11th Lecture on ODE

(MA-1150)

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Indian Institute of Technology Hyderabad

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What have we learnt?

- Second and Higher Order non-Homogeneous ODE with constant coefficient
- General method for finding PI
- Short Method for finding PI

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Today's Class

- Method of Variation of Parameters
- Euler-Cauchy Equations
- Reduction of Order

Method of variation of parameters:

Consider the second order non-homogeneous linear eqn

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \text{ provided } a_0(x) \neq 0$$

we shall discuss a general method of solution called the method of variation of parameters, which can always be used to find a particular integral whenever the complementary function of the equation is known.

At first, we need to find the solution of the corresponding homogeneous eqn -

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \,, \quad a_0(x) \neq 0.$$

Using the methods discussed earlier in the classnote, we can find easily two linearly independent solutions.

Let us consider $y_1(x)$ and $y_2(x)$ be two linearly independent solutions to the homogeneous eqⁿ.

Now the complementary function (C.F.) is given by $\boxed{y_c(x) = c_1 y_1(x) + c_2 y_2(x)}$

where c_1 and c_2 are arbitrary constants.

Note: The idea behind the method of variation of parameters is to vary the parameters A and B.

So in this method, we assume A and B to be functions of x.

Now we consider the particular solution or integral (PI) as

$$\boxed{y_p(x) = A(x)\,y_1(x) + B(x)\,y_2(x)}$$

So, the particular solution of the given nonhomogeneous linear eq^n contains two functions $A(x)$ and $B(x)$

we need to determine $A(x)$ and $B(x)$.

we have the solution (PI) as

$$\boxed{y_p(x) = A(x)\,y_1(x) + B(x)\,y_2(x)}$$

Differentiating w.r.t x

$$\frac{dy_p}{dx} = y_p' = A'y_1 + Ay_1' + B'y_2 + By_2'$$

$$= (Ay_1' + By_2') + (A'y_1 + B'y_2)$$

we shall choose $A(x)$ and $B(x)$ in such a way that

$$\boxed{A'y_1 + B'y_2 = 0} \quad \text{—— (1)}$$

$$y_p' = (Ay_1' + By_2')$$

Differentiating w.r.t x, we have

$$y_p'' = Ay_1'' + A'y_1' + By_2'' + B'y_2'$$

Substituting the expressions for $y_p(x)$, $y_p'(x)$ and $y_p''(x)$ in the given nonhomogeneous linear eqn, we have.

$$a_0(x)\left[Ay_1'' + A'y_1' + By_2'' + B'y_2'\right] + a_1(x)\left[Ay_1' + By_2'\right]$$
$$+ a_2(x)\left[Ay_1 + By_2\right] = r(x)$$

a) $A\left[a_0 y_1'' + a_1 y_1' + a_2 y_1\right] + B\left[a_0 y_2'' + a_1 y_2' + a_2 y_2\right]$

$+ a_0\left[A'y_1' + B'y_2'\right] = r(x)$

Since $y_1(x)$ and $y_2(x)$ are the solutions of homogeneous eqn, we have

$$\boxed{\begin{array}{l} a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \\ \text{and } a_0 y_2'' + a_1 y_2' + a_2 y_2 = 0 \end{array}}$$

So, we get $a_0\left[A'y_1' + B'y_2'\right] = r(x)$

$$A'y_1' + B'y_2' = r(x)/a_0(x) = g(x) \text{ (say)}$$

$$\therefore \boxed{A'y_1' + B'y_2' = g(x)} \quad \text{—} (2)$$

Since $a_0(x)$ and $r(x)$ are continuous functions, $g(x)$ is also continuous. Now solving the equations

$$\boxed{\begin{aligned} A'y_1 + B'y_2 &= 0 \quad \text{—} (1) \\ A'y_1' + B'y_2' &= g(x) \quad \text{—} (2) \end{aligned}}$$

We obtain

$$\boxed{A' = -\frac{g(x)\,y_2}{y_1 y_2' - y_2 y_1'}, \quad B' = \frac{g(x)\,y_1}{y_1 y_2' - y_2 y_1'}}$$

we note that the wronskian $W(y_1, y_2)$ is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Since y_1 and y_2 are the linearly independent solutions of the homogeneous equation, $W(x) \neq 0$.

