Batch : PIONEER (CAT)

Subject : Quantitative Aptitude

Topic : Functions 1 Meaning and Modulus function

DPP–10

1. If $|2x-3|+3|4x-5|-6<5x$, then find the number of integer values that x can assume.

2. If $|3x-4|-2|5x-6|+7>4x$, then find the number of integer values x can assume.

3. If $|6x+5|-4x<11$, then find the number of integral values 5x can assume.

4. Find the range of x where $||x-5|-6|>5$?
(a) $(-\infty, 6)$ or $(16, \infty)$
(b) $(-\infty, -1) \cup (4, 6) \cup (7, 16)$
(c) $(-5, 6) \cup (6, \infty)$
(d) $(-\infty, -6) \cup (4, 6)$ or $(16, \infty)$

5. Solve the inequation:
$$\left|\frac{3}{x-5}\right|>1, x\neq 5$$
(a) $(2, 3)\cup(6, 7)$ (b) $(2, 5)\cup(6, 8)$
(c) $(2, 3)\cup(3, 5)$ (d) $(2, 5)\cup(5, 8)$

6. If $f(x)=\frac{1}{\sqrt{1-x^2}}, g(x)=\frac{1}{x}, 0<x<1$, then find the value of $3f\left(g\left(\frac{5}{4}\right)\right)$.

7. If $f(x)=\frac{2^{-x}}{2^{-x}+2^{x}}$ and $g(x)=\frac{5^{x}}{5^{x}+5^{-x}}$, then find the value of $\left[f(x)+f(-x)\right]^{50}+\left[g(x)+g(-x)\right]^{50}$.

8. Let $f(x)=x.\frac{10^x-1}{10^x+1}$. Then, find the value of $f^3(x)-f^3(-x)$.

9. Let $f(x)=x^2-x$ and $g(x)=x^{-1}$. Then find the value of $f(g(x))+f(x)g(x^3)$.

10. Let $f(x) = 2x + 60$ and $g(x) = x^2 - 17x + 30$, then find the sum of zeroes of $f(g(x))$.

11. If $f(1) = 1$, $f(4) = 73$, $f(9) = 753$, then what is the remainder that $f(15)$ leaves when it's divided by 15?

12. If $f(x)=\sqrt{x-2}+\sqrt{9-x}, g(x)=x+1$, then find the domain of $f(g(x))$.
(a) $(2, 9)$ (b) $[2, 9]$
(c) $(1, 8)$ (d) $[1, 8]$

13. Let $f(x)=x+\frac{1}{x}, g(x)=x^2$, and $h(x) = x^3$. Then, $g(f(x)) + h(f(x)) = f(g(x)) + f(h(x))$ is possible when
1. $f\left(\frac{1}{x}\right)=-\frac{1}{3}f(1)$
2. $f(x)=-\frac{2}{3}$
3. $g\left(\frac{1}{x}\right)=-3f(2)$
4. $g(x)=-\frac{3}{2}$

(a) Only 1 (b) Only 1 and 2
(c) Only 1, 3 and 4 (d) Only 2 and 4

14. If $f(x)=\left(64-x^4\right)^{\frac{1}{4}}$, for $0<x<2\sqrt{2}$, then find the value of $f\left(f\left(\frac{1}{3}\right)\right)$.
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{1}{4}$ (d) $\frac{1}{8}$

15. Let $f(x)=\frac{1}{1-\frac{1}{1-\frac{1}{1-x}}}, x\neq 1$ and $g(x)=\frac{x}{\sqrt{x^2-1}}$,

$x \in (-\infty, -1) \cup (1, \infty)$.

Then find the domain of $g\left(f\left(f\left(\frac{1}{x}\right)\right)\right)$.

(a) $x < 1$ (b) $-1 < x < 1$
(c) $x > 1$ (d) $x < -1, x > 1$

16. Let $f(x) = \frac{25^x}{25^x + 5}, g(x) = f(x) + f(1-x)$, $h(x) = \sqrt{5 + 12x - 9x^2}$ and $k(x) = g(x) + h(x)$. Then find the domain of k(x).

(a) $-\frac{1}{3} \le x \le \frac{3}{5}$ (b) $-\frac{1}{5} \le x \le \frac{3}{5}$
(c) $-\frac{1}{5} \le x \le \frac{5}{3}$ (d) $-\frac{1}{3} \le x \le \frac{5}{3}$

17. Find the range of $f(x) = \frac{3 + 2x + 2x^2}{1 + x + x^2}$.

(a) $\left(\frac{2}{3}, 10\right)$ (b) $\left[\frac{2}{3}, \frac{10}{3}\right]$
(c) [2, 10] (d) $\left[2, \frac{10}{3}\right]$

18. Let g(x) =|x – 1| + |x – 3| + |x – 5|. Then, find the value of x for which the maximum value of g(x) is 12.

(a) 5 (b) 7
(c) 11 (d) 12

19. Let $5f(x^4) + 2f\left(\frac{1}{x^4}\right) = x^4 - 1, x \ne 0$, then $f(x^{16})$ equals:

(a) $\frac{(x^{16}+1)(5x^{16}-2)}{21x^{32}}$
(b) $\frac{(5x^{16}+2)(x^{16}-1)}{21x^{16}}$
(c) $\frac{(4x^{16}+2)(x^{16}-5)}{25x^{16}}$
(d) $\frac{(5x^{16}+16)(3x^{16}-5)}{21x^{16}}$

