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Eight Lecture on ODE

(MA-1150)

What have we learnt so far?

- First Order and First Degree ODE
- Linear Equations
- Non-Linear equations
- First Order but not in First Degree
- Clairaut’s Equation
- Singular Solution & Envelope
- IVP

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Today’s Class

- Linear Second and Higher order Ordinary Differential Equation

Linear Differential Equation

- A linear ordinary differential eqⁿ of order n is written as —

$$\boxed{a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=r(x)} \quad (1)$$

or,

$$\boxed{a_0(x)y^{(n)}+a_1(x)\,y^{(n-1)}+\cdots+a_{n-1}(x)\,y'+a_n(x)y=r(x)}$$

where y is dependent variable and x is independent variable and $a_0(x)\neq 0$

- If $r(x)=0$, then it is called a homogeneous linear equation.

when $r(x)\neq 0$, it is called a non-homogeneous linear eqⁿ.

- $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are functions of x.

Second Order Differential Equation

- Second order linear differential eqⁿ is written as —

$$\boxed{a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = r(x)}$$

This is non-homogeneous eqⁿ. When $r(x)=0$, this eqⁿ reduces to homogeneous eqⁿ.

- If $a_0(x), a_1(x), \ldots, a_n(x)$ are only constants, then the above eqⁿ is known as linear second order constant coefficient equation.

Ex. (i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x^2e^x$

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$

Second Order Differential Equation

Linear Differential operator:

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$$

is said to be a linear differential operator if for any n times differentiable functions $y_1(x)$ and $y_2(x)$ and for any constants c_1 and c_2, it has the property

$$\boxed{L\left[c_1 y_1(x) + c_2 y_2(x)\right] = c_1 L\left[y_1(x)\right] + c_2 L\left[y_2(x)\right]}$$

Second Order Differential Equation

So one can write the nth order linear nonhomogeneous ODE as

$$\boxed{L\,y(x) = r(x)}, \quad x \in I$$

where

$$L \equiv a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$$

Now we want to discuss the fundamental existence and uniqueness theorem on the nth order linear non-homogeneous ordinary differential eqn.

See this theorem in next page.

Linear Differential Equation

Theorem: If the functions $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are continuous over some interval I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the IVP

Existence and uniqueness Theorem

$$\boxed{a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = r(x)}$$

with the initial conditions —

$$y(x_0) = c_1, \quad y'(x_0) = c_2, \ldots, y^{(n-1)}(x_0) = c_n.$$

IVP for nth order linear ODE

where $x_0 \in I$, and $c_1, c_2, \ldots, c_n$ are n known constants.

Note: The interval I may be open i.e, (a,b), closed i.e, $[a,b]$, semi open $(a,b]$ or infinite $(-\infty,\infty)$ or $(0,\infty)$.

Example :

(1) $$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3 y(x) = e^x, \quad x \in (-\infty, \infty)$$

$$y(1) = 1$$
$$y'(1) = 2$$

Solⁿ : Here the coefficients 1, $3x$ and x^3, as well as nonhomogeneous term e^x in this second order differential eqⁿ are all continuous for all values of x, $-\infty < x < \infty$.

Here $x_0 = 1$, which belong to this interval.

$C_1 = 1$ and $C_2 = 2 \in \mathbb{R}$., $a_0(x) = 1 \neq 0$, $x \in \mathbb{R}$.

Then the solution of the given problem exits and its is unique and is defined for all x, $-\infty < x < \infty$.

Example: $x^2y'' - 2xy' + 2y = 6$

with $y(0) = 3$, $y'(0) = 1$ on $(-\infty, \infty)$

Solⁿ:

Here x^2, $-2x$, 2 and 6 are all continuous on $(-\infty, \infty)$.

But $a_0(x) = x^2 = 0$ at $x = 0$

and it is not satisfying $a_0(x) \neq 0$ $\forall x \in (-\infty, \infty)$

So this IVP has no unique solⁿ.

Linear Differential Equation

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Note: This theorem does not give us a procedure to find the solution but guarantees that there exists a unique solution if the conditions stated in the theorem are satisfied.

Next find the corollary to the existence and uniqueness theorem.