Now we can write $\quad A' = -\dfrac{g(x)\,y_2}{W(x)}$

$$B' = \frac{g(x)\,y_1}{W(x)}$$

Integrating, we obtain,

$$A(x) = -\int \frac{g(x)\, y_2(x)}{W(x)}\, dx = X_1(x)$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(x)}\, dx = X_2(x)$$

$$X_1(x) = -\int \frac{g(x)\, y_2(x)}{W(x)}\, dx$$

and

$$X_2(x) = \int \frac{g(x)\, y_1(x)}{W(x)}\, dx$$

The particular integral is

PI $\rightarrow$ $A(x)\, y_1(x) + B(x)\, y_2(x)$

$$= X_1(x)\, y_1(x) + X_2(x)\, y_2(x)$$

The particular integral is

$$\boxed{y_p(x) = X_1(x)\, y_1(x) + X_2(x)\, y_2(x)}$$

and the complementary function is

$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are arbitrary constants.

The general solution is

$$\boxed{y(x) = \text{C.F.} + \text{P.I.} = y_c(x) + y_p(x)}$$

$$= c_1 y_1(x) + c_2 y_2(x) + X_1(x) y_1(x) + X_2(x) y_2(x)$$

Note: The method is applicable both for constant coefficient and variable coefficient in the ODE.

This method can be applied for higher order ODE.

- For example, consider the third order equation

$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = r(x), \quad a_0(x) \neq 0$$

The C.F. is $y_c(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$

where y_1, y_2 and y_3 are the linearly independent solutions of the corresponding homogeneous eq^ns and c_1, c_2, c_3 are arbitrary constants.

Let us consider the particular solution or integral (PI) as $y_p(x) = A(x)\,y_1(x) + B(x)\,y_2(x) + C(x)\,y_3(x)$

Following the procedure discussed earlier, we obtain the required equations for determining $A(x)$, $B(x)$ and $C(x)$ as

$$\begin{cases} A'(x)\,y_1 + B'(x)\,y_2 + C'(x)\,y_3 = 0 & \text{--- } (1) \\ A'(x)\,y_1' + B'(x)\,y_2' + C'(x)\,y_3' = 0 & \text{--- } (2) \\ A'(x)\,y_1'' + B'(x)\,y_2'' + C'(x)\,y_3'' = \dfrac{r(x)}{a_0(x)} = g(x). & \text{--- } (3) \end{cases}$$

Three eqⁿˢ give $A(x)$, $B(x)$ and $C(x)$.

The particular solution is obtained as

$$\Rightarrow \boxed{y_p(x) = A(x)y_1(x) + B(x)y_2(x) + C(x)y_3(x)} \quad \underline{\text{P.I.}}$$

Here $A(x)$, $B(x)$, and $C(x)$ are known and these are obtained from eqⁿˢ ①, ② and ③.

Therefore the general solution is (C.F. + P.I.)

$$\boxed{y(x) = y_c(x) + y_p(x)} = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$$

$$+ A(x)y_1(x) + B(x)y_2(x) + C(x)y_3(x).$$

For n'th order:

The nth order nonhomogeneous linear eqⁿ is given by —

$$\boxed{a_0(x)\,y^{(n)}(x) + a_1(x)\,y^{(n-1)}(x) + \cdots + a_{n-1}(x)\,y'(x) + a_n(x)\,y(x) = r(x)}$$

Let us consider n numbers of linearly independent solutions of the homogeneous linear eqⁿ

$$\boxed{a_0(x)\,y^{(n)}(x) + \cdots + a_n(x)\,y(x) = 0.}$$

The Complementary function (CF) is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x).$$