20. Let $f(0) = 1,\ f(1) = 1,\ f(11) - f(10) = 1$. What is the value of f(20)? (Assume f(x) as a quadratic function.)

(a) 30 (b) 20
(c) 501 (d) 0

21. Find the domain of $= \sqrt{\frac{x^2 - 7x + 10}{x^2 - 5x + 4}}$.

(a) $(-\infty, 2) \cup [5, \infty)$
(b) $(1, 2) \cup (2, 4] \cup [5, \infty)$
(c) $(-\infty, 1) \cup [2, 4) \cup [5, \infty)$
(d) $(-\infty, 1] \cup (2, 4] \cup (5, \infty)$

22. Find the minimum value of $x^2 + 5x^8 + 3x$ when |x – 1| + |x – 2| + |x – 3| = 9.

(a) 5 (b) 4
(c) 3 (d) 2

23. If f(x) = f(x – 1) + f(x + 1) and f(25) + f(26) = 9. Also, f(47) = –5, then what is the value of f(1) + f(2) + f(3)…………+ f(99)?

(a) 10 (b) 12
(c) 9 (d) 16

24. $f(x) = px^2 + 2x + 1$ and $g(x) = x^2 + 6x + 2$ are two quadratic functions such that p < 0. Then the value of p for which there is only one point of intersection between f(x) and g(x) is:

(a) –4 (b) –3
(c) –2 (d) –1

25. Find the range of the function f(x) = (x + 4) (5 – x) (x + 1).

(a) [–2, 3] (b) $[-\infty, 20]$
(c) $[-\infty, +\infty]$ (d) $[-20, \infty]$

26. Let $g(x) = x^2 - \alpha x + \beta$, α is an even positive integer. If one of the roots of g(x) = 0 is a prime number and the other is an even number and $\alpha + 2\beta = 32$, then, g(g(x)) equals:

(a) $x^4 + 8x^3 - 76x^2 + 110x + 50$
(b) $x^4 + 18x^3 - 36x^2 - 12x + 65$
(c) $x^4 - 8x^3 + 38x^2 - 156x + 56$
(d) $x^4 - 16x^3 + 80x^2 - 128x + 60$

27. Let $f(x) = |x-2| + |x-5| + |x-7|$. Find the sum of the maximum and minimum integer values of x when $f(x) \le 15$.

(a) 12 (b) 11
(c) 10 (d) 9

28. Let f is a function such that $f(0) = 3$, $f(1) = 4$, $f(2) = 6$ and $f(x+3) = 3f(x) - f(x+1)$. Then, the value of $f(6)$ is:

(a) 13 (b) –3
(c) 6 (d) 9

29. If $f(x^2 - 9) = 3x^2 + 2a + 3b$, $f(x - 3) = x^3 - 3ax + 2b$ and $g(x) = x^2$, then what is the value of $g\left(\frac{b}{a}\right)$?

(a) –125 (b) 119
(c) 121 (d) 129

30. Consider the following two functions:
$f(x) = 3x^2 - 2x + 5$
$g(x) = 7x^3 + 4x^2 - x + 8$.
Then, the nature of the function $h(x) = f(g(2x-1))$ is:

(a) Even (b) Odd
(c) Neither (d) Can't be determined

Answer Key

1.	**(2)**	**16.**	**(d)**
2.	**(1)**	**17.**	**(d)**
3.	**(22)**	**18.**	**(b)**
4.	**(d)**	**19.**	**(b)**
5.	**(d)**	**20.**	**(b)**
6.	**(5)**	**21.**	**(c)**
7.	**(2)**	**22.**	**(c)**
8.	**(0)**	**23.**	**(a)**
9.	**(0)**	**24.**	**(b)**
10.	**(17)**	**25.**	**(c)**
11.	**(12)**	**26.**	**(d)**
12.	**(d)**	**27.**	**(d)**
13.	**(b)**	**28.**	**(d)**
14.	**(b)**	**29.**	**(c)**
15.	**(b)**	**30.**	**(c)**

Hints & Solutions

1. **(2)**
To solve this inequality, we need to consider two cases, one when $(2x - 3)$ is positive and the other when it's negative.
Case 1: $2x - 3 \geq 0$
In this case, we have $|2x - 3| = 2x - 3$. We can rewrite the inequality as follows:
$(2x - 3) + 3|4x - 5| - 6 < 5x$
Rearranging and simplifying, we get:
$|4x - 5| < x + 3$
Now, we need to consider two sub–cases: when $4x - 5$ is positive and when it's negative.
Sub–case 1: $4x - 5 \geq 0$
In this case, we have $|4x - 5| = 4x - 5$. Substituting this into the inequality, we get:
$4x - 5 < x + 3$
Solving for x, we get:
$3x < 8$
$x < \frac{8}{3}$ …. (i)
Sub–case 2: $4x - 5 < 0$
In this case, we have $|4x - 5| = -(4x - 5)$. Substituting this into the inequality, we get:
$-(4x - 5) < x + 3$
Solving for x, we get:
$5x - 2 > 0$
$5x > 2$
$x > \frac{2}{5}$ …. (ii)
Therefore, the solution for Case 1 is:
$\frac{2}{5} < x < \frac{8}{3}$ …. (iii)

