Ex. 4.9 Let f be a continuous function on the interval $[-a,a]$ for some $a \in \mathbb{R}$, Show that

(i) If f is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(ii) If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

Integration by substitution

Let $f:[a,b]\to\mathbb{R}$ be continuous and let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi([\alpha,\beta])=[a,b]$.

If ϕ is differentiable and ϕ' is integrable on $[\alpha,\beta]$, then

$(f\circ\phi)\phi' : [a,b]\to\mathbb{R}$

is integrable. Furthermore,

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} (f\circ\phi)(t)\,\phi'(t)\,dt$$

You can use it freely!!

We have,
$$\int_{-a}^{a} f(x)\,dx$$
$$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{(By domain additivity)} \longrightarrow (1).$$

Define $\phi:[0,a]\to[-a,0]$ as $\phi(x)=-x$.

Clearly ϕ is differentiable and $\phi'(x)=-1$ is integrable on $[0,a]$ and $\phi([0,a]) = [-a,0]$. with $\phi(0)=0$ and $\phi(a)=-a$.

Since f is continuous, from integration by substitution, we obtain

$$\int_{\phi(0)}^{\phi(a)} f(x)\,dx = \int_{0}^{a} f(\phi(t))\,\phi'(t)\,dt = \int_{0}^{a} f(-t)(-1)\,dt = -\int_{0}^{a} f(-t)\,dt.$$

$$\int_{\phi(0)}^{\phi(a)} f(x)\,dx = \int_{0}^{-a} f(x)\,dx \;(\text{convention})= -\int_{-a}^{0} f(x)\,dx.$$

Thus, $-\int_{-a}^{0} f(x)\,dt = -\int_{0}^{a} f(-t)\,dt$.

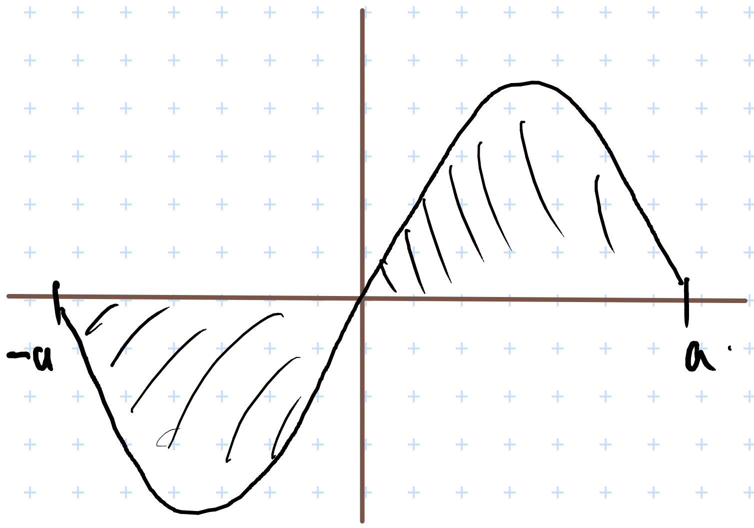
$$\Leftrightarrow \int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-t)\,dt = \begin{cases} \int_{0}^{a} f(t)\,dt & \text{if } f \text{ even} \\ -\int_{0}^{a} f(t)\,dt & \text{if } f \text{ odd.} \end{cases}$$

From (1) we obtain.

$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$ if f is even.

$\int_{-a}^{a} f(t)\,dt = 0$ if f is odd

$\square$



$m_i(f)$

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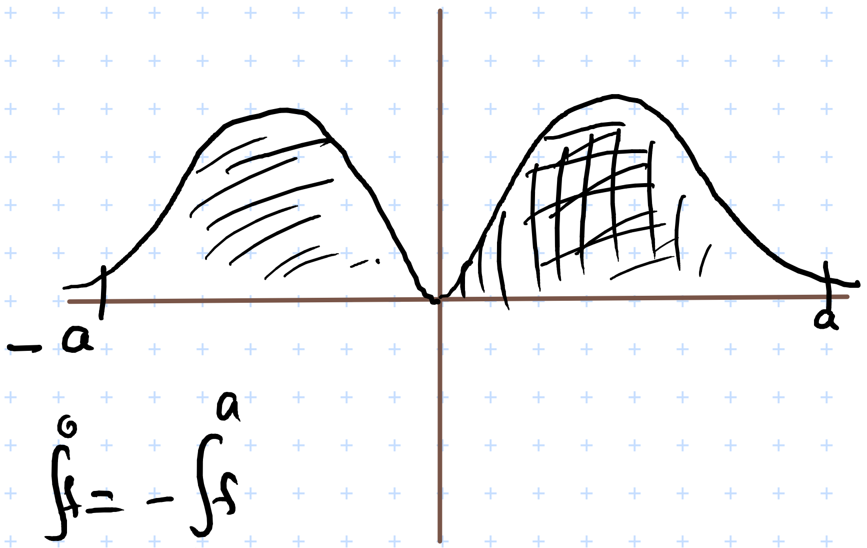