Let us consider the particular integral (PI)

as $y_p(x) = A_1(x) y_1(x) + A_2(x) y_2(x) + \dots$

$+ A_n(x) y_n(x)$

Therefore by similar approach, we obtain n numbers of algebraic equations in terms of $A_1(x), \dots A_n(x)$

$$\begin{cases} A_1'(x)\,y_1(x) + A_2'(x)\,y_2(x) + \cdots + A_n'(x)\,y_n(x) = 0 \\ A_1'(x)\,y_1'(x) + A_2'(x)\,y_2'(x) + \cdots + A_n'(x)\,y_n'(x) = 0 \\ A_1'(x)\,y_1''(x) + A_2'(x)\,y_2''(x) + \cdots + A_n'(x)\,y_n''(x) = 0 \\ \quad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ A_1'(x)\,y_1^{(n-2)}(x) + A_2'(x)\,y_2^{(n-2)}(x) + \cdots + A_n'(x)\,y_n^{(n-2)}(x) = 0 \\ A_1'(x)\,y_1^{(n-1)}(x) + A_2'(x)\,y_2^{(n-1)}(x) + \cdots + A_n'(x)\,y_n^{(n-1)}(x) = \dfrac{r(x)}{a_0(x)} \\ \qquad\qquad\qquad = g(x). \end{cases}$$

From n numbers of eq^{n}s, we can obtained the n unknowns A_1', A_2', ..., A_n'.

Therefore integrating we will have the $A_1(x)$, $A_2(x)$, ..., $A_n(x)$. Hence the particular integral is obtained as

$$y_p(x) = A_1(x)y_1(x) + A_2(x)y_2(x) + \dots + A_n(x)y_n(x)$$

The general solution is obtained as $y(x) = CF + PI = y_c(x) + y_p(x)$.

Example: $y'' + 3y' + 2y = 2e^{x}$, using the method of variation of parameter.

Soln: The corresponding homogeneous equation is $y'' + 3y' + 2y = 0$. The characteristic equation is $m^2 + 3m + 2 = 0$ and its roots are $m = -1, -2$.

Hence the C.F is $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$

$$= c_1 e^{-x} + c_2 e^{-2x}$$

where $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$ are two linearly independent solutions of the homogeneous equation.

Assume the particular solution as

PI $\rightarrow$ $$\boxed{y_p(x) = A(x)e^{-x} + B(x)e^{-2x}}$$

where $$A(x) = -\int \frac{g(x)\, y_2(x)}{W(x)}\, dx$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(x)}\, dx$$

$$g(x) = \frac{r(x)}{a_0(x)} = 2e^{x}, \quad y_1(x) = e^{-x}, \quad y_2(x) = e^{-2x}$$

and $$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x} \; (\neq 0)$$

$$A(x) = -\int \frac{2e^{x}e^{-2x}}{-e^{-3x}}\,dx = e^{2x}$$

$$B(x) = \int \frac{2e^{x}e^{-x}}{-e^{-3x}}\,dx = -\tfrac{2}{3}e^{3x}$$

The general solution is

$$y(x) = \underbrace{A(x)e^{-x} + B(x)e^{-2x}}_{\text{P.I. } (y_p(x))} + \underbrace{c_1 e^{-x} + c_2 e^{-2x}}_{\text{C.F. } (y_c(x))}$$

$$= \underbrace{c_1 e^{-x} + c_2 e^{-2x}}_{\text{C.F.}} + \underbrace{\tfrac{1}{3}e^{x}}_{\text{P.I.}}$$

, where c_1 and c_2 are arbitrary constants.

Example: $y'' + 16y = 32\sec 2x$.

Sol^n $(D^2+16)y = 32\sec 2x$

The Auxiliary eq^n or characteristic eq^n is

$m^2+16=0 \Rightarrow m = \pm 4i$ } These are the characteristic roots.

The C.F. is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 \cos 4x + c_2 \sin 4x$$

where $y_1(x) = \cos 4x$ and $y_2(x) = \sin 4x$ are two linearly independent solutions of the homogeneous equation. By the method of the variation of parameters, we write the particular solution as

PI $\rightarrow$ $$y_p(x) = A(x)\,y_1(x) + B(x)\,y_2(x)$$

where $$A(x) = -\int \frac{g(x)\,y_2(x)}{W(x)}\,dx,$$

$$B(x) = \int \frac{g(x)\,y_1(x)}{W(x)}\,dx,$$

$$g(x) = \frac{r(x)}{a_0(x)} = 32\sec 2x,$$ $y_1(x) = \cos 4x,$
$y_2(x) = \sin 4x,$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4.$$

