भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

11th Lecture on Transform Techniques

(MA-2120)

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What will we learn today?

- Fourier Integral
- Fourier Transform

Summary:

Fourier Series

$f(x)$ is periodic function defined on $[-l, l]$ with period $2l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

Fourier Integral

$f(x)$ is nonperiodic function defined on $(-\infty, \infty)$.

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x \right] d\omega$$

$$A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

Fourier Cosine integral:

If $f(x)$ is even function, then the Fourier Cosine integral representation of $f(x)$ on $[0,\infty)$ is

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos\omega x \, d\omega} \qquad 0 < x < \infty$$

where $A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t \, dt$

$$= 2\int_0^\infty f(t)\cos\omega t \, dt$$

Fourier Sine integral:

If $f(x)$ is odd function, then the Fourier Sine integral representation of $f(x)$ on $[0,\infty)$ is

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty} B(\omega)\sin\omega x\, d\omega}, \quad 0<x<\infty$$

where $B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$

$$= 2\int_0^{\infty} f(t)\sin\omega t\, dt.$$

Ex:

$$f(x) = \begin{cases} 0, & -\infty < x < -\pi \\ -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

Soln:

Here $f(x)$ is odd function.

$A(\omega) = 0$. $B(\omega) = 2\int_0^\infty f(t) \sin \omega t \, dt$

$$= \frac{2(1 - \cos \omega\pi)}{\omega}$$

Fourier sine integral representation of

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos w\pi}{w}\right)\sin wx\,dw.$$

at $x = -\pi$,

$$-\frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos w\pi}{w}\right)\sin w\pi\,dw$$

$$= \frac{f(-\pi+) + f(-\pi-)}{2} = \frac{-1+0}{2}$$

$$= -\frac{1}{2}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{1-\cos\omega\pi}{\omega}\right) \sin\omega\pi \, d\omega$$

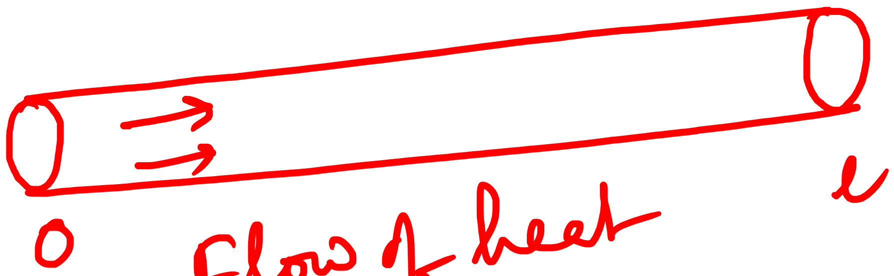
$$= \underline{\pi/4}.$$

Ex: Find the Fourier cosine and sine integral representation of $f(x) = e^{-kx}$, $x \geq 0$, where k is a positive constant.

Sol^n: Try it!

(2) Now you see the applications to the Fourier series and Fourier Integral,

Fourier Series Solution of the Heat equation



Flow of heat in a bar

Initial temperature of the bar is $f(x)$ and end of the bar kept at zero.

BVP:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \; t > 0$$

Initial Condition: $u(x,0) = f(x), \; 0 < x < l$

Boundary Condition: $u(0,t) = u(l,t) = 0, \; t \geq 0$

Fourier method: $u(x,t) = X(x)\,T(t)$.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2t}{l^2}}, \quad 0 \le x \le l$$

Initial condition: $u(x,0) = f(x)$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 \le x \le l$$

This is Fourier half range sine series in $[0,l]$.

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2t}{l^2}}, \quad 0 \le x \le l$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

Temperature distribution in thin, infinite bar :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \; t > 0$$

initial condition $u(x,0) = f(x), \quad -\infty < x < \infty$

Here Fourier integral is used to get the solution $u(x,t)$.

One dimensional wave eqⁿ:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0<x<l, \quad t>0$$

IC: $u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$

BC: $u(0,t) = 0, \quad u(l,t) = 0, \quad t>0$

Here Fourier Series is used to get the solution $u(x,t)$.

Complex form of Fourier integral representation

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(t)\left\{\cos\omega t \cos\omega x + \sin\omega t \sin\omega x\right\} dt\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\, dt\, d\omega$$

Since $\cos\omega(t-x)$ is an even function w.r.t. ω.

we add the term

$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega(t-x)\,dt\,d\omega,$$

the value of this integral is 0.

