Centre of gravity and centroid; Moment of inertia

Centre of gravity: The plane figures (like triangle, quadrilateral, circle etc.) have only areas, but no mass. The centre of area of such figures is known as centroid. It is also known as first moment of area. The method of finding out the centroid of a figure is the same as that of finding out the centre of gravity of a body. In many books, the authors also write centre of gravity for centroid and vice-versa.

Centre of mass:

The *centre of mass* is a position defined relative to an object or system of objects. It is the average position of all the parts of the system, weighted according to their masses.

For simple rigid objects with uniform density, the centre of mass is located at the centroid. For example, the centre of mass of a uniform disc shape would be at its centre. Sometimes the centre of mass doesn't fall anywhere on the object. The centre of mass of a ring for example is located at its centre, where there isn't any material.

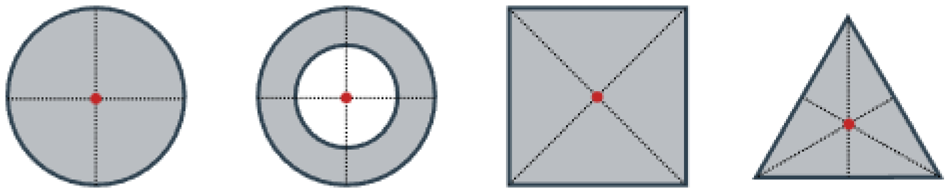
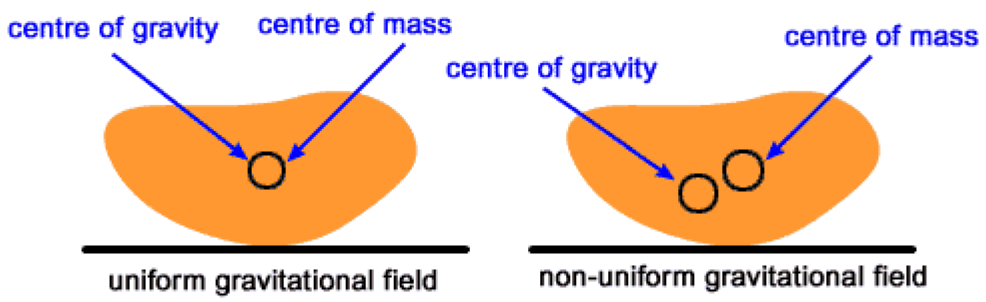


Figure 1: Center of mass for some simple geometric shapes (red dots).

Difference between centre of mass and centre of gravity:

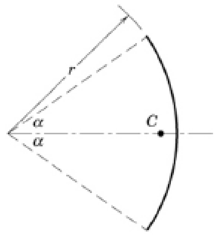
The centre of mass is defined as being the position at which the distribution of mass is equal in all directions, or in other words, the point representing the mean position of all the mass in a massive body. The centre of gravity is the point through which gravity appears to act. For nearly all objects, they are indeed the same point.

However, if the mass is placed in a non-uniform gravitational field, the positions can be different.



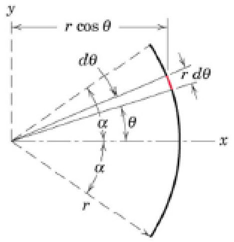
Examples: Centroids

Locate the centroid of the circular arc



Solution: Polar coordinate system is better
Since the figure is symmetric: centroid lies on the x axis

Differential element of arc has length $dL = rd\Theta$
Total length of arc: $L = 2\alpha r$
x-coordinate of the centroid of differential element: $x = r\cos\Theta$

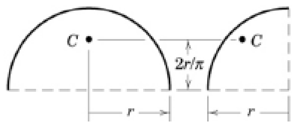


$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$

$$\mathrm{L}\bar{x} = \int x dL \qquad 2\alpha r\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta) r d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

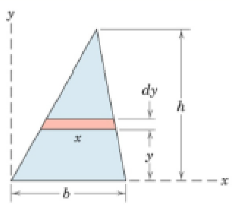
$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$