Therefore we obtain,

$$A(x) = -\int \frac{g(x)\, y_2(x)}{W(x)} = -16\int \sin 2x \, dx$$

$$= 8\cos 2x$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(x)} = \frac{1}{4}\int 32 \sec 2x \cos 4x \, dx$$

$$= 8\int \frac{2\cos^2 2x - 1}{\cos 2x} \, dx$$

$$= 8\int (2\cos 2x - \sec 2x)\,dx$$

$$= 8\sin 2x - 4\log\left|(\sec 2x + \tan 2x)\right|$$

The general solution is $y(x) = PI + CF = y_p(x) + y_c(x)$

$$y(x) = A(x)\cos 4x + B(x)\sin 4x + c_1\cos 4x + c_2\sin 4x$$

$$= c_1\cos 4x + c_2\sin 4x + 8\cos 2x - 4\sin 4x \log\left|(\sec 2x + \tan 2x)\right|$$

where c_1 and c_2 are arbitrary constants.

Example: $y''' - 6y'' + 11y' - 6y = e^{-x}$.

The Auxiliary eqn is $m^3 - 6m^2 + 11m - 6 = 0$.

$m = 1, 2, 3$.

The C.F. is $y_c(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$.

To find PI:

By the method of variation of parameters, we assume the particular solution as

PI $\longrightarrow$ $y_p(x) = A(x) e^{x} + B(x) e^{2x} + C(x) e^{3x}$

we have $g(x) = r(x)/a_0(x) = e^{-x}$.

The eq^{n}s for determining $A(x)$, $B(x)$ and $C(x)$ are

$$\begin{cases} A'e^{x}+B'e^{2x}+C'e^{3x}=0 \\ A'e^{x}+2B'e^{2x}+3C'e^{3x}=0 \\ A'e^{x}+4B'e^{2x}+9C'e^{3x}=e^{-x} \end{cases}$$

The wronskian $W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$

$$= e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \underline{2e^{6x}}.$$

By the Cramer's rule, we obtain -

$$W A' = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^{-x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{4x} \quad \text{g} \quad A' = \frac{e^{4x}}{2e^{6x}} = \frac{1}{2} e^{-2x}.$$

Integrating we have $A = -\frac{1}{4} e^{-2x}$

Similarly we have

$$W B' = \begin{vmatrix} e^{x} & 0 & e^{3x} \\ e^{x} & 0 & 3e^{3x} \\ e^{x} & e^{-x} & 9e^{3x} \end{vmatrix} = -2e^{3x}$$

$$\text{or, } B' = -\frac{2e^{3x}}{2e^{6x}} = -e^{-3x}$$

Integrating $B(x) = \frac{1}{3}e^{-3x}$

$$Wc' = \begin{vmatrix} e^{x} & e^{2x} & 0 \\ e^{x} & 2e^{2x} & 0 \\ e^{x} & 4e^{2x} & e^{-x} \end{vmatrix} = e^{2x} \text{ as } c' = \frac{e^{2x}}{2e^{6x}} = \frac{1}{2}e^{-4x}.$$

$$c(x) = -\frac{1}{8}e^{-4x}$$

The general solution is $y(x) = y_p(x) + y_c(x)$

$$y(x) = A(x)e^{x} + B(x)e^{2x} + c(x)e^{3x} + c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

$$= \underbrace{c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}_{C.F.} - \underbrace{\frac{1}{24} e^{-x}}_{P.I.}$$

OR. One can write

$$\begin{pmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{pmatrix} \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{-x} \end{pmatrix}$$

$$\Rightarrow \quad Ax = b.$$

$$A = \begin{pmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{pmatrix}$$

$$x = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ e^{-x} \end{pmatrix}$$

Now $\boxed{x = A^{-1} b.}$

where $A^{-1} = \dfrac{\operatorname{Adj} A}{\det A}.$

If you find the A^{-1}, you can easily obtain

$$x = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}.$$ i.e, you will get

A', B' and C'.. Then integrating, we can get $A(x)$, $B(x)$, and $C(x)$.

⑦ Cramer's Rule:

Let A be nonsingular matrix $(\det A \neq 0)$.

The Cramer's rule for the solution $Ax = b$ is given by

$$x_i = \frac{|A_i|}{|A|} = \frac{\det A_i}{\det A}, \quad i = 1, 2, \ldots n.$$

where $|A_i|$ is the determinant of the matrix A_i obtained by replacing the ith column of A by the right hand side column vector b.

where: system def^n:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\}$$

we need to solve this system of equations by matrix representation.

