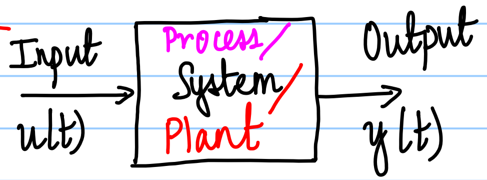
Note Title 02-02-2018

ED2040 Control Systems

System:

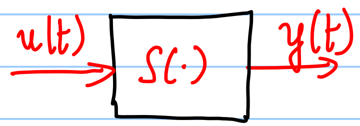


Dynamic System: Time is the independent variable. All variables that are associated with the system are functions of time.

System → (a collection of objects)/(a process) that is under study.

Mathematically, we can visualize a dynamic system as a mapping $S(\cdot)$ from $u(t)$ to $y(t)$.

That is, $y(t) = S(u(t))$.



Classification of Dynamic Systems:

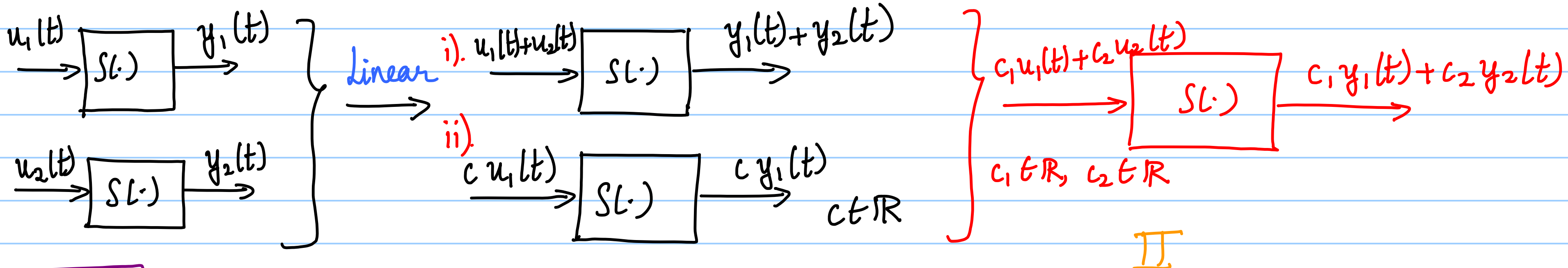
1). SISO vs MIMO: $\boxed{\text{SISO}}$ → Single Input Single Output. ⇔ $u(\cdot): \mathbb{R} \to \mathbb{R}$. $t \in \mathbb{R}$.
$y(\cdot): \mathbb{R} \to \mathbb{R}$.

MIMO → Multiple Input Multiple Output. m inputs & p outputs, $m>1$, $p>1$.

$\underset{\sim}{u}(\cdot): \mathbb{R} \to \mathbb{R}^m$.

$\underset{\sim}{y}(\cdot): \mathbb{R} \to \mathbb{R}^p$.

2). Linear vs Nonlinear:



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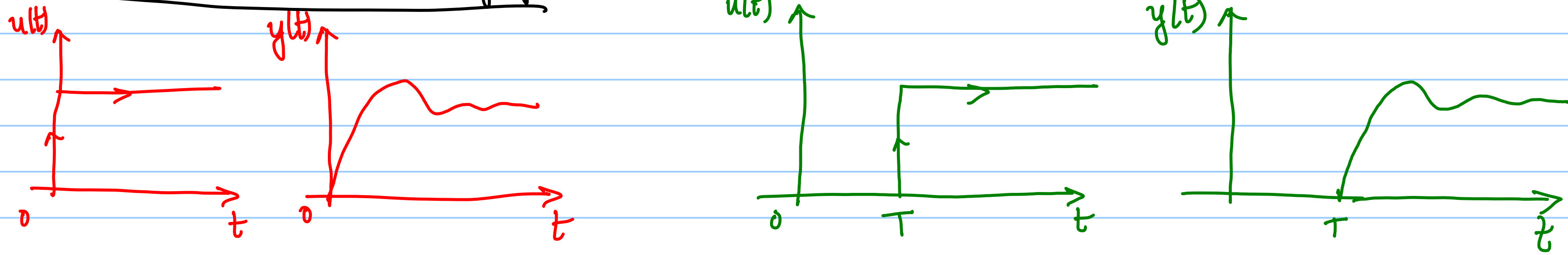
SUPERPOSITION

3). Time Invariant vs Time Varying:



A time invariant system is one that provides the same output for the same input irrespective of when the input is given.

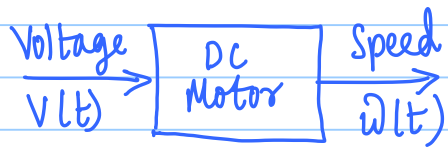
If $y(t) = S(u(t))$, then time invariance $\Rightarrow$ $y(t-T) = S(u(t-T)) \;\forall\, T \in \mathbb{R}$.

4). $\boxed{\text{Causal}}$ vs Non-causal: A causal system is one where the output at any instant of time depends only on past and current inputs.

A Causal system is $\boxed{\text{NON-ANTICIPATIVE.}}$

CLASS OF SYSTEMS: SISO $\underbrace{\text{LINEAR TIME INVARIANT}}_{\text{LTI}}$ CAUSAL DYNAMIC SYSTEMS → USEFUL CLASS OF SYSTEMS TO STUDY.

CONTROL: Making a system behave as desired.



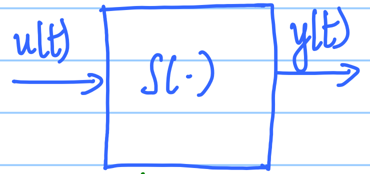
Q: If we wish to achieve a ω_{des}, what is the input voltage that should be provided?

Approach: 1). Develop a mathematical representation for S(·). → MATHEMATICAL MODELLING OF DYNAMIC SYSTEMS

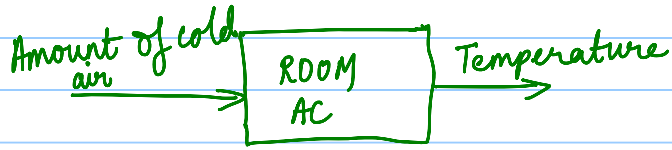
1). Physics based.
2). Empirical.
3). Mixed approach.

2). Analyze the system response.

3). Design the controller.



$y(t) = S(u(t))$.



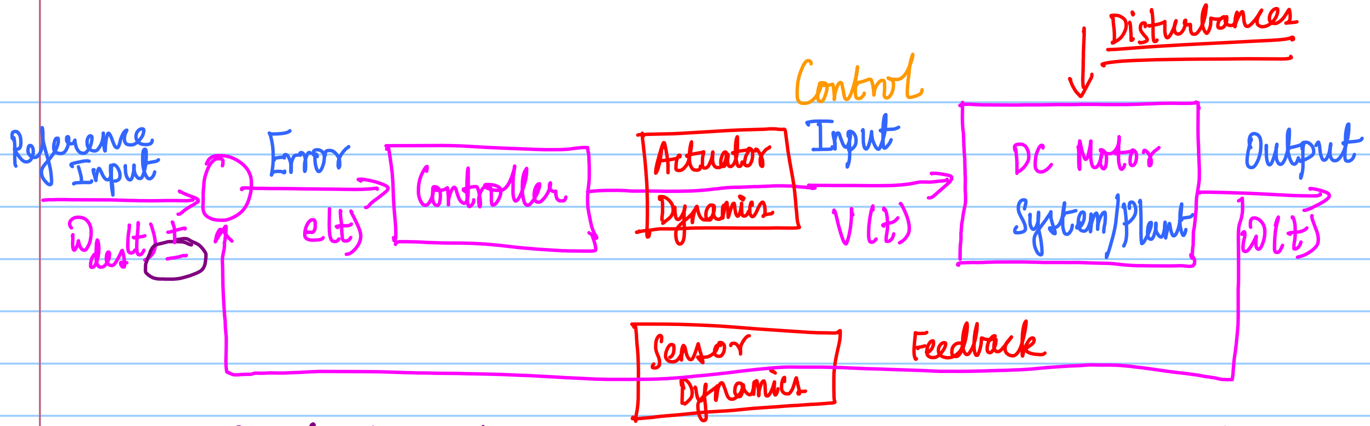
Problem 1). Given $u(t)$ & $y(t)$, find $S(\cdot)$. → SYNTHESIS.

Problem 2). Given $S(\cdot)$ & $u(t)$, find $y(t)$. → ANALYSIS/PREDICTION.

Problem 3): Given $S(\cdot)$ & $y(t)$, find $u(t)$. → CONTROL.

Examples:
- → Room temperature control.
- → Motor speed control.
- → Human body.

Control
- → Open Loop Control, eg.: Ceiling fan.
  - → No feedback.
  - → Cannot tolerate disturbances.
  - → Lower cost and complexity.
- → Closed Loop Control
  - → Feedback → process of measuring variables that need to be regulated so that corrective action can be taken.
  - → Robust to uncertainties, disturbances.
  - → Higher cost and complexity.



Closed Loop Control System with Negative Feedback.

Non-unity feedback vs unity feedback.

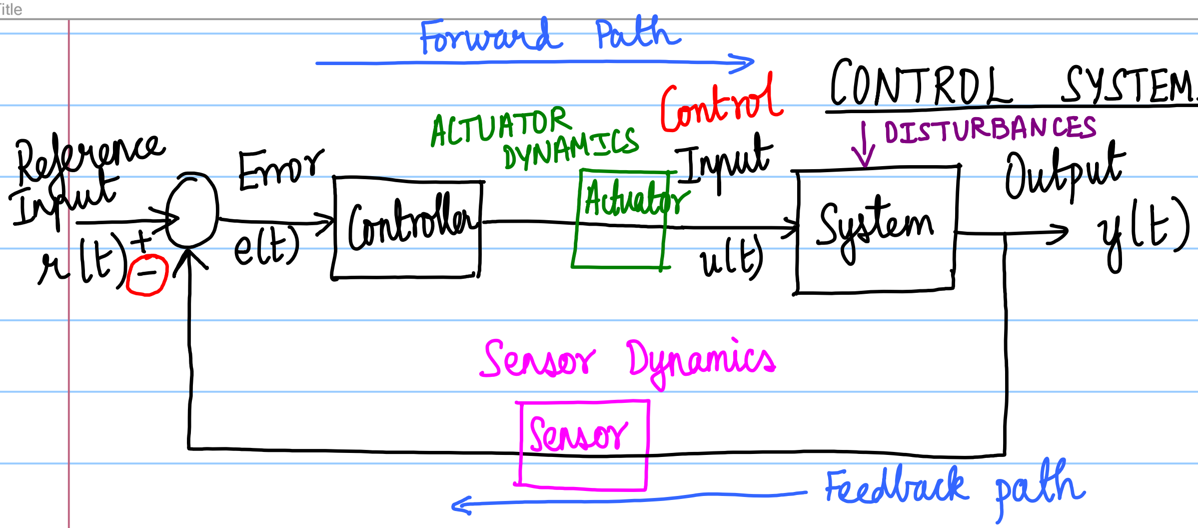
Summary: "Control Systems" $\longrightarrow$ Closed loop Feedback Control of $\underbrace{\text{SISO LTI Causal Dynamic Systems}}$.

The mathematical models typically used to characteriz[e] this class of systems usually take the form of linear Ordinary Differential Equations (ODEs) with constant coefficients.

Mathematical Models: Spatially Homogeneous Continuous Time Dynamic Deterministic Mathematical Models.

Note Title 09-05-2018

CONTROL SYSTEMS



CLOSED LOOP CONTROL SYSTEM WITH NEGATIVE FEEDBACK

Time → independent variable.

SISO CAUSAL LTI DYNAMIC SYSTEMS

↓

LINEAR ODEs with CONSTANT COEFFICIENTS

Laplace Transform → TRANSFER FUNCTION REPRESENTATION

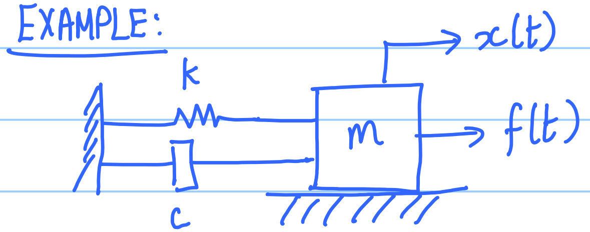
→ STATE SPACE REPRESENTATION

NOTE:

1). Unity vs non-unity feedback: If the mapping in the feedback path is unity (one), then it is called as unity feedback.

2). CONTROL SYSTEM → STABILITY
→ PERFORMANCE

EXAMPLE:



The governing equation of motion is

$$\underbrace{m\ddot{x}(t)}_{\text{Inertia}} + \underbrace{c\dot{x}(t)}_{\text{Viscous dissipation}} + \underbrace{kx(t)}_{\text{Compliance}} = f(t).$$

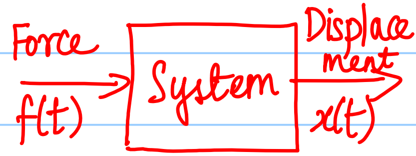
LTI Causal SISO system.

$$\dot{x}(t) = \frac{dx(t)}{dt}.$$

$$\ddot{x}(t) = \frac{d^2x(t)}{dt^2}.$$

Note that the governing eqn. is a 2^{nd} order linear inhomogeneous ODE with constant coefficients.

System response → non-zero initial conditions (ICs).
→ input provided.



Let us consider a scenario where the spring constant changes with time.

$$m\ddot{x}(t) + c\dot{x}(t) + \underbrace{k(t)}\,x(t) = f(t).$$

Linear Time Varying system (LTV)

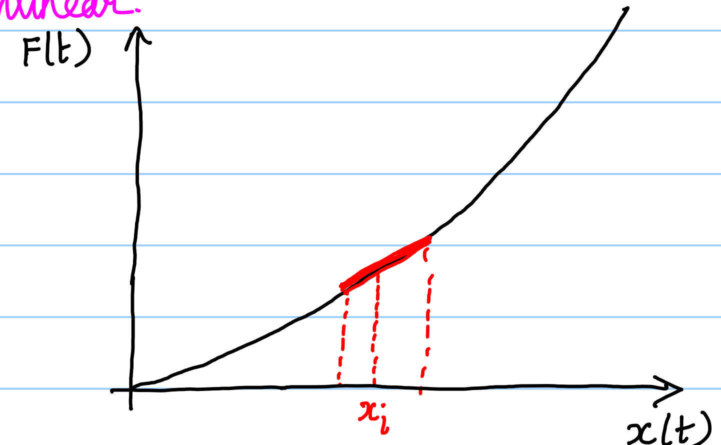
2^{nd} order linear inhomogeneous ODE with time varying coefficients.

Let us consider a scenario where the spring is nonlinear.

The governing equation may be written as

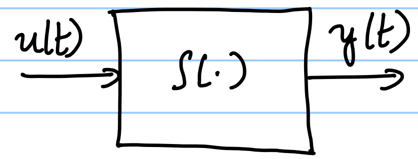
$$m\ddot{x}(t) + c\dot{x}(t) + \underbrace{k x^2(t)}_{} = f(t). \quad \boxed{\text{NONLINEAR}}$$

2^{nd} order nonlinear inhomogeneous ODE with constant coefficients.



(linear approximation?)

Mathematical Preliminaries



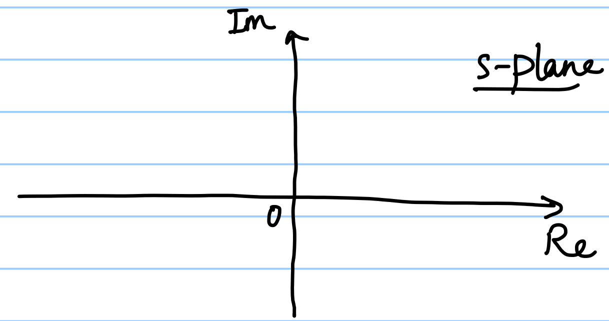
Ordinary Differential Equations: (ODEs)

linear ODEs with $\underbrace{\text{constant coefficients}}_{\text{Time invariance}}$.

First order homogeneous ODE : $\dfrac{dy(t)}{dt} + a\,y(t) = 0.$ $a \to$ constant.

$\Rightarrow y(t) = c\,e^{-at}, \quad c = y(0) \to$ initial condition.

First order inhomogeneous ODE: $\dfrac{dy(t)}{dt} + a\,y(t) = b\,u(t).$



$$y(t) = \underbrace{y(0)e^{-at}}_{\text{Due to initial conditions}} + \underbrace{\int_0^t e^{-a(t-\tau)}\, b\, u(\tau)\, d\tau}_{\text{Due to the input provided}}.$$

Due to initial conditions — FREE RESPONSE

Due to the input provided. — FORCED RESPONSE

Complex Variables: Let us consider a complex variable $s = \sigma + j\omega$, $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}$, $j^2 = -1$.

A complex function $F(s)$ is a complex valued function of s. $F(s) = F_r(\sigma, \omega) + j\,F_i(\sigma, \omega)$.

Eg.: $F(s) = s^2 = (\sigma + j\omega)^2 = \underbrace{\sigma^2 - \omega^2}_{F_r(\sigma,\omega)} + j\underbrace{(2\sigma\omega)}_{F_i(\sigma,\omega)}$.

A complex function $F(s)$ is said to be analytic in a given domain if $F(s)$ and all its derivatives exist in that domain.

Points in the s-domain where $F(s)$ is not analytic are called 'SINGULAR POINTS' OR 'POLES'.

Eg.: $F(s) = \frac{1}{s+1}$. Note that $s=-1$ is a singular point or pole.

Cauchy - Riemann Conditions: i). $\frac{\partial F_r}{\partial \sigma} = \frac{\partial F_i}{\partial \omega}$, ii). $\frac{\partial F_i}{\partial \sigma} = -\frac{\partial F_r}{\partial \omega}$.

Euler Relations:

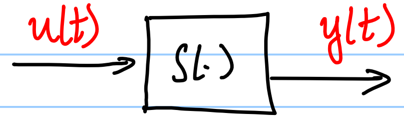
$$\left.\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t), \\ e^{-j\omega t} &= \cos(\omega t) - j\sin(\omega t). \end{aligned}\right\} \Rightarrow \begin{aligned} \cos(\omega t) &= \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \\ \sin(\omega t) &= \frac{e^{j\omega t} - e^{-j\omega t}}{2j}. \end{aligned}$$

Laplace Transform:

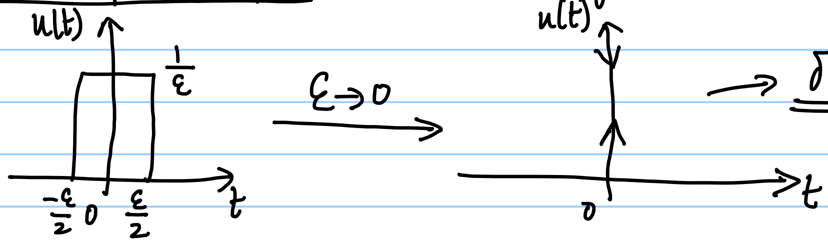
$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt. \longrightarrow \text{BILATERAL LAPLACE TRANSFORM.}$$

$$\mathcal{L}[f(t)] = \int_{0}^{\infty} f(t) e^{-st}\, dt. \longrightarrow \text{UNILATERAL LAPLACE TRANSFORM.}$$

Some Typical Standard Inputs:



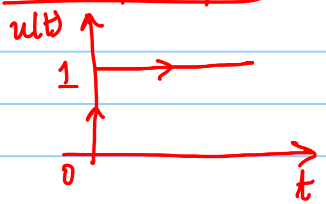
1). Unit Impulse Input: (Dirac Delta function)



$\to \underline{\underline{\delta(t)}}$

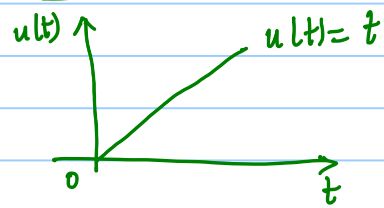
UNIT IMPULSE RESPONSE

2). Unit Step Input:



UNIT STEP RESPONSE

3). UNIT RAMP INPUT:



UNIT RAMP RESPONSE

4). SINUSOIDAL INPUTS:

$u(t) = \cos(\omega t),\ \sin(\omega t).$

FREQUENCY RESPONSE

Control Design → STABILITY → Bounded Input Bounded Output (BIBO) Stability.

Control Design → PERFORMANCE

$$u(t) \to \boxed{S(\cdot)} \to y(t)$$

A system is said to be BIBO stable if given any input $u(t)$ such that $|u(t)| \leq M < \infty \;\; \forall t$, the output of the system is always bounded in magnitude $\forall t$, that is, $|y(t)| \leq N < \infty \;\; \forall t$. Here, M and N are finite positive real numbers.

Return to Laplace Transform:

Properties:

1). Transform of Derivatives: If $\mathcal{L}[f(t)] = F(s)$,

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0). \quad \text{Here, } f(0) = f(t)\Big|_{t=0}. \rightarrow \text{INITIAL CONDITION.}$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0) - \frac{df(t)}{dt}\Big|_{t=0}.$$

$$\dot{f}(t) = \frac{df(t)}{dt}.$$

$$\ddot{f}(t) = \frac{d^2f(t)}{dt^2}.$$

2). Initial Value Theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$.

$$\to \boxed{s} \to$$

3). Final Value Theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$.

$$\to \boxed{\tfrac{1}{s}} \to$$

4). Complex Differentiation Theorem: $\mathcal{L}[t\, f(t)] = -\dfrac{dF(s)}{ds}$.

5). Real Integration Theorem: $\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \dfrac{F(s)}{s}$.

6). Multiplication by an exponential function: $\mathcal{L}[e^{-at} f(t)] = F(s+a)$.

$f(t)$	$F(s)$
$\delta(t)$	1
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

Let $f(t) = \cos(\omega t)$. $\mathcal{L}[e^{-at}\cos(\omega t)] = \dfrac{(s+a)}{(s+a)^2+\omega^2}$.

Similarly, $\mathcal{L}[e^{-at}\sin(\omega t)] = \dfrac{\omega}{(s+a)^2+\omega^2}$.

Linear ODEs with constant coefficients $\xrightarrow{\text{Laplace Transform}}$ Algebraic Equations

Time Domain $\longrightarrow$ s- Domain

POLYNOMIALS

Inverse Laplace Transform: $F(s) = \dfrac{1}{s^2+3s+2} = \dfrac{1}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$.

Residues

A: $*(s+1) \Rightarrow \dfrac{1}{s+2} = A + \dfrac{B(s+1)}{s+2} \xRightarrow{s=-1} 1 = A.$

B: $*(s+2) \Rightarrow \dfrac{1}{s+1} = \dfrac{A(s+2)}{s+1} + B \xRightarrow{s=-2} -1 = B.$

$\Rightarrow \boxed{y(t) = e^{-t} - e^{-2t}.}$

Solve initial value problems:

Consider a system whose governing equation is

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t), \quad y(0)=0, \ \dot{y}(0).$$

Find its unit step response, i.e., find $y(t)$ for $u(t) = 1$. $\Rightarrow U(s) = \frac{1}{s}$.

Take Laplace transform on both sides:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 5[sY(s) - y(0)] + 6Y(s) = U(s).$$

$$\Rightarrow (s^2+5s+6)Y(s) - (s+5)y(0) - \dot{y}(0) = U(s).$$

$$\Rightarrow Y(s) = \underbrace{\frac{(s+5)y(0) + \dot{y}(0)}{(s^2+5s+6)}}_{\substack{\text{Due to initial conditions} \\ \text{FREE RESPONSE}}} + \underbrace{\frac{U(s)}{(s^2+5s+6)}}_{\substack{\text{Due to input provided} \\ \text{FORCED RESPONSE}}}$$

Given $y(0)=0$, $\dot{y}(0)=0$, $V(s)=\frac{1}{s}$.

$$\Rightarrow Y(s)=\frac{1}{s(s^2+5s+6)}=\frac{1}{s(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+3}=\frac{1}{6s}-\frac{1}{2(s+2)}+\frac{1}{3(s+3)}.$$

$\underline{A}$: $*s$: $\frac{1}{(s+2)(s+3)}=A+\frac{Bs}{s+2}+\frac{Cs}{s+3}\xrightarrow{s=0}\frac{1}{6}=A.$

$$\Rightarrow \boxed{y(t)=\frac{1}{6}-\frac{1}{2}e^{-2t}+\frac{1}{3}e^{-3t}.}$$

$\underline{B}$: $*(s+2)$: $\frac{1}{s(s+3)}=\frac{A(s+2)}{s}+B+\frac{C(s+2)}{(s+3)}\xrightarrow{s=-2}-\frac{1}{2}=B.$

$\underline{C}$: $*(s+3)$: $\frac{1}{s(s+2)}=\frac{A(s+3)}{s}+\frac{B(s+3)}{s+2}+C\xrightarrow{s=-3}\frac{1}{3}=C.$

Exercises: Find the inverse Laplace transform of:

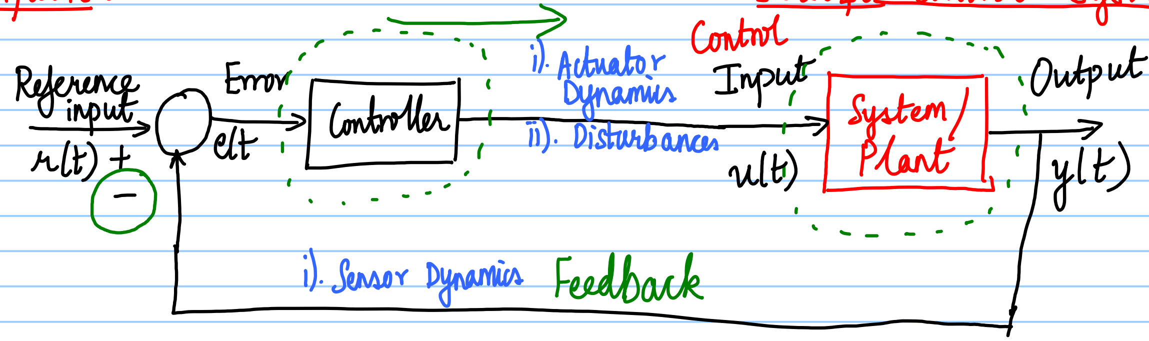
a). $\frac{1}{s^2+2s+2}$, b). $\frac{1}{s^2+2s+1}$, c). $\frac{1}{s^2+s}$, d). $\frac{s}{(s^2+1)^2}$, e). $\frac{1}{s^2+s-2}$.

Note Title 1/23/2018

22/1/2018.

ED2040 Control Systems

TIME → independent time.



Closed Loop Control System with Negative Feedback

Unity Non-unity

SISO Causal LTI Dynamic System

↓

Linear ODEs with constant coefficients.

Laplace Transform

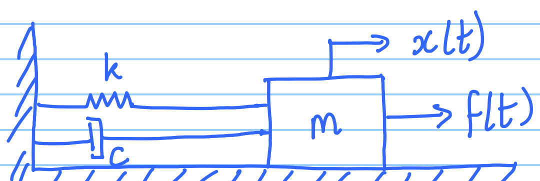
TRANSFER FUNCTION REPRESENTATION

STATE SPACE REPRESENTATION

Control Design → Stability

→ Performance

Example:

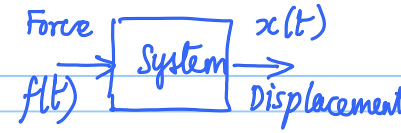


Equation of motion:

$$\underbrace{m\ddot{x}(t)}_{\text{Inertia}} + \underbrace{c\dot{x}(t)}_{\text{Viscous Dissipation}} + \underbrace{kx(t)}_{\text{Compliance}} = f(t).$$

$$\dot{x}(t) = \frac{dx}{dt}(t).$$

$$\ddot{x}(t) = \frac{d^2x}{dt^2}(t).$$

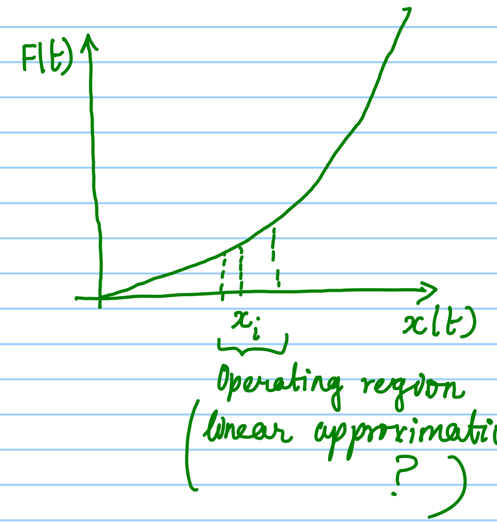


2nd order linear inhomogeneous ODE with constant coefficients.

Let us consider a scenario where the spring constant changes with time.

$$m\ddot{x}(t) + c\dot{x}(t) + k(t)\,x(t) = f(t). \rightarrow \boxed{\text{LTV system}}$$

2nd order linear inhomogeneous ODE with time varying coefficients.



Let us consider a scenario where the spring is nonlinear.

$$m\ddot{x}(t) + c\dot{x}(t) + k\,x^2(t) = f(t). \rightarrow \boxed{\text{NONLINEAR.}}$$

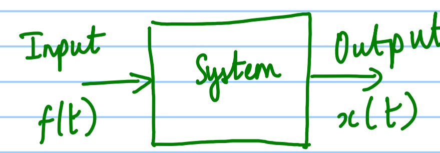
2nd order nonlinear inhomogeneous ODE with constant coefficients.

24/1/2018. Transfer function:

Recall that we consider SISO LTI causal dynamic systems that are characterized by linear ODEs with constant coefficients.

Consider the mass-spring-damper system that is governed by

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t).$$



Take the Laplace transform on both sides

$$m\left[s^2X(s) - s\,x(0) - \dot{x}(0)\right] + c\left[s\,X(s) - x(0)\right] + k\,X(s) = F(s).$$

$$\Rightarrow \left[ms^2 + cs + k\right]X(s) = (ms + c)\,x(0) + m\,\dot{x}(0) + F(s).$$

$$\Rightarrow X(s) = \underbrace{\frac{(ms+c)\,x(0) + m\,\dot{x}(0)}{(ms^2+cs+k)}}_{\substack{\text{Due to non-zero initial conditions} \\ \text{'FREE RESPONSE'}}} + \underbrace{\left(\frac{1}{ms^2+cs+k}\right)F(s)}_{\substack{\text{Due to input} \\ \text{'FORCED RESPONSE'}}}$$

Consider ALL initial conditions to be zero. That is, $x(0) = 0$, $\dot{x}(0) = 0$.

$$\Rightarrow X(s) = \left(\frac{1}{ms^2+cs+k}\right)F(s).$$

$$\Rightarrow \boxed{\frac{X(s)}{F(s)} = \frac{1}{(ms^2+cs+k)}.}$$ Transfer function of the system/plant.
$P(s)$

Def.: In general, Plant transfer function, $P(s) = \dfrac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]} = \dfrac{Y(s)}{U(s)}$ with ALL initial conditions being taken as zero. (SISO system)
Single Input Single Output

$$u(t) \longrightarrow \boxed{\text{System/Plant}} \longrightarrow y(t)$$

1). Synthesis: Obtain S(·).
2). Analysis
3). Control

This implies that $\boxed{Y(s) = P(s)\,U(s)}$ $\Rightarrow$ $y(t) = \int_0^t p(t-\tau)\,u(\tau)\,d\tau$. Here $\boxed{P(s) = \mathcal{L}[p(t)]}$.

Note:

1). If $P(s)$ is known, we can find the system output for any input.

2). Let $u(t) = \delta(t)$ $\rightarrow$ unit impulse input. $\Rightarrow$ $U(s) = 1$. $\Rightarrow$ $Y(s) = P(s)$ $\Rightarrow$ $y(t) = p(t)$.

$p(t)$ $\rightarrow$ impulse response function.

The general governing equation of a n^{th} order SISO LTI causal dynamic system can be written as

$$a_0 \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{(n-1)} y(t)}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m u(t)}{dt^m} + b_1 \frac{d^{(m-1)} u(t)}{dt^{(m-1)}} + \cdots + b_{m-1}\frac{du(t)}{dt} + b_m u(t).$$

Time invariance $\rightarrow$ constant coefficients.

Linear $\rightarrow$ linear in $u(t)$ & $y(t)$.

Causal $\rightarrow$ $n \geq m$. $n > m$ $\rightarrow$ strictly causal.

Eg.: Mass-spring-damper system: $u(t) = f(t)$, $y(t) = x(t)$.

$n = 2$.

$m = 0$.

HW:

Take the Laplace transform on both sides and apply zero initial conditions to obtain

$$P(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{(m-1)} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{(n-1)} + \cdots + a_{n-1}s + a_n}.$$

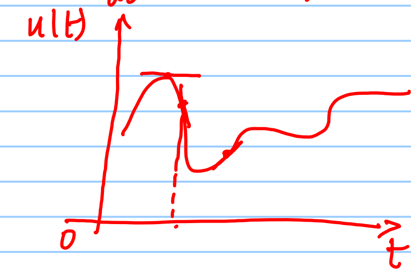
$P(s) = \frac{n(s)}{d(s)}$ $n(s)$ $\rightarrow$ numerator polynomial of order 'm'. $d(s)$ $\rightarrow$ denominator polynomial of order 'n'.

$n \geq m \rightarrow$ Proper transfer function.
$n > m \rightarrow$ Strictly Proper transfer function.

$P(s) = \dfrac{n(s)}{d(s)}$. $\underbrace{\text{Roots of } n(s)}$, i.e., solve $b_0 s^m + b_1 s^{(m-1)} + \dots + b_{m-1} s + b_m = 0$.
$\hookrightarrow$ Zeros of transfer function.

$\underbrace{\text{Roots of } d(s)}$, i.e., $a_0 s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n = 0$.
$\hookrightarrow$ Poles of transfer function.

$y(t) = \dfrac{du(t)}{dt}$. $n = 0$, $m = 1$



Exercises: Determine the plant transfer function, its poles and zero and calculate its unit impulse response.

1). $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$.

2). $\dot{y}(t) + y(t) = u(t)$.

3). $\ddot{y}(t) + \dot{y}(t) - 2y(t) = u(t)$.

Note Title 1/29/2018

29/1/2018. **Transfer Function:**

Exercises:

1). $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$.