Corollary: Let $y(x)$ be a solution of the n^{th} order homogeneous linear differential eqn

$$a_0(x)\frac{d^n y(x)}{dx^n} + a_1(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx}$$

$$+ a_n(x)\,y(x) = 0$$

such that

$$y(x_0) = 0,\ y'(x_0) = 0,\ \ldots\ y^{(n-1)}(x_0) = 0$$

where $x_0 \in I$ and the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ are all continuous on I and $a_0(x) \neq 0$.

Then $y(x) = 0$ ~~be~~ is a trivial solution $\forall x \in I$.

Example:

$$\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$$

$$y(1) = y'(1) = y''(1) = 0,$$

$\therefore$ $y(x) = 0$ is the trivial solution for all x.

Theorem: If $y_1(x), y_2(x), \dots, y_n(x)$ are n solutions of $L[y]=0$, then a linear combination of the solution $c_1y_1+c_2y_2+\dots+c_ny_n$, where $c_1, c_2, \dots, c_n$ are constants, is also a solution of $L[y]=0$.

Proof: By virtue of linearity, we have

$$L[c_1y_1+c_2y_2+\dots+c_ny_n] = c_1L[y_1]+c_2L[y_2] + \dots + c_nL[y_n]$$

Since $y_1, y_2, \dots, y_n$ are the solutions of $L[y]=0$

We have $L[y_1] = 0 = L[y_2] = \cdots = L[y_n]$

Now $\boxed{L[c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n] = 0}$.

which shows that $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is also a solution of $L[y] = 0$.

Example: Let $y_1(x)$ and $y_2(x)$ be a solⁿ of $L[y] = y'' + xy' + y = 0$, $x \in I$

Then $c_1 y_1 + c_2 y_2$ is also a solution to the ODE $L[y]=0$, where c_1, c_2 are constants.

You try it.

⊘ Now we will determine whether the solution to the linear homogeneous ODE are linearly independent.

⊘ Let $y_1(x)$ and $y_2(x)$ be the solutions of 2nd order linear homogeneous ODE $L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, $x \in I$.

⊘ <u>Linear independence of Solution:</u>

Two Solutions $y_1(x)$ and $y_2(x)$ are linearly independent on I if one is not the constant ~~mu~~ multiple of other.

Thats mean $\boxed{c_1 y_1 + c_2 y_2 = 0 \text{ implies } c_1 = c_2 = 0.}$, $\forall$ $x \in I$.

$y_1/y_2 \neq$ Constant.

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Linearly dependent: Two Solutions $y_1(x)$ and $y_2(x)$ are linearly dependent on I if one is the constant multiple of other, i.e,

$\exists$ c_1 and c_2 not both zero, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0. \quad c_1, c_2 \neq 0$$

$\forall x \in I.$

Ex. (i) $y_1(x) = \sin x$, $y_2(x) = \cos x$, $\frac{y_1}{y_2} \neq$ Constant.

So y_1 and y_2 are linearly independent.

(ii) $y_1(x) = \sin 2x$, $y_2(x) = \sin x$, $\frac{y_1}{y_2} = \frac{2 \sin x \cos x}{\sin x}$

$= 2\cos x \;(\neq \text{Const}).$

So, y_1 and y_2 are linearly independent

(iii) $y_1 = 2x^2$ and $y_2 = x^2$, $\Rightarrow y_1/y_2 = 2.$

So, y_1 and y_2 are not linearly independent functions.

These are linearly dependent.

when n solutions $y_1, y_2, \dots y_n$ are linearly independent on I. then we have

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0 \text{ implies}, \quad \forall x \in I$$

$$c_1 = 0, \; c_2 = 0, \dots c_n = 0$$

i.e, all c_i are zero, $i = 1, 2, \dots n$.

But this procedure is lengthy and difficult for finding the values of all c_i, $(i = 1, \dots n)$.

So, we will learn a more efficient procedure to justify the linear dependence or independence.

Theorem: If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ in the linear homogeneous eqn

$$a_0(x)\frac{d^n y}{dx^n} + \cdots + a_n(x)\,y = 0, \quad a_0 \neq 0,$$

are continuous on I and $y_1(x), y_2(x), \ldots, y_n(x)$ are n solutions of this eqn, then –

(i) the wronskian

$$W(x) = W(y_1, y_2, \ldots, y_n; x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 \text{ for all } x \in I.$$

$y' = \frac{dy}{dx}$, $y^{(n-1)} = \frac{d^{n-1}y}{dx^{n-1}}$

$\Leftrightarrow$ $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent on I.

