① Ans: If $\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$, then ~~we have~~ the following statements are true —

(ii) $\lim_{s\to\infty} F(s) = 0$ and (iii) $\int_0^\infty e^{-st} f(t)\,dt$ converges

(i) and (iv) are not true always.

Counterexample: Let $f(t) = t^{-1/2}$. The function is not continuous on any interval $[0,R]$. However Laplace Transform of $f(t)$ exists. $\mathcal{L}[f(t)] = F(s) = \sqrt{\pi/s}$, $s>0$.

So (i) is not possible for the function $f(t) = t^{-1/2}$.

Now $\lim_{s\to\infty} s F(s) = \lim_{s\to\infty} s . F(s) = \lim_{s\to\infty} s . \sqrt{\pi/s} = \infty$.

So for this function $f(t) = t^{-1/2}$, $\lim_{s\to\infty} sF(s)$ is infinite.

Hence (iv) is not true always.

② $\mathcal{L}[\sqrt{t}\cos\sqrt{7t}] = \mathcal{L}\left[t . \frac{\cos\sqrt{7t}}{\sqrt{t}}\right] = -\frac{d}{ds}\left[\mathcal{L}\left[\frac{\cos\sqrt{7t}}{\sqrt{t}}\right]\right]$ [redacted]

Now $\frac{\cos\sqrt{7t}}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left\{1 - \frac{(\sqrt{7t})^2}{2!} + \frac{(\sqrt{7t})^4}{4!} - \frac{(\sqrt{7t})^6}{6!} + \cdots\right\}$

$$= t^{-1/2} - \frac{7t^{1/2}}{2!} + \frac{7^2 t^{3/2}}{4!} - \frac{7^3 t^{5/2}}{6!} + \cdots$$

Take Laplace Transform —

$$\mathcal{L}\left[\frac{\cos\sqrt{7t}}{\sqrt{t}}\right] = \mathcal{L}[t^{-1/2}] - \frac{7}{2}\mathcal{L}[t^{1/2}] + \frac{7^2}{4.3.2}\mathcal{L}[t^{3/2}] - \cdots$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{7}{1.2} . \frac{\Gamma 3/2}{s^{3/2}} + \frac{7^2}{4.3.2} \frac{\Gamma 5/2}{s^{5/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{7}{1.2} . \frac{\frac{1}{2} . \sqrt{\pi}}{s^{3/2}} + \frac{7^2}{4.3.2} . \frac{\frac{3}{2} . \frac{1}{2} . \sqrt{\pi}}{s^{5/2}} - \cdots$$

$$= \sqrt{\pi/s}\left[1 - \frac{7/4s}{1!} + \frac{(7/4s)^2}{2!} - \cdots\right] = \sqrt{\pi/s}\, e^{-7/4s}$$

$$\mathcal{L}[\sqrt{t}\cos\sqrt{7t}] = -\frac{d}{ds}\left[\mathcal{L}\left[\frac{\cos\sqrt{7t}}{\sqrt{t}}\right]\right] = -\frac{d}{ds}\left[\sqrt{\pi/s}\, e^{-7/4s}\right] = \frac{\sqrt{\pi}(2s-7)}{4s^{5/2}} e^{-7/4s}$$

(Ans)

(3) $f(t) = \begin{cases} t, & 0 \le t < \frac{1}{2} \\ (t-1), & \frac{1}{2} \le t < 1 \\ 0, & t \ge 1 \end{cases}$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{1/2} e^{-st}\cdot t\,dt + \int_{1/2}^{1} e^{-st}\cdot(t-1)\,dt + \int_1^\infty e^{-st}\cdot 0\cdot dt$$

$$= -\frac{t e^{-st}}{s}\Big|_0^{1/2} - \int_0^{1/2} \frac{e^{-st}}{-s}\,dt + \frac{(t-1)e^{-st}}{-s}\Big|_{1/2}^{1} - \int_{1/2}^{1}\left[-\frac{e^{-st}}{s}\right]dt$$

