CALCULUS - I

Overview

Join google Classroom

Start Date : 29/11/21
End Date : 30/12/21

M → 11 - 12
W → 10 - 11
T → 9 - 10

Grading: Based on performance in 5 quizzes (online) each worth a 20 points (a total of 100 points).

Quiz Dates. 6th , 13th , 20th , 27th Dec. (Mondays) from 11:10 – 11:40 AM. and 30th Dec. (Thursday) from 9:10 – 9:40 AM.

Doubt Sessions. Wednesdays, 10:00 - 10:55 AM (or by appointment)

Weekly Worksheets. (Not to be graded) It is up to you to solve these practice problems.

References.

1. Thomas's Calculus
by Hass, Heil and Weir

2. Calculus Revisited, Part-I
by Herbert Gross
(available online)

Advanced Texts

3. Principles of Mathematical Analysis
by Walter Rudin

4. Mathematical Analysis
by Tom Apostol.

Main Objectives

1. Real valued functions

   - functions with domain contained in $\mathbb{R}$, and takes values in $\mathbb{R}$

   Real Numbers

   - Natural domains of functions

     e.g. the natural domain of $\sin x$ is $\mathbb{R}$ – $\sin x$ is defined for every real number x.

   - Restricted domain of a function

     - study the behavior of a function on a subset of a $\mathbb{R}$.

     Includes

     - finite sets
     - infinite sets

       Intervals

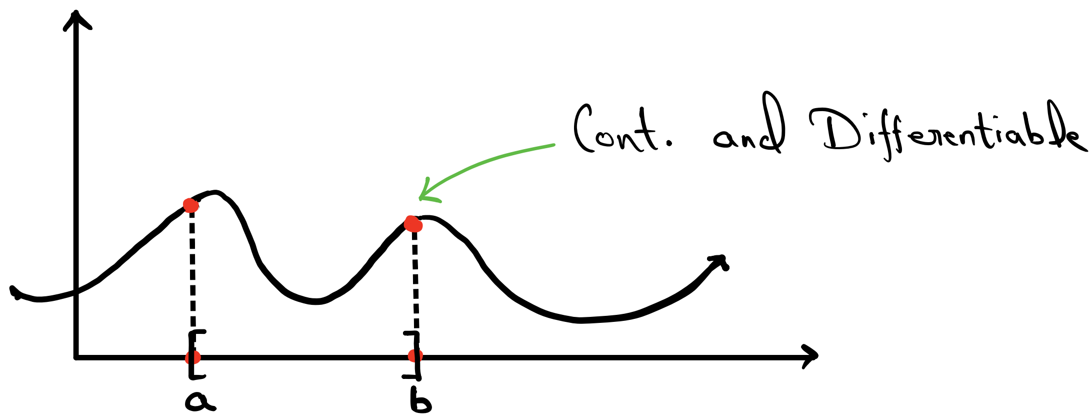
       $[a, b]$

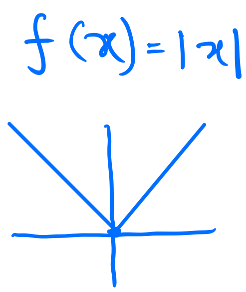
       (a, b)

