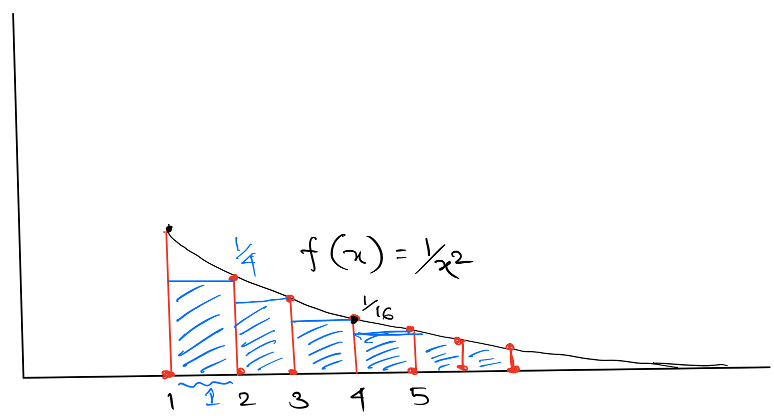
- $S = \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n}$$

So, $S_n \to 2$ & $S = 2$.

- $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$




$$\frac{1}{k^2} < \int_{k-1}^{k} \frac{dx}{x^2}$$

Thus, $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \int_1^2 + \int_2^3 + \cdots + \int_{n-1}^{n} = \int_1^n \frac{dx}{x^2} = 1 - \frac{1}{n}$

$$\Rightarrow \quad S_n - 1 < 1 - \frac{1}{n} \quad \forall \quad n \in \mathbb{N}.$$

$$\Rightarrow \quad S_n < 2 - \frac{1}{n} \quad \forall \quad n \in \mathbb{N}$$

$$< 2 \quad \forall \quad n \in \mathbb{N}.$$

Also, $S_1 < S_2 < S_3 < \cdots$

So, $S_n$ converges (montone bdd. sequence)

- $S = \sum_{n=1}^{\infty} 1$

$$S_n = n \longrightarrow \infty$$

Thus $S$ diverges

- $S = 1 - 1 + 1 - 1 + 1 - \cdots$

$$S_n = \begin{cases} 0 & \text{if } n\text{-even} \\ 1 & \text{if } n\text{-odd} \end{cases}$$

Thus, $S_n$ oscillates, and so does $S$.

Proposition. $S < \infty \iff E_n \longrightarrow 0$. Use Cauchy!

Proof. Exercise. "$\Rightarrow$" $E_n = S - S_n \longrightarrow 0$ as $n \to \infty$.

Thus, if $S < \infty$, then $\forall \ \varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t. $|E_n| < \varepsilon$ $\forall \ n \geq n_0$

# Telescoping Series

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Telescoping

$1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Thus, $S = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots$

## Partial sums.

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$$

$$\vdots \qquad = 1 - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\Rightarrow \quad S_n \longrightarrow 1$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

**Proposition.** If $S = \sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

$\boxed{S<\infty \Rightarrow a_n \to 0}$

*Proof.* $a_n = S_n - S_{n-1} \longrightarrow 0$ since $S_n$ converges. $\square$

How useful is this?

- $S = \sum_{n=1}^{\infty} n$

  Here, $a_n = n \longrightarrow \infty$

  By the Prop. $S$ is not convergent.

- $S = \sum_{n=1}^{\infty} \sin n$

  Since $\sin n$ does not converge $S$ does not converge.

**Nonexample.**

$\boxed{S_n \sim \ln n \to \infty}$

$S = \sum_{n=1}^{\infty} \frac{1}{n}$, here $a_n \to 0$ but $S \to \infty$.

Let $S = a_1 + a_2 + \cdots\cdots$

Define: $\boxed{S^+ = |a_1| + |a_2| + \cdots\cdots}$

Example. $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$

Then $S^+ = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Infinite $\Delta$-inequality.