Case 2: $2x - 3 < 0$
In this case, we have $|2x - 3| = -(2x - 3)$. We can rewrite the inequality as follows:
$-(2x - 3) + 3|4x - 5| - 6 < 5x$
Rearranging and simplifying, we get:
$3|4x - 5| < 7x + 3$
$|4x - 5| < \frac{7x}{3} + 1$
Now, we need to consider two sub–cases again: when $4x - 5$ is positive and when it's negative.

Sub–case 1: $4x - 5 \geq 0$
In this case, we have $|4x - 5| = 4x - 5$. Substituting this into the inequality, we get:
$4x - 5 < \frac{7x}{3} + 1$
Solving for x, we get:
$x < \frac{18}{5}$ …. (iv)

Sub–case 2: $4x - 5 < 0$
In this case, we have $|4x - 5| = -(4x - 5)$. Substituting this into the inequality, we get:
$-(4x - 5) < \frac{7x}{3} + 1$
Solving for x, we get:
$x > \frac{12}{19}$ …. (v)
Therefore, the solution for Case 2 is:
$\frac{12}{19} < x < \frac{18}{5}$ …. (vi)

Hence, combining the conditions (iii) and (vi), we can conclude that,
$\frac{12}{19} < x < \frac{8}{3}$
Hence, the number of integral values x can assume is 2.

2. **(1)**
To solve this inequality, we need to consider two cases, one when $3x - 4$ is positive and the other when it's negative.
Case 1: $3x - 4 \geq 0$
In this case, we have $|3x - 4| = 3x - 4$. We can rewrite the inequality as follows:
$(3x - 4) - 2|5x - 6| + 7 > 4x$
Rearranging and simplifying, we get:
$-2|5x - 6| > x - 3$
$|5x - 6| < \frac{3}{2} - \frac{x}{2}$
Now, we need to consider two sub–cases: when $5x - 6$ is positive and when it's negative.
Sub–case 1: $5x - 6 \geq 0$
In this case, we have $|5x - 6| = 5x - 6$. Substituting this into the inequality, we get:

$5x - 6 < \frac{3}{2} - \frac{x}{2}$

Solving for x, we get:

$x < \frac{15}{11}$ …. (i)

Sub–case 2: $5x - 6 < 0$

In this case, we have $|5x - 6| = -(5x - 6)$. Substituting this into the inequality, we get:

$-(5x - 6) < \frac{3}{2} - \frac{x}{2}$

Solving for x, we get:

$x > 1$ …. (ii)

Therefore, the solution for Case 1 is:

$1 < x < \frac{15}{11}$ …. (iii)

Case 2: $3x - 4 < 0$

In this case, we have $|3x - 4| = -(3x - 4)$. We can rewrite the inequality as follows:

$-(3x - 4) - 2|5x - 6| + 7 > 4x$

Rearranging and simplifying, we get:

$-2|5x - 6| > 7x - 11$

$|5x - 6| < \frac{11}{2} - \frac{7x}{2}$

Now, we need to consider two sub–cases again: when $5x - 6$ is positive and when it's negative.

Sub–case 1: $5x - 6 \geq 0$

In this case, we have $|5x - 6| = 5x - 6$. Substituting this into the inequality, we get:

$5x - 6 < \frac{11}{2} - \frac{7x}{2}$

Solving for x, we get:

$x < \frac{23}{17}$ …. (iv)

Sub–case 2: $5x - 6 < 0$

In this case, we have $|5x - 6| = -(5x - 6)$. Substituting this into the inequality, we get:

$-(5x - 6) < \frac{11}{2} - \frac{7x}{2}$

Solving for x, we get:

$x > \frac{1}{3}$ …. (v)

Therefore, the solution for Case 2 is:

$\frac{1}{3} < x < \frac{23}{17}$ …. (vi)

Hence, combining the conditions (iii) and (vi), we can conclude that,

$\frac{1}{3} < x < \frac{15}{11}$

3. **(22)**

To solve the inequality $|6x + 5| - 4x < 11$, we need to analyze the two possible scenarios based on the absolute value.

Case 1: $(6x + 5) - 4x < 11$ (when $6x + 5 \geq 0$)

$2x + 5 < 11$

$2x < 6$

$x < 3$

Case 2: $-(6x + 5) - 4x < 11$ (when $6x + 5 < 0$)

$-6x - 5 - 4x < 11$

$-10x < 16$

$x > -\frac{8}{5}$

The solution to the inequality is $-\frac{8}{5} < x < 3$.