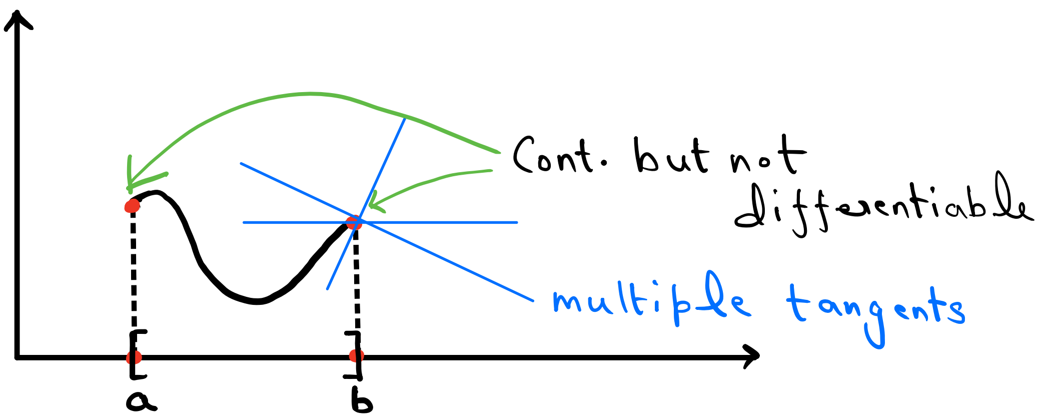
       $[a, b)$

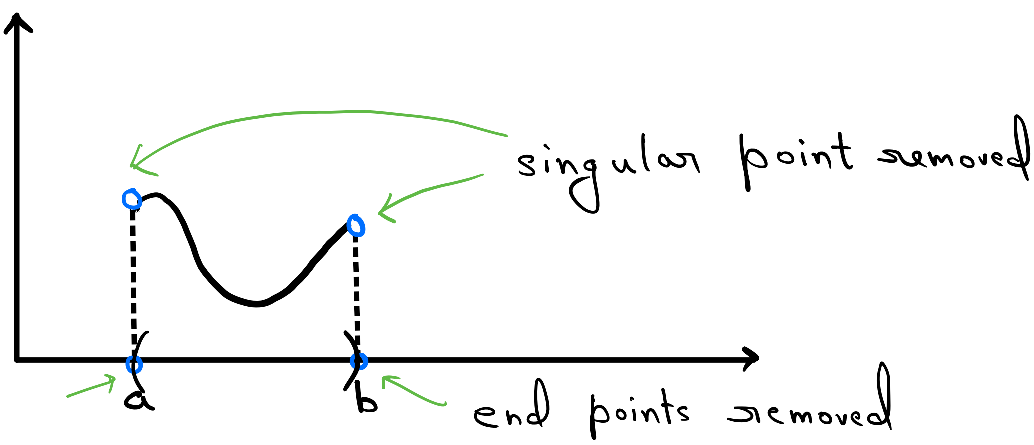
       Rational Numbers

       Irrational numbers









Now the function is differentiable on the open interval (a, b).

2. Numerical sequences & series

- A sequence of real numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

In short $\{a_n\}_{n=1}^{\infty}$

- A series looks like

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

In <u>Sigma</u> notation

$$\sum_{n=1}^{\infty} a_n$$

<u>Main</u> <u>Questions</u>

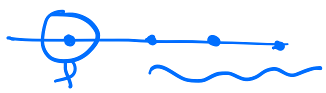
1) Is a sequence $a_1, a_2, \dots, a_n, \dots$ heading towards a limit?

Meaning, is there a real no. ℓ such that $|a_n - \ell| \to 0$ as $n \to \infty$?

or $\lim_{n\to\infty} a_n = \ell$?

Some common scenarios

- Consider the following sequences



- $1, 1, 1, 1, \ldots$ (constant sequence)

  limit $= 1$

- $1, 2, 3, \ldots, 100, 101, \ldots$

  limit $= \infty$

- $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}, \ldots, \frac{1}{10000}, \ldots$

  limit $= 0$

- $1, -1, 1, -1, 1, -1, \ldots$

  limit does not exist!

  $$|a_n - \ell| = \begin{cases} |1+\ell| & \text{if } n\text{-even} \\ |1-\ell| & \text{if } n\text{-odd} \end{cases}$$

  clearly, $1+\ell$ and $1-\ell$ cannot be simultaneously 0.

2) When does the sum $\sum_{n=1}^{\infty} a_n$ exist?

3) If $\sum_{n=1}^{\infty} a_n$ exists, then what is the sum?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Why study sequence and series?

(1) Sequences are building blocks of series.

(2) Series provide a way to find "approximate" values of various "transcendental" functions.

$$\frac{x^2-3x+1}{x^4+x^2+1}$$

Algebraic fn.

What is the value of e?