For a semi-circular arc: $2\alpha = \pi$ → centroid lies at $2r/\pi$

Examples: Centroids

Locate the centroid of the triangle along h from the base



Solution:

$$dA = x dy \qquad \frac{x}{(h-y)} = \frac{b}{h}$$

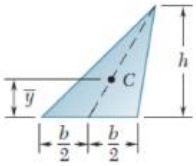
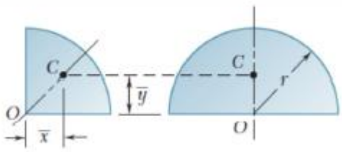
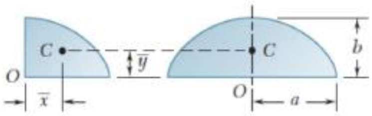
Total Area $A = \frac{1}{2} bh \qquad y = y_c$

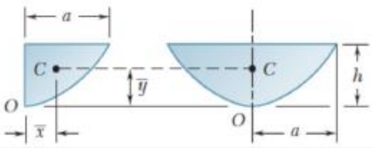
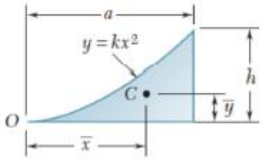
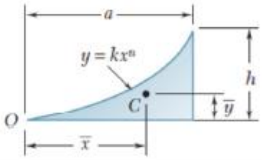
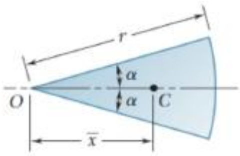
$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

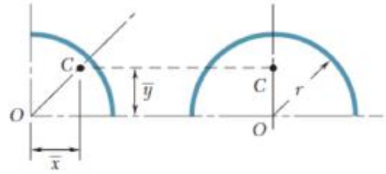
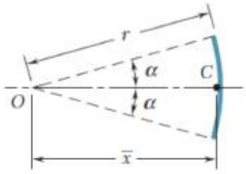
$$\mathrm{A}\bar{y} = \int y_c dA \quad \Rightarrow \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{y} dy = \frac{bh^2}{6}$$

$$\boxed{\bar{y} = \frac{h}{3}}$$

<u>**Centroid for some common shapes:**</u>

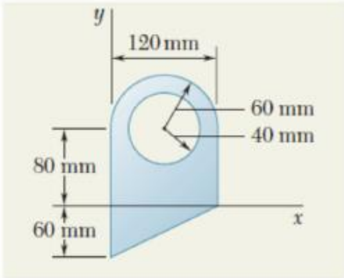
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

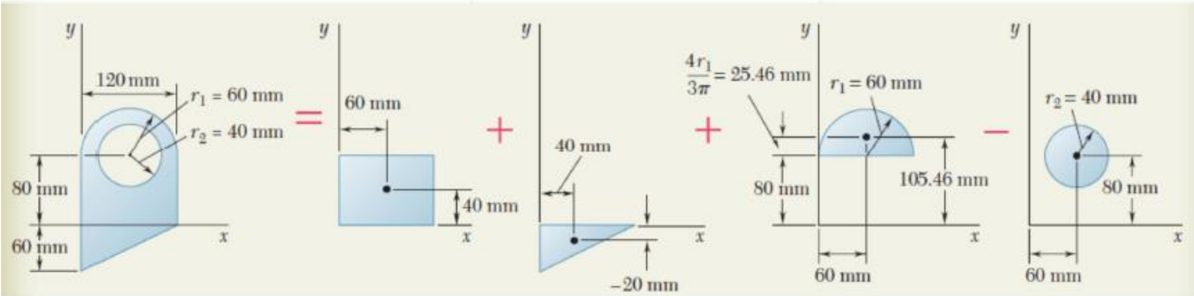
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