$\Rightarrow -8 < 5x < 15$

Hence, 5x can assume 22 integral values. (From –7 to 14)

4. **(d)**

Given that,

$||x - 5| - 6| > 5$

Case 1: When $|x - 5| - 6 > 5$

$\Rightarrow |x - 5| > 5 + 6$

$\Rightarrow |x - 5| > 11$

$\Rightarrow x - 5 < -11$ or, $x - 5 > 11$

$\Rightarrow x < -11 + 5$ or, $x > 11 + 5$

$\Rightarrow x < -6$ or, $x > 16$

Case 2: When $|x - 5| - 6 < -5$

$\Rightarrow |x - 5| < -5 + 6$

$\Rightarrow |x - 5| < 1$

$\Rightarrow -1 < x - 5 < 1$

$\Rightarrow -1 + 5 < x < 1 + 5$

$\Rightarrow 4 < x < 6$

So, the required range is $(-\infty, -6)$ or $(4, 6)$ or $(16, \infty)$.

5. **(d)**

It is given that,

$$\left|\frac{3}{x-5}\right| > 1, x \neq 5$$

$$\Rightarrow \frac{3}{x-5} < -1 \text{ or } \frac{3}{x-5} > 1$$

$$\Rightarrow \frac{x-2}{x-5} < 0 \text{ or } \frac{8-x}{x-5} > 0$$

$\Rightarrow 2 < x < 5$ or $5 < x < 8$, since $x \neq 5$.

The solution set of given inequation $(2,5) \cup (5,8)$.

6. **(5)**

Given that, $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ and $g(x) = \dfrac{1}{x}$

Therefore, $f(g(x)) = f\left(\dfrac{1}{x}\right)$

$$= \frac{1}{\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{x}{\sqrt{x^2-1}}$$

Now,

$$f\left(g\left(\frac{5}{4}\right)\right) = \frac{\frac{5}{4}}{\sqrt{\left(\frac{5}{4}\right)^2 - 1}}$$

$$= \frac{\frac{5}{4}}{\sqrt{\frac{25}{16}-1}}$$

$$= \frac{\frac{5}{4}}{\sqrt{\frac{25-16}{16}}}$$

$$= \frac{\frac{5}{4}}{\frac{3}{4}}$$

$$= \frac{5}{3}$$

Therefore, $3f\left(g\left(\dfrac{5}{4}\right)\right) = 5$

7. **(2)**

Given that,

$$f(x) = \frac{2^{-x}}{2^{-x}+2^{x}} \text{ and } g(x) = \frac{5^{x}}{5^{x}+5^{-x}}$$

Now, $f(x) + f(-x) = \dfrac{2^{-x}}{2^{-x}+2^{x}} + \dfrac{2^{-(-x)}}{2^{-(-x)}+2^{-x}}$

$$= \frac{2^{-x}}{2^{-x}+2^{x}} + \frac{2^{x}}{2^{x}+2^{-x}}$$

$$= \frac{2^{-x}+2^{x}}{2^{-x}+2^{x}}$$

$= 1$

Similarly,

$$g(x) + g(-x) = \frac{5^{x}}{5^{x}+5^{-x}} + \frac{5^{-x}}{5^{-x}+5^{x}}$$

$$= \frac{5^{x}+5^{-x}}{5^{x}+5^{-x}}$$

$= 1$

Now, $[f(x) + f(-x)]^{50} + [g(x) + g(-x)]^{50}$

$= (1)^{50} + (1)^{50}$

$= 1 + 1$

$= 2$

8. **(0)**

Given that,

$$f(x) = x.\frac{10^{x}-1}{10^{x}+1}$$

Then,

$$f(x) - f(-x) = x.\frac{10^{x}-1}{10^{x}+1} - (-x).\frac{10^{-x}-1}{10^{-x}+1}$$

$$= x.\frac{10^{x}-1}{10^{x}+1} + x.\frac{10^{x}\left(10^{-x}-1\right)}{10^{-x}+1}$$