$$= -\frac{1}{2s}e^{-s/2} + \frac{1}{s}\cdot\left[-\frac{e^{-st}}{s}\right]\Big|_0^{1/2} - \frac{1}{2s}e^{-s/2} + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]\Big|_{1/2}^{1}$$

$$= -\frac{1}{s}e^{-s/2} - \frac{1}{s^2}e^{-s/2} + \frac{1}{s^2} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}e^{-s/2}$$

$$= -\frac{1}{s}e^{-s/2} + \frac{1}{s^2}(1 - e^{-s}),\ s>0 \text{ (Ans)}$$

(4) $\mathcal{L}[\sinh at \cos at]$

we know $\mathcal{L}[\sinh at] = \dfrac{a}{s^2-a^2} = F(s)$.

~~[illegible]~~ $\mathcal{L}[e^{iat}\sinh at] = F(s-ia)$ [by shifting property]

$$= \frac{a}{(s-ia)^2-a^2} = \frac{a}{(s^2-2a^2)-2ias}$$

$$= \frac{a\{(s^2-2a^2)+2ias\}}{s^4+4a^4}$$

Now

$$\mathcal{L}[(\cos at + i\sin at)\sinh at] = \frac{a(s^2-2a^2)}{s^4+4a^4} + i\frac{2as}{s^4+4a^4}$$

$$\mathcal{L}[\cos at\sinh at] + i\,\mathcal{L}[\sin at\sinh at]$$

$$= \frac{a(s^2-2a^2)}{s^4+4a^4} + i\frac{2as}{s^4+4a^4}$$

By ~~comp~~ equating real part.

$$\boxed{\mathcal{L}[\sinh at\cos at] = \frac{a(s^2-2a^2)}{s^4+4a^4}}\ \text{(Ans)}$$

(5) $\mathcal{L}\left[\frac{\cos at}{t}\right] = \int_s^\infty \mathcal{L}[\cos at]\,dt$

$= \int_s^\infty \frac{s}{s^2+a^2}\,ds$

$= \frac{1}{2}\log(s^2+a^2)\Big|_s^\infty$

$= \frac{1}{2}\lim_{s\to\infty}\log(s^2+a^2) - \frac{1}{2}\log(s^2+a^2)$

which does not exist since $\lim_{s\to\infty}\log(s^2+a^2)$ is infinite.

Hence $\mathcal{L}\left[\frac{\cos(at)}{t}\right]$ does not exist.

(6) Let $f(t) = \int_0^\infty e^{-tx^2}dx$

$F(s) = \mathcal{L}[f(t)] = \int_0^\infty \mathcal{L}[e^{-tx^2}]\,dx$ [changing the order of the integration and definition of Laplace Transform]

$= \int_0^\infty \frac{dx}{s+x^2} = \frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\Big|_0^\infty$

$= \frac{1}{\sqrt{s}}\cdot\pi/2$

Take Inverse Laplace Transform, we have –

$\mathcal{L}^{-1}[F(s)] = f(t) = \pi/2\,\mathcal{L}^{-1}\left[\frac{1}{s^{1/2}}\right] = \pi/2\cdot\frac{t^{1/2-1}}{\Gamma 1/2}$

$= \frac{1}{2}\sqrt{\pi/t}.$

$\Rightarrow \boxed{\int_0^\infty e^{-tx^2}dx = \frac{1}{2}\sqrt{\pi/t}}$ (Ans).