(ii) $W(x) = W(y_1, y_2, \cdots y_n) = 0$ for all $x \in I$

$\Longleftrightarrow$ $y_1(x), y_2(x), \cdots, y_n(x)$ are linearly dependent solutions.

i.e. (i) $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent solutions on I if and only if $W(x) \neq 0, \forall x \in I$.

(ii) $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly dependent solutions on I if and only if $W(x) = 0 \; \forall x \in I$.

Proof: (i) The Condition is necessary:

Let there be a point $x_0 \in I$ such that

$$W(y_1, y_2, \ldots, y_n; x_0) = W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\ \ldots & \ldots & & \ldots \\ y_1^{(n-1)}(x_0) & y_2^{n-1}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{vmatrix}$$

$$\neq 0. \quad \text{— (1)}$$

Let there exists constants $c_1, c_2, \ldots, c_n$ such that $c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$ — (2)

$= 0, \ \forall x \in I.$

Aim: To show that $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent solutions.

i.e $c_1 = c_2 = \cdots = c_n = 0$.

Differentiating (2) successively w.r.t x, we have

$$\left.\begin{aligned} c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_n y_n'(x) &= 0 \\ c_1 y_1''(x) + c_2 y_2''(x) + \cdots + c_n y_n''(x) &= 0 \\ \cdots \quad \cdots \quad & \cdots \\ c_1 y^{(n-1)}(x) + c_2 y_2''(x) + \cdots + c_n y_n''(x) &= 0 \end{aligned}\right\} \quad (3)$$

At $x = x_0 \in I$, we have from ② and ③,

$$\left.\begin{aligned} &c_1 y_1(x_0) + c_2 y_2(x_0) + \cdots + c_n y_n(x_0) = 0 \\ &c_1 y_1'(x_0) + c_2 y_2'(x_0) + \cdots + c_n y_n'(x_0) = 0 \\ &\ldots \qquad \ldots \qquad \ldots \\ &c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = 0 \end{aligned}\right\} \text{—④}$$

This is a homogeneous algebraic system of eq^{n}s in $c_1, c_2, \ldots, c_n$. Since the determinant

As the coefficients (i.e. $W(x_0)$) does not vanish there exists a unique solution $c_1 = c_2 = \cdots = c_n = 0$.

Hence $y_1(x), y_2(x), \ldots,$ and $y_n(x)$ are linearly independent solutions.

Sufficient Condition: Let $y_1, y_2, \ldots, y_n$ be the linearly independent solutions on I.

Let us assume that $\exists$ a point $x_0 \in I$

Such that $W(y_1, y_2, \ldots, y_n; x_0) = W(x_0) = 0$.

This implies that the system of eqns (4) has a non trivial solution for $c_1, c_2, \ldots$ and c_n. Now using $c_1, c_2, \ldots, c_n$, we define

$$\psi(x) = c_1 y_1(x) + \cdots + c_n y_n(x), \quad x \in I.$$

Since $y_1, y_2, \ldots, y_n$ are linearly independent solutions of $L[y] = 0$, we have

$$L[y_1] = 0 = L[y_2] = \cdots = L[y_n], \quad \forall x \in I$$

Therefore we have

$$L[\psi] = 0, \quad \forall x \in I$$

and $\psi(x_0) = 0, \psi'(x_0) = 0, \ldots, \psi^{(n-1)}(x_0) = 0.$

Thus by uniqueness theorem, we have a solution $\psi(x) = 0, \forall x \in I$

$$\Rightarrow c_1 y_1(x) + \cdots + c_n y_n(x) = 0, \quad \forall x \in I.$$

where $c_1, c_2, \ldots, c_n$ are not all zero.

This contradicts the hypothesis that $y_1, y_2, \ldots y_n$ are linearly independent sol^n. Hence $W(y_1, y_2, \ldots y_n; x) = W(x) \neq 0$ $\forall\, x \in I$.

(Proved)

Ex. Check the solutions $\sin x$ and $\cos x$.

Let $y_1(x) = \sin x$, and $y_2(x) = \cos x$.

The wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x$$

$$= -1 \; (\neq 0)$$

The solutions $\sin x$ and $\cos x$ are linearly independent.