$$= x.\frac{10^{x}-1}{10^{x}+1} + x.\frac{1-10^{x}}{1+10^{x}}$$

$$= x.\frac{10^{x}-1}{10^{x}+1} - x.\frac{10^{x}-1}{10^{x}+1}$$

$= 0$

Now,

$f^3(x) - f^3(-x)$

$= [f(x)]^3 - [f(-x)]^3$

$= [f(x) - f(-x)]^3 + 3f(x)f(-x) \times [f(x) - f(-x)]$

$= 0^3 + (3 \times f(x)f(-x) \times 0)$

$= 0$

9. **(0)**

Given that,

$f(x) = x^2 - x$ and $g(x) = x^{-1}$

Now, $f(g(x)) = f(x^{-1})$

$= (x^{-1})^2 - x^{-1}$

$= x^{-2} - x^{-1}$

$= \frac{1}{x^2} - \frac{1}{x}$

$= \frac{1-x}{x^2}$

$= \frac{-(x-1)}{x^2}$

$= \frac{-(x^2-x)}{x^3}$

$= \frac{-f(x)}{x^3}$

$= -\left(\frac{1}{x}\right)^3 f(x)$

$= -\left(x^{-1}\right)^3 f(x)$

$= -\left(x^3\right)^{-1} f(x)$

$= -g(x^3)f(x)$

Therefore, $f(g(x)) = -g(x^3)f(x)$

$\Rightarrow f(g(x)) + g(x^3)f(x) = 0$

10. **(17)**

Given that,

$f(x) = 2x + 60$ and $g(x) = x^2 - 17x + 30$

Then, $f(g(x)) = f(x^2 - 17x + 30)$

$= 2(x^2 - 17x + 30) + 60$

$= 2x^2 - 34x + 60 + 60$

$= 2x^2 - 34x + 120$

Now, for finding the zeroes of $f(g(x))$, we can have

$f(g(x)) = 0$

$\Rightarrow 2x^2 - 34x + 120 = 0$

$\Rightarrow x^2 - 17x + 60 = 0$

$\Rightarrow x^2 - 12x - 5x + 60 = 0$

$\Rightarrow x(x-12) - 5(x-12) = 0$

$\Rightarrow (x-12)(x-5) = 0$

$\Rightarrow x = 5, 12$

Hence, the sum of the required zeroes of $f(g(x))$ is $(12 + 5) = 17$

11. **(12)**

$f(1) = 1 = 1^3 + 3 \times 1 - 3$

$f(4) = 73 = 4^3 + 3 \times 4 - 3$

$f(9) = 753 = 9^3 + 3 \times 9 - 3$

$\therefore f(n) = n^3 + 3n - 3$

$\therefore f(15) = 15^3 + 3 \times 15 - 3$

So, $\frac{\left(15^3 + 3\times 15 - 3\right)}{15} = \text{rem}(12)$

12. **(d)**

Given that,

$f(x) = \sqrt{x-2} + \sqrt{9-x}$ and $g(x) = x + 1$

Therefore,

$f(g(x)) = f(x + 1)$

$= \sqrt{x+1-2} + \sqrt{9-(x+1)}$

$= \sqrt{x-1} + \sqrt{9-x-1}$

$= \sqrt{x-1} + \sqrt{8-x}$

Now, $f(g(x))$ will exist if

$x - 1 \geq 0$ and $8 - x \geq 0$

i.e., if $x \geq 1$ and $x \leq 8$

Hence, the required domain of $f(g(x))$ is $x \in [1, 8]$

13. **(b)**

Given that,

$f(x) = x + \frac{1}{x}$, $g(x) = x^2$, and $h(x) = x^3$

Now,

$g(f(x)) = g\left(x + \frac{1}{x}\right)$

$= \left(x + \frac{1}{x}\right)^2$

$= x^2 + \frac{1}{x^2} + 2$

$= f\left(x^2\right) + 2$

Also, $h(f(x)) = h\left(x + \frac{1}{x}\right)$

$= \left(x + \frac{1}{x}\right)^3$

$= x^3 + \frac{1}{x^3} + 3x.\frac{1}{x}\left(x + \frac{1}{x}\right)$

$$= f\left(x^3\right) + 3f\left(\frac{1}{x}\right)$$

[Since, $f\left(\frac{1}{x}\right) = \frac{1}{x} + \frac{1}{\frac{1}{x}} = \frac{1}{x} + x = f(x)$]

Now,

$$g\left(f(x)\right) + h\left(f(x)\right) = f\left(x^2\right) + 2 + f\left(x^3\right) + 3f\left(\frac{1}{x}\right)$$

$$= f\left(g(x)\right) + f\left(h(x)\right) + 3f\left(\frac{1}{x}\right) + 2$$

$$= f\left(g(x)\right) + f\left(h(x)\right) + 3f\left(\frac{1}{x}\right) + f(1)$$

[Since, $f(1) = 1 + \frac{1}{1} = 2$]

Therefore, $g\left(f(x)\right) + h\left(f(x)\right) = f\left(g(x)\right) + f\left(h(x)\right)$ is possible when

$$3f\left(\frac{1}{x}\right) + f(1) = 0$$

i.e., $f\left(\frac{1}{x}\right) = -\frac{1}{3} f(1)$

i.e., if $f(x) = -\frac{2}{3}$ [Since, $f(1) = 2$ and $f(x) = f\left(\frac{1}{x}\right)$]

Hence, only 1 and 2 are correct.
So, option b is correct.

14. **(b)**
Given that,

$$f(x) = \left(64 - x^4\right)^{\frac{1}{4}}$$

Then,

$$f\left(\frac{1}{3}\right) = \left(64 - \frac{1}{3^4}\right)^{\frac{1}{4}}$$

$$= \left(64 - \frac{1}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{5183}{81}\right)^{\frac{1}{4}}$$

Now,

$$f\left(f\left(\frac{1}{3}\right)\right) = f\left[\left(\frac{5183}{81}\right)^{\frac{1}{4}}\right]$$

$$= \left[64 - \left(\frac{5183}{81}\right)^{\frac{1}{4}\times 4}\right]^{\frac{1}{4}}$$

$$= \left(64 - \frac{5183}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{1}{81}\right)^{\frac{1}{4}}$$

$$= \left(3^{-4}\right)^{\frac{1}{4}}$$

$$= 3^{-1}$$

$$= \frac{1}{3}$$

15. **(b)**
Given that,

$$f(x) = \frac{1}{1 - \frac{1}{1 - \frac{1}{1-x}}}$$

$$= \frac{1}{1 - \frac{1}{\frac{1-x-1}{1-x}}}$$

$$= \frac{1}{1 - \frac{1-x}{-x}}$$

$$= \frac{1}{1 + \frac{1-x}{x}}$$

$$= \frac{x}{x + 1 - x}$$

$$= x$$

Now,
$f(f(x)) = f(x) = x$

Therefore, $f\left(f\left(\frac{1}{x}\right)\right) = \frac{1}{x}$

So, $g\left(f\left(f\left(\frac{1}{x}\right)\right)\right) = g\left(\frac{1}{x}\right)$

$$= \frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2} - 1}}$$

$$= \frac{x \times \frac{1}{x}}{\sqrt{1 - x^2}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