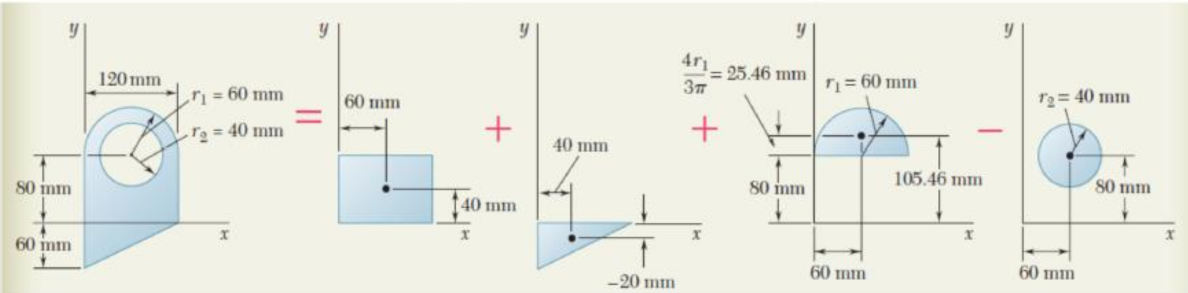
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Examples

For the plane area shown, determine (*a*) the first moments with respect to the x and y axes, (*b*) the location of the centroid.







Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

First Moments of the Area.

$$Q_x = \Sigma\bar{y}A = 506.2 \times 10^3 \text{ mm}^3$$
$$Q_y = \Sigma\bar{x}A = 757.7 \times 10^3 \text{ mm}^3$$

Location of Centroid.

$$\bar{X}\Sigma A = \Sigma\bar{x}A: \quad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$$
$$\bar{X} = 54.8 \text{ mm}$$
$$\bar{Y}\Sigma A = \Sigma\bar{y}A: \quad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$$
$$\bar{Y} = 36.6 \text{ mm}$$

Example: Composite Bodies and Figures

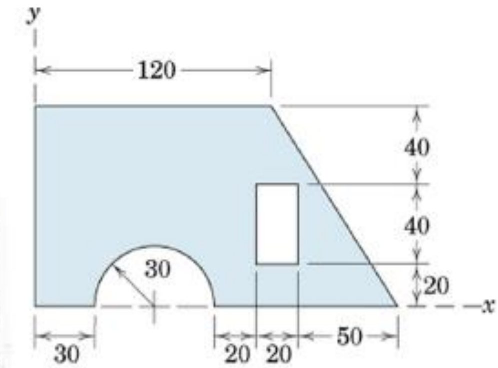
Example:
Locate the centroid of the shaded area

Solution: Divide the area into four elementary shapes: Total Area = $A_1 + A_2 - A_3 - A_4$

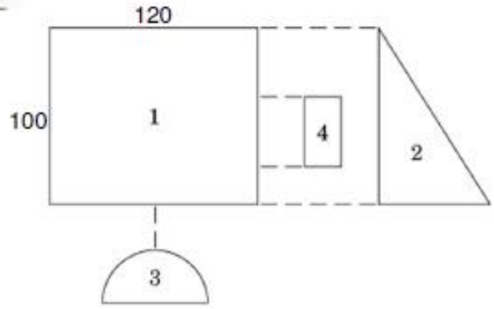
PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959\,000}{12\,790} = 75.0 \text{ mm}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650\,000}{12\,790} = 50.8 \text{ mm}$$



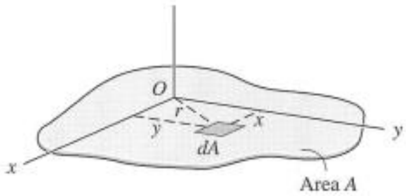
Dimensions in millimeters



Area Moment of Inertia (Second moment of an area)

The Moment of Inertia (I) is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as X-X or Y-Y. It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis (axis of interest). The reference axis is usually a centroidal axis.