(7) $\mathcal{L}^{-1}\left[5/s^2 + \frac{(\sqrt{s}-1)^2}{s^2} - \frac{7}{3s+2}\right]$

$= \mathcal{L}^{-1}\left[5/s^2 + \frac{s-2\sqrt{s}+1}{s^2} - \frac{7}{3s+2}\right]$

$= \mathcal{L}^{-1}\left[\frac{6}{s^2} + \frac{1}{s} - 2\cdot\frac{1}{s^{3/2}} - \frac{7}{3s+2}\right]$

$= 6t + 1 - 2\cdot\frac{t^{1/2}}{\Gamma 3/2} - 7/3\,e^{-2t/3}$

$= 6t+1-4\sqrt{t/\pi} - 7/3\,e^{-2t/3}$ (Ans).

(8) $\mathcal{L}^{-1}\left[\dfrac{s}{s^4+s^2+1}\right]$, $\quad s^4+s^2+1 = (s^2+1)^2 - s^2$

$$= (s^2+s+1)(s^2-s+1)$$

$$\frac{s}{s^4+s^2+1} = \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^4+s^2+1}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1-s}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1+s}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s-\frac{1}{2})^2+(\sqrt{3}/2)^2}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right]$$

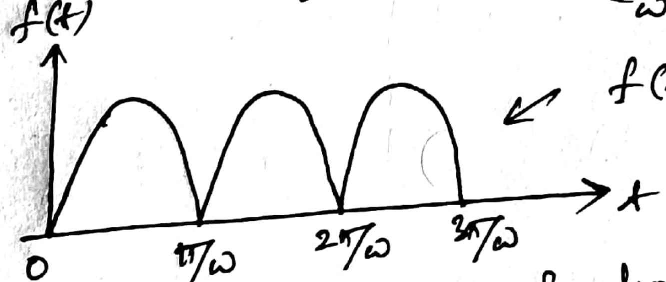
$$= \frac{1}{2}e^{t/2}\mathcal{L}^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right] - \frac{1}{2}e^{-t/2}\mathcal{L}^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]$$

$$= \frac{e^{t/2}}{2}\cdot\frac{1}{\sqrt{3}/2}\sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}e^{-t/2}\frac{1}{(\sqrt{3}/2)}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$= \frac{2}{\sqrt{3}}\,\frac{e^{t/2}-e^{-t/2}}{2}\sin\left(\frac{\sqrt{3}t}{2}\right) = \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)$$

(Ans).

(9) $f(t) = \begin{cases} \sin\omega t, & \frac{2n\pi}{\omega} < t < \frac{(2n+1)\pi}{\omega} \\ -\sin\omega t, & \frac{(2n+1)\pi}{\omega} < t < \frac{(2n+2)\pi}{\omega}, \quad n=0,1,2,\dots \end{cases}$



$f(t) = |\sin\omega t|$

Period $= \pi/\omega$.

Full wave rectified sine function

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt$$

We know
$\int e^{-st}\sin\omega t\,dt$
$= -\dfrac{s\sin\omega t+\omega\cos\omega t}{s^2+\omega^2}$

$$= \frac{1}{1-e^{-\pi s/\omega}}\cdot -\frac{(s\sin\omega t+\omega\cos\omega t)e^{-st}}{s^2+\omega^2}\Bigg|_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\pi s/\omega}}\cdot \omega\left(1+e^{-s\pi/\omega}\right)/s^2+\omega^2$$

$$= \frac{1}{1-e^{-\pi s/\omega}}\cdot\frac{\omega\left(1+e^{-s\pi/\omega}\right)}{\ }$$

$$= \frac{\omega\left(1+e^{-s\pi/\omega}\right)}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)} = \frac{\omega}{s^2+\omega^2}\coth\frac{\pi s}{2\omega} \text{ (Ans)}$$

(10) $\mathcal{L}[f(t)] = \dfrac{2s}{s^2-2s+5} = F(s)$.

$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \dfrac{2s^2}{s^2-2s+5} = 2$

$f'(0) = \lim_{s\to\infty} \left[s^2F(s) - sf(0)\right]$.

$= \lim_{s\to\infty} \left[s^2 \cdot \dfrac{2s}{s^2-2s+5} - 2s\right] = 4$. (Ans)