With the knowledge that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Thus;

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Truncating to first four terms, we get

$$e \sim 2.66667 \sim 2.718\ldots$$

Things to ponder about:

- why should the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ exist?
- why does the truncation give a good approximation?
- why is the sum equal to e?

(3) Sequence and continuity.

$f(x)$ be a function.

Suppose $f(x)$ is defined at $x=a$

Then

$f(x)$ is continuous at $x=a$
if and only if for every
sequence a_n with $\lim_{n\to\infty} a_n = a$,
one has $\lim_{n\to\infty} f(a_n) = f(a)$.

That is,

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

Applications

- Consider the Dirichlet's function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Is $f(x)$ continuous at $x=1$?

Proof by contradiction

So, assume f is cont. at $x=1$.

Consider the sequence

$$a_n = 1 + \frac{1}{2^{n+1/2}} = 1 + \frac{1}{2^n\sqrt{2}}$$

Then a_n is irrational for every n.

so that, $f(a_n) = 1$ for every n

$$\Rightarrow \lim_{n\to\infty} f(a_n) = 1.$$

On the other hand,

$$\lim_{n\to\infty} a_n = 1$$

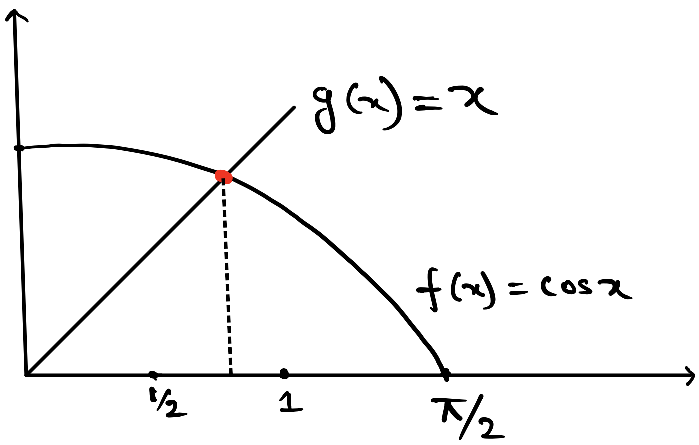
f - continuous means that

$$\underbrace{\lim_{n\to\infty} f(a_n)}_{=1} = \underbrace{f(1)}_{=0}$$

an absurd situation.

So $f(x)$ is a not continuous.

- Solve for x: $\cos x = x$ —— (*)



Guess solution: $x \sim 0.8$

Set $a_0 = 0.8$

$a_1 = \cos a_0$

$a_2 = \cos a_1 = \cos(\cos a_0)$

$\vdots$

$a_{n+1} = \cos a_n = \underbrace{\cos(\cos(\cdots(\cos a_0)\cdots))}_{n\text{-times}}$

$\vdots \qquad \vdots$

Thus, $a_{n+1} = \cos a_n$

$\lim_{n\to\infty} a_n = l$

Take limits on both sides

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \cos a_n$$

$$= \cos\left(\lim_{n\to\infty} a_n\right)$$

Suppose $\lim_{n\to\infty} a_n$ exists! and equal to "l".

Then $\lim_{n\to\infty} a_{n+1} = \lim_{n+1\to\infty} a_{n+1} = \ell$

Thus, $\ell = \cos \ell$

where $\ell = \lim_{n\to\infty} a_n$

That is, ℓ is a solution to (*)

<u>Approximating ℓ</u>

we know $|a_n - \ell| \longrightarrow 0$

as $n \longrightarrow \infty$

Meaning with increasing "n", we get better and better approximations of "ℓ".