The moment of inertia is also known as the **Second Moment of the Area** and is expressed mathematically as:



$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$

Where

y = distance from the x axis to area *dA*
x = distance from the y axis to area *dA*

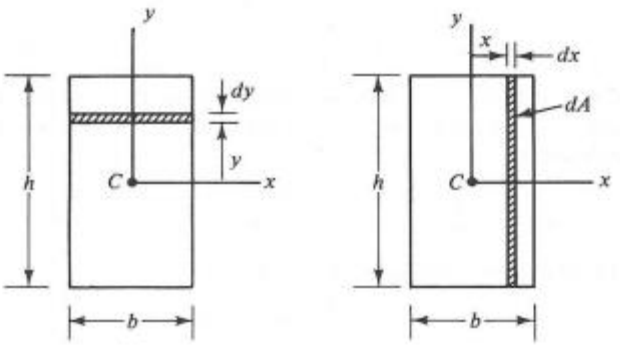


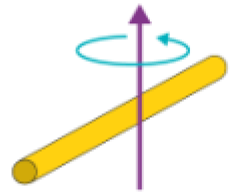
Figure: Example

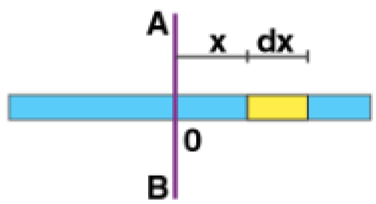
Moment of inertia:

The moment of inertia of continuous mass distribution is found by using the integration technique. If the system is divided into an infinitesimal element of mass 'dm' and if 'x' is the distance from the mass element to the axis of rotation, the moment of inertia is:

$$I = \int r^2 \, dm$$

Consider a uniform rod of mass M and length L and the moment of inertia should be calculated about the bisector AB. Origin is at 0.





The mass element ‘dm’ considered is between x and x + dx from the origin.

As the rod is uniform, mass per unit length (linear mass density) remains constant.

$$\therefore\ M/L = dm/dx$$

$$dm = (M/L)dx$$

Moment of inertia of dm ,

$$dI = dm\ x^2$$

$$dI = (M/L)\ x^2.dx$$

$$I = {}_{-L/2}\int^{+L/2} dI = M/L \times {}_{-L/2}\int^{+L/2} x^2\,dx$$

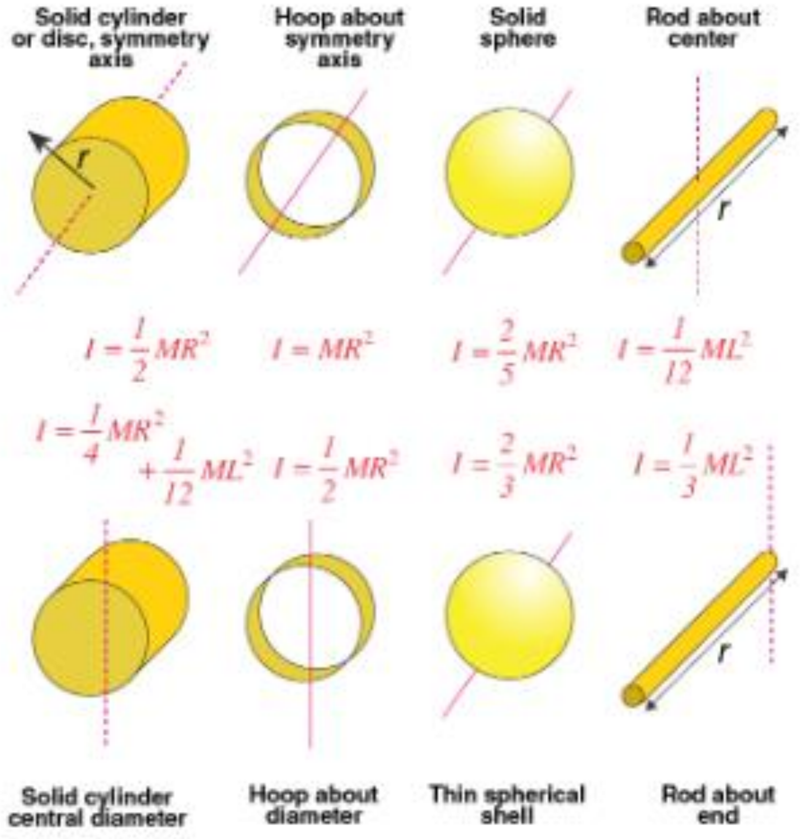
Here, x = -L/2 is the left end of the rod and ‘x’ changes from –L/2 to +L/2, the element covers the entire rod.