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow \boxed{Ax = b.}$$

Cauchy - Euler Equation or Euler - Cauchy Equation

Cauchy - Euler eqⁿ is given by

$$\boxed{x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_{n-1} x \frac{dy}{dx} + p_n y = r(x)}$$

where $p_1, p_2, \dots, p_{n-1}, p_n$ are constants. This is the nth order linear ODE with variable coefficients.

It is known as Euler-Cauchy or Cauchy Euler eqⁿ.

◎ **Note:** To solve this eqⁿ, we need to transform the given eqⁿ with variable coefficients into a linear equation with constant coefficients by means of the substitution $\boxed{x = e^t}$ or $\boxed{t = \log x}$

That's mean, ODE with variable Coefficients

$\downarrow$ [applying a transformation $x = e^t$]

ODE with constant Coefficients

$\Rightarrow$ Then, it can be easily solvable by previous discussed Methods

Example: $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

This is the Cauchy-Euler eqⁿ and it is the eqⁿ with variable coefficients

Solⁿ: we put $\boxed{x = e^t}$, or $\boxed{t = \log x}$

Differentiating w.r.t t, we have

$$\frac{dx}{dt} = e^t = x \Rightarrow \boxed{\frac{dt}{dx} = \frac{1}{x}}$$

$\boxed{x \to t}$

$\boxed{\frac{dy}{dx} \longrightarrow \frac{dy}{dt}}$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x}$$

$$\Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dt}} \Rightarrow \boxed{xDy = \theta y}$$

Consider $D \equiv \frac{d}{dx}$

and $\theta \equiv \frac{d}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\cdot\frac{1}{x}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dt} + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}.$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} \qquad \left[\text{Since } \frac{dt}{dx} = \frac{1}{x}\right]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = -\frac{dy}{dt} + \frac{d^2y}{dt^2} \Rightarrow \boxed{x^2D^2y = \theta(\theta-1)y}$$

Now $\boxed{x^3 D^3 y = \theta(\theta-1)(\theta-2)\, y}$

Making these replacements, we obtain from the given equation —

$$\left\{\theta(\theta-1)(\theta-2) + 3\theta(\theta-1) + \theta + 8\right\} y = 65 \cos t$$

or, $(\theta^3 + 8) y = 65 \cos t$

$$\Rightarrow \boxed{\frac{d^3 y}{dt^3} + 8y = 65 \cos t}$$

This is the non-homogeneous linear ODE with constant coefficients. Now it can be easily solved.

The auxiliary eqn is $m^3+8=0$

$\Rightarrow m=-2,\ 1\pm i\sqrt{3}.$

The C.F is $c_1 e^{-2t} + e^{t}\left(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t\right)$

where c_1, c_2 and c_3 are arbitrary constants.

The particular integral is

$$\frac{1}{\theta^3+8}\, 65\cos t = 65 \cdot \frac{1}{\theta^3+8} \cdot \cos t$$

$$= 65 \cdot \frac{1}{8-\theta} \cos t, \quad \left[\text{Put } \theta^2=-1\right]$$

$$= 65 \cdot \frac{8+\theta}{64-\theta^2} \cos t$$

$$= 65 \cdot \frac{8+\theta}{64+1} \cos t, \quad [\text{Put } \theta^2 = -1]$$

$$= \frac{65}{65} \cdot (8+\theta) \cos t = (8\cos t - \sin t)$$

Now the general solution is

$$y = c_1 e^{-2t} + e^t \left(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t\right) + (8\cos t - \sin t)$$

$\boxed{t = \log x} \Rightarrow$

$$= c_1 \cdot x^{-2} + x \left\{ c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x) \right\} + 8\cos(\log x) - \sin(\log x)$$

Example: $(5+2x)^2\frac{d^2y}{dx^2}-6(5+2x)\frac{dy}{dx}+8y=8(5+2x)^2$

Solⁿ: Put $5+2x=e^t$

Differentiating w.r.t t, we have

or, $2\frac{dx}{dt}=e^t=(5+2x)$

or, $\boxed{\frac{dt}{dx}=\frac{2}{5+2x}}$

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{2}{(5+2x)}$ or, $\frac{dy}{dx}=\frac{2}{5+2x}\frac{dy}{dt}$