Then we have –

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos\omega(t-x) + i\sin\omega(t-x)\right] dt\,d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\omega(t-x)}\, dt\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \right] e^{-i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega)\, e^{-i\omega x}\, d\omega,$$

where $c(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$,

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)\, e^{-i\omega x}\, d\omega \\ \text{where } c(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \end{aligned}}$$

These are the the complex form of Fourier integral representation.

Example: Find the complex form of Fourier integral representation for the function

$$f(x) = \begin{cases} |x|, & -\pi < x < \pi \\ 0, & \text{elsewhere} \end{cases}$$

Verify that it is same as that obtained by using the Fourier Cosine integral representation.

Sol^n:

we have

$$c(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$= \int_{-\pi}^{\pi} |t|\, e^{i\omega t}\, dt = \int_{0}^{\pi} t e^{i\omega t}\, dt - \int_{-\pi}^{0} t e^{i\omega t}\, dt$$

$$= \frac{2}{\omega^2}\cos(\pi\omega) + \frac{2\pi}{\omega}\sin(\pi\omega) - \frac{2}{\omega^2}$$

Therefore,

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2} \left[\cos(\pi\omega) + \pi\omega \sin(\pi\omega) - 1 \right] e^{-i\omega x} \, d\omega$$

Fourier cosine integral:

$$A(\omega) = 2 \int_0^{\pi} t \cos(\omega t) \, dt$$

$$= 2 \left[\frac{\pi}{\omega} \sin \pi\omega + \frac{1}{\omega^2} \left\{ \cos(\pi\omega) - 1 \right\} \right]$$

Here $C(\omega)$ is same as $A(\omega)$.

Fourier Transform :

Fourier Transform is one of the integral transform similar to Laplace Transform.

Fourier Transforms are applied in many areas of Science and engineering.

Let $f(t)$ be piecewise continuous on $(-\infty, \infty)$. Assume that $f(t)$ is absolutely convergent, i.e., $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges.

Then Fourier Transform of $f(t)$ denoted by $\mathcal{F}[f(t)]$ is defined as

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \underline{F(\omega)}$$

Inverse Fourier Transform:

Assume that $\int_{-\infty}^{\infty} |F(\omega)|\, d\omega$ Converges.

Then we define the inverse Fourier transform of $F(\omega)$ as

$$\mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$= f(t).$$

The Fourier Transform and its inverse are called transform pair.

Ex: Find the Fourier Transform of the following functions defined on $(-\infty,\infty)$

$$f(t) = \begin{cases} a, & -1 < t < 0 \\ b, & 0 < t < 1 \\ 0, & \text{otherwise}, \quad a>0, b>0 \end{cases}$$

<u>Solⁿ:</u>

$$\mathcal{F}\left[f(t)\right] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= a\int_{-l}^{0} e^{-i\omega t}\, dt + b\int_{0}^{l} e^{-i\omega t}\, dt$$

$$= -\frac{a}{i\omega}\left(1 - e^{i\omega l}\right) - \frac{b}{i\omega}\left(e^{-i\omega l} - 1\right)$$

$$= \frac{1}{i\omega}\left[(b-a) + a\, e^{i\omega l} - b\, e^{-i\omega l}\right]$$

Example: Find the Fourier Transform of the function

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-\alpha t}, & t \geq 0, \quad \alpha > 0 \end{cases}$$

Soln: The function $f(t)$ has a jump discontinuity at $t = 0$.

$$\int_{-\infty}^{\infty} |f(t)|\, dt = \int_{0}^{\infty} e^{-\alpha t}\, dt = \underline{\frac{1}{\alpha}}.$$

Therefore Fourier Transform of $f(t)$ exists.

Now we have

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \lim_{s \to \infty} \int_0^s e^{-\alpha t}\, e^{-i\omega t}\, dt$$

$$= \lim_{s \to \infty} \left[- \frac{e^{-(\alpha + i\omega)t}}{\alpha + i\omega} \right]_0^s = \frac{1}{\alpha + i\omega}$$

Here $f(t) = e^{-\alpha t} H(t)$, where $H(t)$ is the unit step function.

Hence $F(\omega) = \dfrac{1}{\alpha + i\omega}$ and $f(t) = e^{-\alpha t} H(t)$

form a transform pair.