$$I = M/L \times [x^3/3]^{+L/2}_{-L/2}$$

$$I = ML^2/12.$$

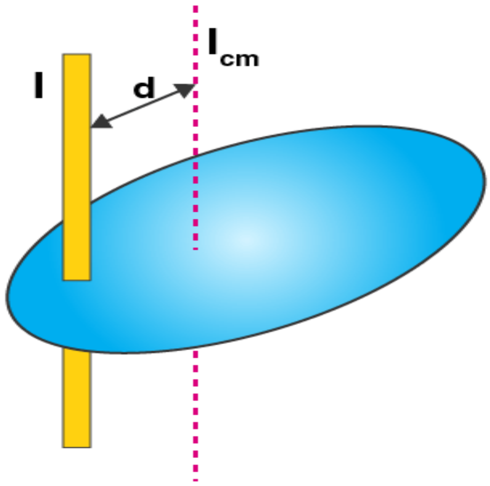
Therefore, the moment of inertia of a uniform rod about a perpendicular bisector (I) = $ML^2/12$.

Mass moment of inertia of simple geometries:



Parallel Axis Theorem

The moment of inertia of an object about an axis through its centre of mass is the minimum moment of inertia for an axis in that direction in space. The moment of inertia about an axis parallel to that axis through the centre of mass is given by,

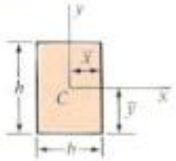


$$I = I_{CM} + Md^2$$

- I is the moment of inertia of the body
- I_{cm} is the moment of inertia about the centre
- M is the mass of the body
- d is the distance between the two axes.

Area Moment of Inertia of simple geometries:

1



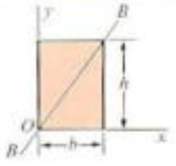
Rectangle (Origin of axes at centroid.)

$A = bh \qquad \bar{x} = \dfrac{b}{2} \qquad \bar{y} = \dfrac{h}{2}$

$I_x = \dfrac{bh^3}{12} \qquad I_y = \dfrac{hb^3}{12}$

$I_{xy} = 0 \qquad I_p = \dfrac{bh}{12}(h^2 + b^2)$

2

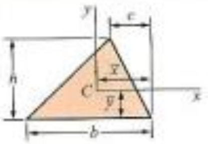


Rectangle (Origin of axes at corner.)

$I_x = \dfrac{bh^3}{3} \qquad I_y = \dfrac{hb^3}{3}$

$I_{xy} = \dfrac{b^2h^2}{4} \qquad I_p = \dfrac{bh}{3}(h^2 + b^2) \qquad I_{BB} = \dfrac{b^3h^3}{6(b^2 + h^2)}$

3



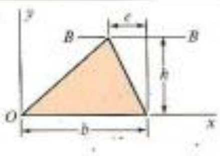
Triangle (Origin of axes at centroid.)

$A = \dfrac{bh}{2} \qquad \bar{x} = \dfrac{b + c}{3} \qquad \bar{y} = \dfrac{h}{3}$

$I_x = \dfrac{bh^3}{36} \qquad I_y = \dfrac{bh}{36}(b^2 - bc + c^2)$

$I_{xy} = \dfrac{bh^2}{72}(b - 2c) \qquad I_p = \dfrac{bh}{36}(h^2 + b^2 - bc + c^2)$

4

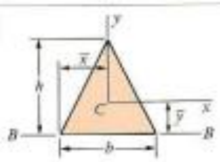


Triangle (Origin of axes at vertex.)