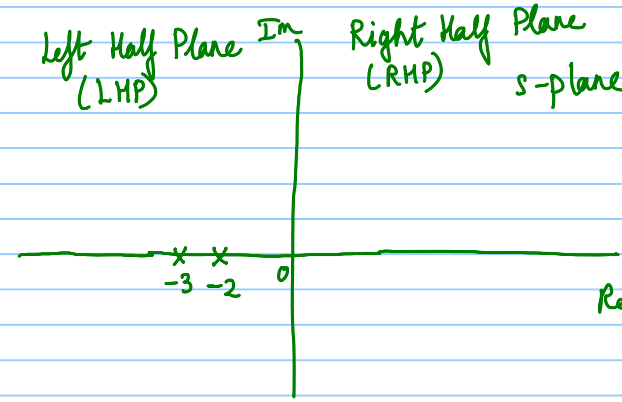
Take the Laplace transform on b/s:

$$s^2 Y(s) - s\,y(0) - \dot{y}(0) + 5[sY(s) - y(0)] + 6Y(s) = U(s).$$

$$[s^2 + 5s + 6]\,Y(s) = [y(0)s + 5y(0) + \dot{y}(0)] + U(s).$$

$$\Rightarrow Y(s) = \underbrace{\left(\frac{y(0)s + 5y(0) + \dot{y}(0)}{s^2 + 5s + 6}\right)}_{\text{Initial conditions}} + \underbrace{\frac{1}{(s^2+5s+6)}U(s)}_{\text{Input}}.$$

FREE RESPONSE FORCED RESPONSE



Take all initial conditions as zero, i.e., $y(0) = 0$, $\dot{y}(0) = 0$.

$$\Rightarrow Y(s) = \frac{1}{(s^2+5s+6)}U(s) \Rightarrow \boxed{\frac{Y(s)}{U(s)} = \frac{1}{s^2+5s+6}} = P(s).$$

$n = 2$ Poles (x): $s^2 + 5s + 6 = 0 \Rightarrow s = -2, -3$.

$m = 0$ Zeros (0): None.

<u>Unit Step Response:</u> $Y(s) = P(s)U(s) = \left(\frac{1}{s^2+5s+6}\right)\frac{1}{s} = \frac{1}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}.$

$U(s) = \frac{1}{s}.$

A: $*s$: $\frac{1}{(s+2)(s+3)} = A + \frac{Bs}{s+2} + \frac{Cs}{s+3} \xrightarrow{s=0} \frac{1}{6} = A.$

B: $*(s+2)$: $\frac{1}{s(s+3)} = \frac{A(s+2)}{s} + B + \frac{C(s+2)}{s+3} \xrightarrow{s=-2} -\frac{1}{2} = B.$

C: $*(s+3)$: $\frac{1}{s(s+2)} = \frac{A(s+3)}{s} + \frac{B(s+3)}{s+2} + C \xrightarrow{s=-3} \frac{1}{3} = C.$

$\Rightarrow Y(s) = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)} \Longrightarrow \boxed{y(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}.}$ UNIT STEP RESPONSE

Poles and zeros can be complex-conjugate pairs.

<u>NOTE:</u>

1). As $t \to \infty$, $y(t) \to \frac{1}{6}$. (STEADY STATE VALUE)

2). Exponents of the exponential terms: $-2, -3$. In general, the $\underbrace{\text{real part of the poles}}_{\text{(poles)}}$ would appear as the exponents.

3). The magnitude of $y(t)$ is bounded for all time.

2). $\ddot{y}(t) + y(t) = u(t)$.

Take the laplace transform on b/s:

$$s^2 Y(s) - s y(0) - \dot{y}(0) + Y(s) = U(s).$$

Take all IC's to be zero:

$$\Rightarrow \boxed{\frac{Y(s)}{U(s)} = \frac{1}{s^2+1}} = P(s).$$

$n = 2$ Poles: $s^2+1=0 \Rightarrow s = \pm j$.

$m = 0$ Zeros: None.

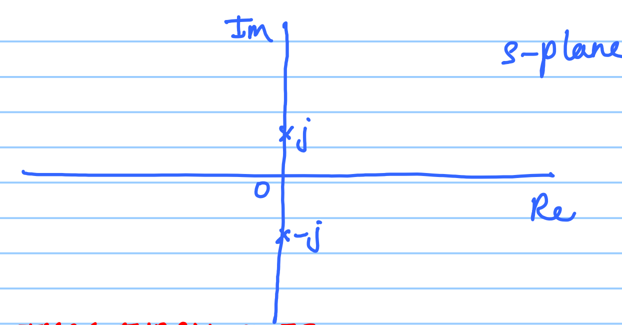
$s = c \pm j,\ c \in \mathbb{R}$.

$(c+j)^2 + 1 = 0$.

$c^2 + 2cj + j^2 + 1 = 0$

$c^2 + 2cj = 0$.

$\Rightarrow c = 0$.



PURELY IMAGINARY POLES.

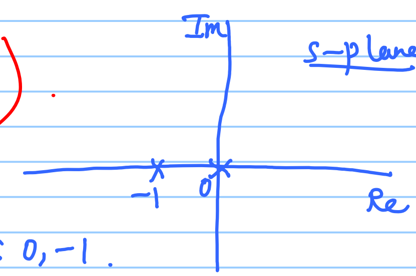
Unit Step Response: $U(s) = \frac{1}{s}$. $Y(s) = P(s)\,U(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$

$$\Rightarrow \boxed{y(t) = 1 - \cos(t).}$$ UNIT STEP RESPONSE.

$y(t) = 1 - \left(\frac{e^{jt} + e^{-jt}}{2}\right)$.

NOTE:

1). As $t \to \infty$, $|y(t)|$ remains bounded.



3). $\ddot{y}(t) + \dot{y}(t) = u(t)$. Plant transfer fn.: $P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+s} = \frac{1}{s(s+1)}$.

$n=2$. Poles: $0, -1$.

$m=0$. Zeros: None.

Unit Step Response: $U(s) = \frac{1}{s}$. $Y(s) = P(s)\,U(s) = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

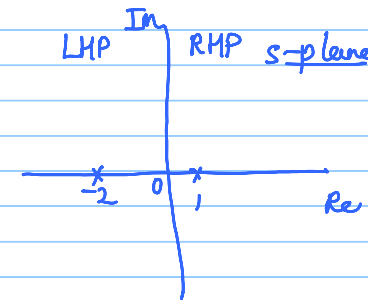
$$\Rightarrow \boxed{y(t) = -1 + t + e^{-t}.}$$ UNIT STEP RESPONSE

NOTE:
1). As $t \to \infty$, $|y(t)| \to \infty$.

4). $\ddot{y}(t) + \dot{y}(t) - 2y(t) = u(t)$. $P(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2+s-2} = \dfrac{1}{(s+2)(s-1)}$.

$n = 2$. Poles: $-2, 1$

$m = 0$. Zeros: None.



Unit Step Response: $U(s) = \dfrac{1}{s}$.

$$\Rightarrow Y(s) = P(s)\,U(s) = \frac{1}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1} = \frac{-1}{2s} + \frac{1}{6(s+2)} + \frac{1}{3(s-1)}.$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{2} + \frac{1}{6}e^{-2t} + \frac{1}{3}e^{t}.}$$ UNIT STEP RESPONSE.

NOTE: As $t \to \infty$, $|y(t)| \to \infty$.

5). $\ddot{y}(t) + y(t) = u(t)$. Find $y(t)$ when $u(t) = \cos(t)$.

6). $\ddot{y}(t) + \dot{y}(t) = u(t)$. Find $y(t)$ when $u(t) = \cos(t)$.

5). $P(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+1}$. $U(s)=\frac{s}{s^2+1}$, $\Rightarrow Y(s)=P(s)\,U(s)=\frac{s}{(s^2+1)^2}$. $\mathcal{L}[t f(t)]=-\frac{dF(s)}{ds}$.

$f(t)=\sin(t)$, $F(s)=\frac{1}{s^2+1}$. $-\frac{dF(s)}{ds}=-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{2s}{(s^2+1)^2} \Rightarrow \boxed{y(t)=\frac{1}{2}t\sin(t).}$

As $t\to\infty$, $|y(t)|\to\infty$.

6). $P(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+s}=\frac{1}{s(s+1)}$. $U(s)=\frac{s}{s^2+1}$ $\Rightarrow Y(s)=P(s)\,U(s)=\frac{s}{s(s+1)(s^2+1)}=\frac{1}{(s+1)(s^2+1)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}$

$Y(s)=\frac{1}{2(s+1)}-\frac{1}{2}\left(\frac{s}{s^2+1}\right)+\frac{1}{2}\left(\frac{1}{s^2+1}\right) \Rightarrow \boxed{y(t)=\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{1}{2}\sin t.}$

As $t\to\infty$, $|y(t)|$ is bounded.

For non-repeating poles on the imaginary axis:

System	Poles	Input	Output
S_1: $\ddot{y}(t)+y(t)=u(t)$	$\pm j$	1	Bounded as $t\to\infty$
		$\cos(t)$	Unbounded as $t\to\infty$
S_2: $\ddot{y}(t)+\dot{y}(t)=u(t)$	$0, -1$	1	Unbounded as $t\to\infty$
		$\cos(t)$	Bounded as $t\to\infty$.

UNSTABLE

MARGINALLY STABLE

CRITICALLY STABLE

$P(s) = \dfrac{1}{s^2(s+1)} \rightarrow$ Poles: $0, 0, -1$ $\rightarrow$ UNSTABLE.

$P(s) = \dfrac{1}{(s^2+1)^2} \rightarrow$ Poles: $+j, +j, -j, -j$ $\rightarrow$ UNSTABLE.

<u>HW:</u> Try $u(t) = 1$, $\cos(t)$.

1). For BIBO stability (asymptotic stability), <u>ALL POLES</u> of the plant transfer fn. must lie in the LHP (i.e., have negative real parts).

2). If there exist even 1 pole of the plant transfer fn. in the RHP (i.e., has a +ve real part), then the plant/system is <u>NOT</u> BIBO stable, i.e., it is UNSTABLE.

3) If there are repeating poles of the plant transfer fn. on the imaginary axis ($j\omega$ axis) with all remaining poles in the LHP, then the system is <u>NOT</u> BIBO stable.

4). If there are non-repeating poles of the plant transfer fn. on the $j\omega$ axis with all remaining poles in the LHP, then the system is
- UNSTABLE.
- CRITICALLY STABLE.

BIBO Stability: $Y(s) = P(s)U(s)$. $\Rightarrow y(t) = \int_0^t p(t-\tau)\, u(\tau)\, d\tau$.

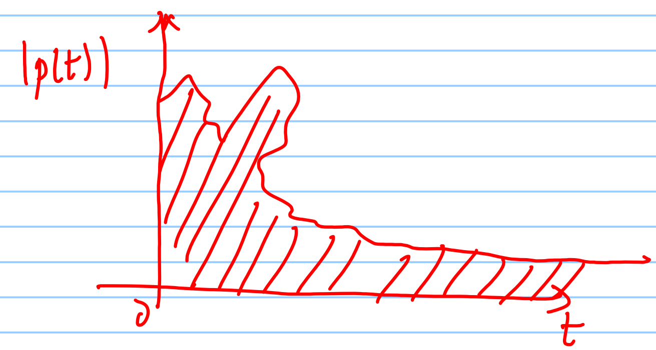
Let $|u(t)| \leq M < \infty \;\; \forall\, t \geq 0$. Here, M is a finite positive real number.

$\Rightarrow \; |y(t)| = \left| \int_0^t p(t-\tau) u(\tau) d\tau \right| \leq \int_0^t |p(t-\tau)|\, |u(\tau)|\, d\tau \leq M \int_0^t |p(t-\tau)|\, d\tau$.

Q: When would $|y(t)|$ be bounded?

Q: When is $\int_0^t |p(t-\tau)| d\tau$ bounded?

A: Only when $\lim_{t\to\infty} |p(t)| = 0$.



$\Rightarrow |y(t)|$ is bounded $\forall\, t$ if $\lim_{t\to\infty} |p(t)| = 0$.

$P(s) = \dfrac{n(s)}{d(s)}$. $\mathcal{L}[p(t)] = P(s)$. The order of the system is $n \Rightarrow n$ poles.

Let there be k distinct poles of $P(s)$. Let the multiplicity of the pole, s_i, $i = 1, \dots, k$, be μ_i.

$$\Rightarrow \sum_{i=1}^{k} \mu_i = n.$$

$$P(s) = \frac{n(s)}{(s+s_1)^{\mu_1}(s+s_2)^{\mu_2}\cdots(s+s_k)^{\mu_k}} = \sum_{l=1}^{k}\sum_{m=0}^{\mu_l-1}\frac{c_{lm}}{(s+s_l)^{m+1}}$$

→ residues

$p(t) = \mathcal{L}^{-1}[P(s)]$.

$$\Rightarrow p(t) = \sum_{l=1}^{k}\sum_{m=0}^{\mu_l-1} c_{lm}\, t^m e^{s_l t}.$$

$s_l = \sigma_l + j\omega_l$.

$e^{s_l t} = e^{(\sigma_l + j\omega_l)t} = e^{\sigma_l t} e^{j\omega_l t}$

$\Rightarrow |e^{s_l t}| = |e^{\sigma_l t}| = e^{\sigma_l t}$

$$\Rightarrow |p(t)| = \left|\sum_{l=1}^{k}\sum_{m=0}^{\mu_l-1} c_{lm}\, t^m e^{s_l t}\right| \leq \sum_{l=1}^{k}\sum_{m=0}^{\mu_l-1} |c_{lm}|\,|t^m|\,|e^{s_l t}|.$$

$$\Rightarrow |p(t)| \leq \sum_{l=1}^{k}\sum_{m=0}^{\mu_l-1} |c_{lm}|\, t^m e^{\sigma_l t}.$$

Now, if $\sigma_l < 0 \;\forall\; l = 1, 2, \ldots, k \Rightarrow \lim_{t\to\infty} |p(t)| = 0$.

$\Rightarrow$ If ALL Poles of the plant transfer function lie in the LHP (i.e., have -ve real parts), then the system is BIBO stable (asymptotically stable).

HW: Q: Is this condition (all poles in LHP) necessary for BIBO stability?

$$P(s) = \frac{1}{(s+1)(s+2)(s+3)^2}$$

$n = 4$, Poles: $-1, -2, -3, -3$.

$k = 3$. $\mu_1 = 1$
$\mu_2 = 1$
$\mu_3 = 2$

$$P(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}.$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \sum_{m=0}^{1}\frac{C_{3m}}{(s+3)^{m+1}}.$$

$p(t) = Ae^{-t} + Be^{-2t} + Ce^{-3t} + Dte^{-3t}$

$|e^{s_l t}| = |e^{\sigma_l t} e^{j\omega_l t}| = |e^{\sigma_l t}|\,|e^{j\omega_l t}|$

$e^{j\omega_l t} = \cos(\omega_l t) + j\sin(\omega_l t)$.

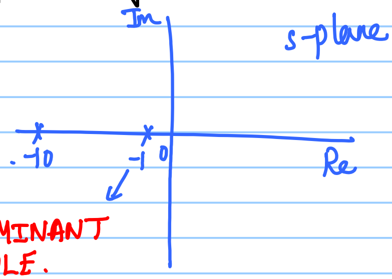
30/1/2018. **Effect of zeros:**

Find the unit step response of these 2 systems.

1). $P(s) = \dfrac{1}{(s+1)(s+10)}$.

$n = 2$. Poles: $-1, -10$.

$m = 0$. Zeros: None.



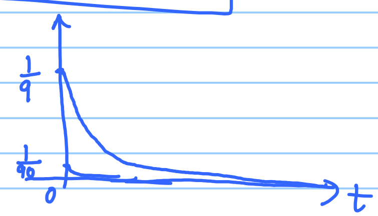
$U(s) = \dfrac{1}{s}$.

$$Y(s) = P(s)\,U(s) = \frac{1}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+10}$$

$$= \frac{1}{10s} - \frac{1}{9(s+1)} + \frac{1}{90(s+10)}.$$

$$\Rightarrow \boxed{y(t) = \frac{1}{10} - \frac{1}{9}e^{-t} + \frac{1}{90}e^{-10t}}.$$

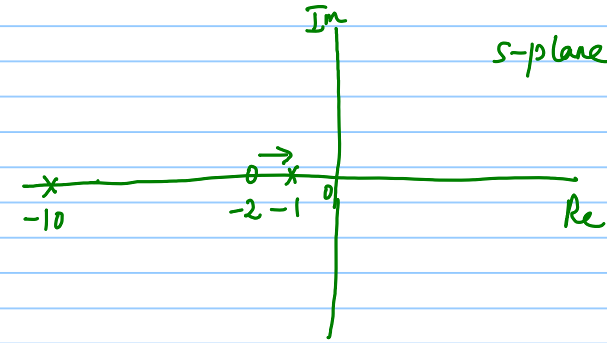
Magnitude: $\frac{1}{9}e^{-t}$ vs $\frac{1}{90}e^{-10t}$



2). $P(s) = \dfrac{s+2}{(s+1)(s+10)}$.

$n = 2$. Poles: $-1, -10$.

$m = 1$. Zeros: -2.



$U(s) = \dfrac{1}{s}$.

$$Y(s) = P(s)\,U(s) = \frac{(s+2)}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+10}$$

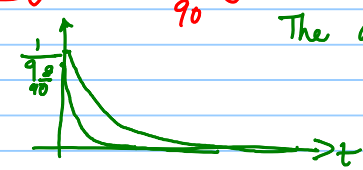
$$= \frac{1}{5s} - \frac{1}{9(s+1)} - \frac{8}{90(s+10)}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{5} - \frac{1}{9}e^{-t} - \frac{8}{90}e^{-10t}}.$$

HW: $P(s) = \begin{cases} \dfrac{s+1.5}{(s+1)(s+10)} \\ \dfrac{s+1.1}{(s+1)(s+10)} \end{cases}$.

Magnitude: $\frac{1}{9}e^{-t}$ vs $\frac{8}{90}e^{-10t}$

The dominance of the pole at -1 is reduced

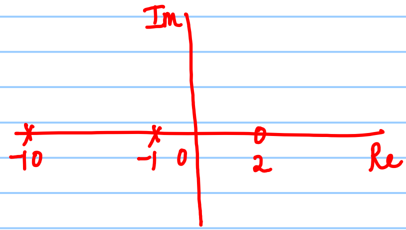


If we shift the zero to -1, $P(s) = \dfrac{\cancel{s+1}}{\cancel{(s+1)}(s+10)} = \left(\dfrac{1}{s+10}\right)$. POLE - ZERO CANCELLATION.

3). $P(s) = \dfrac{s-2}{(s+1)(s+10)}$.

$n = 2$. Poles: $-1, -10$.

$m = 1$. Zeros: 2.



$U(s) = \dfrac{1}{s}$.

$$Y(s) = P(s)U(s) = \frac{s-2}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+10}$$

$$= \frac{-2}{10s} + \frac{3}{9(s+1)} - \frac{12}{90(s+10)}$$

$$\Rightarrow \boxed{y(t) = -\frac{2}{10} + \frac{3}{9}e^{-t} - \frac{12}{90}e^{-10t}}.$$

Zeros in the RHP $\rightarrow$ Non-minimum Phase Zeros.

Example: System with time delay.

$\dot{y}(t) + y(t) = u(t - T_d)$, $T_d \rightarrow$ time delay.

Take Laplace transform on b/s:

$$sY(s) - y(0) + Y(s) = U(s)\,e^{-T_d s}.$$

Take IC to be zero:

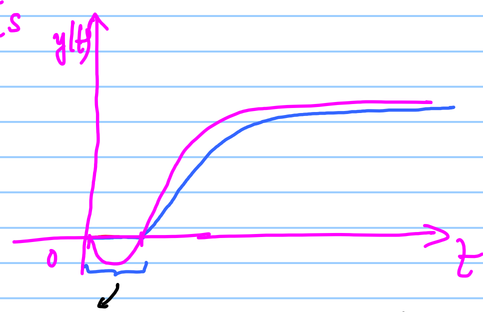
$$P(s) = \frac{Y(s)}{U(s)} = \frac{e^{-T_d s}}{s+1}.$$

Approximation of Time delay (provide T_d is "sufficiently small"):

1). $e^{-T_d s} \approx \dfrac{1}{T_d s + 1}$. $\quad P(s) \approx \dfrac{1}{(s+1)(T_d s+1)}$.

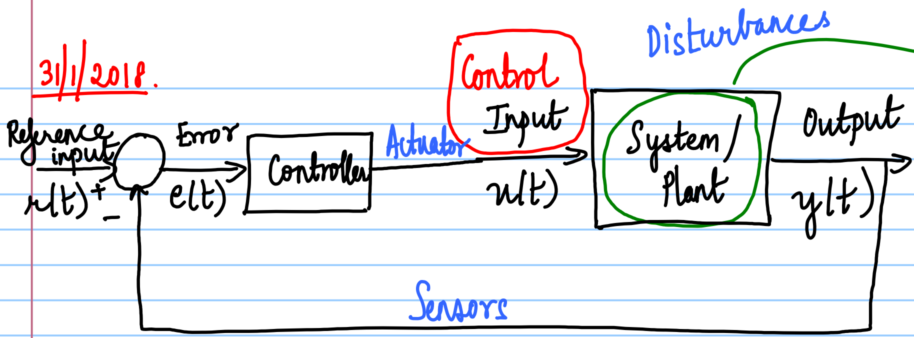
2). $e^{-T_d s} = \dfrac{e^{-T_d s/2}}{e^{T_d s/2}} \approx \dfrac{1 - T_d s/2}{1 + T_d s/2} = \dfrac{2 - T_d s}{2 + T_d s}$ $\longrightarrow$ PADE's 1st order approximation.

$$\Rightarrow P(s) \approx \frac{2 - T_d s}{(s+1)(2+T_d s)}.$$

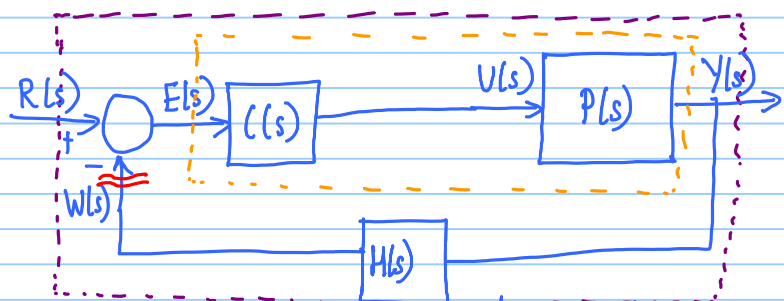


Approximation of Time delay.

31/1/2018.



Closed Loop Control System with Negative Feedback



$H(s) \to$ feedback path transfer function

$H(s) = 1 \to$ unity feedback

$H(s) \neq 1 \to$ non-unity feedback.

$\to$ SISO LTI Causal Dynamic Systems

$\downarrow$

Lumped Parameter Dynamic Deterministic Continuous Time Mathematical Models

(Linear ODEs with constant coefficients)

$\downarrow$ Laplace Transform

Transfer Function

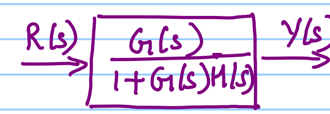
Poles $\to$ stability.

Zeros $\to$ affect dynamic response.

$$Y(s) = P(s)\,U(s) = P(s)\,C(s)\,E(s) = C(s)\,P(s)\,[R(s) - W(s)]$$

$$\Rightarrow Y(s) = C(s)\,P(s)\,R(s) - C(s)\,P(s)\,H(s)\,Y(s).$$

Usually, $G(s) := C(s)\,P(s)$.



$$\Rightarrow Y(s) = G(s)\,R(s) - G(s)\,H(s)\,Y(s).$$

$$\Rightarrow [1 + G(s)H(s)]\,Y(s) = G(s)\,R(s).$$

$$\Rightarrow \boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}.}$$

$\to$ Closed loop transfer function.
[c.l.t.f.]

The poles of the c.l.t.f. are called "CLOSED LOOP POLES".
The zeros " " " " " "CLOSED LOOP ZEROS".

$1 + G(s)H(s) = 0 \Rightarrow$ CLOSED LOOP CHARACTERISTIC EQUATION.

The roots of this equation are the closed loop poles.

The polynomial '$1+G(s)H(s)$' is called as the closed loop characteristic polynomial.

It can be observed that the c.l.t.f. of a positive feedback system is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}.$$

$$W(s) = H(s)\,Y(s) = H(s)\,P(s)\,U(s) = H(s)\underbrace{P(s)\,C(s)}_{G(s)}E(s) = G(s)\,H(s)\,E(s).$$

$$\Rightarrow \boxed{\frac{W(s)}{E(s)} = G(s)H(s).}$$ OPEN LOOP TRANSFER FUNCTION. (o.l.t.f.)

The poles of the o.l.t.f. are called as "OPEN LOOP POLES".
The zeros of the o.l.t.f. are called as "OPEN LOOP ZEROS".

MIMO: → transfer function matrix

$\underset{\sim}{Y}(s) = \underset{\sim}{P}(s)\,\underset{\sim}{U}(s)$

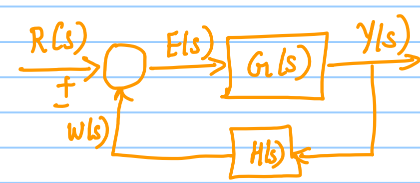
$\underset{\sim}{U}(s) = \underset{\sim}{C}(s)\,\underset{\sim}{E}(s)$

$\underset{\sim}{Y}(s) = \underset{\sim}{P}(s)\,\underset{\sim}{C}(s)\,\underset{\sim}{E}(s)$

$\neq \underset{\sim}{C}(s)\,\underset{\sim}{P}(s)\,\underset{\sim}{E}(s)$

$Y(s) = P(s)\,U(s) = P(s)\,C(s)\,E(s)$

$\Rightarrow Y(s) = G(s)\,E(s)$



Control Design

STABILITY ········ ---> PERFORMANCE

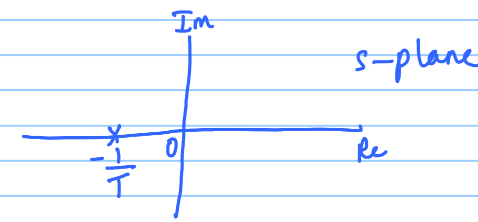
PERFORMANCE SPECIFICATION: Pre-requisite → The system is stable.

Typically, parameters that are used to specify performance are extracted from the unit step response of first order and second order systems.

FIRST ORDER SYSTEMS:

The governing equation of a typical first order system is

$$T\frac{dy(t)}{dt} + y(t) = u(t), \quad T \in \mathbb{R}.$$



Take the Laplace transform on b/s,

$$T[sY(s) - y(0)] + Y(s) = U(s).$$

Assume zero IC ⇒ $\boxed{\frac{Y(s)}{U(s)} = \frac{1}{Ts+1} = P(s).}$

$Ts+1=0.$

$n = 1$, Poles: $-1/T$. ⇒ For stability, $T > 0$.
$m = 0$, Zeros: None.

Q: Calculate the unit step response $\left(U(s)=\frac{1}{s}\right)$, unit ramp response $\left(U(s)=\frac{1}{s^2}\right)$ and the unit impulse response $\left(U(s)=1\right)$ of this system.

5/2/2018. First Order Systems:

Control Design

Stability ----> Performance

Recall that the governing equation is

1 Parameter Model

$$\boxed{T\frac{dy(t)}{dt} + y(t) = u(t).}$$

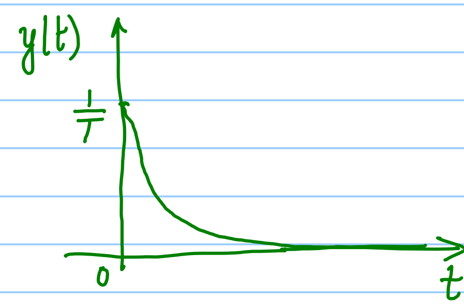
$T \in \mathbb{R}$.

Transfer function, $P(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ts+1}$.

Pole: $-\frac{1}{T}$. $\Rightarrow$ System is stable if $T > 0$

1). Unit impulse response: $u(t) = \delta(t)$, $U(s) = 1$.

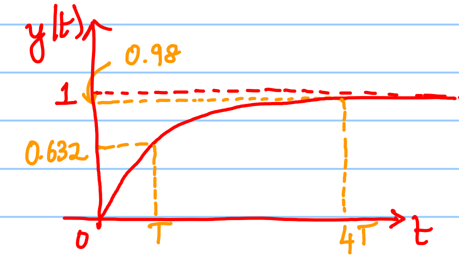
$$\Rightarrow Y(s) = P(s)\,U(s) = \frac{1}{Ts+1} = \frac{1}{T\left(s+\frac{1}{T}\right)} \Rightarrow \boxed{y(t) = \frac{1}{T}e^{-t/T}.}$$



2). Unit step response: $u(t) = 1$, $U(s) = \frac{1}{s}$.

$$\Rightarrow Y(s) = P(s)\,U(s) = \frac{1}{s(Ts+1)} = \frac{A}{s} + \frac{B}{Ts+1} = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}.$$

$$\Rightarrow \boxed{y(t) = 1 - e^{-t/T}.}$$



±2%
±5%

At $t=T$, $y(t=T) = 1-e^{-1} = 0.632$ $\left(63.2\% \text{ of } \underbrace{1}_{\text{steady state value}}\right)$.

$T \Rightarrow$ TIME CONSTANT $\rightarrow$ Time at which the output magnitude is 63.2% of its final steady state value.

At $t=4T$, $y(t=4T) = 1-e^{-4} = 0.982$ (98% of 1).

↳ SETTLING TIME $\rightarrow$ Time taken for the output magnitude to settle within 2% of its final value

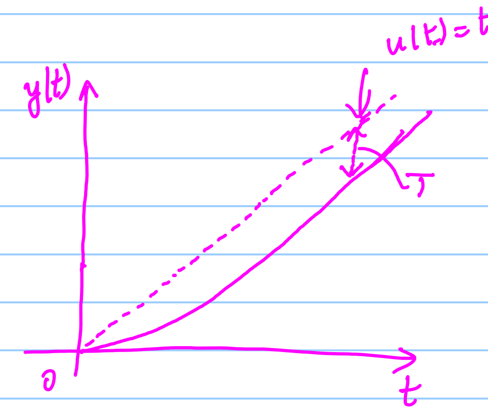
3). Unit ramp response: $u(t) = t$, $U(s) = \frac{1}{s^2}$.

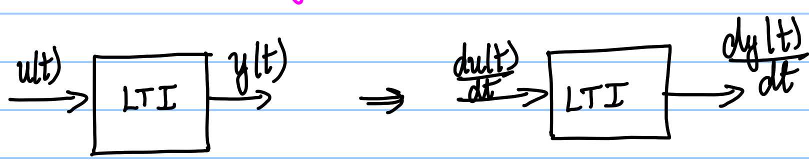
$$Y(s) = P(s)\,U(s) = \frac{1}{s^2(Ts+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{Ts+1} = -\frac{T}{s} + \frac{1}{s^2} + \frac{T^2}{Ts+1}$$

$$\frac{T}{s+\frac{1}{T}}$$

$$\Rightarrow \boxed{y(t) = -T + t + Te^{-t/T} = t - T\left(1-e^{-t/T}\right).}$$

As $t \to \infty$, $y(t) \to t - T$



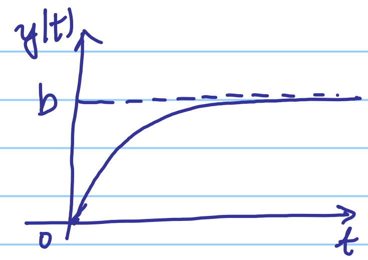


Other Variants:

1). $T\frac{dy(t)}{dt} + y(t) = b\,u(t)$, $b \in \mathbb{R}$, $b \neq 1$.

2 Parameter model

$$\Rightarrow P(s) = \frac{Y(s)}{U(s)} = \frac{b}{Ts+1}. \qquad \begin{matrix} n = 1 \\ m = 0 \end{matrix} \qquad \text{Pole: } -1/T.$$



Unit step response: $U(s) = \frac{1}{s}$. $\Rightarrow Y(s) = P(s)\,U(s) = \frac{b}{s(Ts+1)} \Rightarrow \boxed{y(t) = b\left(1 - e^{-t/T}\right)}$.

Steady state value of unit step response $= \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\frac{b}{\cancel{s}(Ts+1)} = b.$ → STEADY STATE GAIN.

2). $T\frac{dy(t)}{dt} + y(t) = b\,u(t - T_d)$, $b \in \mathbb{R}$, $b \neq 1$, $T_d > 0$, $T_d \in \mathbb{R}$

↳ Time delay.

3 PARAMETER MODEL

$$\Rightarrow P(s) = \frac{Y(s)}{U(s)} = \frac{b\,e^{-T_d s}}{Ts+1} \approx \frac{b(2 - T_d s)}{(Ts+1)(2+T_d s)}$$

↑ PADE's 1st order approximation.

SECOND ORDER SYSTEMS:

The governing equation of 2^{nd} order systems is typically taken as

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_n \frac{dy(t)}{dt} + \omega_n^2\, y(t) = \boxed{\omega_n^2\, u(t)}.$$

$n = 2$
$m = 0$

$\xi \Rightarrow$ Damping Ratio.
$\omega_n \Rightarrow$ Natural Frequency.

For stability, $\xi > 0$, $\omega_n > 0$.

Take the Laplace transform on b/s:

$$s^2 Y(s) - s\,y(0) - \dot{y}(0) + 2\xi\omega_n[sY(s) - y(0)] + \omega_n^2 Y(s) = \omega_n^2 U(s).$$

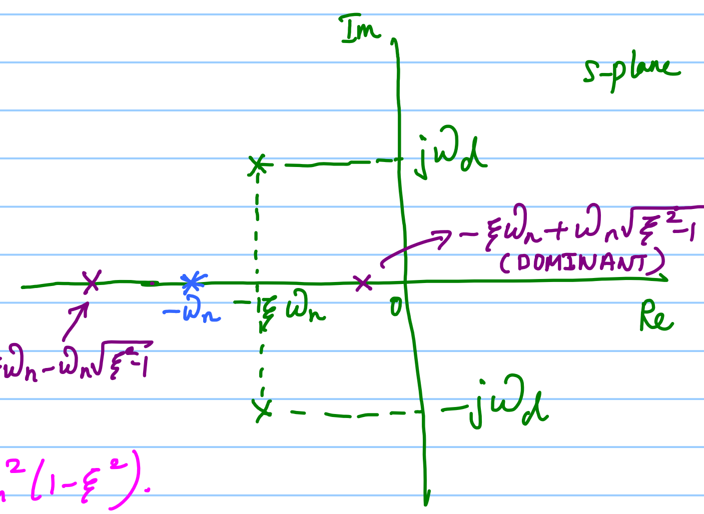
Apply zero ICs:

$$[s^2 + 2\xi\omega_n s + \omega_n^2]\, Y(s) = \omega_n^2\, U(s).$$

$$\Rightarrow \boxed{P(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.}$$

$s^2 + 2\xi\omega_n s + \omega_n^2 = 0.$

Poles: $\underline{s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$.



1). $\underline{0 < \xi < 1}$: Poles: $-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} = -\xi\omega_n \pm j\underbrace{\omega_n\sqrt{1-\xi^2}}$
(UNDER DAMPED)

$\omega_d := \omega_n\sqrt{1-\xi^2} \Rightarrow \omega_d^2 = \omega_n^2(1-\xi^2).$

DAMPED NATURAL FREQUENCY.