Now, $g\left(f\left(f\left(\frac{1}{x}\right)\right)\right)$ will exist if

$1 - x^2 > 0$

i.e., if $x^2 < 1$

i.e., if $-1 < x < 1$.

16. **(d)**

Given that,

$$f(x) = \frac{25^x}{25^x + 5}$$

Then, $f(1-x) = \frac{25^{1-x}}{25^{1-x} + 5}$

$$= \frac{25.25^{-x}}{25.25^{-x} + 5}$$

$$= \frac{25}{25 + 5.25^x}$$

$$= \frac{5}{5 + 25^x}$$

Now,

$g(x) = f(x) + f(1 - x)$

$$= \frac{25^x}{25^x + 5} + \frac{5}{5 + 25^x}$$

$$= \frac{25^x + 5}{25^x + 5}$$

$= 1$

So, $k(x) = g(x) + h(x)$

$$= 1 + \sqrt{5 + 12x - 9x^2}$$

$$= 1 + \sqrt{-\left(9x^2 - 12x - 5\right)}$$

$$= 1 + \sqrt{-\left(9x^2 - 15x + 3x - 5\right)}$$

$$= 1 + \sqrt{-(3x + 1)(3x - 5)}$$

Now, k(x) will exist if

$$-3(3x + 1)(3x - 5) \geq 0$$

$$\Rightarrow (3x + 1)(3x - 5) \leq 0$$

$$\Rightarrow -\frac{1}{3} \leq x \leq \frac{5}{3}$$

17. **(d)**

Let $y = \frac{3 + 2x + 2x^2}{1 + x + x^2}$

$$\Rightarrow y\left(1 + x + x^2\right) = 3 + 2x + 2x^2$$

$$\Rightarrow y + xy + x^2y = 3 + 2x + 2x^2$$

$$\Rightarrow x^2y - 2x^2 + xy - 2x + y - 3 = 0$$

$$\Rightarrow x^2(y - 2) + x(y - 2) + y - 3 = 0 \quad \ldots\ldots\text{(i)}$$

Since, the discriminant of (i) ≥ 0, so

$(y - 2)^2 - 4(y - 2)(y - 3) \geq 0$

$$\Rightarrow (y - 2)(y - 2 - 4y + 12) \geq 0$$

$$\Rightarrow (y - 2)(10 - 3y) \geq 0$$

$$\Rightarrow (y - 2)(3y - 10) \leq 0$$

$$\Rightarrow (y - 2)\left(y - \frac{10}{3}\right) \leq 0$$

$$\Rightarrow 2 \leq y \leq \frac{10}{3}$$

$$\Rightarrow 2 \leq f(x) \leq \frac{10}{3}$$

18. **(b)**

To find the maximum value of x in the inequality $|x-1| + |x-3| + |x-5| \leq 12$, we will consider the different possible cases for the absolute value expressions and find the intervals where the inequality is satisfied. Then, we'll find the maximum value of x from these intervals.

First, let's analyze the different cases for the absolute value expressions:

$|x-1|$: This expression will change sign when $x < 1$

$|x-3|$: This expression will change sign when $x < 3$

$|x-5|$: This expression will change sign when $x < 5$

Based on these critical points (1, 3, and 5), we will have 4 intervals to analyze:

Interval 1: $x < 1$

In this interval, all absolute value expressions are negative, so we have:
$-(x-1) - (x-3) - (x-5) \le 12$
Solving this inequality, we get:
$-3x + 9 \le 12$
$-3x \le 3$
$x \ge -1$ …. (i)

Interval 2: $1 \le x < 3$
In this interval, the expression $|x-1|$ is positive, while the other two are negative:
$(x-1) - (x-3) - (x-5) \le 12$
Solving this inequality, we get:
$-x + 7 \le 12$
$-x \le 5$
$x \ge -5$ …. (ii)

Interval 3: $3 \le x < 5$
In this interval, the expressions $|x-1|$ and $|x-3|$ are positive, while $|x-5|$ is negative:
$(x-1) + (x-3) - (x-5) \le 12$
Solving this inequality, we get:
$x + 1 \le 12$
$x \le 11$ ……. (iii)

Interval 4: $x \ge 5$
In this interval, all absolute value expressions are positive, so we have:
$(x-1) + (x-3) + (x-5) \le 12$
Solving this inequality, we get:
$3x - 9 \le 12$
$3x \le 21$
$x \le 7$