$I_x = \dfrac{bh^3}{12} \qquad I_y = \dfrac{bh}{12}(3b^2 - 3bc + c^2)$

$I_{xy} = \dfrac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \dfrac{bh^3}{4}$

5



Isosceles triangle (Origin of axes at centroid.)

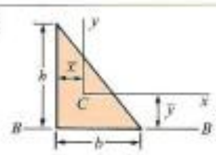
$A = \dfrac{bh}{2} \qquad \bar{x} = \dfrac{b}{2} \qquad \bar{y} = \dfrac{h}{3}$

$I_x = \dfrac{bh^3}{36} \qquad I_y = \dfrac{hb^3}{48} \qquad I_{xy} = 0$

$I_p = \dfrac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \dfrac{bh^3}{12}$

(Note: For an equilateral triangle, $h = \sqrt{3}b/2$.)

6



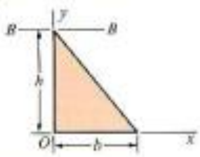
Right triangle (Origin of axes at centroid.)

$A = \dfrac{bh}{2} \qquad \bar{x} = \dfrac{b}{3} \qquad \bar{y} = \dfrac{h}{3}$

$I_x = \dfrac{bh^3}{36} \qquad I_y = \dfrac{hb^3}{36} \qquad I_{xy} = -\dfrac{b^2h^2}{72}$

$I_p = \dfrac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \dfrac{bh^3}{12}$

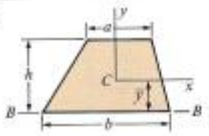
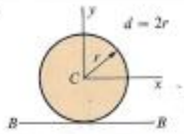
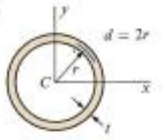
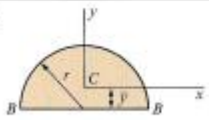
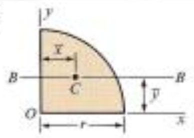
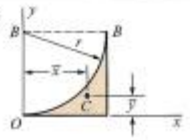
7



Right triangle (Origin of axes at vertex.)

$I_x = \dfrac{bh^3}{12} \qquad I_y = \dfrac{hb^3}{12} \qquad I_{xy} = \dfrac{b^2h^2}{24}$

$I_p = \dfrac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \dfrac{bh^3}{4}$

8		**Trapezoid** (Origin of axes at centroid.) $A = \frac{h(a+b)}{2}$ $\quad \bar{y} = \frac{h(2a+b)}{3(a+b)}$ $\quad I_x = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$ $\quad I_{BB} = \frac{h^3(3a+b)}{12}$
9		**Circle** (Origin of axes at center.) $A = \pi r^2 = \frac{\pi d^2}{4}$ $\quad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$ $\quad I_{xy} = 0$ $\quad I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$ $\quad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$
10		**Circular ring** (Origin of axes at center.) Approximate formulas for case when t is small. $A = 2\pi r t = \pi d t$ $\quad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$ $\quad I_{xy} = 0$ $\quad I_p = 2\pi r^3 t = \frac{\pi d^3 t}{4}$
11		**Semicircle** (Origin of axes at centroid.) $A = \frac{\pi r^2}{2}$ $\quad \bar{y} = \frac{4r}{3\pi}$ $\quad I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4$ $\quad I_y = \frac{\pi r^4}{8}$ $\quad I_{xy} = 0$ $\quad I_{BB} = \frac{\pi r^4}{8}$
12		**Quarter circle** (Origin of axes at center of circle.) $A = \frac{\pi r^2}{4}$ $\quad \bar{x} = \bar{y} = \frac{4r}{3\pi}$ $\quad I_x = I_y = \frac{\pi r^4}{16}$ $\quad I_{xy} = \frac{r^4}{8}$ $\quad I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488r^4$
13		**Quarter-circular spandrel** (Origin of axes at vertex.) $A = \left(1 - \frac{\pi}{4}\right)r^2$ $\quad \bar{x} = \frac{2r}{3(4-\pi)} \approx 0.7766r$ $\quad \bar{y} = \frac{(10 - 3\pi)r}{3(4-\pi)} \approx 0.2234r$ $\quad I_x = \left(1 - \frac{5\pi}{16}\right)r^4 \approx 0.01825r^4$ $\quad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right)r^4 \approx 0.1370r^4$