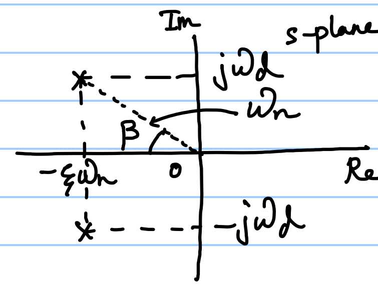
2). $\xi = 1$: Poles: $-\omega_n, -\omega_n$. $P(s) = \dfrac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} = \dfrac{\omega_n^2}{(s+\omega_n)^2}$

(CRITICALLY DAMPED)

3). $\xi > 1$: Poles: $-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$.

(OVER DAMPED)

Note that $\boxed{\cos\beta = \xi.}$



Under Damped System: $P(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$. Poles: $-\xi\omega_n \pm j\omega_d$.

$(s+\xi\omega_n)^2 - \xi^2\omega_n^2 + \omega_n^2$
$= (s+\xi\omega_n)^2 + \omega_d^2$

Let us calculate its unit step response. $U(s) = \frac{1}{s}$.

$$Y(s) = P(s)U(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{1}{s} - \frac{(s+2\xi\omega_n)}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} = \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$\dfrac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}$

$$\Rightarrow \boxed{y(t) = 1 - e^{-\xi\omega_n t}\cos(\omega_d t) - \frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_d t).}$$

The performance specifications of 2^{nd} order systems are motivated/derived from the above expression.

Mass-Spring-Damper System:

$$P(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k} = \frac{1/m}{s^2+\frac{c}{m}s+\frac{k}{m}}.$$

$\Rightarrow \omega_n^2 = \frac{k}{m}$, $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$.

$\Rightarrow 2\xi\omega_n = \frac{c}{m} \Rightarrow \xi = \frac{c}{2m\omega_n}$

$\Rightarrow \boxed{\xi = \frac{c}{2\sqrt{mk}}.}$

HW: Consider $u(t)=0$, $y(0)\neq 0$, $\dot{y}(0)\neq 0$, $\xi = 0$.

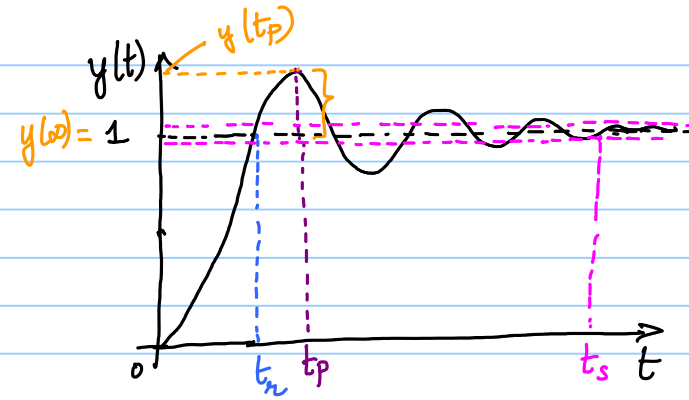
$\ddot{y}(t) + \omega_n^2 y(t) = 0$.

Apply the Laplace transform & calculate the free response.

$\mathcal{L}[e^{-at}\cos(bt)] = \dfrac{s+a}{(s+a)^2+b^2}$.

$\mathcal{L}[e^{-at}\sin(bt)] = \dfrac{b}{(s+a)^2+b^2}$.

1). Rise Time (t_r): Time taken for the system output/response to reach 100% of its final value for the first time. [0% – 100%].



$$y(t=t_r)=1 = 1 - e^{-\xi\omega_n t_r}\left[\cos(\omega_d t_r) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t_r)\right].$$

$$\Rightarrow e^{-\xi\omega_n t_r}\left[\cos(\omega_d t_r) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t_r)\right] = 0.$$

$$\Rightarrow \cos(\omega_d t_r) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t_r) = 0.$$

$$\Rightarrow \tan(\omega_d t_r) = -\frac{\sqrt{1-\xi^2}}{\xi} \Rightarrow \boxed{t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\sqrt{1-\xi^2}}{\xi}\right) = \frac{\pi-\beta}{\omega_d}}.$$

$\tan\beta = \frac{\omega_d}{\xi\omega_n} = \frac{\sqrt{1-\xi^2}}{\xi}$

$\cos\beta = \xi$.

As $\xi\uparrow$, $t_r\uparrow$. As $\xi\downarrow$, $t_r\downarrow$.

2). Peak Time (t_p): $\frac{dy(t)}{dt} = 0.$

$$\Rightarrow \cancel{\xi\omega_n e^{-\xi\omega_n t}\cos(\omega_d t)} + e^{-\xi\omega_n t}\omega_d\sin(\omega_d t) + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_d t) - \cancel{\frac{\xi\omega_d}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\cos(\omega_d t)} = 0.$$

$$\Rightarrow e^{-\xi\omega_n t}\left[\omega_d + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\right]\sin(\omega_d t) = 0. \Rightarrow \sin(\omega_d t) = 0 \Rightarrow t = \frac{n\pi}{\omega_d},\ n = 1,2,3,\ldots$$

$$\frac{d^2y(t)}{dt^2} = \left[\omega_d + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\right]\left[-\xi\omega_n e^{-\xi\omega_n t}\sin(\omega_d t) + e^{-\xi\omega_n t}\omega_d\cos(\omega_d t)\right].$$

At $t = \frac{n\pi}{\omega_d}$

$$\frac{d^2y(t)}{dt^2} = \left[\omega_d + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\right]\left[e^{-\frac{\xi\omega_n n\pi}{\omega_d}}\omega_d\cos(n\pi)\right]$$

For $n=1$, $\frac{d^2y(t)}{dt^2} < 0$. $\Rightarrow$ $\boxed{t_p = \frac{\pi}{\omega_d}}$.

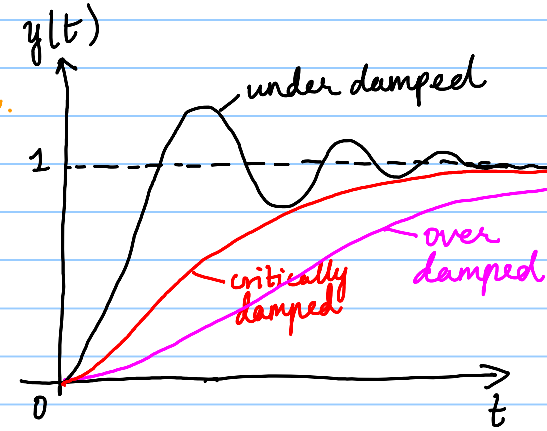
3). Maximum Peak Overshoot: $M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} * 100\%$

$\boxed{\xi = 0.4 \text{ to } 0.8.}$

$y(t_p) = 1 + e^{-\frac{\xi\omega_n\pi}{\omega_d}} = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$, $y(\infty) = 1$

$\Rightarrow \boxed{M_p = \frac{\left(1+e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}\right) - 1}{1} * 100\% = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} * 100\%.}$

As $\xi \uparrow$, $M_p \downarrow$.



UNIT STEP RESPONSE

4). Settling Time:

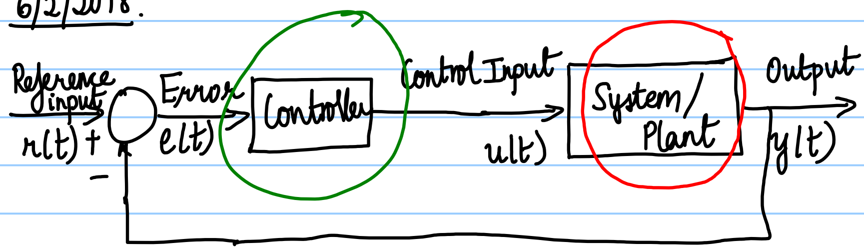
$$t_s = \begin{cases} \frac{4}{\xi\omega_n}, & \pm 2\% \text{ band.} \\ \frac{3}{\xi\omega_n}, & \pm 5\% \text{ band.} \end{cases}$$

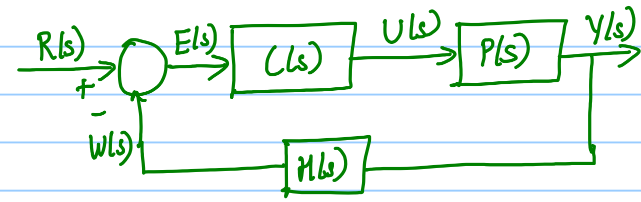
HW:

1). Find the unit step response of critically damped and overdamped 2nd order systems.

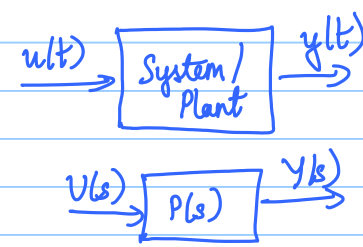
2). Find the unit impulse response of underdamped, critically damped and overdamped 2nd order systems.

6/2/2018.





- We have looked at the "System/Plant" block till now.
- Learnt about transfer function representation.



- Response
  - Free response → due to non-zero initial conditions.
  - Forced response → due to inputs provided.

- Poles → affect the "nature" of system response (bounded, unbounded, oscillatory, etc.).
- Zeros → affect the "shape" of " " (dominance of poles, residues, etc.).
- Higher Order System : $P(s) = \dfrac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\cdots(s+p_n)}$. → The poles located closer to the $j\omega$-axis dominate the system response. But, the presence of zeros would influence their "dominance".

$P(s) \leftrightarrow G_p(s)$
$C(s) \leftrightarrow G_c(s)$

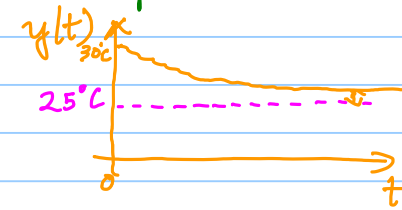
Controller:

$e(t) \rightarrow \boxed{\text{Controller}} \rightarrow u(t)$ $\qquad$ $E(s) \rightarrow \boxed{C(s)} \rightarrow U(s)$ $\qquad$ $C(s) \rightarrow$ Controller Transfer Function.

$$\boxed{U(s) = C(s)\,E(s).}$$

1). **"On-Off" Controller ("Bang-Bang"):** $\quad u(t) = \begin{cases} u_1, & \text{if } e(t) < e_1, \\ u_2, & \text{otherwise}. \end{cases}$

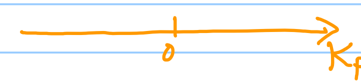
2). **Proportional (P) Controller:** $\quad \boxed{u(t) = K_p\, e(t)}$, $\quad K_p \rightarrow$ Proportional controller Parameter. $K_p \in \mathbb{R}$.
(Proportional Gain).



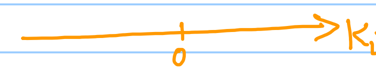
$\Rightarrow \quad U(s) = K_p\, E(s) \quad \Rightarrow \quad \boxed{C(s) = K_p.}$

→ Simple
→ leads to non-zero steady state tracking error in certain classes of systems.

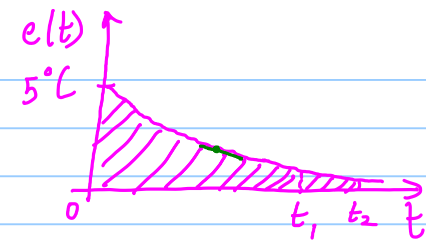
3). **Integral (I) Controller:** $\quad \boxed{u(t) = K_i \int_0^t e(\tau)\,d\tau,}$ $\quad K_i \in \mathbb{R}$, $\quad K_i \rightarrow$ Integral Gain.



$\Rightarrow \quad U(s) = \dfrac{K_i}{s} E(s). \quad \Rightarrow \quad \boxed{C(s) = \dfrac{K_i}{s}}.$ $\qquad$ open loop pole: 0.

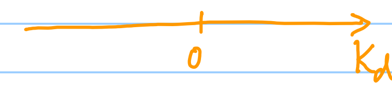


→ Can lead to zero steady state error.
→ Introduces an open loop pole at 0.
→ Actuator Saturation.



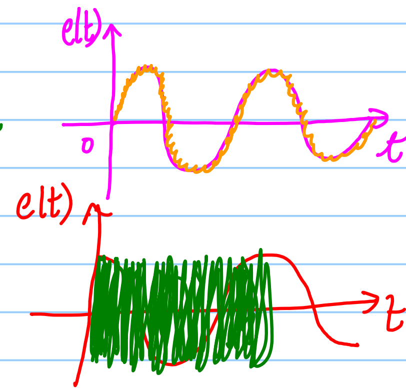
4). **Derivative (D) Controller:** $u(t) = K_d \frac{de(t)}{dt}$, $K_d \to$ Derivative Gain, $K_d \in \mathbb{R}$.

$\Rightarrow U(s) = K_d\, s\, E(s) \Rightarrow \boxed{C(s) = K_d s.}$ → open loop zero: 0



→ Anticipatory.
→ $e(t) = \sin(t) + 0.01 \sin(100t)$. → Low magnitude high frequency noise.

$\frac{de(t)}{dt} = \cos(t) + \cos(100t)$.



$\frac{K_d s}{Ts+1}$

O.l.t.f. $= G(s)H(s) = C(s)P(s)H(s)$.

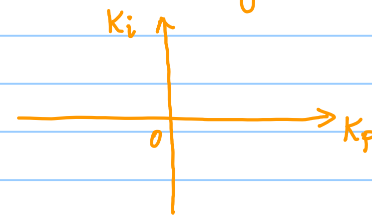
7/2/2018

5). **Proportional Integral (PI) Controller:** $u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau$.

$\Rightarrow \boxed{C(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}.}$

→ open loop pole: 0
→ open loop zero: $-K_i/K_p$.

Controller Parameter Space → used to indicate the feasible region(s) of the controller gain(s) to achieve closed loop stability and performance.



6). <u>Proportional Derivative (PD) Controller:</u> $u(t) = K_p e(t) + K_d \frac{de(t)}{dt}$.

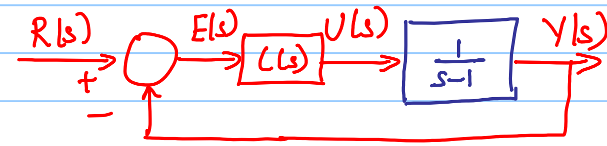
$\Rightarrow \boxed{C(s) = K_p + K_d s.}$ → open loop zero: $-\frac{K_p}{K_d}$.

7). <u>Proportional Integral Derivative (PID) Controller:</u> $u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \frac{de(t)}{dt}$.

$\Rightarrow \boxed{C(s) = K_p + \frac{K_i}{s} + K_d s.}$ → 2 open zeros
open loop pole: 0.

<u>Examples:</u> Let us consider the design of a unity negative feedback closed loop control system.

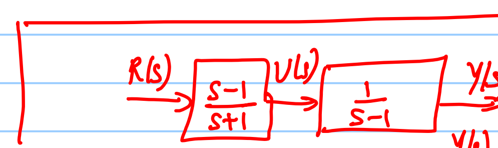
1). $P(s) = \frac{1}{(s-1)}$. 1^{st} order plant, unstable since $s=1$ is its pole.



Given, $H(s) = 1$.

a). Let $C(s) = K_p$.

$\Rightarrow G(s) = C(s)\,P(s) = \frac{K_p}{(s-1)}$.

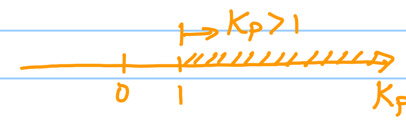


$= \frac{1}{s+1}$

$Y(s)/R(s) = \frac{\cancel{s-1}}{s+1} \times \frac{1}{\cancel{s-1}}$

$\Rightarrow$ c.l.t.f. $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{\frac{K_p}{(s-1)}}{1+\frac{K_p}{(s-1)}} = \dfrac{K_p}{s+(K_p-1)}$.

$\Rightarrow$ Closed Loop Characteristic Equation:

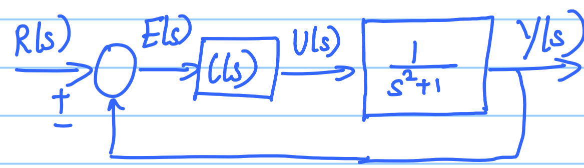


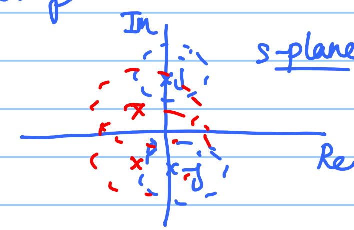
$$s+(K_p-1)=0 \quad \Rightarrow \text{Closed loop pole: } s = 1-K_p.$$

$\Rightarrow$ For closed loop stability, $\boxed{K_p>1.}$

2). $P(s) = \dfrac{1}{s^2+1}$. 2^{nd} order plant. Poles: $\pm j$. BIBO unstable / Marginally (Critically) Stable.

a). $C(s) = K_p$.





$\Rightarrow G(s) = C(s)P(s) = \dfrac{K_p}{s^2+1}$.

c.l.t.f. $= \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{\frac{K_p}{s^2+1}}{1+\frac{K_p}{s^2+1}} = \dfrac{K_p}{s^2+K_p+1}$.

c.l. char. eqn.: $s^2 + K_p + 1 = 0$. $\Rightarrow$ $s = \pm\sqrt{-(K_p+1)}$.

If $K_p > -1$, c.l. poles on the $j\omega$-axis.

If $K_p = -1$, c.l. poles at 0, 0.

If $K_p < -1$, one c.l. pole each on the -ve real axis & +ve real axis.

$\Rightarrow$ Proportional control would not achieve c.l. stability for this plant with unity negative feedback.

b). $C(s) = K_p + \dfrac{K_i}{s} = \dfrac{K_p s + K_i}{s}$.

$$\Rightarrow G(s) = C(s)\,P(s) = \frac{K_p s + K_i}{s(s^2+1)} = \frac{K_p s + K_i}{s^3 + s}.$$

$$\text{c.l. t.f.} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\dfrac{K_p s + K_i}{s^3+s}}{1 + \dfrac{K_p s + K_i}{s^3+s}} = \frac{K_p s + K_i}{s^3 + (K_p+1)s + K_i}.$$

c.l. char. eqn.: $s^3 + (K_p+1)s + K_i = 0$.

Consider a polynomial $d(s) = a_0 s^n + a_1 s^{(n-1)} + \dots + a_{n-1}s + a_n$. All coefficients $a_0, \dots, a_n$ are real numbers.

Def.: A polynomial is said to be 'Hurwitz' if all its roots have negative real parts.

Q: How do we evaluate whether a polynomial $d(s)$ is Hurwitz.

Note Title 2/5/2018

12/2/2018.

Consider a polynomial $d(s) = a_0 s^n + a_1 s^{(n-1)} + \dots + a_{n-1}s + a_n$. All coefficients $a_0, \dots, a_n$ are real numbers.

Def.: A polynomial is said to be 'Hurwitz' if all its roots have negative real parts.

Q: How do we evaluate whether a polynomial $d(s)$ is Hurwitz?

n=1: $d(s) = a_0 s + a_1$. This would be Hurwitz when $a_0 \neq 0$, $a_1 \neq 0$ AND both a_0 and a_1 have the same sign.

$$\boxed{s = -\frac{a_1}{a_0}.}$$

$a_0 = 1 \rightarrow$ monic polynomial

n=2: $d(s) = a_0 s^2 + a_1 s + a_2$. This would be Hurwitz when a_0, a_1 and a_2 are

- non-zero, AND
- of the same sign.

n=3: $d(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$.

roots → MATLAB

roots([1 2 3 1])

$d_1(s) = s^3 + 2s^2 + 3s + 1 \rightarrow$ Roots: all 3 roots are in LHP.

$d_2(s) = s^3 + 2s^2 + s + 3 \rightarrow$ Roots: 1 root in LHP + 2 roots in RHP.

For polynomials of order 3 and more, a NECESSARY condition for them to be 'Hurwitz' is that all coefficients must be non-zero and of the same sign.

$$d_3(s) = s^3 + 2s^2 + 3s.$$
$$= s(s^2+2s+3).$$

For sufficiency, we construct the 'ROUTH's ARRAY'.

ROUTH's ARRAY:

1^{st} COLUMN

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \ \cdots \\
s^{(n-1)} & a_1 & a_3 & a_5 & a_7 \ \cdots \\
s^{(n-2)} & b_1 & b_2 & b_3 \ \cdots & \\
s^{(n-3)} & c_1 & c_2 & c_3 \ \cdots & \\
\vdots & \vdots & & & \\
s^2 & & & & \\
s^1 & & & & \\
s^0 & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}.$$

SUFFICIENT CONDITION: The number of sign changes in the first column of the Routh's array should be zero.

NOTE: The number of sign changes in the first column of the

ROUTH'S CRITERION

Routh's array is equal to the number of roots of $d(s)$ in RHP.

a). $d_1(s) = s^3 + 2s^2 + 3s + 1.$

s^3	1	3
s^2	2	1
s^1	$5/2$	
s^0	1	

No sign changes.
$\Rightarrow$ All 3 roots of $d_1(s)$ have $-$ve real parts.

b). $d_2(s) = s^3 + 2s^2 + s + 3$

s^3	1	1
s^2	2	3
s^1	$-1/2$	
s^0	3	

2 sign changes in the 1st column of the Routh's array.
$\Rightarrow$ 2 roots in RHP + 1 root in LHP.

ROUTH'S STABILITY CRITERION:

Consider a system whose transfer function is $\frac{n(s)}{d(s)}$. This system would be BIBO stable (asymptotically stable) iff:

a). All coefficients of $d(s)$ are non-zero and of the same sign, and

b). The first column of the Routh's array constructed using the coefficients of $d(s)$ has all entries being non-zero and of the same sign.

13/2/2018.

Case I: The first element in any row of the Routh's array is zero with the other remaining elements being non-zero. $\longrightarrow$ Replace the zero entry with a "small" positive real number ε and continue the process.

↳ If there is a sign change from the entry above ε in the 1^{st} column of the Routh's array to the entry below ε, then there are root(s) in the RHP.

↳ If not, then there is a pair of roots on the $j\omega$ axis.

$$d(s) = s^4 + 3s^3 + 3s^2 + 3s + 2.$$

Roots: $-1, -2, \pm j$.

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 2 \\ s^3 & 3 & 3 & \\ s^2 & 2 & 2 & \\ s^1 & 0 = \varepsilon & & \\ s^0 & 2 & & \end{array}$$

$\Rightarrow$ There is no sign change from the entry above ε to the one below ε in the first column of the Routh's array.

$\Rightarrow$ There is a pair of roots on the imaginary axis.

MATLAB:
'poly'
poly([-2 1 1])

$$d(s) = s^3 - 3s + 2$$

Roots: $-2, 1, 1$.

$$\begin{array}{c|cc} s^3 & 1 & -3 \\ s^2 & 0 = \varepsilon & 2 \\ s^1 & \frac{-3\varepsilon - 2}{\varepsilon} & \\ s^0 & 2 & \end{array}$$

$\Rightarrow$ 2 sign changes $\Rightarrow$ 2 roots in RHP.

Case II: What happens when an entire row of the Routh's array is zero? $\Rightarrow$ Radially opposite roots.

$$d(s) = s^7 + 6s^6 + 11s^5 + 6s^4 + 4s^3 + 24s^2 + 44s + 24.$$

Roots: $-1, -2, -3, -1 \pm j, 1 \pm j$.

s^7	1	11	4	44
s^6	6	6	24	24
s^5	10	0	40	
s^4	6	0	24	
s^3	0 : 24	0 : 0		
s^2	$0 = \varepsilon$	24		
s^1	$-576/\varepsilon$			
s^0	24			

① ②

The roots of $P_a(s)$ give us the radially opposite roots.

Auxilliary Polynomial,

$$P_a(s) = 6s^4 + 24$$

$$\frac{dP_a(s)}{ds} = 24s^3$$

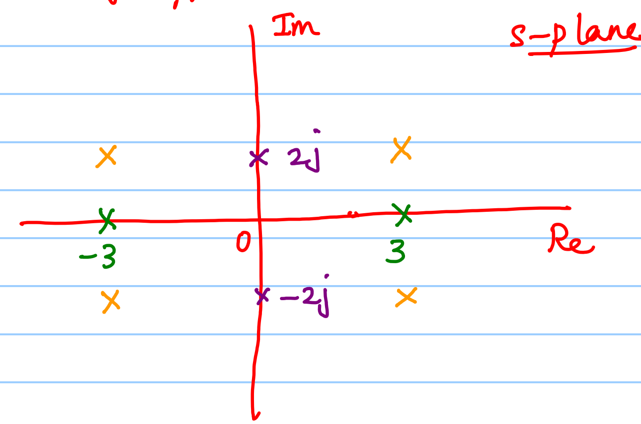
$\Rightarrow$ 2 sign changes $\Rightarrow$ 2 roots in RHP.

$$6s^4 + 24 = 0 \Rightarrow s^4 + 4 = 0.$$

$$\underbrace{(s^2+2s+2)}_{-1\pm j}(s^2-2s+2) = 0.$$

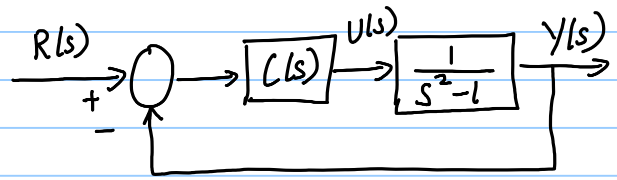
$-1 \pm j$: $\underbrace{}_{\text{LHP}}$ $1 \pm j$: $\underbrace{}_{\text{RHP}}$

Radially opposite roots:

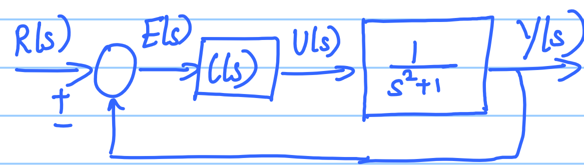


Application of Routh's Stability Criterion:

$$P(s) = \frac{1}{(s^2-1)}.$$



2). $P(s) = \frac{1}{s^2+1}$. 2^{nd} order plant. Poles: $\pm j$. BIBO unstable / Marginally (Critically) Stable.



a). $C(s) = K_p$.

$$\Rightarrow G(s) = C(s)P(s) = \frac{K_p}{s^2+1}.$$

$$\text{c.l.t.f.} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K_p}{s^2+1}}{1+\frac{K_p}{s^2+1}} = \frac{K_p}{s^2+K_p+1}.$$

c.l. char. eqn.: $s^2 + K_p + 1 = 0$. $\Rightarrow$ $s = \pm\sqrt{-(K_p+1)}$.

If $K_p > -1$, c.l. poles on the $j\omega$-axis.

If $K_p = -1$, c.l. poles at 0, 0.

If $K_p < -1$, one c.l. pole each on the $-$ve real axis & $+$ve real axis.

$\Rightarrow$ Proportional Control would not achieve c.l. stability for this plant with unity negative feedback.

b). $C(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}$.

$\Rightarrow$ $G(s) = C(s)\,P(s) = \frac{K_p s + K_i}{s(s^2+1)} = \frac{K_p s + K_i}{s^3 + s}$.

$$\text{c.l. t.f.} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K_p s + K_i}{s^3+s}}{1 + \frac{K_p s + K_i}{s^3+s}} = \frac{K_p s + K_i}{s^3 + (K_p+1)s + K_i}.$$

c.l. char. eqn.: $s^3 + (K_p+1)s + K_i = 0$. $\rightarrow$ The coefficient of s^2 is zero.

$\Rightarrow$ No values of K_p & K_i can stabilize the closed loop system.

c). $C(s) = K_p + K_d s.$

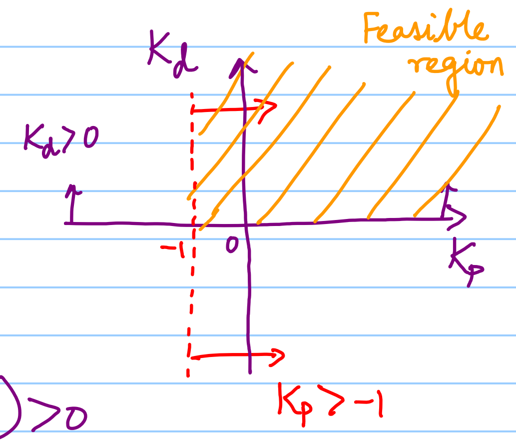
$\Rightarrow G(s) = C(s)\,P(s) = \dfrac{K_d s + K_p}{s^2+1}.$

$$\Rightarrow \text{c.l.t.f.} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K_d s + K_p}{s^2+1}}{1+\dfrac{K_d s + K_p}{s^2+1}} = \frac{K_d s + K_p}{s^2 + K_d s + (K_p+1)}.$$

c.l. char. eqn.: $s^2 + K_d s + (K_p+1) = 0.$

$\Rightarrow$ The c.l. system would be BIBO stable iff $K_d > 0,\ (K_p+1) > 0$

$\Rightarrow \boxed{K_d > 0 \text{ AND } K_p > -1.}$



Note Title 2/19/2018

19/2/2018.

Performance Specifications:

Q: How can we convert performance specifications into desired region(s) of closed loop poles?

It is desired that the "dominant dynamics" of a closed loop control system behave like that of an underdamped 2nd order system with a settling time of atmost 4 s and a maximum peak overshoot of atmost 10%.

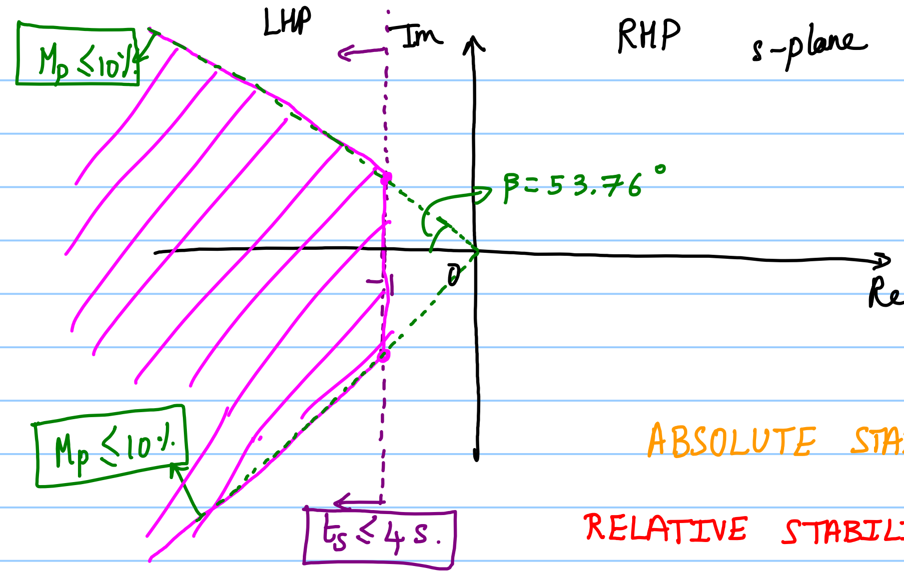
Recall that $t_s = \frac{4}{\xi\omega_n} \leq 4 \Rightarrow \xi\omega_n \geq 1 \Rightarrow \boxed{-\xi\omega_n \leq -1.}$

Recall that the real part of the poles of an underdamped 2nd order system is $-\xi\omega_n$.

$$M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100\% \leq 10\%$$

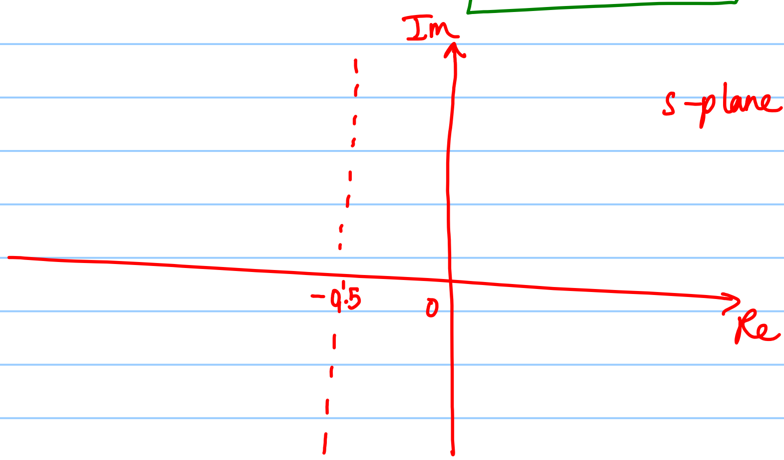
$$\Rightarrow e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \leq 0.1 \Rightarrow \xi \geq 0.5911$$

$$\xi = \cos(\beta) \geq 0.5911 \Rightarrow \boxed{\beta \leq 53.76^\circ}$$



ABSOLUTE STABILITY

RELATIVE STABILITY



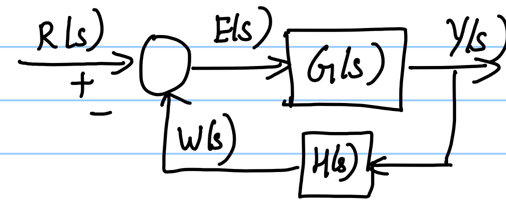
$$\hat{s} := s - 0.5$$

$$\Rightarrow s = \hat{s} + 0.5$$

Exercise: $s^2 + (K_p - 3)s + 2$. For absolute stability, $K_p > 3$.

Find the range of K_p to ensure relative stability, as desired above.

Analysis of Steady State Errors:



Recall that the c.l.t.f. $= \dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$.

$G(s) = C(s)P(s)$.

Let us assume that the closed loop is stable.

$G(s) \to$ Forward Path Tr. Fn.

$H(s) \to$ Feedback Path Tr. Fn.

$G(s)H(s) \to$ Open Loop Tr. Fn.

Note that $\dfrac{E(s)}{R(s)} = \dfrac{1}{1+G(s)H(s)}$. $\Rightarrow$ $\boxed{E(s) = \dfrac{R(s)}{1+G(s)H(s)}}$.

$\Rightarrow$ Steady State Error, $e_{ss} = \lim\limits_{t\to\infty} e(t) = \lim\limits_{s\to 0} sE(s) = \lim\limits_{s\to 0} \dfrac{sR(s)}{1+G(s)H(s)}$.

Let $G(s)H(s) = \dfrac{b_0 s^m + b_1 s^{(m-1)} + \dots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{(n-1)} + \dots + a_{n-1}s + a_n} = \dfrac{K\left(\hat{b}_0 s^m + \hat{b}_1 s^{(m-1)} + \dots + \hat{b}_{m-1}s + 1\right)}{s^N\left(\hat{a}_0 s^p + \hat{a}_1 s^{(p-1)} + \dots + \hat{a}_{p-1}s + 1\right)}$.

$\boxed{\text{TYPE 'N' SYSTEM}}$

$\boxed{N+p=n}$

$N=0 \to$	'TYPE 0'	SYSTEM
$N=1 \to$	'TYPE 1'	SYSTEM
$N=2 \to$	'TYPE 2'	SYSTEM

$\boxed{K \to \text{OPEN LOOP GAIN}}$

$s^3 + 3s^2 + 4s = s(s^2+3s+4)$

$= 4s\left(\frac{s^2}{4} + \frac{3}{4}s + 1\right)$

$$G(s)H(s) = \frac{s+4}{s^3+3s^2+5s} = \frac{4(0.25s+1)}{5s(0.2s^2+0.6s+1)} = \frac{0.8(0.25s+1)}{s(0.2s^2+0.6s+1)}.$$

21/2/2018. Recall that $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$.