Let $\{a_n\}_n$ be a sequence of numbers s.t.

$$S = \sum_n a_n \text{ exists.}$$

$S < \infty$
$\Rightarrow |S| \leq S^+$

Then $|S| \leq S^+$

Recall the finite $\Delta$-inequality
$|a_1 + a_2 + \cdots + a_n|$
$\leq |a_1| + |a_2| \cdots + |a_n|$

In other words,

$$\left|\sum_n a_n\right| \leq \sum_n |a_n|.$$

Proof. By the usual $\Delta$-inequality,

$$|S_n| \leq S_n^+ = \sum_{k \leq n} |a_k| \leq \sum_n |a| = S^+$$

Taking limit $n \to \infty$ on both sides of $|S_n| \leq S^+$,

we get $|S| \leq S^+$.

**Definition.** A series $S = \sum_n a_n$ is said to be **Absolutely Convergent** if $S^+ = \sum_n |a_n|$ converges.

$\boxed{S \text{ is absolutely convergent if } S^+ < \infty}$

**Proposition.** An absolutely convergent series is convergent.

$\cancel{|S| \leq S^+}$

$\boxed{S^+ < \infty \Rightarrow S < \infty}$

**Proof.** Warning: Infinite $\Delta$-inequality does not apply here!

Given: $\{S_n^+\}_n$ - convergent

To show: $\{S_n\}_n$ - convergent

Now, $\{S_n^+\}$ - convergent $\Rightarrow \{S_n^+\}$ - Cauchy

Thus, given $\varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t

$$|S_m^+ - S_n^+| < \varepsilon \quad \forall \; m, n \geq n_0$$

That is,

$$\Big| |a_1| + |a_2| + \cdots + |a_m| - |a_1| - |a_2| - \cdots - |a_n| \Big| < \varepsilon \quad \forall \; m, n \geq n_0$$

Assume $m > n$

Then $(*)$ implies that

$$|a_{n+1}| + \cdots + |a_m| < \varepsilon \quad \forall\, m, n \geq n_0$$

Now,

$$\begin{aligned} |S_m - S_n| &= |a_{n+1} + a_{n+2} + \cdots + a_m| \\ &\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon \\ &\qquad \forall\, m, n \geq n_0 \end{aligned}$$

$\Rightarrow \{S_n\}_n$ is Cauchy

$\Rightarrow \{S_n\}_n$ is convergent. $\square$

<u>Definition.</u> A series $S = \sum_n a_n$ is said to be <u>conditionally</u> <u>convergent</u> if $S < \infty$ but $S^+$ does not converge.

$S$ is conditionally convergent if $S < \infty$ but $S^+ = \infty$

<u>Example.</u> Let $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$= \ln 2$$

But $S^+ = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty.$

## Testing a series for convergence

Let $S = \sum_n a_n$

**Step 1.** Compute: $\lim_{n\to\infty} a_n$

If this limit $\neq 0$

then stop.

Conclude: $S$ is not convergent.

If $\lim_{n\to\infty} a_n$ exists, then

procede to step 2.

**Step 2.** Test $S$ for absolute convergence via

- Integral Test
- Comparison Test
- Ratio Test
- Root Test

If the series is conditionally convergent (i.e. not absolutely convergent),

then procede to <u>step 3</u>.

<u>Step 3</u>. Test for conditional convergence via

- Alternating series Test.

<u>Some</u> <u>background</u> <u>for</u> <u>step 2</u>.

<u>Proposition</u>. Let $S = \sum_n a_n$ with $a_n \geq 0 \ \forall n$.

Then either $0 \leq S < \infty$ or $S$ diverges

whence, we write $S = \infty$.

<u>Proof</u>. Let $S_n = \sum_{k \leq n} a_k$.

Then $S_{n+1} \geq S_n \quad \forall \quad n \in \mathbb{N}$.

$\Rightarrow \{S_n\}$ is a monotone increasing sequence.

Such sequences either converge or diverge.

Finally, if $S < \infty$, then $S \geq 0$ since $S_n \geq 0 \ \forall n$.

## General Examples

Investigate the convergence of of the following series by analyzing their partial sum sequences.

1. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

2. $\frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{2^3} + \cdots$

3. $\left(\sin \frac{\pi}{1}\right)^2 + \left(\sin \frac{\pi}{2}\right)^2 + \left(\sin \frac{\pi}{3}\right)^2 + \cdots$

4. $\cos \pi + \cos 2\pi + \cos 3\pi + \cdots$