Interval 1: $x \ge -1$ (for $x < 1$)
Interval 2: $x \ge -5$ (for $1 \le x < 3$)
Interval 3: $x \le 11$ (for $3 \le x < 5$)
Interval 4: $x \le 7$ (for $x \ge 5$)

We're looking for the maximum value of x that satisfies the inequality. Since the inequality is non–strict (i.e., it contains the "less than or equal to" sign), we can also consider the boundary points of the intervals.
In Interval 1, the maximum value for x is < 1 (the boundary point of this interval). In Interval 2, the maximum value for x is < 3 (the boundary point of this interval). In Interval 3, the maximum value for x is 11 (from the constraint $x \le 11$). Finally, in Interval 4, the maximum value for x is 7 (from the constraint $x \le 7$).
Comparing the maximum values from all intervals, we find that the overall maximum value of x is 11, which is from Interval 3, but at x = 11, the given inequation is not satisfied.
So, the required maximum value of x = 7.

19. **(b)**
Given that,
$$5f\left(x^4\right)+2f\left(\frac{1}{x^4}\right)=x^4-1$$
Replace x by x^4, then we have
$$5f\left(x^{16}\right)+2f\left(\frac{1}{x^{16}}\right)=x^{16}-1 \quad \text{……. (i)}$$
Replace x by $\frac{1}{x}$, then we have
$$5f\left(\frac{1}{x^{16}}\right)+2f\left(x^{16}\right)=\frac{1}{x^{16}}-1 \quad \text{…….. (ii)}$$
Now, multiplying (i) by 5 and (ii) by 2 and then subtracting them, we get
$$25f\left(x^{16}\right)+10f\left(\frac{1}{x^{16}}\right)-10f\left(\frac{1}{x^{16}}\right)-4f\left(x^{16}\right)$$
$$=5x^{16}-5-\frac{2}{x^{16}}+2$$
$$\Rightarrow 21f\left(x^{16}\right)=5x^{16}-\frac{2}{x^{16}}-3$$
$$\Rightarrow f\left(x^{16}\right)=\frac{5x^{32}-2-3x^{16}}{21x^{16}}$$
$$=\frac{5x^{32}-5x^{16}+2x^{16}-2}{21x^{16}}$$
$$=\frac{5x^{16}\left(x^{16}-1\right)+2\left(x^{16}-1\right)}{21x^{16}}$$
$$=\frac{\left(5x^{16}+2\right)\left(x^{16}-1\right)}{21x^{16}}$$

20. **(b)**
Let $f(x) = ax^2 + bx + c$
$f(0) = 1 \Rightarrow c = 1$
$\therefore\ f(x) = ax^2 + bx + 1$
$f(1) = 1 \Rightarrow a + b = 0$ ... (i)

$f(11) - f(10) = 1$

$\Rightarrow (121a + 11b + 1) - (100a + 10b + 1) = 1$

$\Rightarrow 21a + b = 1$ ... (ii)

From (i) and (ii),

$a = \frac{1}{20}, b = -\frac{1}{20}$

$\Rightarrow f(20) = 400a + 20b + 1 = 20 - 1 + 1 = 20$

Option (b) is correct.

21. **(c)**

Given that,

$$f(x) = \sqrt{\frac{x^2 - 7x + 10}{x^2 - 5x + 4}}$$

$$= \sqrt{\frac{x^2 - 5x - 2x + 10}{x^2 - 4x - x + 4}}$$

$$= \sqrt{\frac{x(x-5) - 2(x-5)}{x(x-4) - 1(x-4)}}$$

$$= \sqrt{\frac{(x-2)(x-5)}{(x-1)(x-4)}}$$

Now, f(x) will exist if

$(x - 2)(x - 5) \geq 0$ and $(x - 1)(x - 4) > 0$

i.e., if $x \leq 2$, $x \geq 5$ and $x < 1$, $x > 4$

i.e., if $x < 1$, $2 \leq x < 4$, $x \geq 5$

i.e., if $x \in (-\infty, 1) \cup [2, 4) \cup [5, \infty)$

22. **(c)**

$|x - 1| + |x - 2| + |x - 3| = 9$

The value of the expression at x = 3 is 3, and the value at x = 1 is also 3, while at x = 2 is 2.

So, the value of the expression equals 9 when x is greater than 3 or when x is less than 1.

<u>**Case 1:**</u> $x > 3$

$x - 1 + x - 2 + x - 3 = 9$

or $x = \frac{15}{3} = 5$

<u>**Case 2:**</u> $x < 1$

$1 - x + 2 - x + 3 - x = 9$

$\Rightarrow 6 - 3x = 9$

$\Rightarrow 3x = -3$

$\Rightarrow x = -1$

Now, the minimum value of $x^2 + 5x^8 + 3x$ can be obtained at $x = -1$.

The required minimum value of $x^2 + 5x^8 + 3x = (-1)^2 + 5(-1)^8 + 3(-1) = 3$

23. **(a)**

Here any term is equal to the sum of its neighbours except for the first and the last term.