$\Rightarrow \boxed{(5+2x)\frac{dy}{dx}=2\frac{dy}{dt}}$

$$\boxed{(5+2x)Dy = 2\theta y}$$ where $D \equiv \frac{d}{dx}$ and $\theta \equiv \frac{d}{dt}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{2}{5+2x}\cdot\frac{dy}{dt}\right)$$

$$= 2\cdot\frac{d}{dx}\left(\frac{1}{5+2x}\right)\frac{dy}{dt} + \frac{2}{5+2x}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= 2\cdot\frac{0-2}{(5+2x)^2}\cdot\frac{dy}{dt} + \frac{2}{5+2x}\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}$$

$$= -\frac{4}{(5+2x)^2}\frac{dy}{dt} + \frac{4}{(5+2x)^2}\frac{d^2y}{dt^2}$$

$$(5+2x)^2 \frac{d^2y}{dx^2} = 4\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$D \equiv \frac{d}{dx}$ and $\theta \equiv \frac{d}{dt}$.

$$\Rightarrow \boxed{(5+2x)^2 D^2 y = 4\,\theta(\theta-1)y}$$

The given equation becomes –

$$\{4\theta(\theta-1) - 6\cdot 2\theta + 8\}y = 8e^{2t}$$

or, $$(4\theta^2 - 16\theta + 8)y = 8e^{2t}$$

or, $$\boxed{(\theta^2 - 4\theta + 2)y = 2e^{2t}}$$

This is the non-homogeneous linear ODE with constant coefficients.

The auxiliary eq^n is $m^2-4m+2=0$

$\Rightarrow m = 2 \pm \sqrt{2}$

The C.F. is

$$\boxed{y_c(t) = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t}}$$

The P.I. is $\dfrac{1}{\theta^2-4\theta+2} \cdot 2e^{2t}$ [Put $\theta=2$]

$= -e^{2t}$

The general solution is

$= c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t} - e^{2t}.$

$$= e^{2t}\left(c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}\right) - e^{2t}$$

$$(5+2x) = e^t$$
$$\Rightarrow t = \log(5+2x)$$

$$= (5+2x)^2\left\{c_1 (5+2x)^{\sqrt{2}} + c_2 (5+2x)^{-\sqrt{2}}\right\} - (5+2x)^2$$

Reduction of order:

Now we will see that if one solution is given to the linear homogeneous ODE, another solution can be obtained by reducing its order. For this we need a transformation to reduce its order from the given ODE. See the theorem —

Reduction of order :

Theorem : Let $u(x)$ be a nontrivial solution of the nth order homogeneous linear ODE –

$$\boxed{a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0}$$

The transformation $y = u(x)v(x)$ reduces the above eq^n to an $(n-1)$ order homogeneous linear ODE in the dependent variable $w = \frac{dv}{dx}$.

Idea:

Linear homogeneous n^{th} order ODE with variable coefficients

Transformation

$$\boxed{y = u(x)\, v(x)}$$

where $u(x)$ is the given solution to the given ODE.

Here $v(x)$ is unknown function of x and it is required to find.

Linear homogeneous $(n-1)^{th}$ order ODE with variable coefficients

In this ODE, one order is reduced.

Theorem: Let $u(x)$ be nontrivial solution of the second order homogeneous linear ODE —

$$\boxed{a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0} \quad — (1)$$

The transformation $y = u(x)\,v(x)$ reduces eq^n (1) to the first order homogeneous linear ODE

$$\boxed{a_0(x)\,u(x)\,\frac{dw}{dx} + \left[2a_0(x)\,u'(x) + a_1(x)\,u(x)\right]w(x) = 0} \quad (2)$$

in the dependent variable w, where

$$w = \frac{dv}{dx}.$$

The particular solution

$$\boxed{w = \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[u(x)]^2}}$$

& eqn (2) gives the function $v(x)$, where

$$\boxed{v(x) = \int w(x)\,dx}.$$

The function $\boxed{f(x) = u(x)v(x)}$ is then a solution of the second order eq^n.

Therefore the solutions $u(x)$ and $f(x)$ are linearly independent solutions and hence the general solution is

$$\boxed{c_1 u(x) + c_2 f(x)}$$

where c_1 and c_2 are constant.

