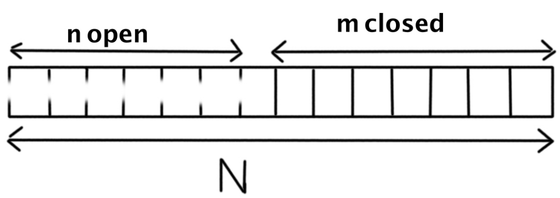
EP1108 - Assignment - Modern Statistical Mechanics & Lasers

1) (5 points) From a canonical ensemble we know that the probability of finding a microstate ν is proportional to $P_\nu \propto \Omega(E - E_\nu)$. For $E_\nu \ll E$ use the Taylor expansion and the fact that $\beta = (1/\Omega)(\partial\Omega/\partial E)|_{N,V}$ to prove that $P_\nu \propto \exp(-\beta E_\nu)$.

2) (10 points) Consider a chess board with $2V \times 2V$ squares with the usual pattern of alternating black and white squares (V white squares and V black squares in each row/column). Suppose N rooks are placed on this board such that no rook is attacked by any other rook (rooks travel freely in vertical and horizontal lines on the board, so any 2 rooks placed in same vertical or horizontal line will attack each other). Find the number of microstates of this system $\Omega(V, N)$

3) (10 points) A DNA molecule can be considered as long chain of links which can be open or closed. Open links have energy Δ while closed links have energy 0. A link can be open only if the link to its left is also open (ignore the left-right symmetry for simplicity), as shown in the figure. If n are open and m are closed, then $m + n = N$. Calculate the



canonical partition function $Q(N, \beta)$. If $\Delta = 0.001eV$ find out how many links are open on average at room temperature in the limit of large N.

4) (10 points) Imagine a new type of particles are found called "bions". These particles occupy a quantum state only in even numbers, i.e., only 0,2,4,6,... particles are found in a particular quantum state. Find their grand canonical partition function and their average occupation number

5) (10 points) Consider a quantum system which has 2 energy levels, $0.5eV$, $1.0eV$ and it is populated by fermions. Take its grand canonical ensemble. It has a temperature of $300K$ and a chemical potential of $0.7eV$. Calculate the average number of particles in both the energy levels.

6) (5 points) Consider a system whose free energy is given as following. Calculate the order of phase transition which occurs at $T = T_c$

$$G = \begin{cases} a\left(1 - \frac{T}{T_c}\right)^2 & \text{if } T < T_c \\ a\left(1 - \frac{T}{T_c}\right)^2 + b\left(1 - \frac{T}{T_c}\right)^3 & \text{if } T > T_c \end{cases} \tag{1}$$

7) (10 points) Consider a gas laser which has a wavelength of 6328Å in vacuum. The upper level has a lifetime of $10^{-10}s$ under spontaneous decay. Take the refractive index of the lasing medium to be $n_0 = 1$

i) Calculate the energy gap between the 2 lasing energy levels in eV

ii) Calculate the Einstein A and B coefficients

iii) Suppose the cavity length of the optical resonator is 100cm. Neglecting Doppler/thermal broadening, calculate how many resonant modes will be present in the laser beam

8) (10 points) A lasing cavity is operating in a steady state condition, emitting a laser light of constant intensity $10mW/m^2$, with length $10cm$ and cross sectional area $1cm^2$. The energy difference of the lasing levels is 0.1eV. One mirror has 100% reflectivity, while the other mirror has 99% reflectivity. How many photons are present in the resonating cavity? Assuming all input power is converted to laser beam, how much power does this laser consume?