Steady State Error to a Unit Step Reference Input:

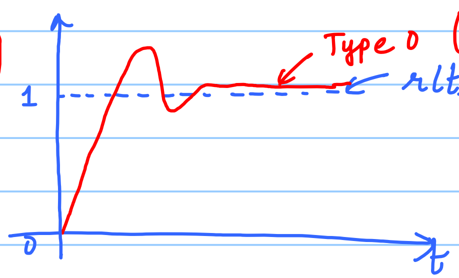
$$r(t) = 1 \Rightarrow R(s) = \frac{1}{s} \Rightarrow e_{ss} = \lim_{s\to 0}\frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0}\frac{1}{1+G(s)H(s)} = \frac{1}{1+G(0)H(0)}.$$

Static Position Error Constant, $K_{pe} := \lim_{s\to 0} G(s)H(s)$.

$$\Rightarrow e_{ss} = \frac{1}{1+K_{pe}}.$$

N	K_{pe}	e_{ss}
0	K	$\frac{1}{1+K}$
1	∞	0
2	∞	0

TRACKING ERROR



Eg.: $P(s) = \frac{1}{s^2+s+1}$.

$H(s) = 1$.

$C(s) = \frac{K_i}{s}$

If $P(s) = \frac{1}{s(s+1)}$, we donot require Integral term in $C(s)$.

Steady State Error to a Unit Ramp Reference Input:

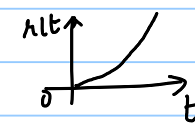
$$r(t)=t \Rightarrow R(s)=\frac{1}{s^2} \Rightarrow e_{ss}=\lim_{s\to 0}\frac{s\,R(s)}{1+G(s)H(s)} = \lim_{s\to 0}\frac{1}{s(1+G(s)H(s))} = \lim_{s\to 0}\left[\frac{1}{s\,G(s)H(s)}\right].$$

Static Velocity Error Constant, $K_{ve} := \lim_{s\to 0} s\,G(s)H(s)$.

$$\Rightarrow \quad e_{ss} = \frac{1}{K_{ve}}.$$

N	K_{ve}	e_{ss}
0	0	∞
1	K	$1/K$
2	∞	0

Steady State Error to a Unit Parabolic Reference Input:



$$r(t)=\frac{t^2}{2}, \Rightarrow R(s)=\frac{1}{s^3} \Rightarrow e_{ss}=\lim_{s\to 0}\frac{s\,R(s)}{1+G(s)H(s)} = \lim_{s\to 0}\frac{1}{s^2(1+G(s)H(s))} = \lim_{s\to 0}\frac{1}{s^2G(s)H(s)}$$

Static Acceleration Error Constant, $K_{ae} := \lim_{s \to 0} s^2 G(s) H(s)$

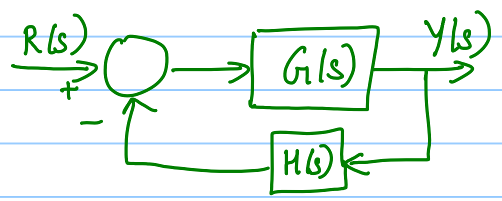
$\Rightarrow \quad e_{ss} = \dfrac{1}{K_{ae}}$.

Type	K_{ae}	e_{ss}
0	0	∞
1	0	∞
2	K	1/K
3	∞	0

$\boxed{\text{ROOT LOCUS}}$

→ Locus of the closed loop poles as a system parameter is varied.

Closed loop tr. fn. $= \dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$.



Closed loop characteristic eqn. : $1 + G(s)H(s) = 0$.

The roots of this equation are the closed loop poles.

$$1 + G(s)H(s) = 0 \Rightarrow G(s)H(s) = -1$$

$\rightarrow |G(s)H(s)| = 1 \rightarrow$ MAGNITUDE CONDITION

$\rightarrow \angle G(s)H(s) = \pm 180^\circ (2k+1), \; k = 0, 1, 2, \ldots$

$\hookrightarrow$ ANGLE CONDITION (PHASE)

Let us consider that the open loop transfer function takes the form

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$1 + G(s)H(s) = (s+p_1)(s+p_2)\cdots(s+p_n) + K(s+z_1)\cdots(s+z_m) = 0.$$

'm' open loop zeros $\Rightarrow -z_1, -z_2, \ldots, -z_m$.

'n' open loop poles $\Rightarrow -p_1, -p_2, \ldots, -p_n$.

$\Rightarrow$ There would be 'n' closed loop poles.

Q: How would the n closed loop poles vary w.r.t. the open loop poles and the open loop zeros as the parameter 'K' is varied?

The root locus provides us with the corresponding graphical visualization.

We would first consider the case when K is non-negative, i.e., $K \geq 0$.

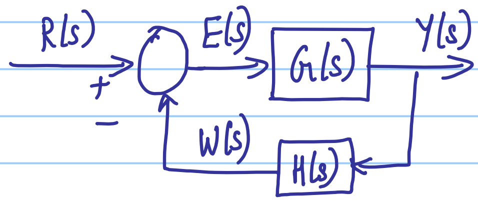
Q: What happens to the closed loop poles as $K \to 0$ and $K \to \infty$?

Q: How do we check if a root of a polynomial has a multiplicity greater than 1 ?

Note Title 2/26/2018

26/2/2018. Root Locus.

Q: How do the closed loop poles change when a system parameter is varied?



$G(s) = C(s)P(s)$.

Recall that $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$ → CLOSED LOOP TRANSFER FN.

$1+G(s)H(s) = 0$. → CLOSED LOOP CHARACTERISTIC EQUATION.

$G(s)H(s) = -1$

→ $|G(s)H(s)| = 1$ → MAGNITUDE CONDITION

→ $\boxed{\angle G(s)H(s) = \pm 180^\circ(2k+1),\ k = 0,1,2,\ldots}$ → ANGLE CONDITION.

Typically, the open loop transfer function can be expressed as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}.$$

$-z_1, -z_2, \ldots, -z_m$ → open loop zeros.

$-p_1, -p_2, \ldots, -p_n$ → open loop poles.

Q: How would the closed loop poles change as K is varied.

Let us first consider the scenario when $K \geq 0$.

$$1 + \frac{K(s+z_1)\dots(s+z_m)}{(s+p_1)\dots(s+p_n)} = 0.$$

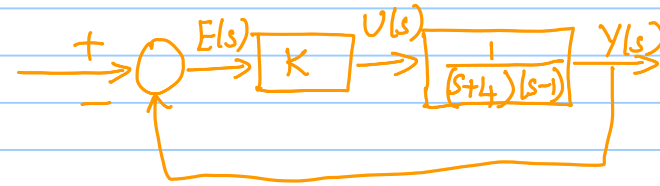
ROOT LOCUS $\rightarrow$ Locus of the closed loop poles, as K is varied.

$\rightarrow$ The root locus would have n branches.

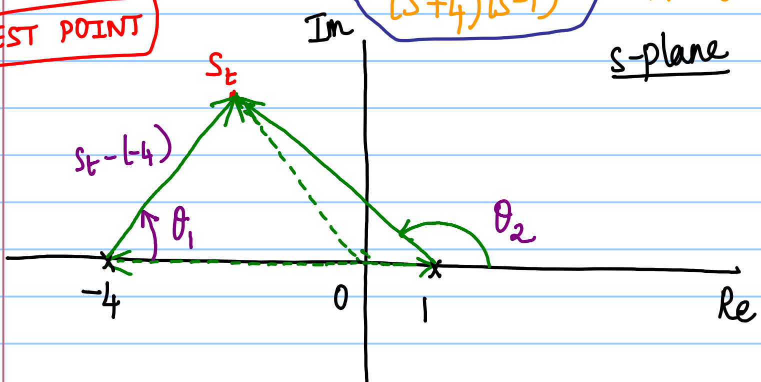
Eg.: $\boxed{G(s)H(s) = \frac{K}{(s+4)(s-1)}}$

$n = 2$. O.L. poles: $-4, 1$

$m = 0$ O.L. zeros: Nil.



TEST POINT



$s_t+4 = s_t-(-4)$

Q: How do we determine if the test point s_t lies on the root locus?

$$\angle G(s_t)H(s_t) = \angle \frac{K}{(s_t+4)(s_t-1)} = \underbrace{\angle K}_{0} - \underbrace{\angle s_t+4}_{\theta_1} - \underbrace{\angle s_t-1}_{\theta_2}$$

$$= \boxed{-\theta_1 - \theta_2 \overset{?}{=} \pm 180^\circ(2k+1),} \quad k = 0, 1, 2, \dots$$

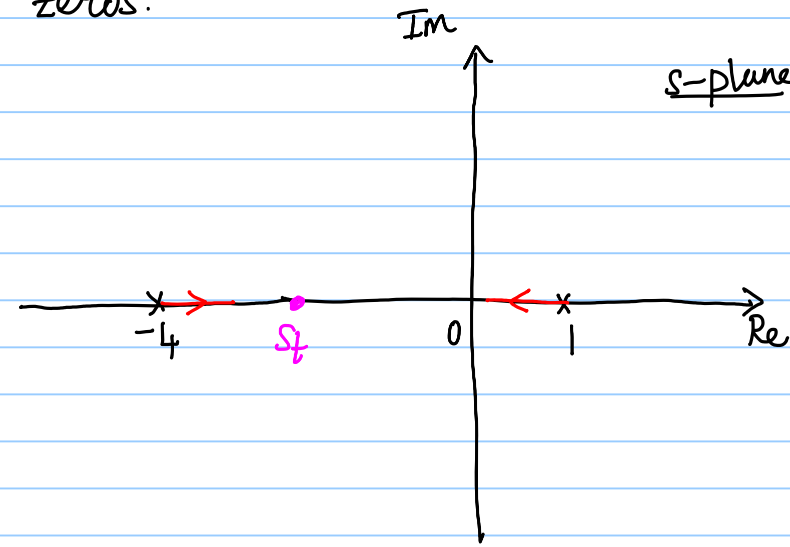
$\boxed{n - m = 2}$

$\boxed{\text{rlocus}}$

If $\theta_1 \approx \theta_2$ as $s_t \to \infty$,

$$\theta_{asym} = \pm \frac{180^\circ(2k+1)}{2}.$$

Step 1: Locate the open loop poles and open loop zeros.



open loop poles: $-4, 1$. $n=2$.

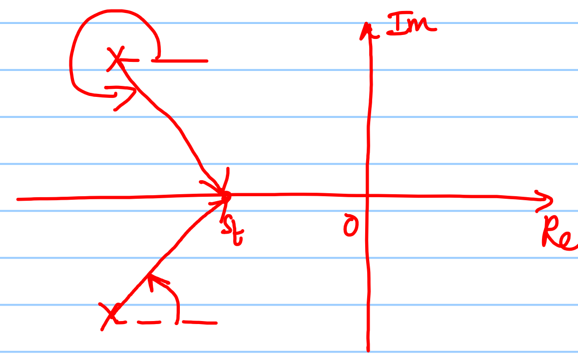
open loop zeros: Nil. $m=0$.

The root locus would have 2 branches.

$$1+\frac{K}{(s+4)(s-1)}=0. \Rightarrow \boxed{(s+4)(s-1)+K=0.}$$

$$\frac{1}{K}(s+4)(s-1)+1=0.$$

$(0s+1)\,(0s+1)$

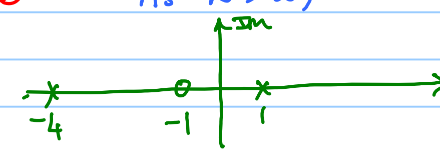


$$1+\frac{K(s+1)}{(s+4)(s-1)}=0.$$

$$\Rightarrow \frac{1}{K}(s+4)(s-1)+(s+1)=0.$$

$(0s+1)$

As $K\to\infty$, $s\to -1$.



✓ $(-\infty,-4)$

✗ $(-4,-1)$

$(-1,1)$ ✓

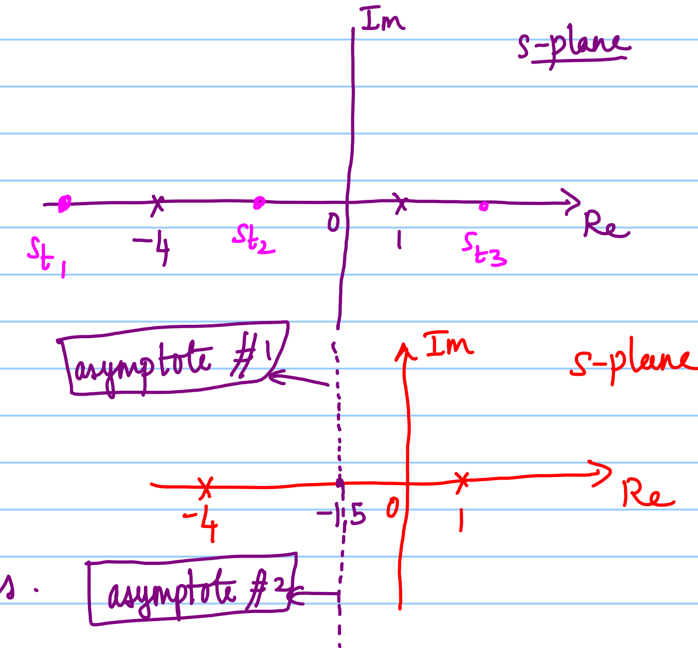
$(1,\infty)$ ✗

Step 2: Locate the parts of the real axis that lie on the root locus.

$(-\infty, -4) \rightarrow \times$

$(-4, 1) \rightarrow \checkmark$

$(1, \infty) \rightarrow \times$



$(-4, 1)$ lies on the root locus.

27/2/2018. o.l. poles: $-4, 1$; o.l. zeros: NIL.

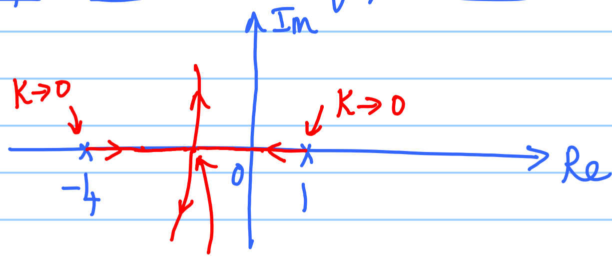
Step 3: Asymptotes. $(n-m) = 2-0 = 2 \Rightarrow 2$ asymptotes.

Angle made by the 2 asymptotes $= \pm \dfrac{180^\circ(2k+1)}{(n-m)} = \pm 90^\circ(2k+1)$, $k = 0, 1, 2, \ldots$

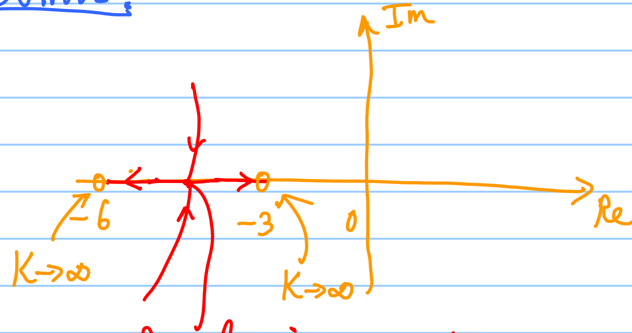
$+90^\circ, -90^\circ\ (270^\circ)$.

Point of intersection of the 2 asymptotes $= \dfrac{(-4+1)-(0)}{2} = -1.5$

Step 4: Break-away / Break-in points:



Break-away Point



Break-in point

$d_1(s) = (s+1)(s+2)$

$d_1'(s) = (s+1) + (s+2).$

$d_2(s) = (s+1)^2 (s+2)$

$d_2'(s) = 2(s+1)(s+2) + (s+1)^2$

Let $G(s)H(s) = K\dfrac{A(s)}{B(s)}$. $A(s) = (s+z_1)\cdots(s+z_m)$.

$B(s) = (s+p_1)\cdots(s+p_n)$.

The closed loop characteristic equation is $\boxed{1 + K\dfrac{A(s)}{B(s)} = 0.}$

$$P_{cl}(s) = 1 + K\frac{A(s)}{B(s)} \Rightarrow P_{cl}'(s) = \frac{dP_{cl}(s)}{ds} = +K\frac{\left(A'(s)B(s) - B'(s)A(s)\right)}{[B(s)]^2} = 0.$$

$$\Rightarrow \boxed{A'(s)B(s) - B'(s)A(s) = 0.}$$

→ The roots of this eqn. (s_b) are potential break-away / break-in points.

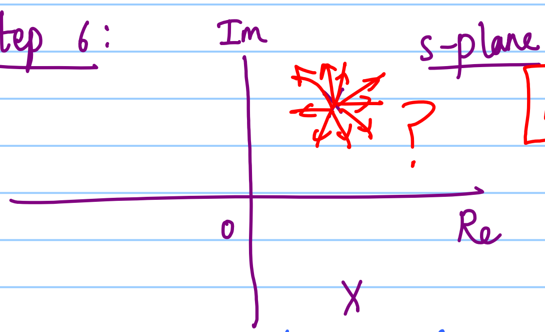
Then, calculate $K = -\frac{B(s)}{A(s)}\Big|_{s=s_b}$. Only those values of s_b that result in $K>0$ would lie on the root locus.

$A(s) = 1, \quad B(s) = (s+4)(s-1) \Rightarrow A'(s) = 0, \quad B'(s) = 2s+3.$

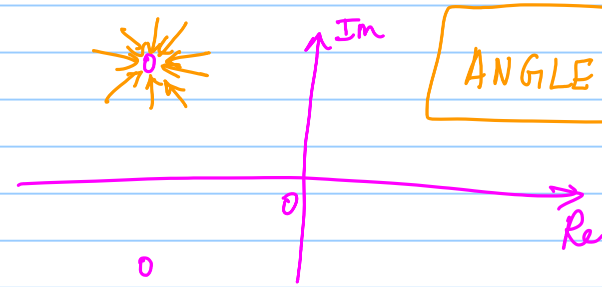
$\Rightarrow A'(s)\,B(s) - B'(s)\,A(s) = 0 \Rightarrow -(2s+3) = 0 \Rightarrow \boxed{s_b = -1.5.}$

$$K\Big|_{s=s_b} = -\left[\frac{B(s)}{A(s)}\right]_{s=s_b} = -\left[\frac{(s_b+4)(s_b-1)}{1}\right] = -\left[(2.5)(-2.5)\right] = \boxed{6.25} > 0.$$

Step 6:



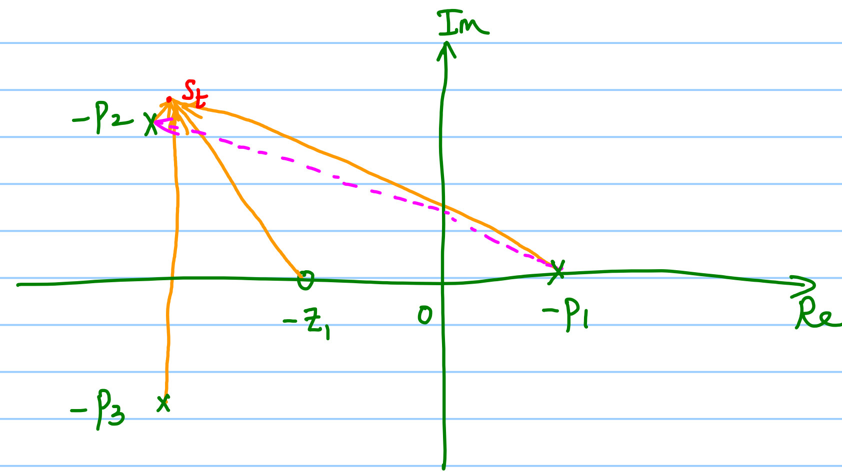
ANGLE OF DEPARTURE



ANGLE OF ARRIVAL

Step 6 applies only when we have complex open loop poles / open loop zeros.

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)}, \quad n=3, \quad m=1.$$



Angle of departure from $-p_2$:

$$\angle G(s_t)H(s_t) = \pm 180^\circ(2k+1), \quad k=0,1,2,\dots$$

$$\underbrace{\angle K}_{\substack{0^\circ \\ (K>0)}} + \angle s_t+z_1 - \left[\angle s_t+p_1 + \angle s_t+p_2 + \angle s_t+p_3\right]$$

$$\theta_{dep,(-p_2)} = \pm 180^\circ(2k+1).$$

As $s_t \to -p_2$ (s_t is "very" close to $-p_2$). $\angle s_t+p_1 \approx \angle -p_2+p_1$

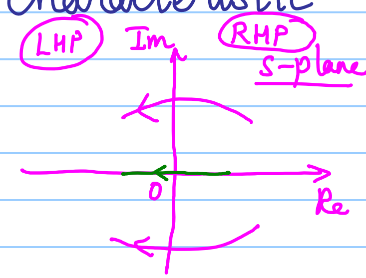
$$\theta_{dep,(-p_2)} = 180^\circ + \angle -p_2+z_1 - \left[\angle -p_2+p_1 + \angle -p_2+p_3\right]$$

$$\theta_{dep,(-p_3)} = -\theta_{dep,(-p_2)}$$

28/2/2018

Step 6: Determine the "cross-over" points (points where the root locus cuts the imaginary axis).

These are found by substituting $\boxed{s = j\omega}$ in the closed loop characteristic equation.



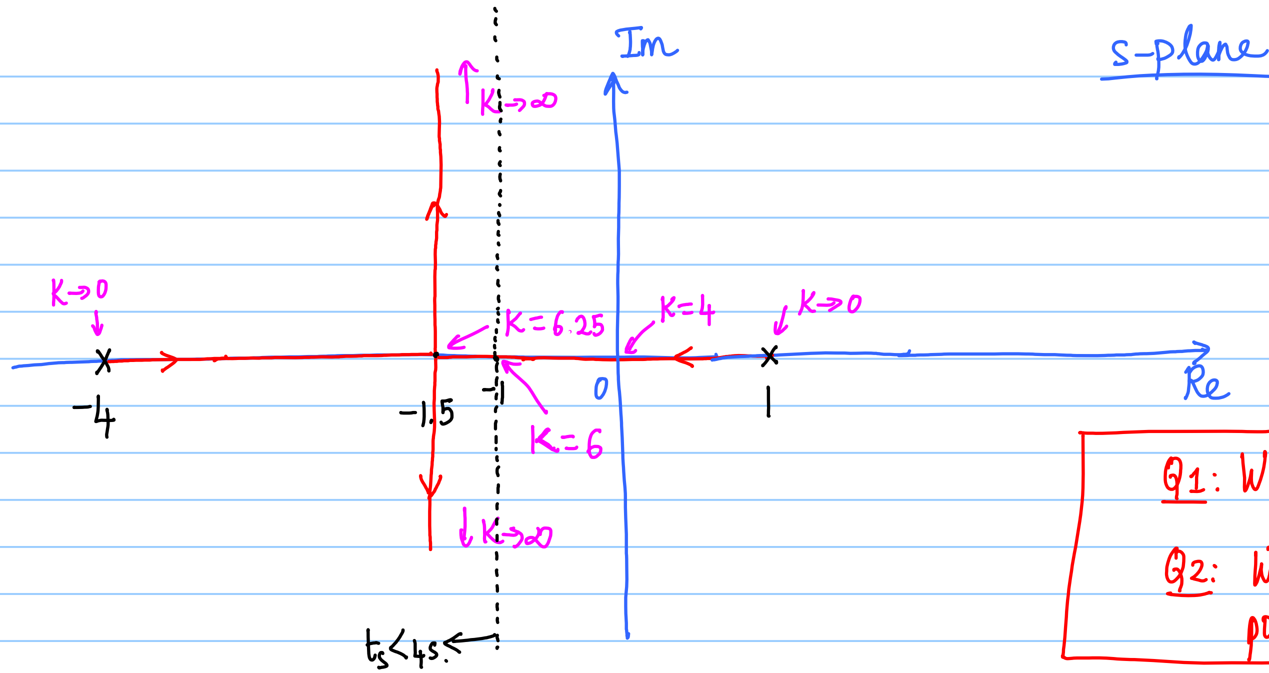
For this example, $G(s)H(s) = \dfrac{K}{(s+4)(s-1)} = \dfrac{K}{s^2+3s-4}$.

The closed loop characteristic equation is $\boxed{1 + G(s)H(s) = 0.}$

$$\Rightarrow \quad 1 + \frac{K}{s^2+3s-4} = 0. \quad \Rightarrow \quad \boxed{s^2 + 3s + (K-4) = 0.}$$

Substitute $\boxed{s = j\omega}$ $\Rightarrow \; -\omega^2 + 3j\omega + (K-4) = 0 \;\Rightarrow\; \underbrace{[-\omega^2 + K - 4]} + j\underbrace{[3\omega]} = 0.$

$\Rightarrow \boxed{\omega = 0} \Rightarrow K - 4 = 0 \Rightarrow \boxed{K = 4.} \Rightarrow s = 0$ is a cross-over point where $K = 4$.



$s^2 + 3s + K - 4 = 0.$

Substitute $s = -1$,

$K = 6.$

Q1: What happens when $K < 0$?

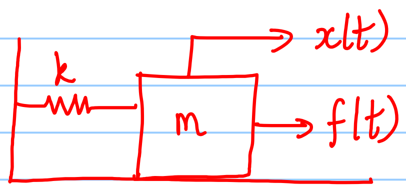
Q2: What would happen with positive feedback?

The closed loop system stable $\forall \; K > 4$.

Q: What range(s) of K would provide a settling time less than 4 s?

$$t_s = \frac{4}{\xi\omega_n} < 4 \Rightarrow \xi\omega_n > 1 \Rightarrow \boxed{-\xi\omega_n < -1.} \Rightarrow \boxed{K > 6}$$

Example:

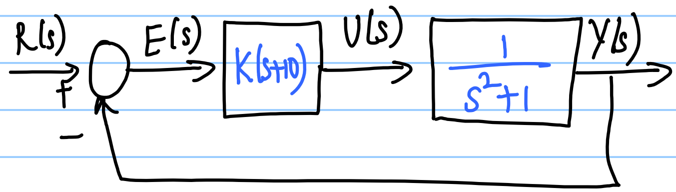


$m\ddot{x}(t) + kx(t) = f(t)$. $P(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{ms^2+k}$.

Let $m = 1$ kg, $k = 1$ N/m $\Rightarrow P(s) = \dfrac{1}{s^2+1}$.

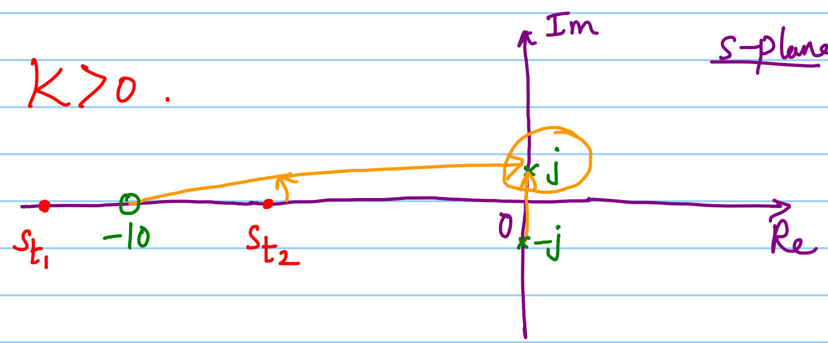
Let us design a PD controller to stabilize the plant with unit negative feedback.

$C(s) = K_d s + K_p = K_d\left(s + \dfrac{K_p}{K_d}\right) = K(s+10)$ $\left[\text{Let } K_d = K, \ \dfrac{K_p}{K_d} = 10\right]$.



$\Rightarrow G(s)H(s) = \dfrac{K(s+10)}{s^2+1}$ $\left(= \dfrac{K A(s)}{B(s)}\right)$

Q: Plot the locus of closed loop poles for $K > 0$.



Step 1: $n = 2$ O.L. poles: $\pm j$.

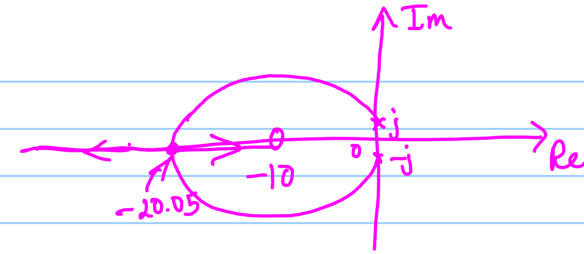
$m = 1$ O.L. zero: -10.

Step 2: $(-\infty, -10)$ ✓

$(-10, \infty)$ ✗

Step 3: $(n-m) = 1$ asymptote.

$\measuredangle$ of asymptote $= \pm \dfrac{180^\circ (2k+1)}{n-m} = 180^\circ$.



Point of intersection of the asymptote $= \dfrac{[+j-j]-[-10]}{1} = 10$. $\Rightarrow$ The asymptote is along the $-$ve real axis.

Step 4: Break-away / Break-in points:

$A(s) = s+10$, $B(s) = s^2+1$, $A'(s) = 1$, $B'(s) = 2s$.

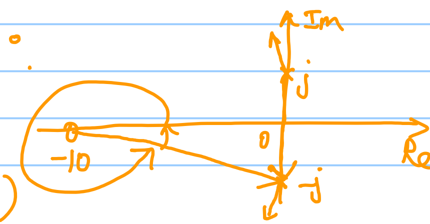
$A'(s)\,B(s) - A(s)\,B'(s) = 0 \Rightarrow s^2+1-(s+10)\,2s = 0 \Rightarrow -s^2-20s+1=0$.

$s^2+20s-1=0 \Rightarrow s_b = \begin{cases} -20.05 \Rightarrow K = 40.1 \checkmark \\ 0.05 \Rightarrow K = -0.1 \times \end{cases}$

$\Rightarrow s = -20.05$ is a break-in point where $K = 40.1$.

Step 5: Angle of departure from $+j = 180^\circ - [90^\circ] + \tan^{-1}(0.1) = 95.71^\circ$.

" " " " $-j = 180^\circ - [270^\circ] + 360^\circ - \tan^{-1}(0.1) = 264.29^\circ$ (-95.71°)

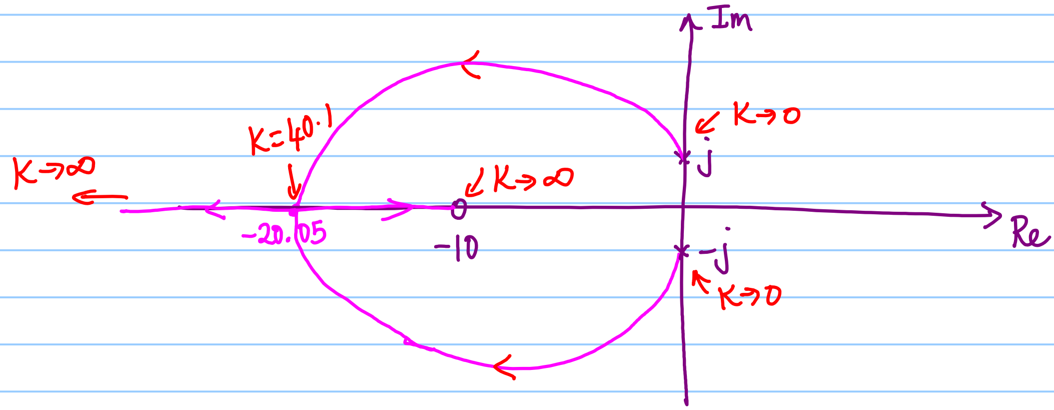


Step 6: **Cross-over point:** $1 + \frac{K(s+10)}{s^2+1} = 0 \Rightarrow s^2 + Ks + 10K + 1 = 0.$

$s = j\omega \Rightarrow [-\omega^2 + 10K + 1] + j(K\omega) = 0.$

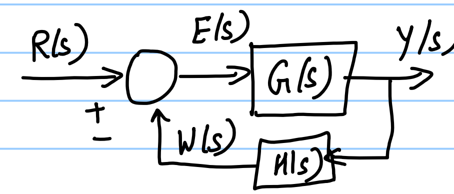
→ $K = 0 \Rightarrow \omega = \pm 1.$ → open loop poles.

→ $\omega = 0 \Rightarrow K = -0.1$ ✗



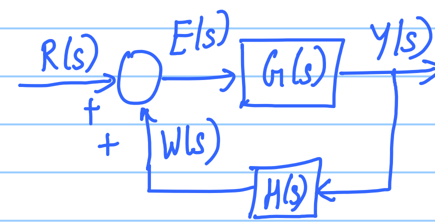
Note Title 05-Mar-18

5/3/2018. $\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$. $1+G(s)H(s) = 0$.



$$G(s)H(s) = \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}. \quad K>0.$$

Q: What happens in the case of positive feedback?



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}. \quad 1-G(s)H(s) = 0.$$

↳ Closed Loop Characteristic Equation.

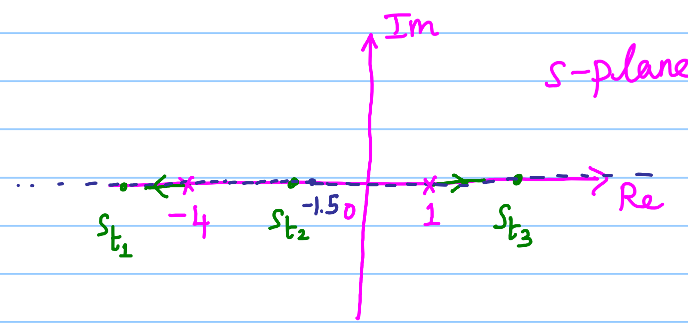
$\Rightarrow \quad G(s)H(s) = 1$. → $|G(s)H(s)| = 1$.

→ $\angle G(s)H(s) = \pm 360^\circ k, \quad k = 0, 1, 2, \ldots$

Still, $G(s)H(s) = \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}, \quad K>0.$

Step 4: $1 - \frac{K A(s)}{B(s)} = 0.$ $A'(s)\,B(s) - A(s)\,B'(s) = 0. \Rightarrow K\Big|_{s=s_b} = \frac{B(s)}{A(s)}\Big|_{s=s_b}.$

Example: $G(s)\,H(s) = \frac{K}{(s+4)(s-1)}$, $K > 0.$



Step 1: $n = 2$. o.l. poles: $-4, 1$
$m = 0$ o.l. zeros: Nil.

Step 2: $(-\infty, -4)$ ✓

$(-4, 1)$ ✗

$(1, \infty)$ ✓

Step 3: $(n-m) = 2$ asymptotes.

Angle of asymptotes $= \frac{\pm 360^\circ k}{(n-m)}$, $k = 0, 1, 2, \ldots$ $= 0^\circ, 180^\circ.$

Point of intersection of the asymptotes $= \dfrac{(-4+1)-(0)}{2} = -1.5.$

Step 4: $A(s) = 1,\ B(s) = (s+4)(s-1),\ A'(s) = 0,\ B'(s) = 2s+3.$

$A'(s)\,B(s) - B'(s)\,A(s) = 0 \Rightarrow -(2s+3) = 0 \Rightarrow s_b = -1.5.$ ✗

$$K\Big|_{s=s_b} = \frac{B(s)}{A(s)}\Big|_{s=s_b} = -6.25 \quad ✗$$

$\Rightarrow$ No break-away / break-in points.