Proof: Let $u(x)$ be a nontrivial solution of the second order homogeneous linear ODE —

$$\boxed{a_0(x)y'' + a_1(x)y' + a_2(x)y = 0} \quad \text{— (1)}$$

Let us consider the transformation

$\boxed{y(x) = u(x)v(x)}$, where $u(x)$ is the known solution of (1) and $v(x)$ is a function of x that will be determined.

$$y = u(x)v(x)$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\& \quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

From (1), we have

$$a_0(x)\left[uv'' + 2u'v' + vu''\right] + a_1(x)\left[uv' + vu'\right] + a_2(x)uv = 0$$

or,
$$a_0(x)u(x)v'' + \left[2a_0(x)u' + a_1(x)u(x)\right]v'$$
$$+ \left[a_0(x)u'' + a_1 u' + a_2 u\right]v = 0 \quad \xrightarrow{} 0$$

or,
$$\boxed{a_0(x)u(x)v'' + \left[2a_0(x)u' + a_1(x)u(x)\right]v' = 0}$$

Let $w = \frac{dv}{dx} \Rightarrow$
$$\boxed{a_0(x)u(x)\frac{dw}{dx} + \left[2a_0(x)u' + a_1(x)u(x)\right]w = 0} \quad \text{--- (3)}$$

This is a first order homogeneous linear differential eqn in the dependent variable w. The eqn is separable.

we have $\dfrac{dw}{w} = -\left[2\,\dfrac{u'(x)}{u(x)} + \dfrac{a_1(x)}{a_0(x)}\right]dx$

Integrating →

$$w = \frac{c\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}}{[u(x)]^2},$$ where c is arbitrary constant.

This is the general solution of eq^n (3).

So choosing a particular value of c as 1, we will get the particular sol^n

$$W(x) = \frac{dv}{dx} = \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[u(x)]^2}$$

Integrating

$$\boxed{v(x) = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[u(x)]^2} dx.}$$

Finally we obtain

$$y(x) = u(x)\int \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[u(x)]^2}\,dx.$$

We denote it $f(x)$ as a solution.

$$W\big(u(x), f(x)\big) = \begin{vmatrix} u(x) & f(x) \\ u'(x) & f'(x) \end{vmatrix} = \begin{vmatrix} u(x) & u(x)v \\ u'(x) & u(x)v' + u'(x)v \end{vmatrix}$$

$$= [u(x)]^2 v'$$

$$= [u(x)]^2 v' = \exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right] \neq 0$$

So, $u(x)$ and $f(x)$ are two linearly independent solutions.

Now the general solution is

$$\boxed{y(x) = c_1 u(x) + c_2 f(x)}$$

Example: If $y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

find a linearly independent solution by reducing the order.

Solⁿ: Here $\boxed{y = x}$ satisfies the given ODE.

Let $\boxed{y(x) = xv}$ be a transformation.

Then $y' = xv' + v$, $y'' = xv'' + 2v'$

From the given ODE, we have

$$(x^2+1)(xv''+2v') - 2x(xv'+v) + 2xv = 0$$

or $$x(x^2+1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0.$$

Let $w = \dfrac{du}{dx}$ and we have the first order homogeneous linear ODE

$$x(x^2+1)\frac{dw}{dx} + 2w = 0$$

$$\Rightarrow \frac{dw}{w} = \left(-\frac{2}{x} + \frac{2x}{x^2+1}\right)dx$$

$$\Rightarrow w(x) = \frac{c\,(x^2+1)}{x^2}.$$

choosing $c=1$, we reach at particular solution as $w(x) = \frac{x^2+1}{x^2}$.

$$\Rightarrow \frac{dv}{dx} = \frac{x^2+1}{x^2}$$

$$\Rightarrow \boxed{v(x) = x - \frac{1}{x}}$$

Now we have $y(x) = u(x)v(x) = \underline{x^2-1}$

we can easily check that x and (x^2-1) are two linearly independent solutions since wronskian of x and (x^2-1) is non-zero.

The general sol^n is

$$y(x) = c_1 x + c_2 (x^2-1).$$

Ex If $y = x$ is the solution of the reduced eqn of

$$x^2 \frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0$$

find a linearly independent solution by reducing its order.

Ans: General solution is

$$\boxed{y(x) = c_1 x e^x + c_2 x.}$$