Therefore $$\boxed{\mathcal{F}\left[e^{-\alpha t} H(t)\right] = \frac{1}{\alpha + i\omega}}.$$

$$\mathcal{F}^{-1}\left[\frac{1}{\alpha + i\omega}\right] = e^{-\alpha t} H(t).$$

Example: Find the Fourier transform of the function $f(t) = e^{-a|t|}$, $-\infty < t < \infty$, $a > 0$. Write the inverse transform.

Sol^n:

$$f(t) = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}$$

Therefore, $\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

$$= \int_{-\infty}^{0} e^{at} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t} dt$$

$$= \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}$$

Inverse transform —

$$\mathcal{F}^{-1}\left[\frac{2a}{a^2+\omega^2}\right] = e^{-a|t|} .$$

Ex: Find the Fourier transform of e^{-at^2}, $a>0$

Soln: $$\mathcal{F}\left[e^{-at^2}\right] = \int_{-\infty}^{\infty} e^{-at^2} \cdot e^{-i\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[t^2 + (i\omega t/a)\right]}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[\left\{t + i\omega/2a\right\}^2 + \omega^2/4a^2\right]}\, dt$$

$$= e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-a\left[t + i\omega/2a\right]^2}\, dt$$

Set $\sqrt{a}\left(t + \frac{i\omega}{2a}\right) = \tau$.

$$= e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-\tau^2} \frac{d\tau}{\sqrt{a}}$$

$$= \sqrt{\pi}/\sqrt{a}\; e^{-\omega^2/4a}$$

Since $\int_{-\infty}^{\infty} e^{-\tau^2} d\tau = 2 \int_{0}^{\infty} e^{-\tau^2} d\tau = \sqrt{\pi}$.

Amplitude Spectrum:

The graph of $(\omega, |F(\omega)|)$ is called the amplitude spectrum and ω is called the frequency of the transform.

Properties of the Fourier Transform:

(1) Linearity of Fourier Transform:

$$\mathcal{F}[a f(t) + b g(t)] = a \mathcal{F}[f(t)] + b \mathcal{F}[g(t)]$$

provided that the Fourier transform of $f(t)$ and $g(t)$ exist.

(2) Frequency shifting: (Translation)

Theorem: If $\mathcal{F}[f(t)] = F(\omega)$ and ω_0 is any real number, then

$$\boxed{\mathcal{F}[e^{i\omega_0 t} f(t)] = F(\omega-\omega_0)}$$

Proof: From the definition,

$$\mathcal{F}[e^{i\omega_0 t} f(t)] = \int_{-\infty}^{\infty} e^{i\omega_0 t}\, e^{-i\omega t} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t} f(t)\, dt = F(\omega-\omega_0).$$

Inverse Transform:

$$\mathcal{F}^{-1}\left[F(\omega-\omega_0)\right] = e^{i\omega_0 t} f(t).$$

③ Shifting on t-axis:

Theorem: If $\mathcal{F}[f(t)] = F(\omega)$ and t_0 is any real number then

$$\boxed{\mathcal{F}[f(t-t_0)] = F(\omega)\, e^{-i\omega t_0}}$$

Proof: From the definition, we get,

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-i\omega t}\, dt$$

$$= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(t-t_0)\, e^{-i\omega(t-t_0)}\, dt$$

Let $t-t_0 = \tau$. Then

$$\mathcal{F}[f(t-t_0)] = e^{-i\omega t_0} \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= e^{-i\omega t_0} F(\omega)$$

Inverse Tranform:

$$\boxed{\mathcal{F}^{-1}\left[e^{-i\omega t_0} F(\omega)\right] = f(t-t_0)}$$

Ex: Find $\mathcal{F}^{-1}\left[\dfrac{e^{4i\omega}}{3+i\omega}\right]$.

Solⁿ: we have $\mathcal{F}\left[e^{-3t} H(t)\right] = \dfrac{1}{3+i\omega} = F(\omega)$

$$\mathcal{F}^{-1}\left[\frac{1}{3+i\omega}\right] = e^{-3t} H(t) = f(t)$$

using shift theorem, we get

$$\mathcal{F}^{-1}\left[\frac{e^{-4i\omega}}{3+i\omega}\right] = \mathcal{F}^{-1}\left[e^{-4i\omega} \cdot F(\omega)\right]$$

$$= f(t-(-4)) = f(t+4)$$

$$= e^{-3(t+4)} H(t+4)$$

$$= \begin{cases} 0, & t < -4 \\ e^{-3(t+4)}, & t \geq -4 \end{cases}.$$