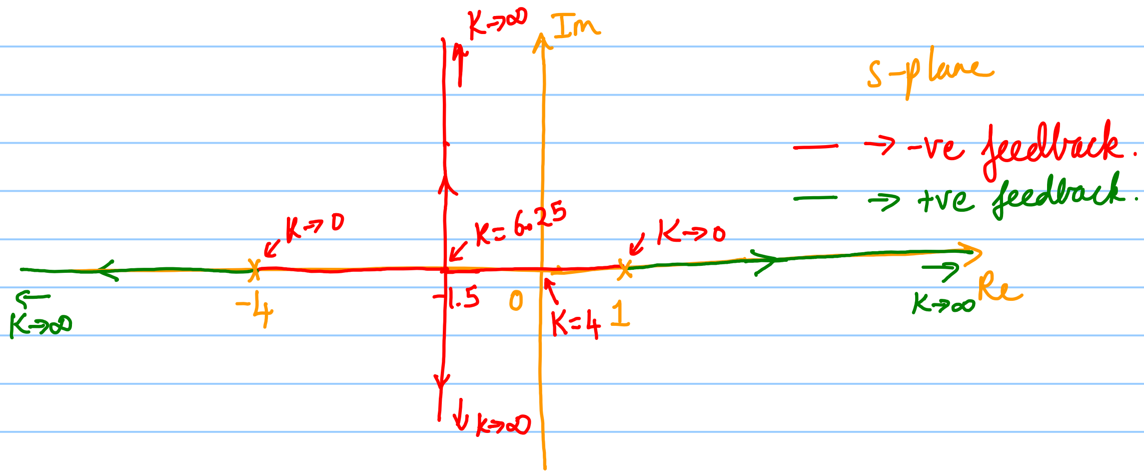
Step 5: Not applicable.

Step 6: $1 - G(s)\,H(s) = 0 \Rightarrow 1 - \dfrac{K}{(s+4)(s-1)} = 0 \Rightarrow s^2 + 3s - K - 4 = 0.$

$s = j\omega \Rightarrow -\omega^2 + j(3\omega) - K - 4 = 0 \Rightarrow [-\omega^2 - K - 4] + j(3\omega) = 0.$

$\Rightarrow \omega = 0 \Rightarrow K = -4.$ ✗

$\Rightarrow$ No Cross-Over Points.



With positive feedback, the closed loop system is unstable $\forall\ K>0$.

HW: $G(s)\,H(s) = \dfrac{K(s+10)}{s^2+1}$, $K>0$. → Plot the root locus in the case of positive feedback.

Q: What happens when $K<0$ in the case of negative feedback?

$$G(s)\,H(s) = \frac{K\,(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}, \quad K<0.$$

Closed Loop Characteristic Equation: $1 + G(s)H(s) = 0$

$$1 + \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = 0. \xrightarrow{\hat{K} := -K} 1 - \frac{\hat{K}(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = 0, \quad \hat{K} > 0.$$

HW: Repeat

$$\left.\begin{array}{l} \text{i). } G(s)H(s) = \dfrac{K}{(s+4)(s-1)}, \quad K<0, \\ \text{ii). } G(s)H(s) = \dfrac{K(s+10)}{(s^2+1)}, \quad K<0, \end{array}\right\} \text{ with negative feedback.}$$

Q: What happens when $K<0$ in the case of positive feedback?

$$1 - \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = 0 \xrightarrow{\hat{K} := -K} 1 + \frac{\hat{K}(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = 0, \quad \hat{K} > 0.$$

HW: Repeat with this case.

Set 1 $\rightarrow$ Set of steps with the angle condition involving odd multiples of 180°.

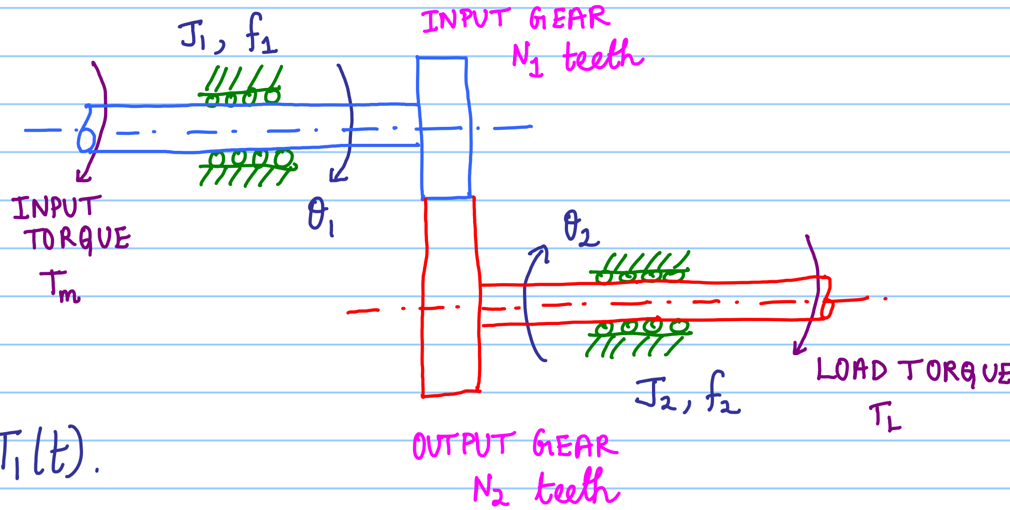
Set 2 $\rightarrow$ " " " " " even " " .

	Negative Feedback	Positive Feedback
$K > 0$	SET 1	SET 2
$K < 0$	SET 2 ($\hat{K} := -K$)	SET 1 ($\hat{K} := -K$)

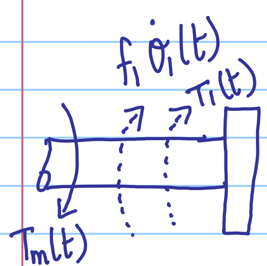
$$G(s)\,H(s) = \frac{K\,(s+z_1)\,\cdots\,(s+z_m)}{(s+p_1)\,\cdots\,(s+p_n)}.$$

06/03/18. **GEAR TRANSMISSION:**

Task 1: Derive the governing equation of motion. Then, obtain the transfer function relating the input $T_m(t)$ and the output $\theta_2(t)$.



$$T_m(t) \longrightarrow \boxed{\text{System}} \longrightarrow \theta_2(t)$$



$$J_1\ddot{\theta}_1(t) = T_m(t) - f_1\dot{\theta}_1(t) - T_1(t).$$

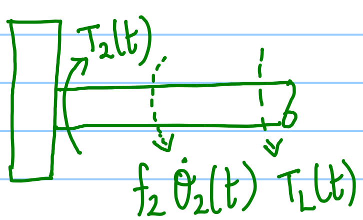
$$J_1\ddot{\theta}_1(t) + f_1\dot{\theta}_1(t) = T_m(t) - T_1(t). \quad \text{--- (1)}.$$

Note that

$$\frac{\theta_2(t)}{\theta_1(t)} = \frac{N_1}{N_2}.$$

$$\Rightarrow \theta_1(t) = \left(\frac{N_2}{N_1}\right)\theta_2(t).$$

$$\Rightarrow \dot{\theta}_1(t) = \left(\frac{N_2}{N_1}\right)\dot{\theta}_2(t).$$



$$J_2\ddot{\theta}_2(t) = T_2(t) - f_2\dot{\theta}_2(t) - T_L(t).$$

$$\Rightarrow J_2\ddot{\theta}_2(t) + f_2\dot{\theta}_2(t) = T_2(t) - T_L(t). \quad \text{--- (2)}.$$

$\eta_t \rightarrow$ efficiency

Neglecting energy losses: $T_1(t)\,\theta_1(t) = T_2(t)\,\theta_2(t)$.

$$\Rightarrow \boxed{\ddot{\theta}_1(t) = \left(\frac{N_2}{N_1}\right)\ddot{\theta}_2(t).}$$

$$\Rightarrow T_1(t)\,\frac{\theta_1(t)}{\theta_2(t)} = T_2(t) \Rightarrow \boxed{T_2(t) = T_1(t)\left(\frac{N_2}{N_1}\right)}.$$

Eq. 1 $* \left(\frac{N_2}{N_1}\right) \Rightarrow$

$$J_1\left(\frac{N_2}{N_1}\right)\ddot{\theta}_1(t) + f_1\left(\frac{N_2}{N_1}\right)\dot{\theta}_1(t) = \left(\frac{N_2}{N_1}\right)T_m(t) - T_1(t)\left(\frac{N_2}{N_1}\right).$$

$$\Rightarrow J_1\left(\frac{N_2}{N_1}\right)^2\ddot{\theta}_2(t) + f_1\left(\frac{N_2}{N_1}\right)^2\dot{\theta}_2(t) = \left(\frac{N_2}{N_1}\right)T_m(t) - T_L(t). \quad -3.$$

2 + 3 $\Rightarrow$

$$\boxed{\left[J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right]\ddot{\theta}_2(t) + \left[f_1\left(\frac{N_2}{N_1}\right)^2 + f_2\right]\dot{\theta}_2(t) = \left(\frac{N_2}{N_1}\right)T_m(t) - T_L(t).}$$

<u>Transfer Function</u>: Take Laplace transform and apply zero IC:

$$\mathcal{L}[\ddot{\theta}_2(t)] = s^2\,\theta_2(s) - s\,\theta_2(0) - \dot{\theta}_2(0).$$

$$\mathcal{L}[\dot{\theta}_2(t)] = s\,\theta_2(s) - \theta_2(0).$$

$$\Rightarrow \quad \theta_2(s) = \underbrace{\left(\frac{(N_2/N_1)}{\left[J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right]s^2 + \left[f_1\left(\frac{N_2}{N_1}\right)^2 + f_2\right]s}\right)T_m(s)}_{\text{Due to the input}} \underbrace{- \left(\frac{1}{\left[J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right]s^2 + \left[f_1\left(\frac{N_2}{N_1}\right)^2 + f_2\right]s}\right)T_L(s)}_{\text{Due to the load torque}}.$$

Neglect the load torque T_L.

$$\Rightarrow \quad P(s) = \frac{\theta_2(s)}{T_m(s)} = \frac{(N_2/N_1)}{\left[J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right]s^2 + \left[f_1\left(\frac{N_2}{N_1}\right)^2 + f_2\right]s} = \frac{a}{s^2 + bs} = \frac{a}{s(s+b)}.$$

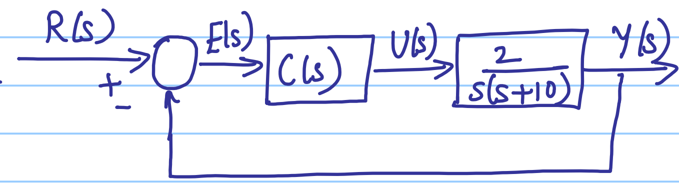
$$a = \frac{(N_2/N_1)}{J_1\left(\frac{N_2}{N_1}\right)^2 + J_2}, \quad b = \frac{f_1\left(\frac{N_2}{N_1}\right)^2 + f_2}{J_1\left(\frac{N_2}{N_1}\right)^2 + J_2}.$$

Let us take $a = 2$, $b = 10$.

$$\Rightarrow \quad P(s) = \frac{2}{s(s+10)}.$$

TASK 2: Design a stable unity negative feedback control system that satisfies: settling time $< 2\,s$, $M_p < 10\%$.

7/3/18. $C(s) = K_p$. $\Rightarrow \dfrac{Y(s)}{R(s)} = \dfrac{\dfrac{2K_p}{s(s+10)}}{1+\dfrac{2K_p}{s(s+10)}} = \dfrac{2K_p}{s^2+10s+2K_p}$.



Closed loop characteristic eqn.: $s^2+10s+2K_p = 0$.

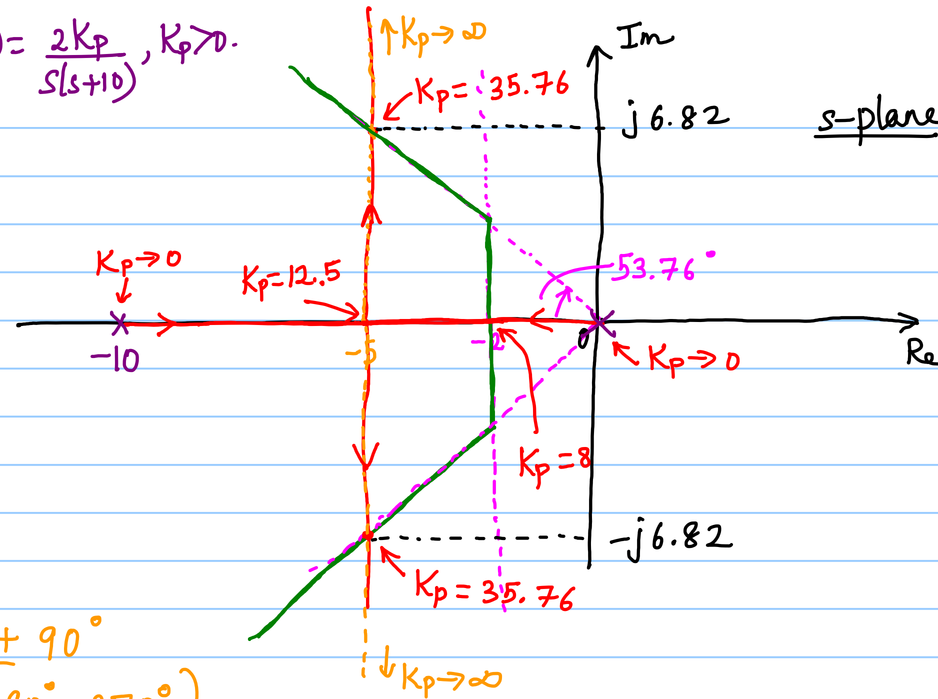
$\Rightarrow$ $\boxed{K_p > 0 \text{ for closed loop stability.}}$

Consider $G(s)H(s) = \boxed{\dfrac{2K_p}{s(s+10)}}$, $K_p > 0$.

$t_s < 2s$, $\dfrac{4}{\xi\omega_n} < 2 \Rightarrow \xi\omega_n > 2 \Rightarrow -\xi\omega_n < -2$

$M_p < 10\% \Rightarrow e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} < 0.1 \Rightarrow \xi > 0.591, \Rightarrow \beta < 53.76^\circ$

$G(s)H(s) = \frac{2K_p}{s(s+10)}$, $K_p > 0$.



Let us construct the root locus.

Step 1: $n = 2$ o.l. poles: $0, -10$.
$m = 0$ o.l. zeros: Nil.

Step 2: $(-\infty, -10)$ ✗
$(-10, 0)$ ✓
$(0, \infty)$ ✗

Step 3: $(n-m) = 2$ asymptotes.

Angle of the asymptotes $= \frac{\pm 180^\circ (2k+1)}{(n-m)} = \pm 90^\circ$
$(90^\circ, 270^\circ)$

Point of intersection of the asymptotes $= \frac{(0-10)-(0)}{2} = -5$.

$G(s)H(s) = K_p \frac{A(s)}{B(s)}$
$= K_p \frac{2}{s(s+10)}$

Step 4: $A(s) = 2$, $B(s) = s(s+10)$, $A'(s) = 0$, $B'(s) = 2s+10$.
$A'(s)B(s) - B'(s)A(s) = 0 \Rightarrow -2(2s+10) = 0 \Rightarrow s_b = -5$.

$$K_p\Big|_{s=s_b} = -\frac{B(s)}{A(s)}\Big|_{s=s_b} = -\frac{(-5*5)}{2} = 12.5 > 0$$

$\Rightarrow$ $s_b = -5$ is a break-away point.

Step 5: Does not apply.

Step 6: Cross - over points: Closed loop characteristic eqn.:

$$1 + \frac{2K_p}{s(s+10)} = 0 \Rightarrow \boxed{s^2 + 10s + 2K_p = 0} \xrightarrow{s=j\omega} -\omega^2 + j(10\omega) + 2K_p = 0.$$

$$\Rightarrow (2K_p - \omega^2) + j(10\omega) = 0 \Rightarrow \omega = 0 \Rightarrow K_p = 0.$$

$\Rightarrow$ For $\forall$ $K_p > 0$, the root locus does not cross the $j\omega$ axis.

Consider $s^2 + 10s + 2K_p = 0 \xrightarrow{s=-2} K_p = 8.$

For satisfying closed loop stability AND performance, $K_p \in (8, 35.76)$.

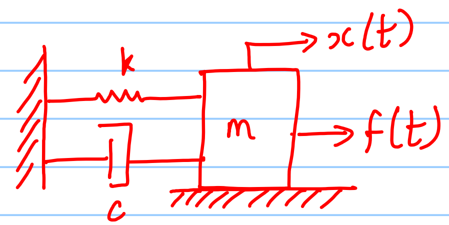
HW: Plot the root locus when $K_p < 0$.

$$G(s)\,H(s) = \frac{2K_p}{s(s+10)}.$$

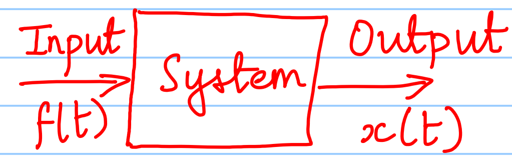
Note Title 12-Mar-18

12/03/18. STATE SPACE REPRESENTATION:

Recall the mass-spring-damper system whose governing equation is



$$\boxed{m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t).}$$ $n=2$



Its transfer function is

$$P(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k}.$$

Basic Idea: To re-write a n^{th} order ODE as a set of n 1^{st} order ODEs.

STATE VARIABLES: They are those variables whose knowledge at each and every instant of time is sufficient to completely characterize the system.

Q: How does one represent a n^{th} order system in the state space form?

1). Select n state variables.

2). Write n 1^{st} order ODEs, each one characterizing the time evolution of the n

state variables.

Select 2 state variables. Let $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$.

Then, ① $\dot{x}_1(t) = \dot{x}(t) = x_2(t)$.

STATE EQUATIONS

② $\dot{x}_2(t) = \ddot{x}(t) = -\frac{k}{m}x(t) - \frac{c}{m}\dot{x}(t) + \frac{1}{m}f(t) = -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + \frac{1}{m}f(t)$

Let STATE VARIABLE VECTOR, $\underset{\sim}{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

In general, the dimension of the state vector $\underset{\sim}{x}(t)$ is n.

$$\dot{\underset{\sim}{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{\underset{\sim}{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\underset{\sim}{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\underset{\sim}{b}} f(t).$$

→ STATE EQUATION

VECTOR ODE

STATE MATRIX

[(nxn) in general]

INPUT VECTOR

The general state equation for a SISO LTI causal dynamic system is

$$\boxed{\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\,\underset{\sim}{x}(t) + \underset{\sim}{b}\,u(t).}$$

Output $y(t) = x(t) = x_1(t) = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{c}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\underset{\sim}{x}(t)}$ $\rightarrow$ OUTPUT EQUATION

OUTPUT VECTOR

The general output equation for a SISO LTI causal dynamic system is

$$\boxed{y(t) = \underset{\sim}{c} \cdot \underset{\sim}{x}(t) + \underbrace{d\,u(t)}.}$$

DIRECT TRANSMISSION TERM $\rightarrow$ $m = n$

$y(t) = \underset{\sim}{c}^T \underset{\sim}{x}(t) + d\,u(t)$

$\underset{\sim}{c}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$

STATE SPACE REPRESENTATION OF A SISO LTI CAUSAL DYAMIC SYSTEM:

$$\boxed{\begin{aligned} \dot{\underset{\sim}{x}}(t) &= \underset{\sim}{A}\,\underset{\sim}{x}(t) + \underset{\sim}{b}\,u(t), \\ y(t) &= \underset{\sim}{c} \cdot \underset{\sim}{x}(t) + d\,u(t). \end{aligned}}$$

$\rightarrow$ STATE EQUATION

$\rightarrow$ OUTPUT EQUATION

$\{\underset{\sim}{A}, \underset{\sim}{b}, \underset{\sim}{c}, d\} \rightarrow$ Realization of a system.

STATE SPACE REPRESENTATION OF A MIMO LTI CAUSAL DYNAMIC SYSTEM:

$$\boxed{\begin{aligned} \dot{\underset{\sim}{x}}(t) &= \underset{\sim}{A}\,\underset{\sim}{x}(t) + \underset{\sim}{B}\,\underset{\sim}{u}(t), \\ \underset{\sim}{y}(t) &= \underset{\sim}{C}\,\underset{\sim}{x}(t) + \underset{\sim}{D}\,\underset{\sim}{u}(t). \end{aligned}}$$

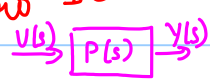
Consider 'p' inputs and 'q' outputs, $p>1$, $q>1$.

$\underset{\sim}{A} \rightarrow (n \times n)$, $\underset{\sim}{B} \rightarrow (n \times p)$ (INPUT MATRIX), $\underset{\sim}{C} \rightarrow (q \times n)$ (OUTPUT MATRIX), $D \rightarrow (q \times p)$ (DIRECT TRANSMISSION MATRIX)

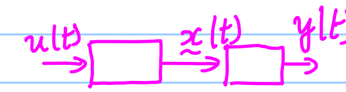
$\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\} \rightarrow$ Realization of a system.

LTI Causal Dynamic Systems

Laplace Transform with zero IC



Re-write as n 1^{st} order ODEs



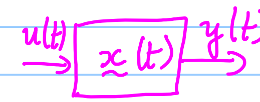
TRANSFER FUNCTION REPRESENTATION

→ EXTERNAL REPRESENTATION.

→ Analysis in the s-domain.

→ UNIQUE.

STATE SPACE REPRESENTATION



→ INTERNAL REPRESENTATION.

→ Analysis in the t-domain.

→ NON-UNIQUE.

Consider SISO LTI causal dynamic systems.

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\,\underset{\sim}{x}(t) + \underset{\sim}{b}\,u(t),$$

$$y(t) = \underset{\sim}{c}\,.\,\underset{\sim}{x}(t) + d\,u(t).$$

Take the Laplace transform, $s\underset{\sim}{X}(s) - \underset{\sim}{x}(0) = \underset{\sim}{A}\,\underset{\sim}{X}(s) + \underset{\sim}{b}\,U(s).$

Let the ICs be zero, i.e., $\underset{\sim}{x}(0) = \underset{\sim}{0} \Rightarrow s\underset{\sim}{X}(s) = \underset{\sim}{A}\,\underset{\sim}{X}(s) + \underset{\sim}{b}\,U(s).$

$$\Rightarrow [s\underset{\sim}{I} - \underset{\sim}{A}]\,\underset{\sim}{X}(s) = \underset{\sim}{b}\,U(s) \Rightarrow \boxed{\underset{\sim}{X}(s) = (s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{b}\,U(s)}$$

$$Y(s) = \underset{\sim}{c}\,.\,\underset{\sim}{X}(s) + d\,U(s) = \underset{\sim}{c}\,.\,(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{b}\,U(s) + d\,U(s).$$

$$\Rightarrow Y(s) = \underbrace{\left[\underset{\sim}{c}\,.\,(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{b} + d\right]}_{P(s)}\,U(s).$$

$$\Rightarrow \boxed{P(s) = \underset{\sim}{c}\,.\,(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{b} + d.}$$

→ Check this relationship for the mass-spring-damper system

→ Calculate the poles of the transfer fn. and the eigenvalues of the state matrix.

13/3/2018.

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad \underset{\sim}{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$s\underset{\sim}{I} - \underset{\sim}{A} = \begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{c}{m} \end{bmatrix}. \quad \det(s\underset{\sim}{I} - \underset{\sim}{A}) = s^2 + \frac{c}{m}s + \frac{k}{m} \; = \frac{ms^2+cs+k}{m}. \quad (s\underset{\sim}{I} - \underset{\sim}{A})^{-1} = \begin{bmatrix} s+\frac{c}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\left(\frac{1}{\det(s\underset{\sim}{I}-\underset{\sim}{A})}\right).$$

$$\underset{\sim}{c}\cdot(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{b} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s+\frac{c}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ 1/m \end{bmatrix}\right\}\overbrace{\left(\frac{m}{ms^2+cs+k}\right)}^{\frac{1}{\det(s\underset{\sim}{I}-\underset{\sim}{A})}} = \frac{m}{ms^2+cs+k}\left\{\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\cdot\underbrace{\begin{bmatrix} 1/m \\ s/m \end{bmatrix}}\right\} = \frac{1}{ms^2+cs+k}.$$

Poles of transfer fn.: $ms^2 + cs + k = 0 \Rightarrow \boxed{s = \frac{-c \pm \sqrt{c^2-4mk}}{2m}}.$

Eig. $(\underset{\sim}{A})$: $\det(\lambda\underset{\sim}{I} - \underset{\sim}{A}) = 0 \Rightarrow \lambda^2 + \frac{c}{m}s + \frac{k}{m} = 0 \Rightarrow \boxed{\lambda = \frac{-c \pm \sqrt{c^2-4mk}}{2m}}$

Poles of P(s) are the same as eig$(\underset{\sim}{A})$.

MINIMAL REALIZATIONS

FREQUENCY RESPONSE: Deals with the response of the system when a sinusoidal input is provided to it.

Let us consider a STABLE LTI causal SISO dynamic system whose transfer fn. is $P(s)$.

$$Y(s) = P(s)U(s).$$

$$U(s) \longrightarrow \boxed{P(s)} \longrightarrow Y(s)$$

Let us consider $u(t) = U_0 \sin(\omega t) \Rightarrow U(s) = \dfrac{U_0\,\omega}{s^2+\omega^2}$.

Let $P(s) = \dfrac{n(s)}{d(s)} = \dfrac{n(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$. All poles of $P(s)$ lie in the LHP.

$$\Rightarrow \quad Y(s) = P(s)\,U(s) = \underbrace{\left(\frac{n(s)}{d(s)}\right)}_{P(s)} \frac{U_0\,\omega}{s^2+\omega^2} = \frac{a_1}{s+j\omega} + \frac{a_2}{s-j\omega} + \frac{n_1(s)}{d(s)}.$$

a_1: $\quad * \, (s+j\omega) \Rightarrow \dfrac{P(s)\,U_0\,\omega}{(s-j\omega)} = a_1 + \dfrac{a_2(s+j\omega)}{(s-j\omega)} + \dfrac{n_1(s)\,(s+j\omega)}{d(s)}.$

$$\xrightarrow{s=-j\omega} \quad \frac{P(-j\omega)\,U_0\,\cancel{\omega}}{-2j\cancel{\omega}} = a_1 \Rightarrow a_1 = -\frac{U_0}{2j}\,P(-j\omega)$$

$$P(j\omega) = |P(j\omega)|\, e^{j\phi}, \quad \boxed{\phi = \angle P(j\omega)} \Rightarrow P(-j\omega) = |P(j\omega)|\, e^{-j\phi}.$$

$$\Rightarrow a_1 = \frac{-U_0}{2j} |P(j\omega)|\, e^{-j\phi}.$$

$\underline{a_2:}$ $* (s-j\omega) \Rightarrow$ $$\frac{P(s)\, U_0\, \omega}{(s+j\omega)} = \frac{a_1 (s-j\omega)}{(s+j\omega)} + a_2 + \frac{n_1(s)\,(s-j\omega)}{d(s)}.$$

$$\xrightarrow{s=j\omega} \frac{P(j\omega)\, U_0 \cancel{\omega}}{2j\cancel{\omega}} = a_2 \Rightarrow a_2 = \frac{U_0}{2j} P(j\omega) = \frac{U_0}{2j} |P(j\omega)|\, e^{j\phi}.$$

$$Y(s) = \frac{a_1}{s+j\omega} + \frac{a_2}{s-j\omega} + \frac{n_1(s)}{d(s)} \Rightarrow y(t) = a_1 e^{-j\omega t} + a_2 e^{j\omega t} + \underbrace{\sum_{l=1}^{k} \sum_{m=0}^{(\mu_l - 1)} c_{lm} t^m e^{p_l t}}_{\text{As } t\to\infty, \text{ these terms tend to } 0.}.$$

$$\lim_{t\to\infty} y(t) = a_1 e^{-j\omega t} + a_2 e^{j\omega t} = -\frac{U_0}{2j} |P(j\omega)|\, e^{-j\phi} e^{-j\omega t} + \frac{U_0}{2j} |P(j\omega)|\, e^{j\phi} e^{j\omega t}.$$

$$\lim_{t\to\infty} y(t) = V_0 |P(j\omega)| \left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right]$$

$$\Rightarrow \boxed{\lim_{t\to\infty} y(t) = \boxed{V_0 |P(j\omega)|} \sin(\omega t + \phi).}$$

STEADY STATE OUTPUT

→ The steady state output is a sinusoidal signal of the same frequency as that of the input, but scaled in magnitude by $|P(j\omega)|$ and shifted in phase by ϕ ($\angle P(j\omega)$). This is a property of stable LTI systems.

$P(j\omega)$ → sinusoidal transfer function.

Eg.: $P(s) = \dfrac{1}{s+1}$, $P(j\omega) = \dfrac{1}{j\omega+1} = \dfrac{1-j\omega}{1+\omega^2} = \dfrac{1}{1+\omega^2} - j\dfrac{\omega}{1+\omega^2}$.

$$|P(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}, \quad \angle P(j\omega) = -\tan^{-1}(\omega).$$

$$P(j\omega) = \frac{n_1(j\omega)\dots n_m(j\omega)}{d_1(j\omega)\dots d_n(j\omega)}. \quad |P(j\omega)| = \frac{|n_1(j\omega)| \dots |n_m(j\omega)|}{|d_1(j\omega)| \dots |d_n(j\omega)|}, \quad \angle P(j\omega) = \angle n_1(j\omega) + \dots + \angle n_m(j\omega) - [\angle d_1(j\omega) + \dots + \angle d_n(j\omega)].$$

Q: How can one visualize $P(j\omega)$ as ω is varied?

→ $\boxed{\text{Bode Plot}}$ → MAGNITUDE PLOT → $|P(j\omega)|$ vs ω.
→ PHASE PLOT → $\angle P(j\omega)$ vs ω.

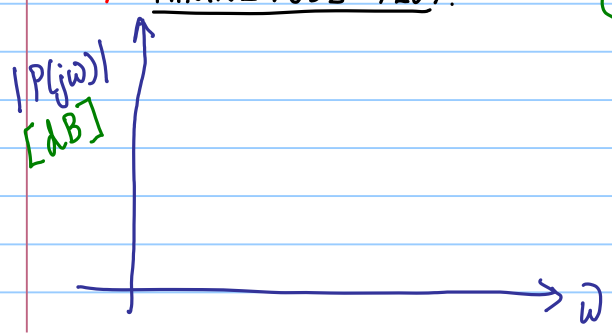
→ Nyquist Plot → $Re(P(j\omega))$ vs $Im(P(j\omega))$ in the complex plane ($P(s)$ plane).

→ Nichols Plot → ?

14/3/18. BODE PLOT :

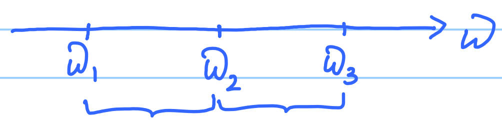
$P(j\omega)$ → $|P(j\omega)|$ → MAGNITUDE ; $\angle P(j\omega)$ → PHASE } are functions of ω.

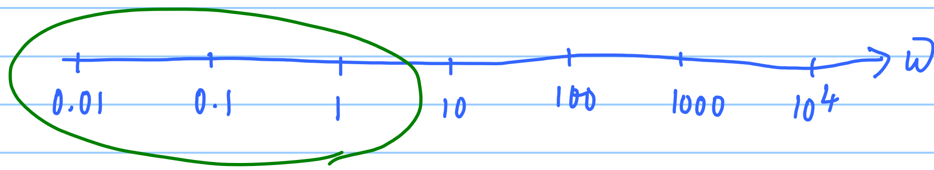
i) MAGNITUDE PLOT. $20\log_{10}|P(j\omega)|$ dB



→ Logarithmic Scale ($\log_{10}$)

Linear: $\omega_2 - \omega_1 = \omega_3 - \omega_2$.
Logarithmic: $\frac{\omega_2}{\omega_1} = \frac{\omega_3}{\omega_2}$.





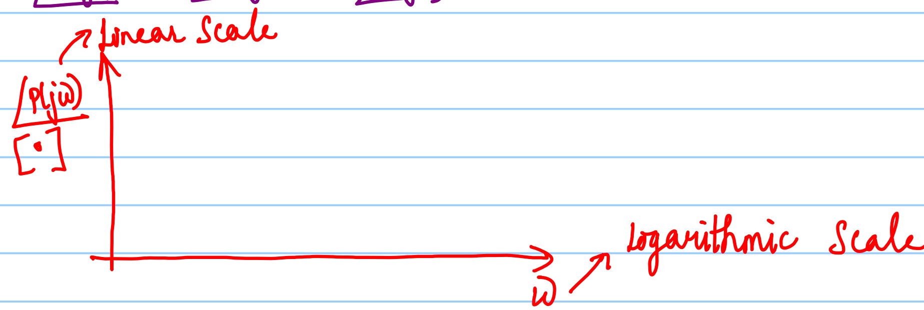
→ Larger range of ω

→ Low frequency region can be expanded.

$$P(j\omega) = \frac{P_1(j\omega)\,P_2(j\omega)}{P_3(j\omega)} \Rightarrow \log_{10}|P(j\omega)| = \log_{10}|P_1(j\omega)| + \log_{10}|P_2(j\omega)| - \log_{10}|P_3(j\omega)|.$$

$$\angle P(j\omega) = \angle P_1(j\omega) + \angle P_2(j\omega) - \angle P_3(j\omega).$$

ii) PHASE PLOT:



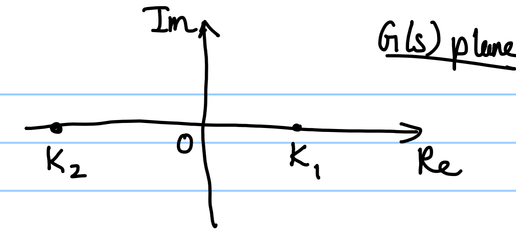
"Building Blocks" : $K,\ \frac{1}{s},\ s,\ \frac{1}{Ts+1},\ Ts+1,\ \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2},\ \frac{s^2+2\xi\omega_n s+\omega_n^2}{\omega_n^2}.$

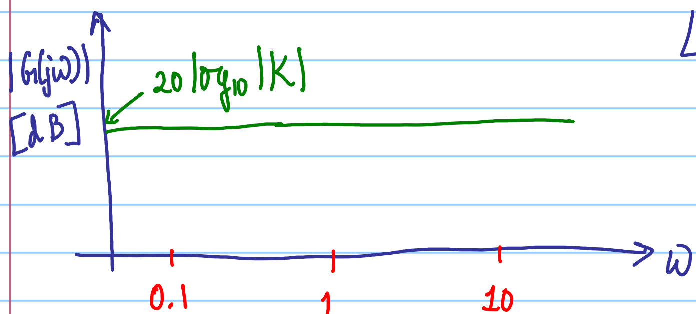
$K,\ \frac{1}{s},\ s,\ \frac{1}{Ts+1},\ Ts+1,\ \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2},\ \frac{s^2+2\xi\omega_n s+\omega_n^2}{\omega_n^2}.$

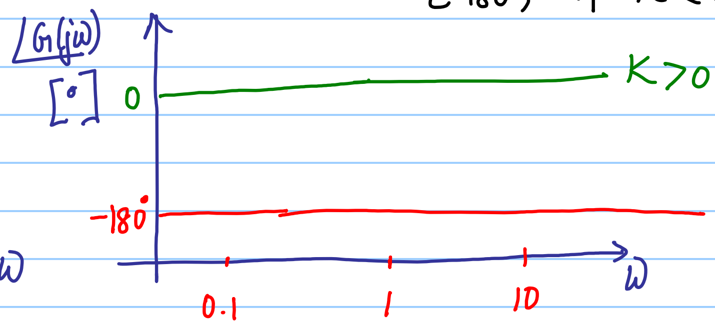
1). $G(s) = K. \quad K \in \mathbb{R}.$ $\quad G(j\omega) = K$, $\quad 20\log_{10}|G(j\omega)|_{dB} = 20\log_{10}|K|$ dB

$|G(j\omega)| = |K|$

$$\angle G(j\omega) = \begin{cases} 0^\circ, & \text{if } K > 0 \\ -180^\circ, & \text{if } K < 0. \end{cases}$$







'DECADE':

At a given frequency ω,

$[\omega, 10\omega] \rightarrow$ one DECADE ABOVE ω

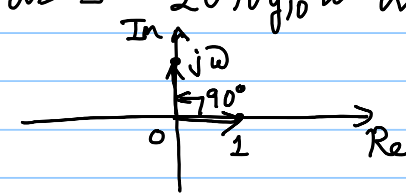
$[0.1\omega, \omega] \rightarrow$ one DECADE BELOW ω.