$$\int_{-a}^{0} f = -\int_{0}^{a} f$$

5.4 (a)

$$\frac{1}{n^{17}} \sum_{i=1}^{n} i^{16} \longrightarrow \frac{1}{17}.$$

Used in 5.5.

5.5 Does $\lim_{n\to\infty} \frac{1}{n^{18}} \sum_{i=1}^{n} i^{16}$ exist?

Define $s_n = \frac{1}{n^{17}} \sum_{i=1}^{n} i^{16} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{16}$.

On the interval $[0,1]$, we define the partition

$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$. and $f(x) = x^{16}$ on $[0,1]$.

Since $f(x)$ is continuous, it is integrable.

Since $\mu(P_n) = \frac{1}{n} \to 0$. Thus.

$$s_n = S(P_n, f) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{16} \longrightarrow \int_0^1 x^{16}\, dx \overset{FTC(2)}{=} \frac{1}{17}.$$

Let $t_n = \frac{1}{n}$. Now $t_n \to 0$ as $n \to \infty$.

Thus $\frac{1}{n^{18}} \sum_{i=1}^{n} i^{16} = s_n t_n \to 0$.

$\left.\begin{array}{l} s_n \to \frac{1}{17} \\ t_n \to 0 \end{array}\right\} \Rightarrow s_n t_n \to 0.$

$\frac{1}{n} \sum_{i=1}^{n} i = \frac{n(n+1)}{2n} = \frac{n+1}{2}$ f_x

$f(x) = x^{16}$

$s_n = \boxed{\frac{1}{n} \sum \left(\frac{i}{n}\right)^{16}}$

$t_n = \frac{1}{n} \to 0$

$\left.\begin{array}{l} a_n \to a \\ b_n \to b \end{array}\right\} \Rightarrow a_n b_n \to ab$

$\{0, \frac{1}{n^2}, \frac{2}{n^2}, \dots, \frac{n}{n^2}\}$, $\frac{n}{n^2} = 1/n$

$\mu(P_n) \to 0$

$0 \qquad 1/n$

x^{16}

$\sum f\left(\frac{i}{n^2}\right) \frac{1}{n^2}$

$= \frac{1}{n^2} \sum \left(\frac{i}{n^2}\right)^{16}$

n^{32}

$\frac{1}{n^2} \sum \left(\frac{i}{n}\right)^{16}$

6.8(d) $\displaystyle\int_4^\infty \frac{2}{t^{3/2}-1}\,dt.$

Evaluate for Assignment.

$\displaystyle\int_a^\infty \frac{1}{t^p}$

Let $f(t) = \dfrac{2}{t^{3/2}-1}$

$g(t) = \dfrac{2}{t^{3/2}}$

$$\frac{f(t)}{g(t)} = \frac{\frac{2}{t^{3/2}-1}}{\frac{2}{t^{3/2}}} = \frac{t^{3/2}}{t^{3/2}-1} \to 1 \neq 0$$

$t \to \infty$: $\dfrac{1}{1-\frac{1}{t^{3/2}}} \to 1$, $\dfrac{1}{t^{3/2}} \to 0$

By limit comparison test the integral is convergent.

$$\frac{2}{t^{3/2}-1} \le \frac{2}{t^{5/4}} \qquad t \geq 4.$$

$$\Leftrightarrow\ t^{5/4} \le t^{3/2}-1.$$

$$\Leftrightarrow\ t^{3/2}-t^{5/4} \geq 1$$

$$\Leftrightarrow\ t^{5/4}\left(t^{1/4}-1\right) \geq 1.$$

$$t^{5/4}\left(t^{1/4}-1\right)$$

$$\geq 4^{5/4}\cdot\left(4^{1/4}-1\right)$$

$$= 4\cdot\sqrt{2}\left(\sqrt{2}-1\right) \geq 1.$$

$$\frac{2}{t^{3/2}-1} \le \frac{4}{t^{3/2}}$$

$$2\,t^{3/2} \le 4\,t^{3/2}-4.$$

$$\Leftrightarrow\ t^{3/2} \geq 2.$$