$a_1 = 0.69670670$

$a_2 = 0.76695963$

$a_3 = 0.72002385$

$a_4 = 0.75178999$

Picture starts getting clearer now

$a_5 = 0.730467564$

$a_6 = 0.744862516$

$a_7 = 0.735181105$

$a_8 = 0.741709294$

$$a_9 = 0.737314924$$

$$a_{10} = 0.740276408$$

It can be inferred that

a_{odd} is moving away from 0.73

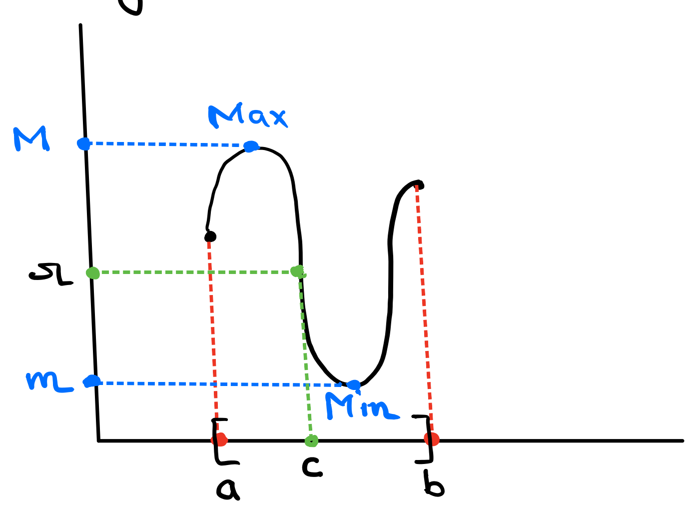
and a_{even} is getting closer to 0.74

Thus, a good approximate solution to (*) is $x = 0.739$

Check! $\cos(0.739) = 0.7391424$

3. Analytic properties of functions.

- Limits and continuity
- Epsilon-Delta definition of limit
- Continuity on an interval



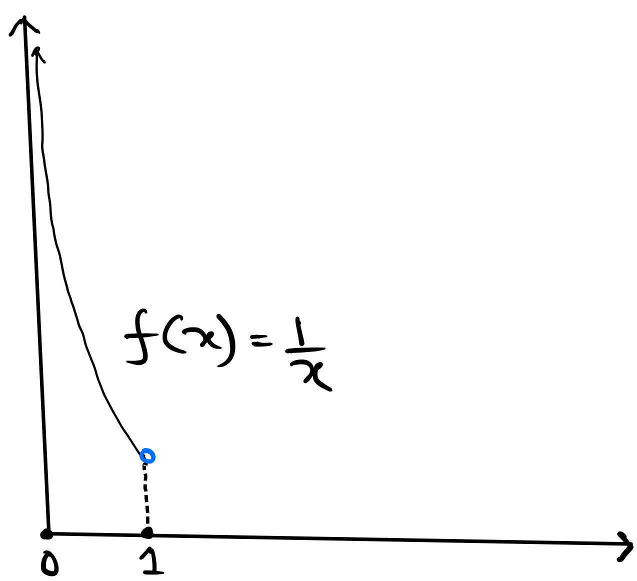
- A cont. function on a closed interval $[a,b]$ is bounded

- The range $f([a,b])$ is a <u>closed</u> interval $[m,M]$

  $\Rightarrow$ if $r \in [m,M]$, then there is a $c \in [a,b]$ s.t.

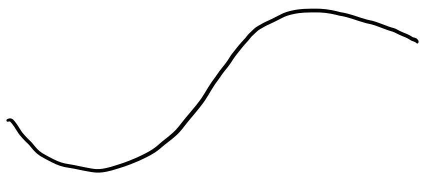
  $$f(c) = r$$

Continuity on an open interval

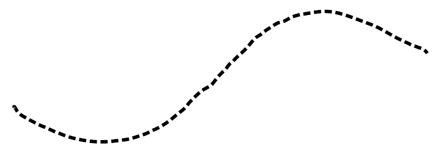


$f(x)$ is cont. on $(0,1)$, but <u>not</u> <u>bounded</u>.

Differentiation



Smooth function



small tangent segments making up a smooth function.

Derivative measures the slope of these tangent segments

Use of derivatives

- Optimization problems
- Understanding graphs of functions
- Meanvalue Theorem
- Darboux Theorem - The derivative of a differentiable function satisfies the intermediate value Theorem. (note: the derivative need not be continuous).

$\cos x = x$ has a <u>unique</u> solution

$x \approx 0.739$

<u>Integrable</u>

$[a, b]$

(5) pts. of discont.