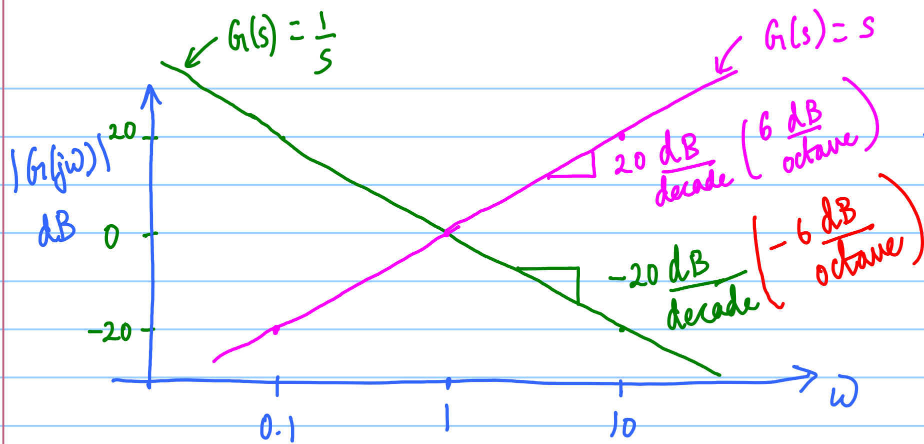
OCTAVE: $[\omega, 2\omega] \rightarrow$ one OCTAVE ABOVE ω.

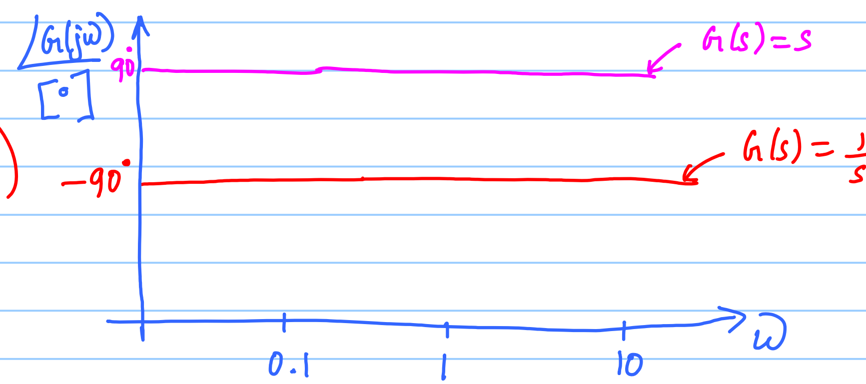
$[0.5\omega, \omega] \rightarrow$ one " BELOW ω.

2). $G(s) = \frac{1}{s}$. $\quad G(j\omega) = \frac{1}{j\omega}$. $\Rightarrow |G(j\omega)| = \frac{1}{\omega}$, $\quad 20\log_{10}|G(j\omega)|$ dB $= -20\log_{10}\omega$ dB.

$\angle G(j\omega) = -90^\circ$.







At $(10\ \omega)$, $-20\log_{10}(10\omega)$ dB $= \left[\underbrace{-20}\ \underbrace{-20\log_{10}\omega}\right]$ dB

At (2ω), $-20\log_{10}(2\omega)$ dB $= (-6 - 20\log_{10}\omega)$ dB.

3). $G(s) = s \qquad G(j\omega) = j\omega \Rightarrow |G(j\omega)| = \omega \Rightarrow 20\log_{10}|G(j\omega)|$ dB $= 20\log_{10}\omega$ dB.

$\angle G(j\omega) = 90°$.

Q: What do you observe about factors that are reciprocals of each other?

$$P(s) = \frac{s+2}{s^3+4s^2+3s} = \frac{s+2}{s(s^2+4s+3)} = \frac{s+2}{s(s+1)(s+3)} = \frac{2\left(\frac{s}{2}+1\right)}{3s(s+1)\left(\frac{s}{3}+1\right)}$$

$$= \left(\frac{2}{3}\right)\left(\frac{s}{2}+1\right)\left(\frac{1}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{1}{\frac{s}{3}+1}\right).$$

Note Title 19-Mar-18

19/3/2018. **Bode Plot:** K, s, $\frac{1}{s}$.

4). $G(s) = \frac{1}{Ts+1}$, $T > 0$. $G(j\omega) = \frac{1}{1+j(T\omega)} = \frac{1-j(T\omega)}{1+T^2\omega^2}$.

$|G(j\omega)| = \boxed{\frac{1}{\sqrt{1+T^2\omega^2}}}$, $\angle G(j\omega) = -\tan^{-1}(T\omega)$.

At low frequencies, $\omega \ll \frac{1}{T}$ ($T\omega \ll 1$), $|G(j\omega)| \to 1$ (0 dB). → LOW FREQUENCY ASYMPTOTE.

At high frequencies, $\omega \gg \frac{1}{T}$ ($T\omega \gg 1$), $|G(j\omega)| \to \frac{1}{T\omega}$ $(-20\log_{10}(T\omega)\ \text{dB})$ → HIGH FREQUENCY ASYMPTOTE.

Note that the low frequency asymptote and the high frequency asymptote intersect at $\omega = \frac{1}{T}$.

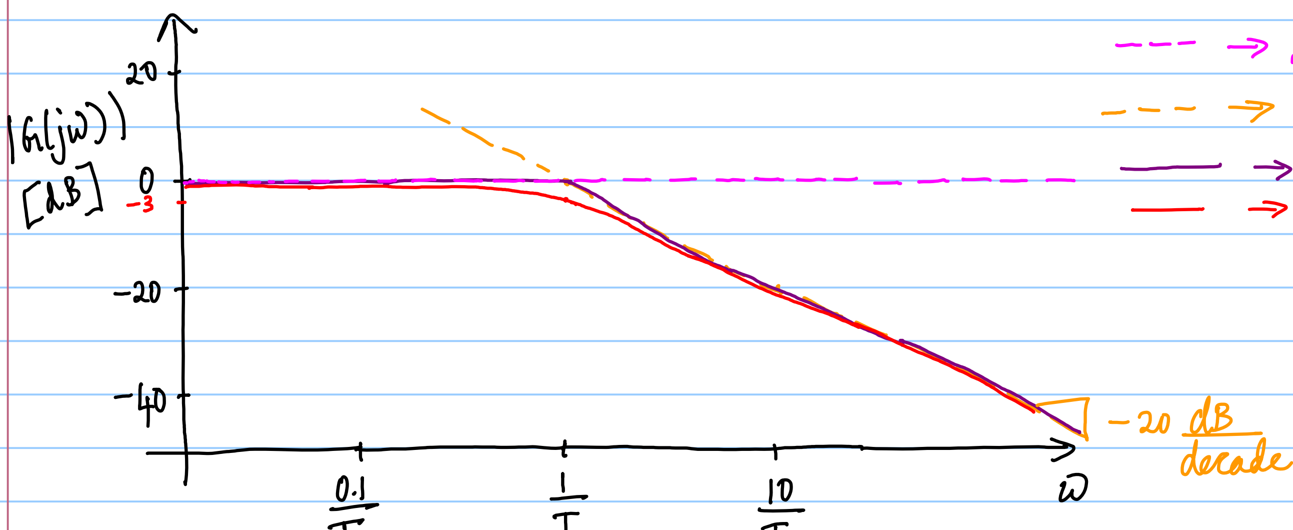
↑ CORNER FREQUENCY / BREAK FREQUENCY.

Q: What is the slope of the high frequency asymptote?

A: At a frequency one decade above ω,

$$-20\log_{10}(T(10\omega)) = -20\log_{10}(10) - 20\log_{10}(T\omega) = -20 - 20\log_{10}(T\omega)\ dB.$$

$\Rightarrow$ The slope of the high frequency asymptote is $-20\ \frac{dB}{decade}$.



$-20\log(T\omega)$ dB

$$|G(j\omega)| = -20\log_{10}\left(\sqrt{1+T^2\omega^2}\right)\ dB.$$

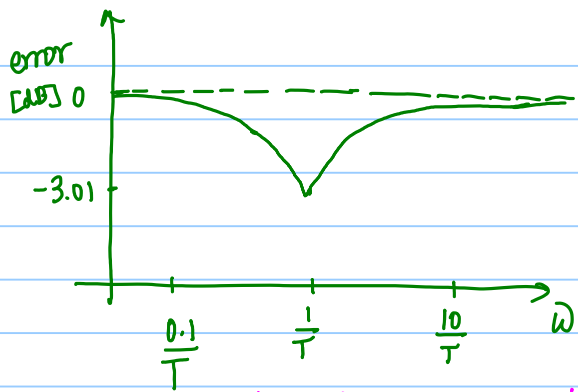
--- → Low frequency asymptote.

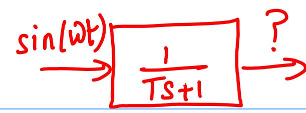
--- → High frequency asymptote.

—— → Asymptotic approximation of $\left|\frac{1}{Tj\omega+1}\right|$

—— → Actual

ω	$\lvert G(j\omega)\rvert$ (dB)	Asymptote value (dB)
$\frac{1}{T}$	-3.01	0
	Difference $= -3.01$ dB	
$\frac{2}{T}$	-6.99	-6.02
	-0.97 dB	
$\frac{0.5}{T}$	-0.97	0
	-0.97 dB	





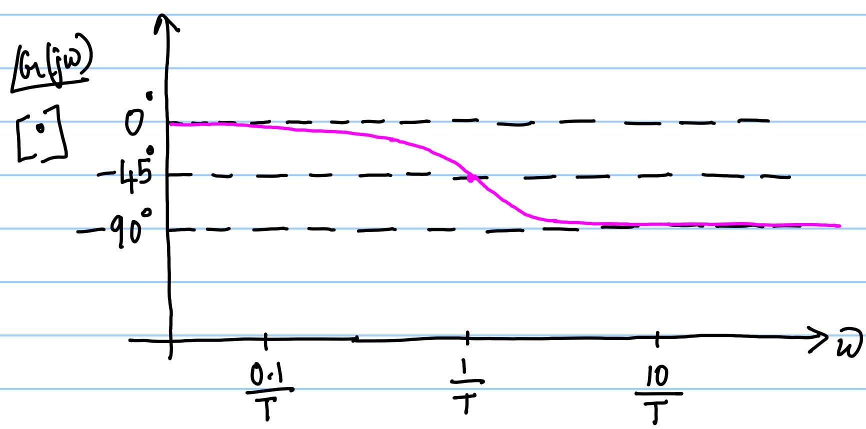
HIGH FREQUENCY ROLL-OFF

$\frac{10}{T}$ $\quad \underbrace{-20.04 \quad\quad -20}_{-0.04\ dB}$

$\frac{0.1}{T}$ $\quad \underbrace{-0.04 \quad\quad 0}_{-0.04\ dB}$

PHASE PLOT: $\quad \angle G(j\omega) = -\tan^{-1}(T\omega).$

ω	$\angle G(j\omega)$
$\to 0$	$\to 0^\circ$
$\frac{1}{T}$	-45°
$\to \infty$	$\to -90^\circ$



HW. Plot the phase plot using a linear scale & logarithmic scale for ω.

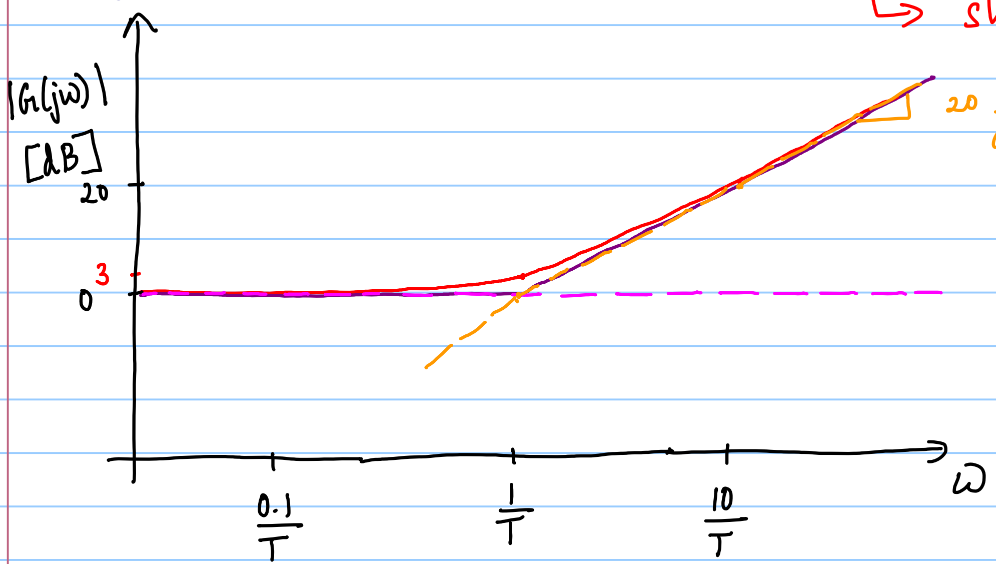
5). $G(s) = Ts+1,\ T>0$. $G(j\omega) = Tj\omega + 1$, $|G(j\omega)| = \sqrt{1+T^2\omega^2}$ $\left(20\log_{10}\sqrt{1+T^2\omega^2}\ dB\right)$.

$\angle G(j\omega) = \tan^{-1}(T\omega)$.

Low Frequency Asymptote $\rightarrow$ 0 dB.

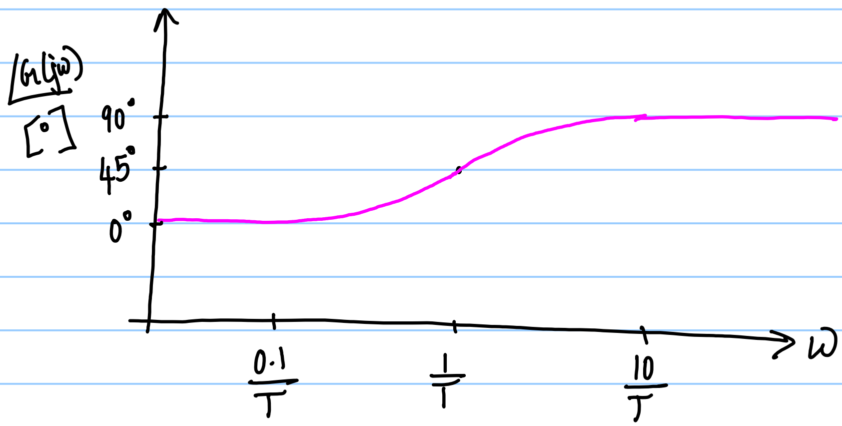
High " " $\rightarrow$ $20\log_{10}(T\omega)$ dB $\rightarrow$ Intersect at $\omega = \frac{1}{T}$. $\rightarrow$ Corner Frequency

$\hookrightarrow$ slope of high frequency asymptote $= 20\ \frac{dB}{decade}$.



HW: Calculate the error in magnitude (as before)

Phase Plot: $\angle G(j\omega) = \tan^{-1}(T\omega)$



20/3/2018.

6). $G(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \dfrac{1}{\left(\frac{s}{\omega_n}\right)^2 + \left(\frac{2\xi}{\omega_n}\right)s + 1}$. $G(j\omega) = \dfrac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\left(\frac{2\xi\omega}{\omega_n}\right)}$

$|G(j\omega)| = \dfrac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$, $\angle G(j\omega) = -\tan^{-1}\left(\dfrac{\frac{2\xi\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$.

$$|G(j\omega)| = -20\log_{10}\left(\sqrt{\underbrace{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2}_{\sim\left(\frac{\omega}{\omega_n}\right)^4}+\left(\frac{2\xi\omega}{\omega_n}\right)^2}\right)\text{dB}.$$

LOW FREQUENCY ASYMPTOTE

$|G(j\omega)| \to 0$ dB, when $\omega \ll \omega_n$.

$|G(j\omega)| \to -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$ dB, when $\omega \gg \omega_n$.

HIGH FREQUENCY ASYMPTOTE

Q: Where do the 2 asymptotes intersect? At $\omega = \omega_n$. $\to$ CORNER FREQUENCY.

Q: What is the slope of the high frequency asymptote?

$$-40\log_{10}\left(\frac{10\omega}{\omega_n}\right)\text{dB} = -40\text{ dB} - 40\log_{10}\left(\frac{\omega}{\omega_n}\right)\text{dB}.$$

Slope of the high frequency asymptote $= -40\ \frac{\text{dB}}{\text{decade}}$.

Q: When is $|G(j\omega)|$ a maximum?

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}}.$$

Let $f(\omega) = \left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2$

HW: Evaluate $\frac{d^2f(\omega)}{d\omega^2}$.

$$\frac{df(\omega)}{d\omega} = -\cancel{2}\left(1-\frac{\omega^2}{\omega_n^2}\right)\frac{\cancel{2}\omega}{\cancel{\omega_n^2}} + \cancel{2}\left(\frac{\cancel{2}\xi\omega}{\cancel{\omega_n}}\right)\frac{2\xi}{\cancel{\omega_n}} = 0.$$

$$\left[-\left(1-\frac{\omega^2}{\omega_n^2}\right)+2\xi^2\right]\omega = 0.$$

$\omega = 0$, $\xi \geq \frac{1}{\sqrt{2}}$.

$2\xi^2 = 1-\frac{\omega^2}{\omega_n^2} \Rightarrow \underbrace{\omega = \omega_n\sqrt{1-2\xi^2}}_{\text{exists only when } 1-2\xi^2>0}$.

$\Rightarrow \xi < \frac{1}{\sqrt{2}}$.

$$\boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}.}$$

RESONANT FREQUENCY

$0 < \xi < \frac{1}{\sqrt{2}}$.

Undamped 2nd order system:

$$P(s) = \frac{\omega_n^2}{s^2+\omega_n^2}.$$

$$u(t) = \sin(\omega_n t) \Rightarrow Y(s) = \frac{\omega_n^2\,\omega_n}{(s^2+\omega_n^2)^2}.$$

$|y(t)| \to \infty$ as $t \to \infty$.

1). Natural Frequency, ω_n.

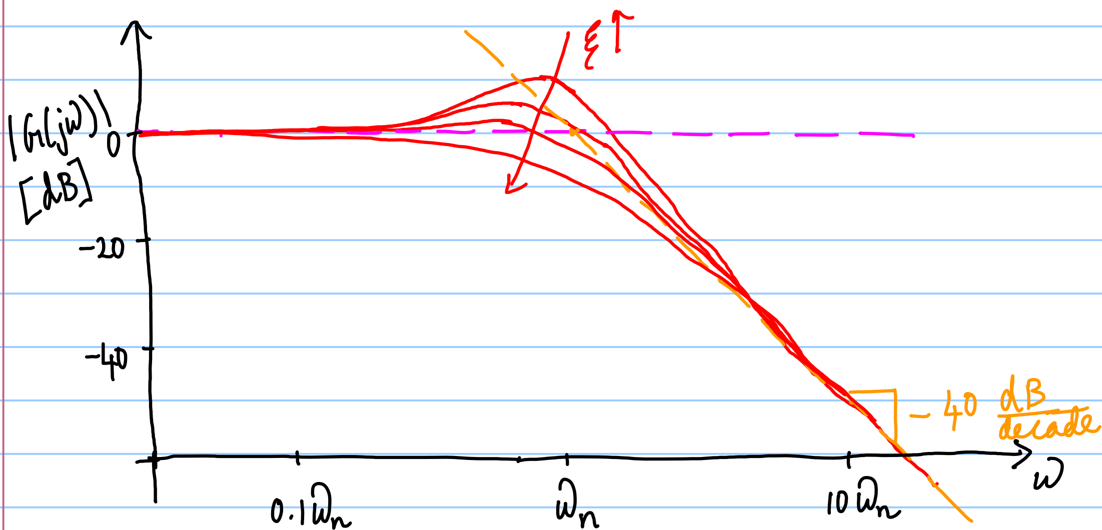
2). Damped Natural Frequency, $\omega_d := \omega_n\sqrt{1-\xi^2}$.

3). Resonant Frequency, $\omega_r := \omega_n\sqrt{1-2\xi^2}$.

TACOMA NARROWS BRIDGE

$$|G(j\omega)|_{max} = |G(j\omega_r)| = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\angle G(j\omega_r) = -\tan^{-1}\left(\frac{\sqrt{1-2\xi^2}}{\xi}\right).$$



MAGNITUDE PLOT

$$G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$G(j\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_n^2}}.$$

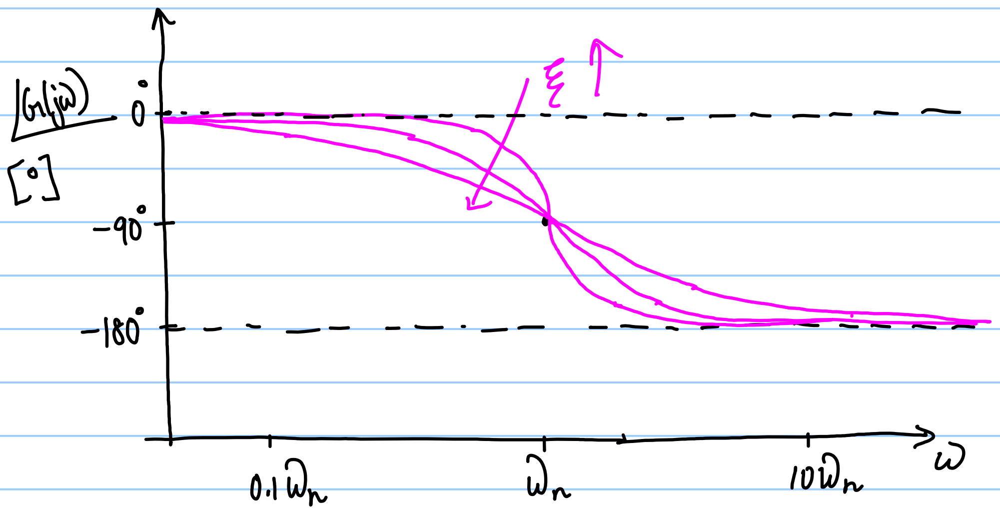
$$|G(j\omega)| = \frac{1}{\left|1 - \frac{\omega^2}{\omega_n^2}\right|}.$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{2\xi\omega/\omega_n}{1 - \omega^2/\omega_n^2}\right)$$

0° as $\omega \to 0$

-90° as $\omega = \omega_n$

-180° as $\omega \to \infty$



7). $G(s) = \dfrac{s^2 + 2\xi\omega_n s + \omega_n^2}{\omega_n^2}$. HW.

Minimum Phase and Non-minimum phase system:

Consider 2 stable systems whose transfer functions are

$$G_1(s) = \frac{1+Ts}{1+T_1 s}, \quad T>0, \quad T_1>0, \quad \text{and} \quad (1+Ts)\left(\frac{1}{1+T_1 s}\right)$$

$(1+Ts)$: 20 $\frac{dB}{decade}$; $\frac{1}{1+T_1 s}$: $-20 \frac{dB}{decade}$

$$G_2(s) = \frac{1-Ts}{1+T_1 s}, \quad T>0, \quad T_1>0.$$

$$G_1(j\omega) = \frac{1+j(T\omega)}{1+j(T_1\omega)}. \qquad G_2(j\omega) = \frac{1-j(T\omega)}{1+j(T_1\omega)}.$$

$$|G_1(j\omega)| = \frac{\sqrt{1+T^2\omega^2}}{\sqrt{1+T_1^2\omega^2}}. \xleftrightarrow{\text{SAME}} |G_2(j\omega)| = \frac{\sqrt{1+T^2\omega^2}}{\sqrt{1+T_1^2\omega^2}}.$$

$$\angle G_1(j\omega) = \tan^{-1}(T\omega) - \tan^{-1}(T_1\omega). \qquad \angle G_2(j\omega) = -\tan^{-1}(T\omega) - \tan^{-1}(T_1\omega).$$

$$\angle G_1(j\omega) \to \boxed{0^\circ} \text{ as } \omega \to \infty. \qquad \angle G_2(j\omega) \to \boxed{-180^\circ} \text{ as } \omega \to \infty.$$

21/3/18.

In general, $G(s) = \frac{n(s)}{d(s)}$. $n(s) \longrightarrow$ order m, $d(s) \longrightarrow$ order n

The slope of the magnitude plot of $G(j\omega)$ would tend to $\boxed{-20(n-m)\ \frac{dB}{decade}}$ as $\omega \to \infty$.

But, the phase of $G(j\omega)$ would tend to $-90^\circ(n-m)$ only for MINIMUM PHASE SYSTEMS.

Example: $G(s) = \frac{s}{(s+1)(s+10)} = \frac{s}{10(s+1)\left(\frac{s}{10}+1\right)} = \frac{0.1\ s}{(s+1)(0.1s+1)} = (0.1)\ (s)\left(\frac{1}{s+1}\right)\left(\frac{1}{0.1s+1}\right)$

a). $0.1 \rightarrow$ Magnitude $= -20$ dB, Phase $= 0^\circ$.

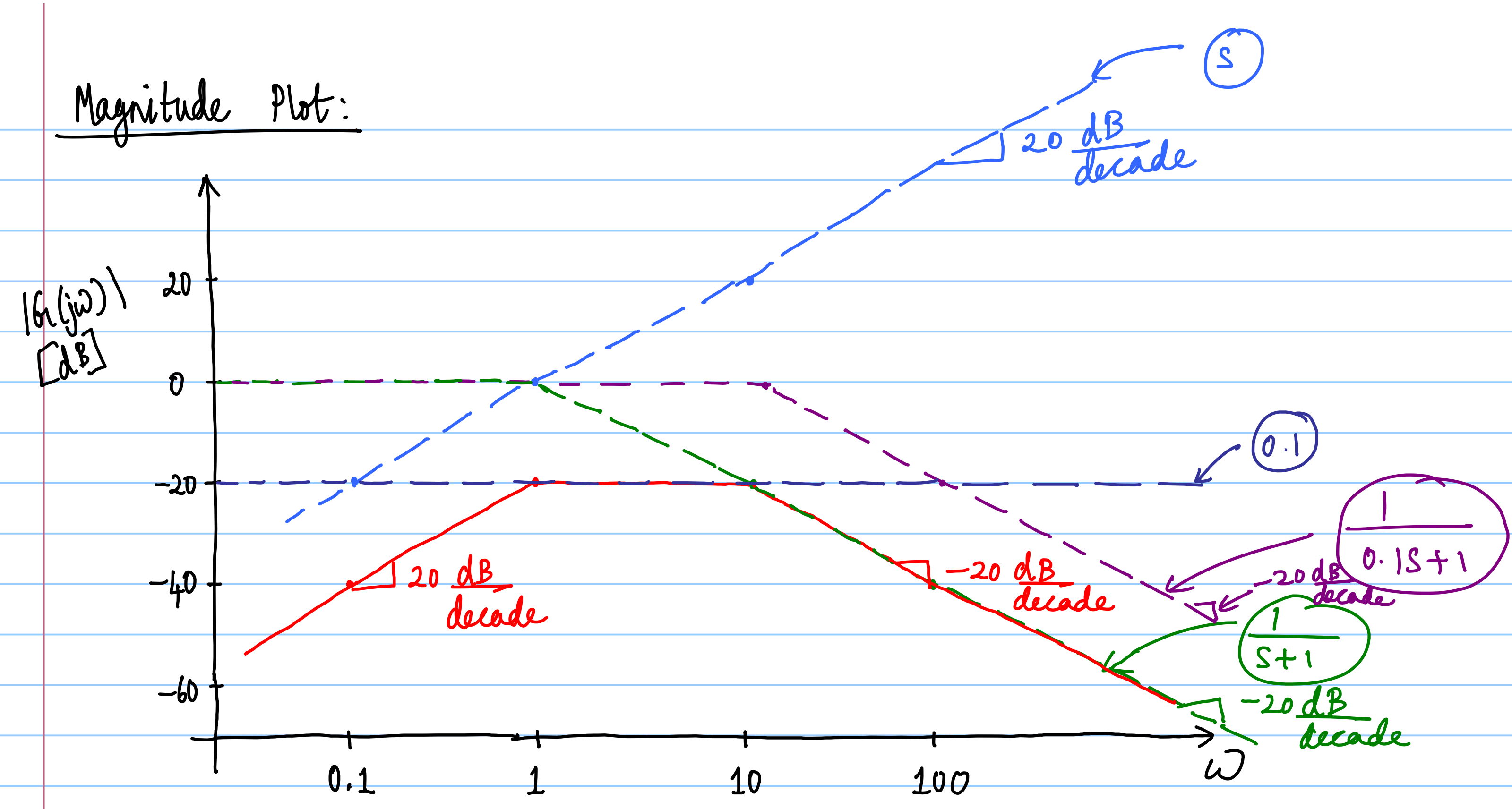
b). s: Magnitude $= 20\log_{10}(\omega)$ dB, Phase $= 90^\circ$.

c). $\frac{1}{s+1}$: Corner frequency $= 1$ rad/s.

d). $\frac{1}{0.1s+1}$: Corner frequency $= 10$ rad/s.

bode

Magnitude Plot:



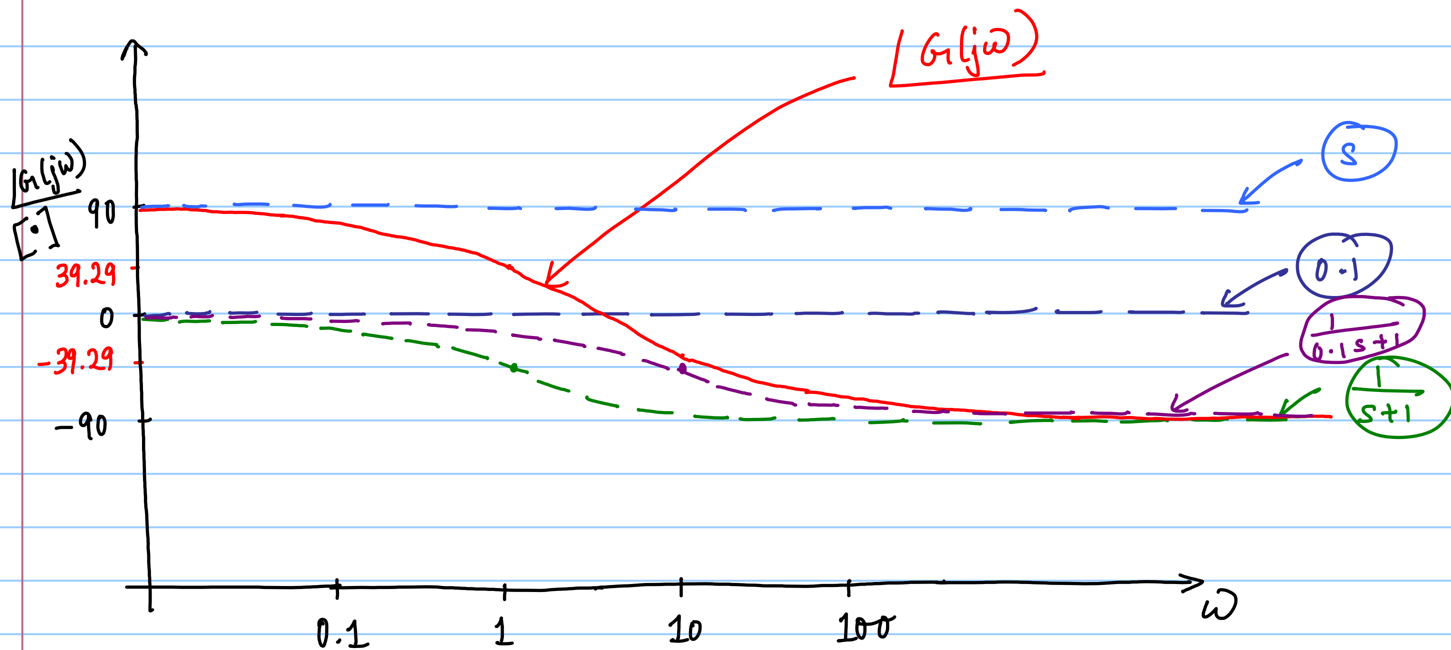
$\omega < 1 \rightarrow (0.1\, s)$

$1 < \omega < 10 \rightarrow \dfrac{0.1\, s}{(s+1)}$

$\omega > 10 \rightarrow \dfrac{0.1\, s}{(s+1)(0.1s+1)}$

$\rightarrow \dfrac{0.1\, s}{(s+1)(0.1s+1)}$

Phase Plot:



$$G(s) = \frac{0.1\ s}{(s+1)(0.1s+1)}.$$

n = 2
m = 1

$$\angle G(j\omega) = 90° - \tan^{-1}(\omega) - \tan^{-1}(0.1\omega)$$

At $\omega = 1$,

$$\angle G(j\omega) = 90° - \tan^{-1}(1) - \tan^{-1}(0.1)$$

$$= 39.29°$$

At $\omega = 10$,

$$\angle G(j\omega) = 90° - \tan^{-1}(10) - \tan^{-1}(1)$$

$$= -39.29°$$

As $\omega \to \infty$,

$$\angle G(j\omega) \to -90°.$$

As $\omega \to 0$,

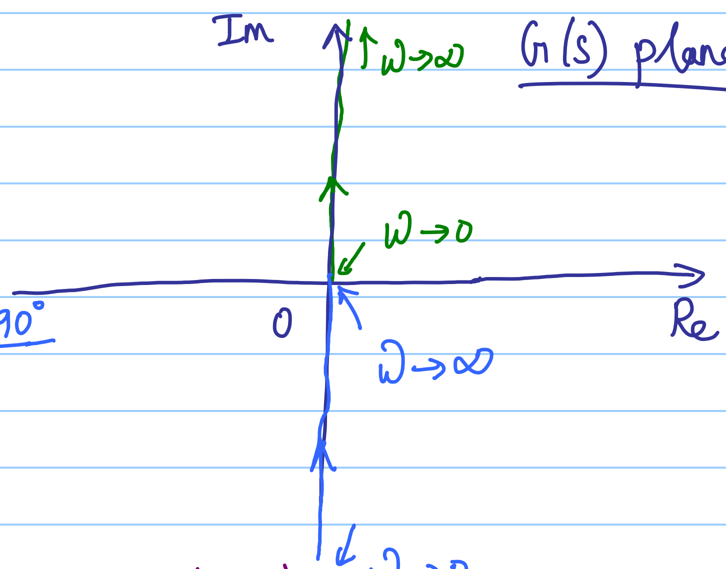
$$\angle G(j\omega) \to 90°.$$

Q: Using the Bode plot, how can one calculate the values of K_{pe}, K_{ve} and K_{ae} (Steady state error analysis)?