Radius of Gyration

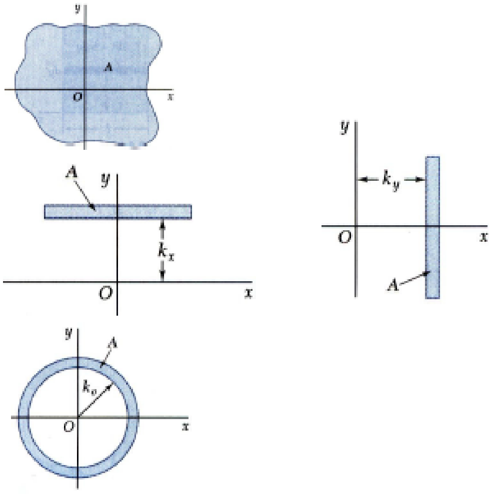
The radius of gyration of a mass with respect to a particular axis is the square root of the quotient of the moment of inertia divided by the mass. It is the distance at which the entire mass must be assumed to be concentrated in order that the product of the mass and the square of this distance will equal the moment of inertia of the actual mass about the given axis. In other words, the radius of gyration describes the way in which the total mass is distributed around its centroidal axis. If more mass is distributed further from the axis, it will have greater resistance to buckling. The most efficient column section to resist buckling is a circular pipe because it has its mass distributed as far away as possible from the centroid.

If the moment of inertia (I) of a body of mass m about an axis be written in the form:

$$I = Mk^2$$

Here, k is called radius of gyration of body about the given axis. It represents the radial distance from the given axis of rotation where the entire mass of the body can be assumed to be concentrated so that its rotational inertia remains unchanged

Radius of Gyration of an Area



- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

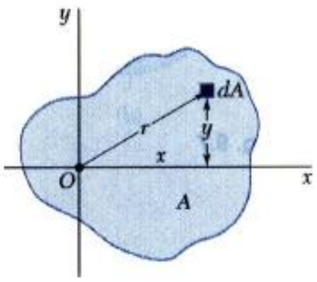
$$J_O = I_z = k_O^2 A = k_z^2 A \qquad k_O = k_z = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_z^2 = k_x^2 + k_y^2$$

Radius of Gyration, *k* is a measure of distribution of area from a reference axis
Radius of Gyration is different from centroidal distances

Polar Moment of Inertia:

Polar Moment of Inertia



- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = I_z = \int r^2 dA$$

- The polar moment of inertia is related to the rectangular moments of inertia,

$$J_0 = I_z = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Moment of Inertia of an area is purely a mathematical property of the area and in itself has no physical significance.

Perpendicular axis theorem

It states, If IXX and IYY be the moments of inertia of a plane section about two perpendicular axis meeting at O, the moment of inertia IZZ about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:

$$I_{zz} = I_{xx} + I_{yy}$$

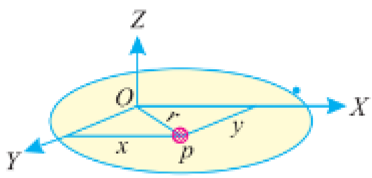


Figure: Perpendicular axis theorem

Note: Please go through the attached additional problems and solutions of surprise test for more information on problem solving.

References:

1. *Engineering mechanics by S. Timoshenko*
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3. *J. L. Meriam and L. G. Kraige, Engineering Mechanics, Vol I - Statics, Vol II - Dynamics, 5th Ed., John Wiley, 2002*
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5. *F. P. Beer and E. R. Johnston, Vector Mechanics for Engineers, Vol I - Statics, Vol II - Dynamics, 3rd Ed., Tata McGraw Hill, 2000*
6. *R. C. Hibbler, Engineering Mechanics, Vols. I and II, Pearson Press, 2002*