Here $\sin x$ and $\cos x$ are the linearly independent solutions of the 2nd order ODE $\frac{d^2y}{dx^2} + y = 0$

we will see this later.

Theorem:

Consider the nth order linear homogeneous ODE $L[y] = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$ $\forall x \in I$, where $a_0(x) \neq 0$, and all the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous on I. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be the linearly independent solutions of the given ODE. Therefore

$$W(y_1, y_2, \ldots, y_n; x) = W(x)$$

$$= c\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}, \quad \forall x \in I.$$

where c is constant.

Proof: Let the 2nd order linear homogeneous ODE —

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \quad x \in I.$$

Here $W(y_1, y_2; x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= y_1 y_2' - y_1' y_2$$

$$W' = y_1 y_2'' - y_1'' y_2.$$

Since y_1 and y_2 are the sol^n of

$a_0 y'' + a_1 y' + a_2 y = 0,$

we have

$$a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \quad \text{—} \; (1)$$

$$a_0 y_2'' + a_2 y_2' + a_2 y_2 = 0 \quad \text{—} \; (2)$$

From (1) $\quad y_1'' = -\dfrac{a_1}{a_0} y_1' - \dfrac{a_2}{a_0} y_1$

From (2) $\quad y_2'' = - a_2/a_0 \, y_2' - \dfrac{a_2}{a_0} y_2$

$$W'(x) = y_1\left(-\frac{a_1}{a_0}y_2' - \frac{a_2}{a_0}y_2\right)$$

$$- y_2\left(-\frac{a_1}{a_0}y_1' - \frac{a_2}{a_0}y_1\right)$$

$$= -\frac{a_1}{a_0}\left(y_1 y_2' - y_1' y_2\right)$$

$$= -\frac{a_1}{a_0} W(x)$$

or, $$W'(x) + \frac{a_1(x)}{a_0(x)} W(x) = 0$$

First order linear homogeneous eqn in W.

$$W(x) = c\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}.$$

where c is an arbitrary constants

(Proved)

Example: Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the given ODE $y'' + A(x)y' + B(x)y = 0$. Then we have $y_1(x)y_2'(x) - y_2(x)y_1'(x) = Ke^{-\int A(x)dx}$ where K is constant. Here $A(x)$ and $B(x)$ are continuous f^n on I.

Sol^n: The given ODE is

$$y'' + A(x)y' + B(x)y = 0$$

where $a_0(x) = 1$, $a_1(x) = A(x)$, $a_2(x) = B(x)$.

Now by applying the theorem on wronskian $W(x)$, we have —

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = K e^{-\int A(x)\,dx}$$

[By theorem]

$$\Rightarrow y_1(x) y_2'(x) - y_2(x) y_1'(x) = K e^{-\int A(x)\,dx}$$ —

Abel's formula: Theorem:

Consider nth order linear homogeneous ODE $L[y] = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$ — (1)

$x \in I$, where $a_0(x) \neq 0$, $a_0, a_1, \ldots$ and a_n are all continuous functions on I. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be the linearly independent solution of given ODE existing on I containing a point x_0. Then we have

$$W(y_1, y_2, \ldots, y_n; x)$$

$$= W(y_1, y_2, \ldots, y_n; x_0)\, e^{-\int_{x_0}^{x} \frac{a_1(x)}{a_0(x)}\,dx}$$

Proof. Try it similarly for 2nd order linear ODE.

Example: Let $y_1(x)$ and $y_2(x)$ be the linearly independent solution to the differential eqn $y'' - 2xy' + \sin(e^{2x})y = 0$, $x \in [0,1]$ with $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 1$. Then find the wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ at $x = 1$.

Solⁿ:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

$$W(y_1, y_2)(x) = y_1 y_2' - y_2 y_1'$$

$$W(y_1, y_2)(0) = y_1(0) y_2'(0) - y_2(0) y_1'(0)$$

$$= \underline{-1.}$$

Here $a_0(x) = 1$, $a_1(x) = -2x$, $a_2(x) = \sin e^{2x}$.

So, $\dfrac{a_1(x)}{a_0(x)} = -2x$. Here $x_0 = 0$

Apply the Abel's formula we have

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} \frac{a_1(x)}{a_0(x)} dx}$$

$$W(1) = W(0)\, e^{-\int_0^1 -2x\, dx}$$

$$= -1 \cdot e^1$$

$$= -e.$$

(Ans).