Note Title 27-Mar-18

27/3/18. **Nyquist Plot:** → Plot of $G(j\omega)$ in the complex plane.

→ Plot of $Re[G(j\omega)]$ vs $Im[G(j\omega)]$ as ω is varied from 0 to ∞.



a). $G(s) = \frac{1}{s}$, $G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega} = 0 + j\left(-\frac{1}{\omega}\right) = \frac{1}{\omega}\angle -90^\circ$

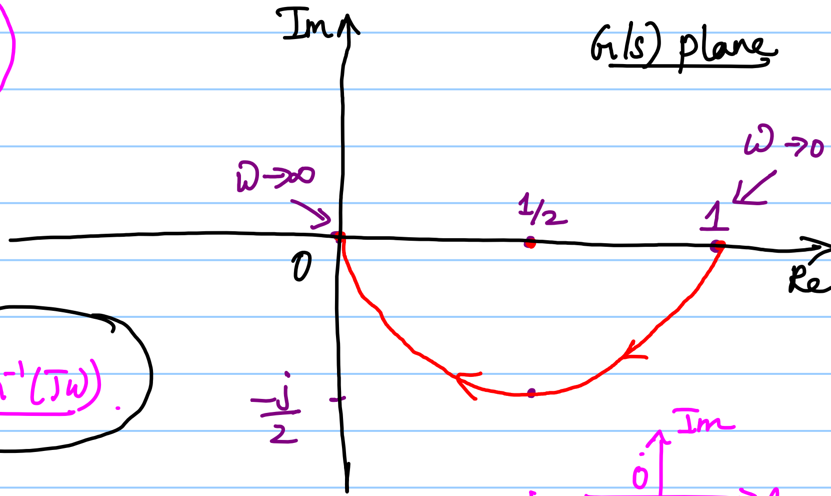
b). $G(s) = s$, $G(j\omega) = j\omega = 0 + j(\omega) = \omega \angle 90^\circ$

c). $G(s) = \frac{1}{Ts+1}$ $(T>0)$ $\Rightarrow G(j\omega) = \frac{1}{1+j(T\omega)} = \frac{1}{1+T^2\omega^2} - j\frac{T\omega}{1+T^2\omega^2} = \frac{1}{\sqrt{1+T^2\omega^2}} \angle -\tan^{-1}(\omega T)$

ω	$Re[G(j\omega)]$	$Im[G(j\omega)]$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
→ 0	1	0	1	0°
$\frac{1}{T}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	-45°

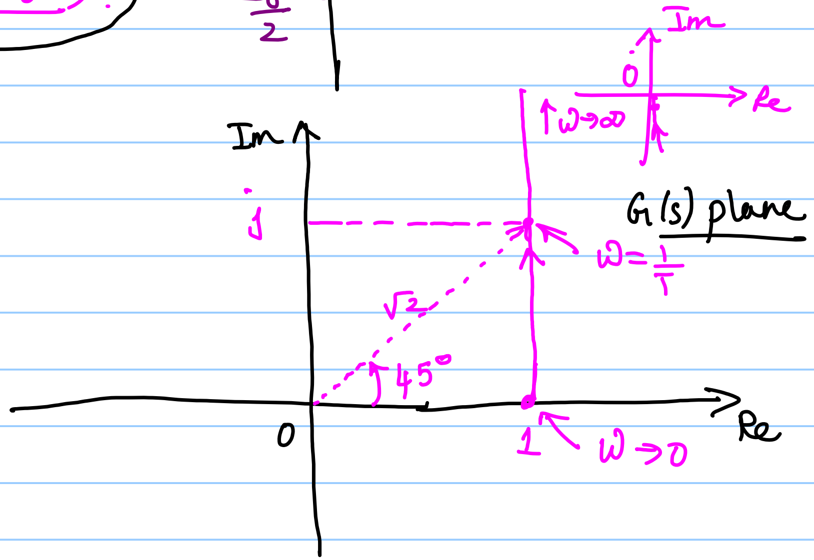
$\to \infty$	0	0	0	-90°

$$\left[\frac{1}{1+T^2\omega^2} - \frac{1}{2}\right]^2 + \left[\frac{-T\omega}{1+T^2\omega^2}\right]^2 = \left(\frac{1}{2}\right)^2.$$



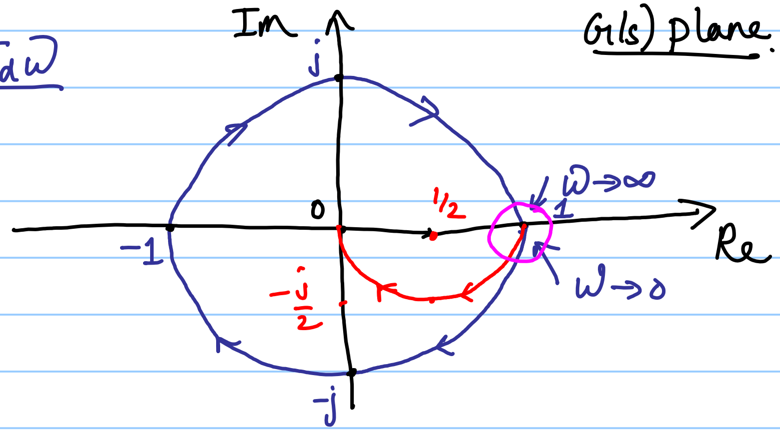
d). $G(s) = Ts + 1 \Rightarrow G(j\omega) = 1 + j(T\omega) = \sqrt{1+T^2\omega^2}\,\angle \tan^{-1}(T\omega)$.

ω	$Re[G(j\omega)]$	$Im[G(j\omega)]$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\to 0$	1	0	1	0°
$= \frac{1}{T}$	1	1	$\sqrt{2}$	45°
$\to \infty$	1	∞	∞	90°



e). <u>Time Delay / Transport Lag</u>: $G(s) = e^{-T_d s}$, $T_d \to$ Time Delay. $\mathcal{L}[u(t-T_d)] = U(s)e^{-T_d s}$.

$$G(j\omega) = e^{-jT_d\omega} = \cos(T_d\omega) - j\sin(T_d\omega) = 1\angle{-T_d\omega}$$



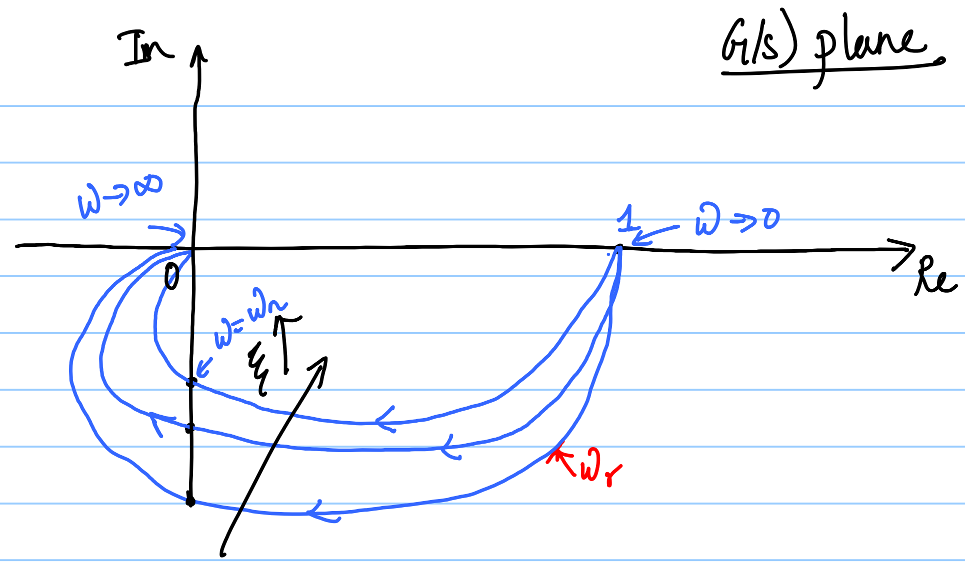
Consider $G(j\omega) = \frac{1}{1+j(T_d\omega)}$.

For $T_d\omega \ll 1$ OR $\omega \ll \frac{1}{T_d}$, $\frac{1}{1+j(T_d\omega)} \approx 1 - j(T_d\omega)$,

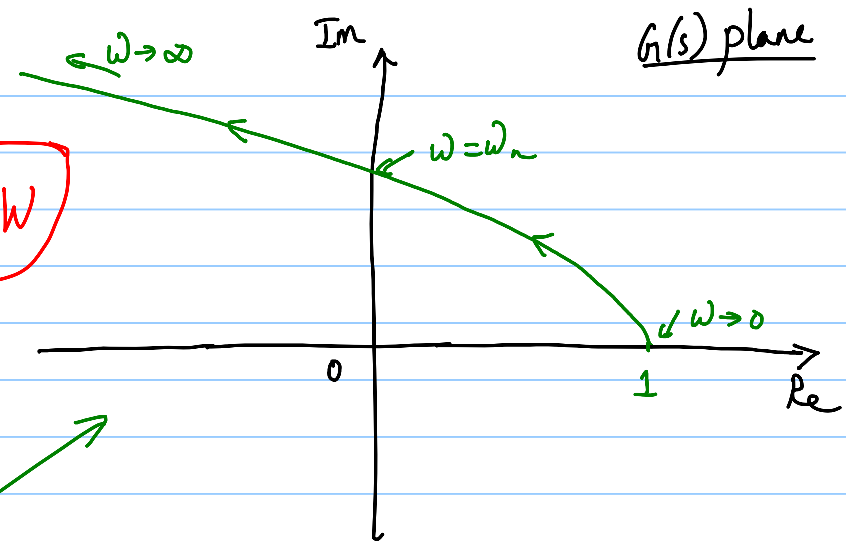
$$e^{-jT_d\omega} \approx 1 - jT_d\omega.$$

f). $G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \Rightarrow G(j\omega) = \frac{1}{1-\frac{\omega^2}{\omega_n^2}+j\left(\frac{2\xi\omega}{\omega_n}\right)} = \frac{1-\frac{\omega^2}{\omega_n^2}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2} - j\,\frac{\frac{2\xi\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}$

$$= \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}}\angle{-\tan^{-1}\left(\frac{\frac{2\xi\omega}{\omega_n}}{1-\omega^2/\omega_n^2}\right)}.$$



HW



g). $G(s) = \dfrac{s^2 + 2\xi\omega_n s + \omega_n^2}{\omega_n^2} \Rightarrow G(j\omega) = 1 - \dfrac{\omega^2}{\omega_n^2} + j\,\dfrac{2\xi\omega}{\omega_n} = \sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(\dfrac{2\xi\omega}{\omega_n}\right)^2}\ \angle \tan^{-1}\left(\dfrac{\frac{2\xi\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$

28/3/18 **Example:** $G(s) = \frac{s+1}{s+10}$. Construct its Nyquist Plot.

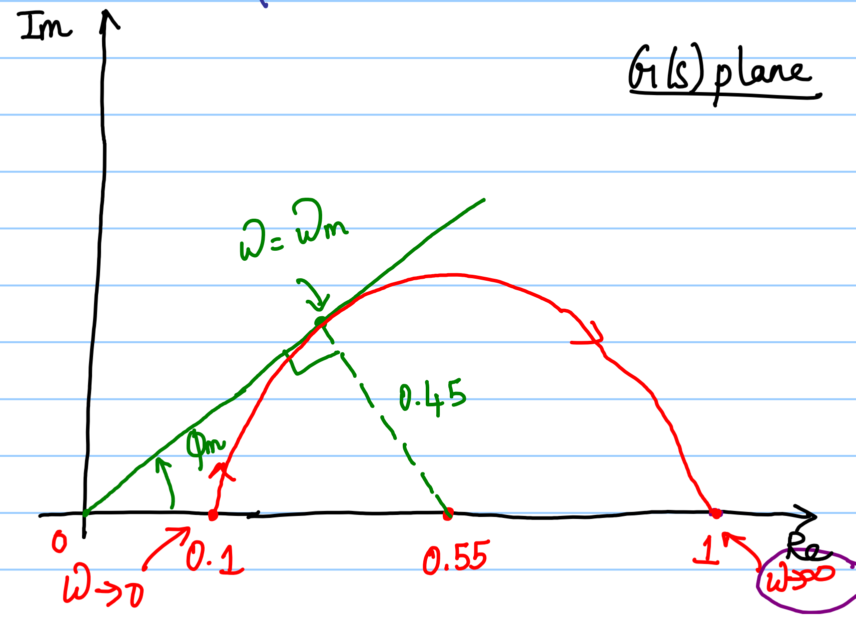
$$G(j\omega) = \frac{1+j\omega}{10+j\omega} = \frac{(1+j\omega)(10-j\omega)}{\omega^2+100} = \boxed{\left(\frac{10+\omega^2}{\omega^2+100}\right)} + j\boxed{\left(\frac{9\omega}{\omega^2+100}\right)}.$$

$$|G(j\omega)| = \sqrt{\left(\frac{10+\omega^2}{\omega^2+100}\right)^2 + \left(\frac{9\omega}{\omega^2+100}\right)^2}, \quad \angle G(j\omega) = \tan^{-1}\left(\frac{9\omega}{10+\omega^2}\right).$$

ω	$Re(G(j\omega))$	$Im(G(j\omega))$
$\to 0$	0.1	0
$\to \infty$	1	0

for $\boxed{0<\omega<\infty,}$ $Re[G(j\omega)] > 0$, $Im[G(j\omega)] > 0$

HW: $$\left[\left(\frac{10+\omega^2}{\omega^2+100}\right) - 0.55\right]^2 + \left[\frac{9\omega}{\omega^2+100}\right]^2 = (0.45)^2.$$



$\phi_m \rightarrow$ maximum phase of $|G(j\omega)|$, ω_m is the corresponding frequency.

$$\sin\phi_m = \frac{0.45}{0.55}, \quad \boxed{\phi_m = 54.9^\circ.}$$

Phase > 0, +ve phase angle, "PHASE LEAD".

Phase < 0, −ve phase $\measuredangle$, "PHASE LAG".

$\underline{\omega_m}$:

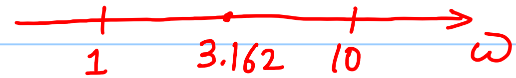
$$\tan\phi_m = \frac{9\,\omega_m}{10+\omega_m^2}. \Rightarrow \boxed{\omega_m = 3.162\ \frac{rad}{s}.}$$

HW

HW: Plot the Bode diagram of $G(s) = \frac{s+1}{s+10}$. $n = 1$, $m = 1$

$$G(s) = \frac{s+1}{s+10} = \frac{s+1}{10\left(\frac{s}{10}+1\right)} = \frac{0.1\,(s+1)}{\left(\frac{s}{10}+1\right)} = (0.1)\underbrace{(s+1)}_{\text{Corner Freq.} = 1\,\frac{rad}{s}}\underbrace{\left(\frac{1}{\frac{s}{10}+1}\right)}_{\text{Corner Freq.} = 10\,\frac{rad}{s}}.$$

$$\log_{10}\sqrt{1*10} = \frac{1}{2}\left[\log_{10}(1*10)\right] = \frac{1}{2}\left[\log_{10}(1) + \log_{10}(10)\right]$$

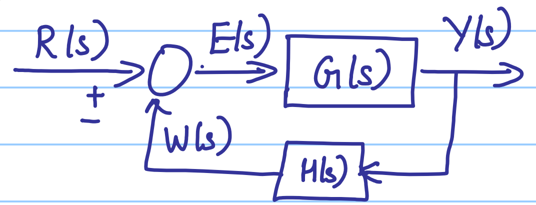


$$G(s) = \frac{n(s) \to \text{order } m}{d(s) \to \text{order } n} \qquad \text{As } \omega \to \infty,\ |\cdot| \to -20(n-m)\ \frac{dB}{decade},\ \angle \to \underbrace{-90^\circ (n-m)}_{\text{min. phase system.}}$$

NYQUIST STABILITY CRITERION

Recall that

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \to \text{Closed loop tr. fn.}$$



$1 + G(s)H(s) = 0 \to$ closed loop characteristic eqn.

closed loop characteristic polynomial $= 1 + G(s)H(s)$.

Let $G(s)H(s) = \frac{n_o(s)}{d_o(s)} \Rightarrow 1 + G(s)H(s) = 1 + \frac{n_o(s)}{d_o(s)} = \boxed{\frac{d_o(s) + n_o(s)}{d_o(s)}}$

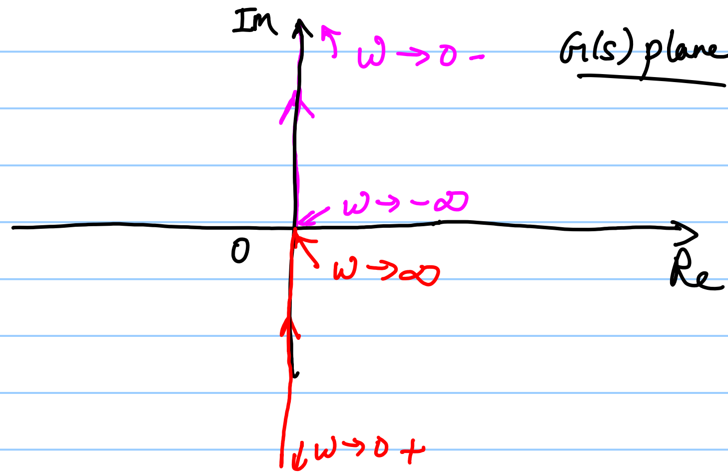
$$\Rightarrow \boxed{\begin{array}{l} \text{Poles of } (1+G(s)H(s)) \to \text{OPEN LOOP POLES.} \\ \text{Zeros of } (1+G(s)H(s)) \to \text{CLOSED LOOP POLES.} \end{array}}$$

Q: Given the open loop transfer function $G(s)H(s)$, can we comment about the stability of the closed loop system using the Nyquist Plot of the open loop transfer function $G(s)H(s)$?

A: Yes. 'MAPPING THEOREM' $\longrightarrow$ 'NYQUIST STABILITY CRITERION'.

Q: Let the plot of $G(j\omega)H(j\omega)$ be given for $\omega \in [0,\infty)$. How can one obtain the plot of $G(j\omega)H(j\omega)$ for $\omega\ (-\infty, 0]$?

$$G(s) = \frac{1}{s} \Rightarrow G(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega}.$$

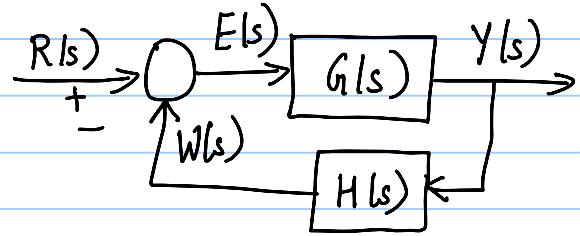


Note Title 02-Apr-18

2/4/18.

NYQUIST STABILITY CRITERION

Recall that $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$.



Closed Loop Characteristic Polynomial $= 1 + G(s)H(s)$.

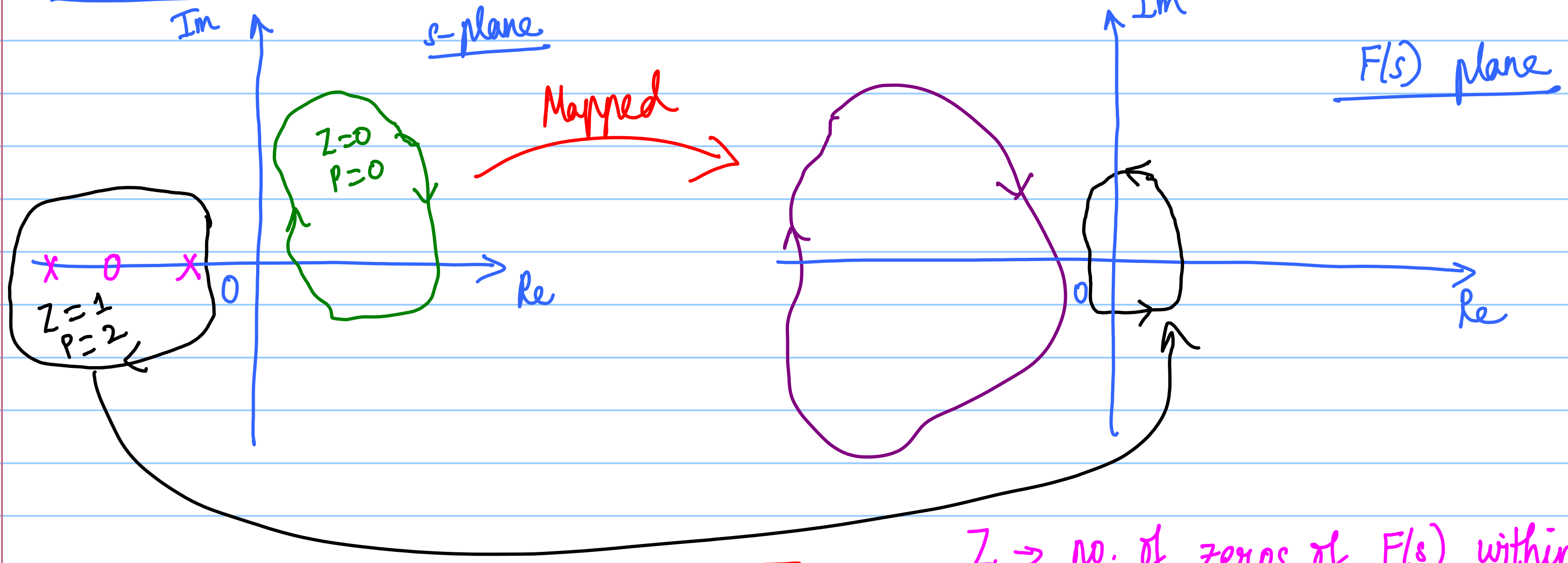
Open loop transfer function, $G(s)H(s) = \dfrac{n_o(s)}{d_o(s)}$.

$$\Rightarrow \quad 1 + G(s)H(s) = 1 + \frac{n_o(s)}{d_o(s)} = \frac{\boxed{d_o(s) + n_o(s)}}{d_o(s)}.$$

Zeros of $1+G(s)H(s) \rightarrow$ Closed loop poles.

Poles of $1+G(s)H(s) \rightarrow$ Open loop poles.

MAPPING THEOREM: Consider a complex-valued F(s).



$$\boxed{N_c = Z - P.}$$

Z → no. of zeros of F(s) within the closed contour in the s-plane.

P → no. of poles of F(s) within the closed contour in the s-plane.

N_c → no. of clockwise encirclements of the origin in the F(s) plane.

Application of Mapping Theorem to Stability Analysis:

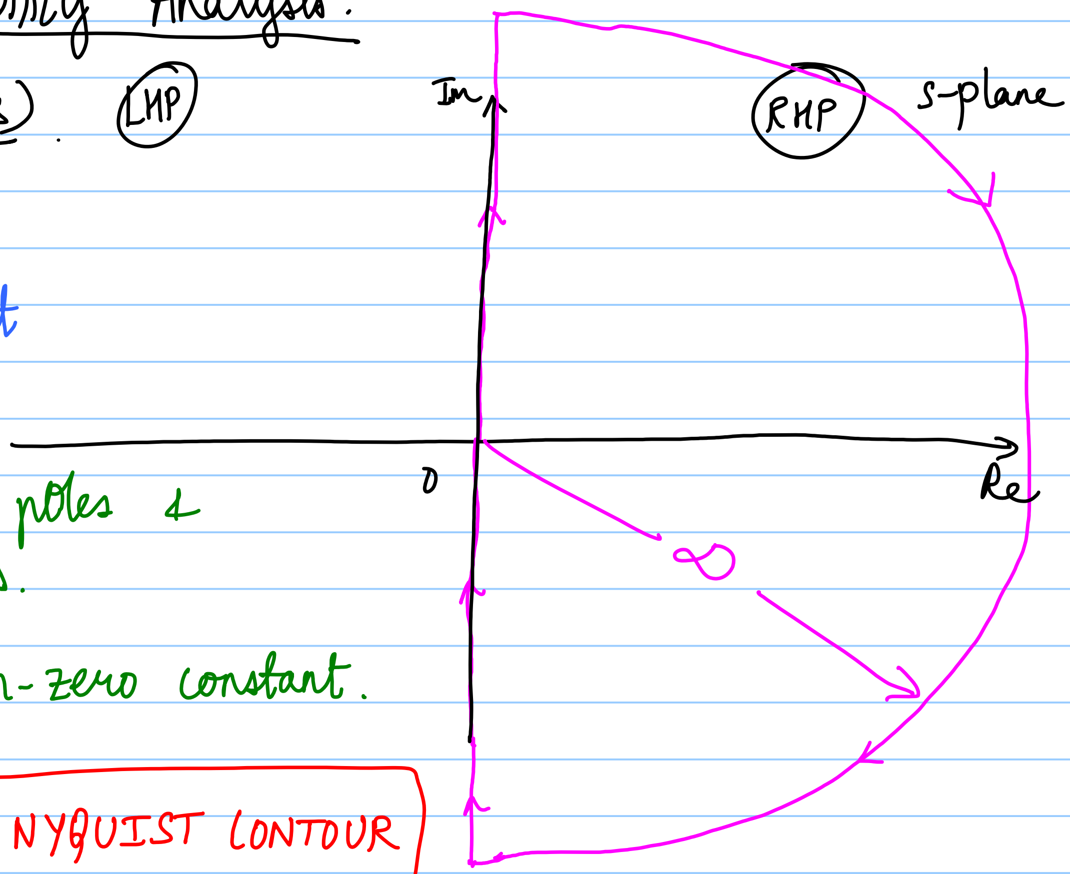
Consider $F(s) = 1 + G(s)H(s) = \dfrac{d_o(s) + n_o(s)}{d_o(s)}$. LHP

Let the closed contour in the s-plane be a semi-circle of infinite radius that sweeps the entire RHP.

→ We assume that none of the open loop poles & open loop zeros lie on the $j\omega$-axis.

→ We assume $\lim_{s\to\infty} G(s)H(s)$ is either zero or a non-zero constant.

$$\boxed{N_c = Z - P.}$$



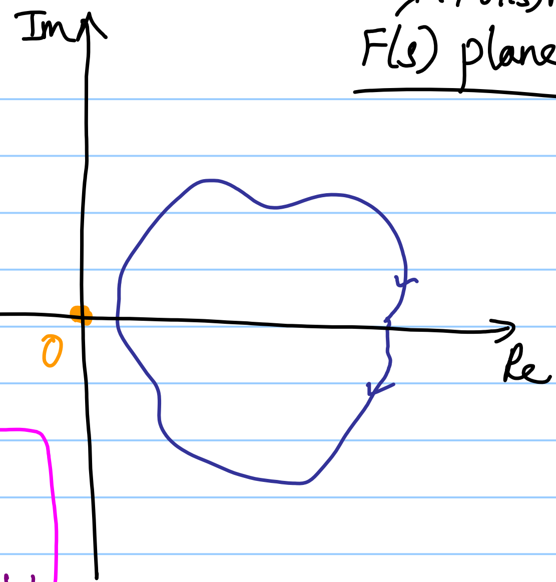
NYQUIST CONTOUR

For closed loop stability, we want $Z = 0$.

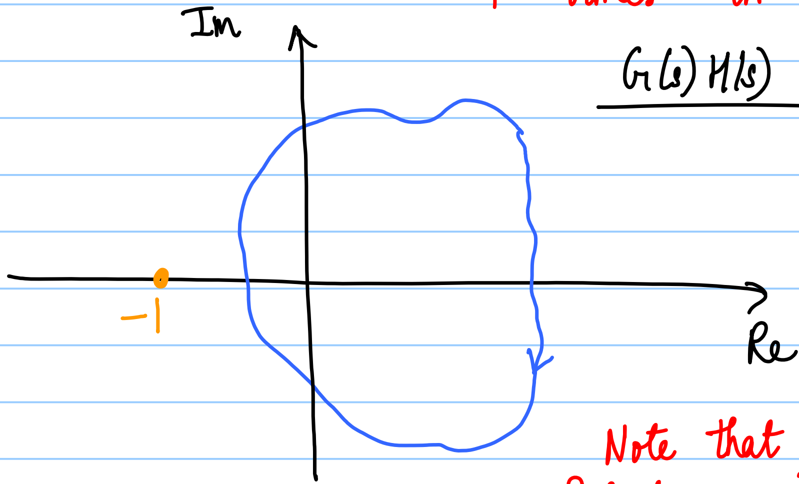
$\Rightarrow$ $\boxed{N_c = -P.}$ $\Rightarrow$ The contour in the $1+G(s)H(s)$ plane should encircle the origin P times in the CCW direction.

$\nearrow 1+G(s)H(s)$

F(s) plane



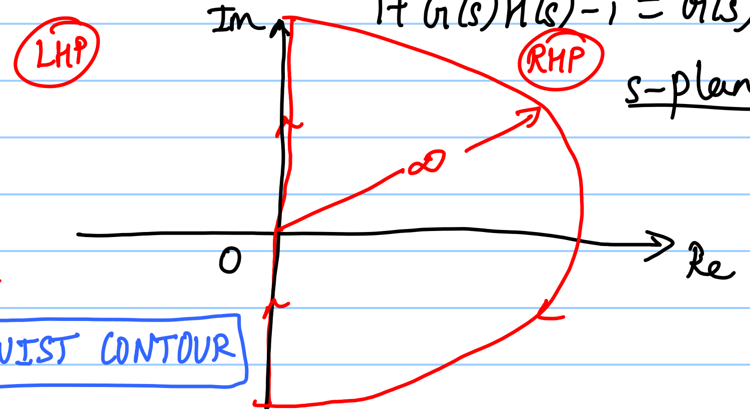
G(s) H(s) plane



The contour in the G(s) H(s) plane should encircle the -1 point P times in the CCW direction.

$1+G(s)H(s)-1 = G(s)H(s)$

s-plane



Note that s is of the form $j\omega$, $\omega \in (-\infty, \infty)$ in this contour.

NYQUIST CONTOUR

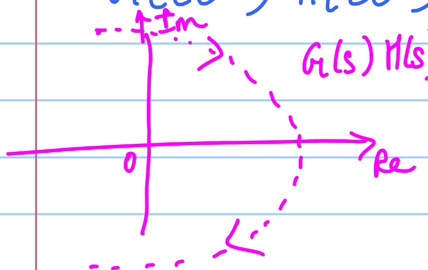
$\Rightarrow$ We are going to be interested in the plot of $G(j\omega)H(j\omega)$, $\omega \in (-\infty, \infty)$.

4/4/2018. Q: What happens if $G(s)H(s)$ has poles and/or zeros on the $j\omega$ axis?

Eg.: Consider $G(s)H(s) = \dfrac{1}{s(s+1)}$ → Poles = 0, −1 ⇒ $G(j\omega)H(j\omega) = \dfrac{1}{j\omega(j\omega+1)}$.

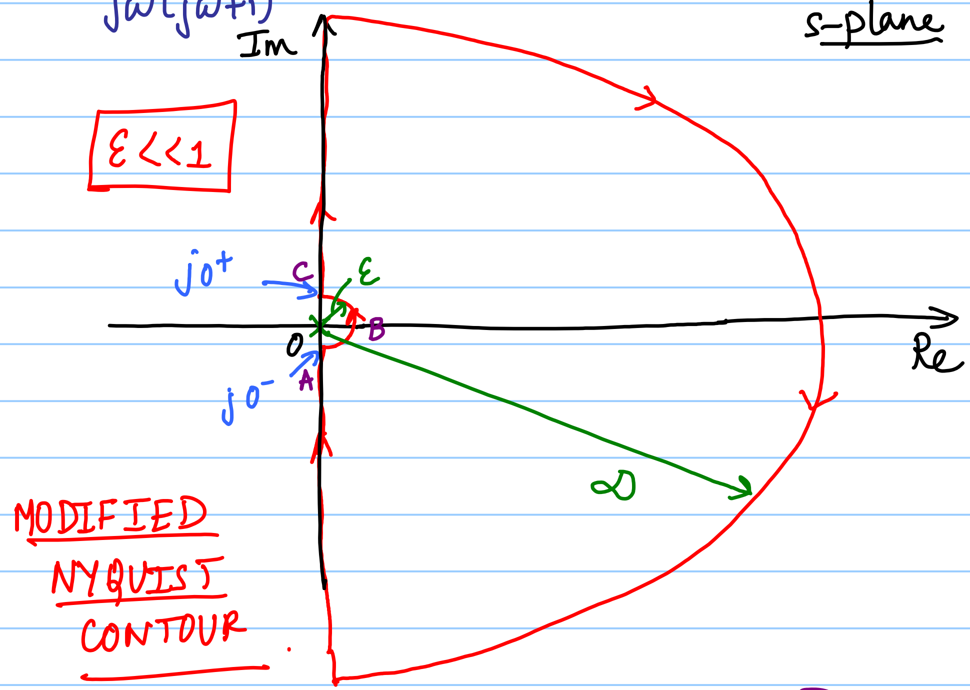
ABC: For $s \in (j0^-, j0^+)$, $s = \varepsilon e^{j\theta}$, $\theta \in [-90^\circ, 90^\circ]$.

$$G(\varepsilon e^{j\theta})H(\varepsilon e^{j\theta}) \simeq \frac{1}{\varepsilon e^{j\theta}} = \frac{1}{\varepsilon} e^{-j\theta}.$$



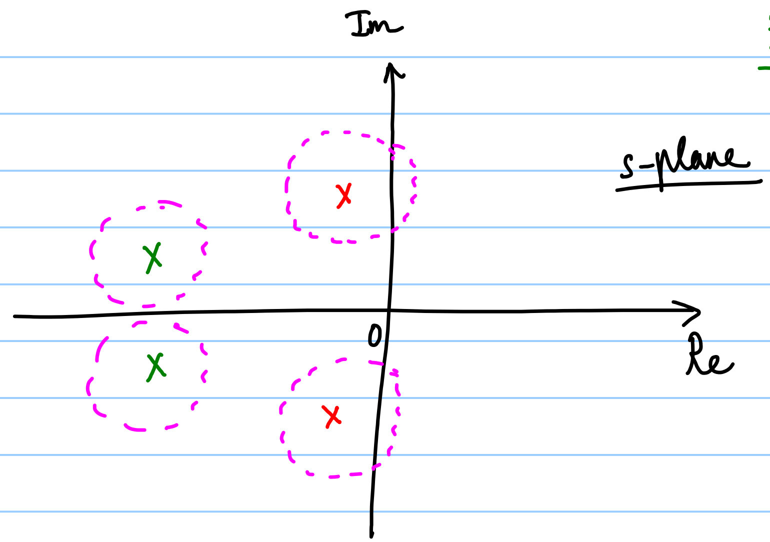
Magnitude → high

Phase → $[90^\circ, -90^\circ]$.



⇒ The corresponding mapped path in the $G(s)H(s)$ plane is a semi-circle of infinite radius in the CW direction.

Please note that, in this example, $P = 0$. For closed loop stability, $N_c \overset{?}{=} 0$.
HW



x → System 1.
x → System 2.

RELATIVE STABILITY

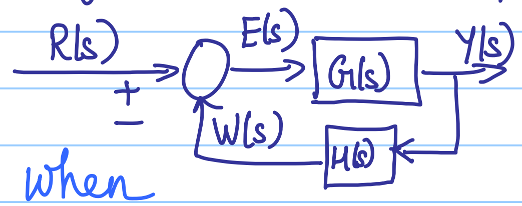
ABSOLUTE STABILITY

We can observe that system 2 is MORE tolerant to parametric uncertainties, unmodeled dynamics, modelling errors, etc.

$\Rightarrow$ System 2 is RELATIVELY MORE STABLE than System 1.

Q: What are measure(s) of the "degree of stability" of a closed loop system?

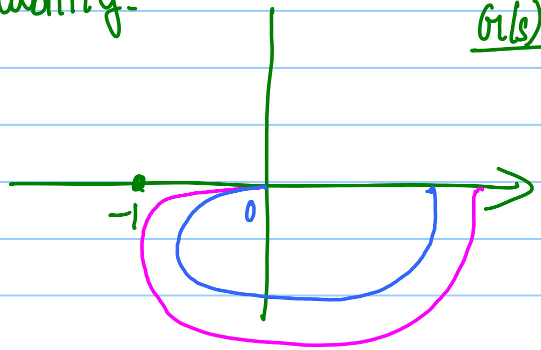
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}. \qquad 1+G(s)H(s) = 0.$$



The closed loop system would become unstable when its poles cross the $j\omega$-axis, i.e., $1+G(j\omega)H(j\omega) = 0$ under this scenario.
In this scenario, $G(j\omega)H(j\omega) = -1$.

$G(j\omega)\,H(j\omega) \Rightarrow$ Open loop sinusoidal transfer function.

The closeness of the $G(j\omega)\,H(j\omega)$ to the -1 point in the $G(s)H(s)$ plane is indocatove of relative stability.



-1 in $G(s)H(s)$ plane $\rightarrow$ Magnitude $= 1$, Phase $= -180^\circ$

Gain Cross Over Frequency (ω_g): $\quad |G(j\omega_g)\,H(j\omega_g)| = 1 \quad (0\ dB)$.

Phase Cross Over Frequency (ω_p): $\quad \angle G(j\omega_p)\,H(j\omega_p) = -180^\circ$.

Eg.: $\quad G(s)\,H(s) = \dfrac{1}{s(s+1)}. \qquad G(j\omega)\,H(j\omega) = \dfrac{1}{j\omega(j\omega+1)}.$

$$\Rightarrow \quad \frac{1}{\omega_g\sqrt{\omega_g^2+1}} = 1 \quad \Rightarrow \quad \omega_g\sqrt{\omega_g^2+1} = 1 \quad \Rightarrow \quad \omega_g^4 + \omega_g^2 = 1.$$

$$\Rightarrow \quad \omega_g^4 + \omega_g^2 - 1 = 0. \longrightarrow \boxed{\omega_g = 0.7862 \ \frac{rad}{s}}.$$

$$\angle G(j\omega)H(j\omega) = 0^\circ - \left[90^\circ + \tan^{-1}(\omega)\right] = -90^\circ - \tan^{-1}(\omega).$$

$$\Rightarrow \quad \boxed{\omega_p = \infty.}$$

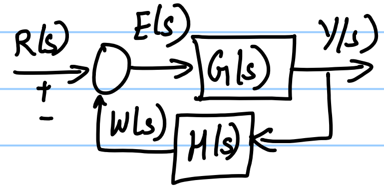
Note Title 09-Apr-18

09/04/2018.

RELATIVE STABILITY

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}.}$$

$$1 + G(s)H(s) = 0.$$

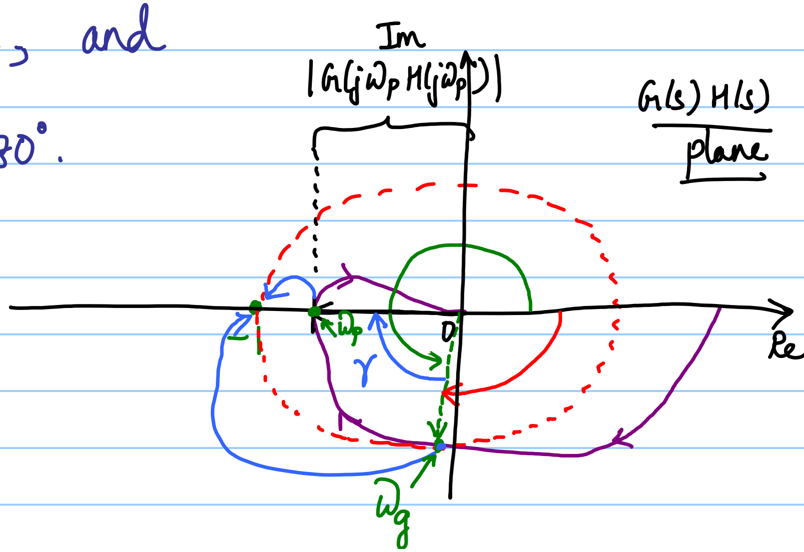


Recall that

i). the closeness of $G(j\omega)H(j\omega)$ locus to the -1 point is indicative of relative stability

ii). Gain Cross-over Frequency (ω_g): $|G(j\omega_g)H(j\omega_g)| = 1$, and

iii). Phase Cross-over Frequency (ω_p): $\angle G(j\omega_p)H(j\omega_p) = -180°$.



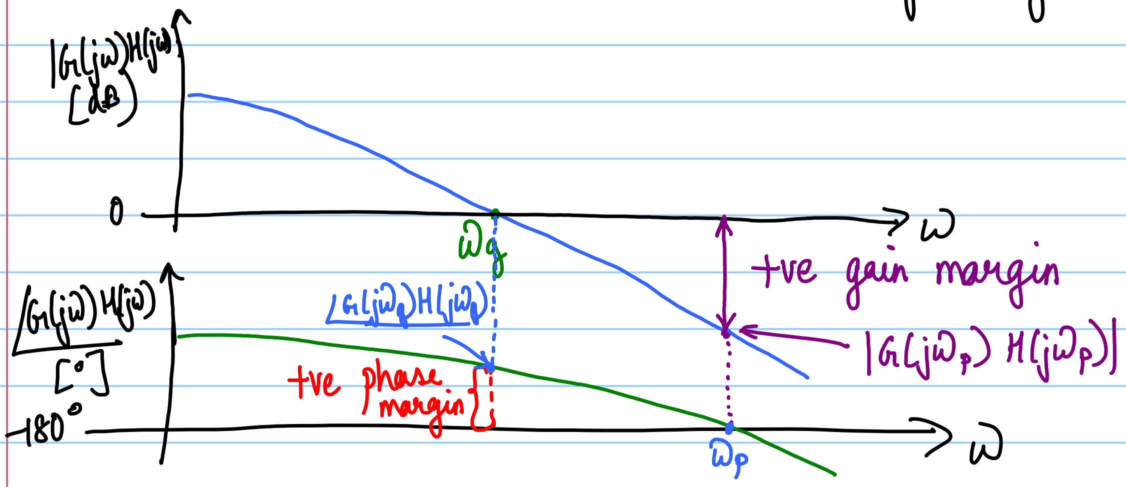
Gain Margin (K_g): It is the reciprocal of $|G(j\omega)H(j\omega)|$ at $\omega = \omega_p$.

$$K_g = \frac{1}{|G(j\omega_p)H(j\omega_p)|} \quad \left(-20\log_{10}|G(j\omega_p)H(j\omega_p)|\ dB\right).$$

Phase Margin (γ): It is the additional amount of phase lag that needs to be added to the phase of the system at $\omega = \omega_g$ to bring the system to the verge of instability.

$$\gamma = \angle G(j\omega_g)H(j\omega_g) - (-180^\circ) = 180^\circ + \angle G(j\omega_g)H(j\omega_g).$$

NOTE: Given a system whose open loop tr. fn. does not have poles/zeros in RHP, both the gain margin (in dB) and the phase margin (in °) should be +ve to ensure the stability of the closed loop system.



Eg.: $G(s)H(s) = \dfrac{1}{s(s+1)}$ $\quad |G(j\omega)H(j\omega)| = \dfrac{1}{j\omega(j\omega+1)}$. $\quad \angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}(\omega)$.

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega\sqrt{\omega^2+1}}.$$

Recall that $\omega_g = 0.7862 \ \frac{rad}{s}$, $\omega_p = \infty$.

$$\Rightarrow \gamma = 180^\circ + \angle G(j\omega_g)H(j\omega_g) = 180^\circ + \left[-90^\circ - \tan^{-1}(0.7862)\right] = \boxed{51.83^\circ}.$$

$$\boxed{K_g = +\infty \ dB.}$$

Consider $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$.

$\left|\dfrac{Y(j0)}{R(j0)}\right|$

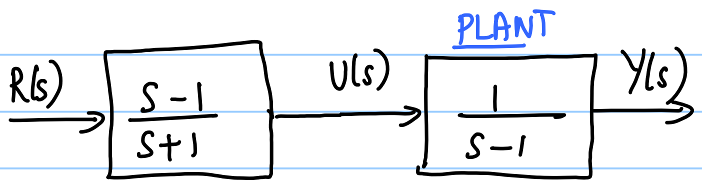
The cut-off frequency (ω_c) is the frequency at which the magnitude of the closed loop frequency response transfer fn. $\left|\dfrac{Y(j\omega)}{R(j\omega)}\right|$ is 3 dB below its zero frequency value.
(low frequency asymptotic value)

The frequency range $0 \leq \omega \leq \omega_c$ is called the "bandwidth" of the system.

Cut-off Rate: It is the slope of the log-magnitude curve at the cutoff frequency.

Higher bandwidth $\Rightarrow$ faster transient response.
$\Rightarrow$ but increased cost.

Performance specifications: Gain margin, phase margin, bandwidth, cut-off rate, cut-off frequency, resonant peak (M_r), resonant frequency (ω_r).



$$Y(s) = \left(\frac{1}{s-1}\right) U(s) = \left(\frac{1}{\cancel{s-1}}\right)\left(\frac{\cancel{s-1}}{s+1}\right) R(s) = \left(\frac{1}{s+1}\right) R(s).$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \left(\frac{1}{s+1}\right).$$

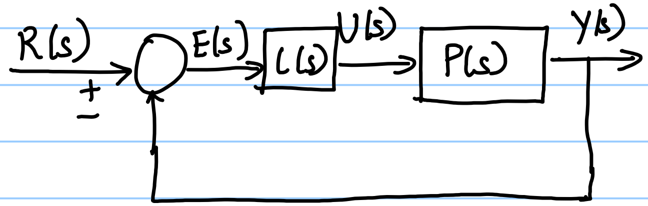
POLE-ZERO CANCELLATION IN THE RHP.

$$Y(s) = \left(\frac{1}{s-1+\varepsilon}\right)\left(\frac{s-1}{s+1}\right) R(s)$$

10/4/2018.

LEAD COMPENSATION

Consider unity negative feedback. $\Rightarrow H(s) = 1$



→ It adds sufficient phase lead to reduce any excessive phase lag associated with the uncompensated system.

open loop transfer function = $G(s)H(s) = C(s)P(s)$.

→ usually improves the transient response, but may amplify high frequency noise.

The transfer function of the lead compensator is

$$C(s) = \frac{K_c \alpha (Ts+1)}{(\alpha Ts+1)} = \frac{K_c\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\alpha T}\right)}, \quad \text{where } 0<\alpha<1, \; T>0, \; K_c>0.$$

Note that the lead compensator introduces an open loop zero at $-\frac{1}{T}$ and an open loop pole at $-\frac{1}{\alpha T}$.

$$C(j\omega) = \frac{K_c\alpha\,(Tj\omega+1)}{(\alpha Tj\omega+1)}. \Rightarrow |C(j\omega)| = \frac{|K_c\alpha|\sqrt{T^2\omega^2+1}}{\sqrt{\alpha^2T^2\omega^2+1}}.$$

Corner frequencies: $\frac{1}{T}, \frac{1}{\alpha T}$.

$$\Rightarrow \angle C(j\omega) = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega).$$

Note that $\angle C(j\omega) \geq 0 \;\forall\, \omega$.

Further, $$C(j\omega) = \frac{K_c\alpha\,(1+jT\omega)}{(1+j\alpha T\omega)} * \frac{(1-j\alpha T\omega)}{(1-j\alpha T\omega)} = \frac{K_c\alpha\,(1+\alpha T^2\omega^2)}{(1+\alpha^2T^2\omega^2)} + j\,\frac{K_c\alpha T\omega(1-\alpha)}{(1+\alpha^2T^2\omega^2)}.$$

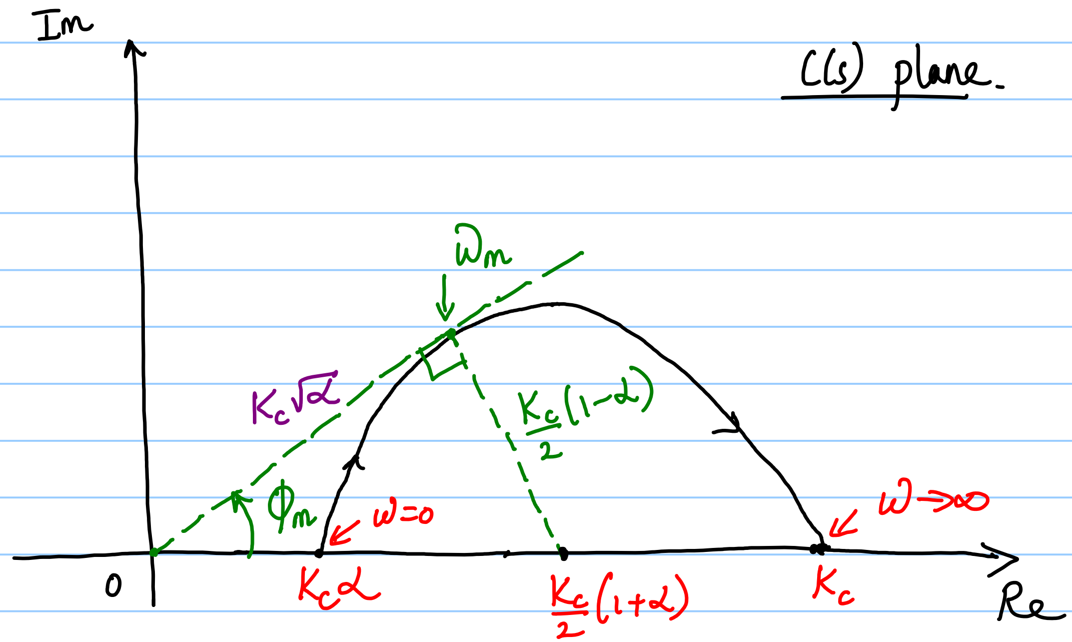
Note that $\boxed{C(j0) = K_c\alpha}$, $\boxed{C(j\infty) = K_c}$.

HW: Show that

$$\left[\frac{K_c\alpha\,(1+\alpha T^2\omega^2)}{(1+\alpha^2T^2\omega^2)} - \frac{K_c}{2}(1+\alpha)\right]^2 + \left[\frac{K_c\alpha T\omega(1-\alpha)}{(1+\alpha^2T^2\omega^2)}\right]^2 = \left[\frac{K_c}{2}(1-\alpha)\right]^2.$$

$\Rightarrow$ The locus of $C(j\omega)$ is a semi-circle of radius $\frac{K_c}{2}(1-\alpha)$ that is centered as $\left[\frac{K_c}{2}(1+\alpha),\, 0\right]$.

$\phi_m \rightarrow$ maximum phase lead provided by the lead compensator at a frequency ω_m.



Note that $\sin\phi_m = \dfrac{\frac{K_c}{2}(1-\alpha)}{\frac{K_c}{2}(1+\alpha)} = \dfrac{(1-\alpha)}{(1+\alpha)}$.

$$\Rightarrow \tan\phi_m = \frac{\frac{K_c}{2}(1-\alpha)}{K_c\sqrt{\alpha}} = \frac{(1-\alpha)}{2\sqrt{\alpha}}.$$

Find ω_m. Recall that

$$\angle C(j\omega) = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega).$$

$$\Rightarrow \phi_m = \tan^{-1}(T\omega_m) - \tan^{-1}(\alpha T\omega_m).$$

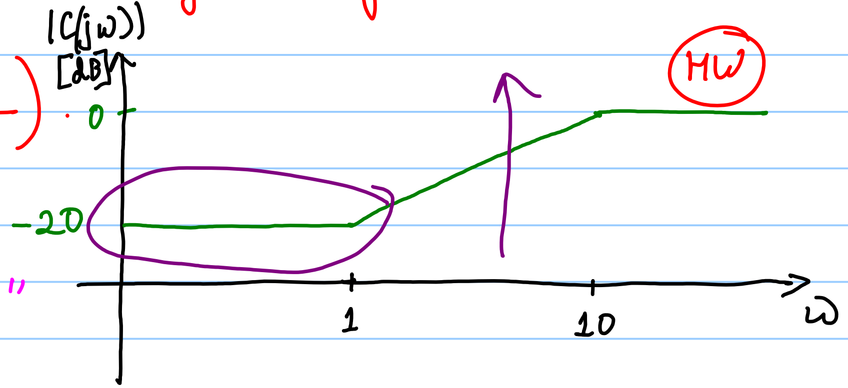
$$\Rightarrow \frac{T\omega_m - \alpha T\omega_m}{1+\alpha T^2\omega_m^2} = \frac{(1-\alpha)}{2\sqrt{\alpha}}.$$

$$\Rightarrow \omega_m^2 (1-\alpha)\alpha T^2 - \omega_m \left[2T\sqrt{\alpha}(1-\alpha) \right] + (1-\alpha) = 0.$$

HW Solve this eqn. to obtain $\boxed{\omega_m = \frac{1}{(\sqrt{\alpha})T}}$. $\Rightarrow$ ω_m is the geometric mean of $\frac{1}{T}$ & $\frac{1}{\alpha T}$.

Example: Consider $K_c = 1$, $\alpha = 0.1$, $T = 1$. Plot the Bode diagram of $C(s)$.

$$C(s) = \frac{K_c \alpha (Ts+1)}{(\alpha Ts + 1)} = \frac{0.1(s+1)}{(0.1s+1)} = (0.1)(s+1)\left(\frac{1}{0.1s+1}\right).$$



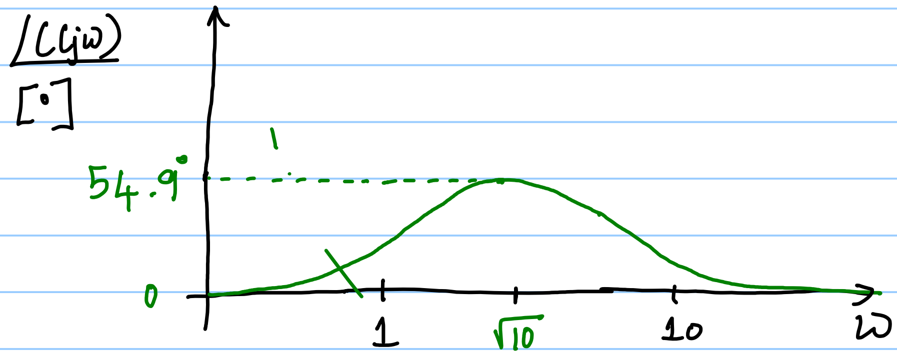
We observe from the log-magnitude plot that

i). the magnitude of the open loop tr. fn. at "low" frequencies is decreased $\Rightarrow$ potentially lead to increase in steady state errors (depending on the system type).

ii). If we increase K_c to address i), then note the high frequency components would be amplified.

Here, $\omega_m = \frac{1}{\sqrt{\alpha T}} = \sqrt{10}$.

$\sin \phi_m = \frac{1-\alpha}{1+\alpha} = \frac{0.9}{1.1} \Rightarrow \boxed{\phi_m = 54.9^\circ}$

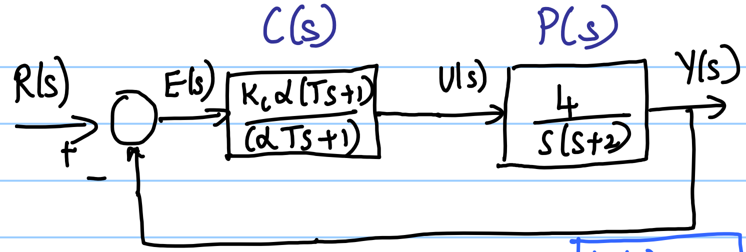


11/4/18. Example: Consider a system / plant whose transfer function is $P(s) = \frac{4}{s(s+2)}$. Design a lead compensator such that $K_{ve} = 20\ s^{-1}$, γ (phase margin) is at least 50° and the gain margin is at least 10 dB. Consider unity negative feedback.

Task: Find the value of K_c, α & T that would stabilize the closed loop system while meeting the above performance requirements.

$$G(s) = C(s)\,P(s) = \frac{(Ts+1)}{(\alpha Ts+1)} \overbrace{\boxed{\underbrace{K_c\alpha}_{K} \left(\frac{4}{s(s+2)}\right)}}^{P_1(s)}$$

K, i.e., $K := K_c\alpha$.



$\boxed{H(s) = 1.}$

$$\Rightarrow \; G(s) = \frac{(Ts+1)}{(\alpha Ts+1)} P_1(s), \text{ where } P_1(s) = K\,P(s), \; K = K_c\alpha.$$

Step 1: Adjust K such that the desired value of K_{ve} is obtained.

Recall that $K_{ve} = \lim_{s\to 0} s\,G(s)\,H(s) = \lim_{s\to 0} \cancel{s} \frac{(Ts+1)}{(\alpha Ts+1)} K \left(\frac{4}{\cancel{s}(s+2)}\right) = 2K.$

Since K_{ve} should be $20\,s^{-1}$, $2K = 20 \Rightarrow \boxed{K = 10.} \Rightarrow \boxed{K_c\alpha = 10.}$

Thus, $P_1(s) = K\,P(s) = 10\left(\frac{4}{s(s+2)}\right) = \frac{40}{s(s+2)}.$

$$\Rightarrow \; G(s) = \overbrace{\left[\frac{(Ts+1)}{(\alpha Ts+1)}\right]}^{1} \overbrace{\left[\frac{40}{s(s+2)}\right]}^{2}.$$

$|G(j\omega)| = \underbrace{|1|}_{>1}\,|2| = |1|\,dB + |2|\,dB.$

Phase Margin, $\gamma = 180^\circ + \angle G(j\omega_g)H(j\omega_g).$
$= 180^\circ + \angle 1 + \angle 2$

$\angle 2$ becomes more −ve as $\omega \uparrow$

$$\angle P_1(j\omega) = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right).$$

$\Rightarrow \omega_p = \infty.$

Step 2: Find α.

Let us first consider $\boxed{P_1(j\omega) = \frac{40}{j\omega(j\omega+2)}}$. Find its phase margin.

$\Rightarrow K_g = \infty$ dB.

The uncompensated system has ∞ gain margin.

First find its gain cross-over frequency.

$$\Rightarrow \left|\frac{40}{j\omega_g(j\omega_g+2)}\right| = 1 \Rightarrow \frac{40}{\omega_g\sqrt{\omega_g^2+4}} = 1 \Rightarrow \omega_g^4 + 4\omega_g^2 - 1600 = 0. \Rightarrow \omega_g = 6.17 \frac{rad}{s}.$$

$$\Rightarrow \angle P_1(j\omega_g) = \boxed{-90^\circ - \tan^{-1}\left(\frac{\omega_g}{2}\right)} = -162.04^\circ.$$

$\Rightarrow$ Available phase margin for $P_1(j\omega)$ $\left(\text{without } \frac{(Ts+1)}{(\alpha Ts+1)}\right)$ $= 180^\circ + \angle P_1(j\omega_g) = 180^\circ - 162.04^\circ = \boxed{17.96^\circ}.$

$\Rightarrow$ The uncompensated system does not meet the phase margin specification. At first glance, the lead compensator should provide a phase of $50^\circ - 17.96^\circ = 32.04^\circ$.

But, With compensation, $|G(j\omega)| = |1|_{dB} + |2|_{dB}$. With compensation, the value of the gain cross-over frequency would increase. Hence, $\angle 2$ would become more $-$ve at the new gain cross-over frequency. Thus, we would design the lead compensator to provide an additional phase of 5° to 6° to account for this effect.

Hence, let us design the lead compensator such that it provides a maximum phase of φ_m of 38°.

$$\Rightarrow \varphi_m = 38° \Rightarrow \sin\varphi_m = \left(\frac{1-\alpha}{1+\alpha}\right) \Rightarrow \boxed{\alpha = 0.238.}$$

$K = K_c\alpha = 10.$

$$\Rightarrow \boxed{K_c = \frac{10}{\alpha} = 42.02.}$$

Step 3: Find T.

Recall that $\boxed{\omega_m = \frac{1}{\sqrt{\alpha}\,T}.}$

Now, $$\left|\frac{1+j\omega T}{1+j\alpha\omega T}\right|_{\omega=\frac{1}{\sqrt{\alpha}T}} = \frac{1}{\sqrt{\alpha}}.$$ HW

We want ω_m to be the new gain cross-over frequency (ω_{gn}).

$$G(s) = \left(\frac{Ts+1}{\alpha Ts+1}\right)\left(\frac{40}{s(s+2)}\right).$$

$$G(j\omega) = \underbrace{\left(\frac{1+jT\omega}{1+j\alpha T\omega}\right)}\left(\frac{40}{j\omega(j\omega+2)}\right).$$

$\Rightarrow \quad = \frac{1}{\sqrt{\alpha}} = 2.05$

$\Rightarrow$ At the new gain cross-over frequency, ω_{gn},

$$\left|\frac{40}{j\omega_{gn}(j\omega_{gn}+2)}\right| = \frac{1}{2.05}. \Rightarrow \boxed{\omega_{gn} = 8.9456\ \frac{rad}{s}.}$$

HW

We want $\omega_m = \omega_{gn} \Rightarrow \frac{1}{\sqrt{\alpha}T} = 8.9456 \Rightarrow \boxed{T = 0.229.}$

$$\Rightarrow \boxed{C(s) = \frac{K_c\alpha(Ts+1)}{(\alpha Ts+1)} = \frac{10(0.229s+1)}{(0.0545s+1)}.}$$

The open loop transfer fn. of the compensated closed loop system is

$$G(s)H(s) = C(s)\,P(s) = \frac{10\,(0.229s+1)}{(0.0545s+1)}\,\frac{4}{s(s+2)}.$$

HW: Check that the above design satisfies the prescribed performance requirements.

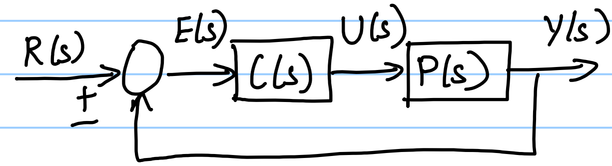
Note that $\omega_{gn} \uparrow \Rightarrow$ bandwidth $\uparrow$ with compensation. But we needed to introduce a high open loop gain to ensure that low frequency components were not attenuated

$\hookrightarrow$ cost $\uparrow$.

16/4/2018.

LAG COMPENSATION.

We consider unity negative feedback.



A lag compensator is typically use to attenuate of high frequency components of the system's response.

The controller transfer fn of a lag compensator is

$$C(s) = \frac{K_c \beta (Ts+1)}{(\beta Ts+1)} = \frac{K_c \left(s + \frac{1}{T}\right)}{\left(s + \frac{1}{\beta T}\right)}, \quad \beta > 1, \; T > 0, \; K_c > 0.$$

open loop pole $\Rightarrow -\frac{1}{\beta T}$.

open loop zero $\Rightarrow -\frac{1}{T}$.

2 corner frequencies $\Rightarrow \frac{1}{T}, \frac{1}{\beta T}$.

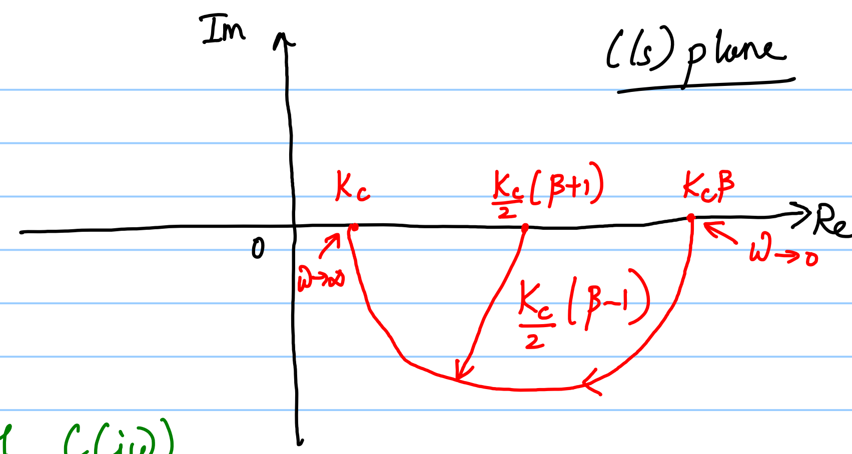
Thus,
$$C(j\omega) = \frac{K_c \beta (1 + jT\omega)}{(1 + j\beta T\omega)} = \frac{K_c \beta (1 + \beta T^2 \omega^2)}{(1 + \beta^2 T^2 \omega^2)} - j \frac{K_c \beta T \omega (\beta - 1)}{1 + \beta^2 T^2 \omega^2}.$$

Note that, $C(j0) = K_c \beta$, $C(j\infty) = K_c$.

Note that

$$\left[\frac{K_c\beta(1+\beta T^2\omega^2)}{(1+\beta T^2\omega^2)} - \frac{K_c}{2}(\beta+1)\right]^2 + \left[\frac{K_c\beta T\omega(\beta-1)}{1+\beta^2T^2\omega^2}\right]^2$$

$$= \left[\frac{K_c}{2}(\beta-1)\right]^2$$



HW Take $K_c = 1$, $\beta = 10$, $T = 1$. Plot the Bode diagram of $C(j\omega)$.

$$C(s) = \frac{K_c\beta(Ts+1)}{(\beta Ts+1)} = \frac{10\,(s+1)}{(10s+1)} = (10)\,(s+1)\left(\frac{1}{10s+1}\right).$$

→ acts like a low pass filter.

→ Low frequency gain ↑ ⇒ lower steady state error.

HW → Plot the phase plot.

→ Gain cross-over frequency ↓ for the compensation system ⇒ reduced bandwidth.

Lag-Lead Compensation.

Lead compensator → improves the stability margins but may decrease the steady state accuracy.

Lag compensator → attenuates the high frequency components, but ↓ the bandwidth.

The transfer function of a lag-lead compensator is

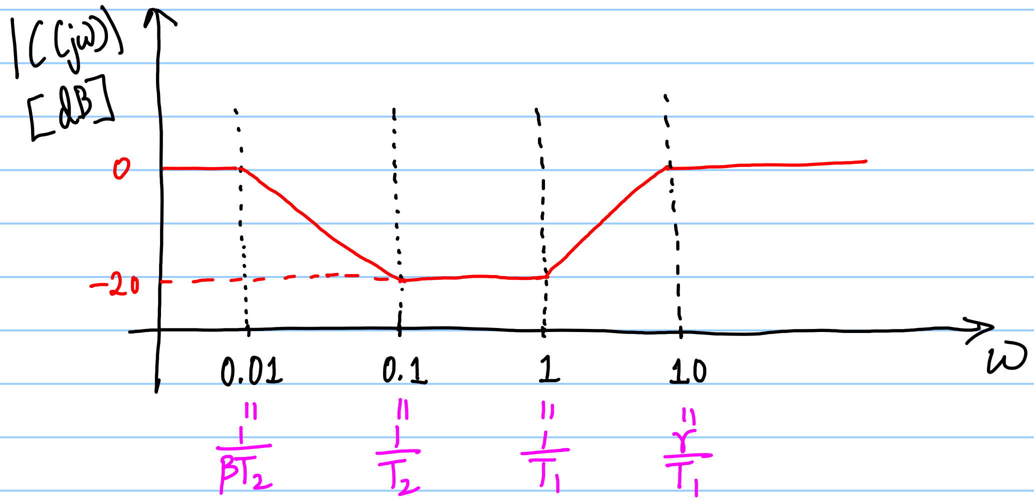
$$C(s) = K_c \underbrace{\frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{\gamma}{T_1}\right)}}_{\text{Lead}} \underbrace{\frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_2}\right)}}_{\text{Lag}}, \quad \gamma>1,\ \beta>1,\ T_1>0,\ T_2>0,\ K_c>0.$$

$\left[\frac{1}{T_1}, \frac{\gamma}{T_1}\right] \leftarrow$ Lead

Lag $\rightarrow \left[\frac{1}{\beta T_2}, \frac{1}{T_2}\right]$

Frequently, we choose $\gamma = \beta$.

HW: Choose $K_c = 1$, $\gamma = 10$, $\beta = 10$, $T_1 = 1$, $T_2 = 10$. Plot the Nyquist plot and the Bode plot of $C(j\omega)$.



COURSE SUMMARY

- Concept of dynamic systems → classifications. Identified the class of systems to be studied → LTI Causal SISO Dynamic Systems.
- Characterization of dynamic systems → ODEs, Laplace Transform.
  Transfer fn. ⟷ State space representation.
- Transfer fn. → Poles, Zeros, Free response, forced response.
- Stability → BIBO stability, Stability Criteria.
- Performance → 1st order systems, 2nd order systems, steady state error analysis.
- PID Controllers.
- Root Locus → control design.
- Frequency response → Bode plot, Nyquist plot, Nyquist stability criterion.

- Relative Stability $\Rightarrow$ Gain margin, phase margin.
- Lead, Lag, Lag-Lead Compensation.