So, if f(1) = a, f(2) = b, then f(3) = b–a, f(4) = –a, f(5) = –b and f(6) = a–b, f(7) =a, f(8) =b and so on…

Thus, terms repeat after a gap of 6 i.e., there is a cyclicity of 6

So, f(25) = a and f(26) = b or a +b = 9

Also, f(47) = f(5) = –b = –5. Or b = 5 or a = 4

The sum of first six terms is zero, i.e., groups of 6 terms starting from the first will be zero. So, sum up–to 96 terms will be zero. Thus, we need to calculate f(97) +f(98)+f(99) = a + b + b–a = 2b = 2 × 5 = 10

So, f(1) + f(2) + f(3)…………+ f(99) = 10

Hence option (a) is correct.

24. **(b)**

Equating the two functions we get: $px^2 + 2x + 1 = x^2 + 6x + 2$

Or, $(p - 1)x^2 - 4x - 1 = 0$

As there is only one point of intersection, so the above equation must have equal roots, or the discriminant must be equal to zero.

$4^2 + 4(p - 1) = 0$

$4 + p - 1 = 0$

or, $p = -3$

Hence option (b) is correct.

25. **(c)**

At x = 0, the value of the function is 20 and this value rejects the first option. Taking some higher values of x, we realize that on the positive side, the value of the function will become negative when we take x greater than 5 since the value of (5 – x) would be negative. Also, the value of f(x) would start tending to –∞ as we take the bigger value of x. Similarly, on the negative side, when we take the value of x lower than –4, f(x) becomes positive and when we take it further away from 0 on the negative side, the value of f(x) would continue tending to + ∞.

Hence, option (c) is the answer.

26. **(d)**

Let m, n be the roots of $x^2 - \alpha x + \beta = 0$, then $m + n = \alpha$.

Therefore, one root is 2, as one of the roots is prime and the other is an even number, but α is an even integer.

[since m is prime and n is even, so m must be 2]

Therefore, $f(2) = 0$

i.e., $2\alpha - \beta = 4$ and $\alpha + 2\beta = 32$

Solving both of these equations, we have

So, $\alpha = 8, \beta = 12$

Hence, the function becomes

$g(x) = x^2 - 8x + 12$

So, $g(g(x)) = g(x^2 - 8x + 12)$

$= (x^2 - 8x + 12)^2 - 8(x^2 - 8x + 12) + 12$

$= x^4 - 16x^3 + 80x^2 - 128x + 60$

27. **(d)**

To find the maximum value of x in the inequality $|x-2| + |x-5| + |x-7| \le 15$, we will consider the different possible cases for the absolute value expressions and find the intervals where the inequality is satisfied. Then, we'll find the maximum value of x from these intervals.

First, let's analyze the different cases for the absolute value expressions:

$|x-2|$: This expression will change sign when $x < 2$

$|x-5|$: This expression will change sign when $x < 5$

$|x-7|$: This expression will change sign when $x < 7$

Based on these critical points (2, 5, and 7), we will have 4 intervals to analyze:

Interval 1: $x < 2$

Then, we have

$-(x-2) - (x-5) - (x-7) \le 15$

$\Rightarrow -3x + 2 + 5 + 7 \le 15$

$\Rightarrow -3x + 14 \le 15$

$\Rightarrow -3x \le 1$

$\Rightarrow x \ge -1/3$

Interval 2: $2 \le x < 5$

Then, we have

$(x - 2) - (x-5) - (x - 7) \le 15$

$\Rightarrow -x + 10 \le 15$

$\Rightarrow -x \le 5$

$\Rightarrow x \ge -5$

Interval 3: $5 \le x < 7$

Then, we have

$(x - 2) + (x-5) - (x - 7) \le 15$

$\Rightarrow x \le 15$

Interval 3: $x \ge 7$

$(x - 2) + (x-5) + (x - 7) \le 15$

- $3x - 14 \le 15$
- $3x \le 29$
- $x \le 29/3 = 9.67$ (approx.)

Hence, the maximum integer value of x in $|x - 2| + |x - 5| + |x - 7| \le 15$ is 9 and the minimum integer value of x in $|x - 2| + |x - 5| + |x - 7| \le 15$ is 0.

So, the required sum = 9 + 0 = 9

28. **(d)**

It is given that,

$f(x + 3) = 3f(x) - f(x + 1)$ …… (1)

Substituting x=0, we have

$f(3) = 3f(0) - f(1)$

$= 3 \times 3 - 4$

$= 5$

Again, substituting x=1, we have

$f(4) = 3f(1) - f(2)$

$= 3 \times 4 - 6$

$= 6$

Similarly, substituting x = 3, we have

$f(6) = 3f(3) - f(4)$

$= 3 \times 5 - 6$

$=9$

Hence, option (d) is the correct answer.

29. **(c)**

Given that,

$$f\left(x^2 - 9\right) = 3x^2 + 2a + 3b$$

At $x^2 = 9$, we have

$f(0) = 3 \times 9 + 2a + 3b$

$\Rightarrow f(0) = 27 + 2a + 3b$ ….. (i)

Also, $f\left(x - 3\right) = x^3 - 3ax + 2b$

At x = 3, we have

$f(0) = 3^3 - 3a(3) + 2b$

$\Rightarrow f(0) = 27 - 9a + 2b$ ….. (ii)

Equating equations (i) and (ii), we have

$27 + 2a + 3b = 27 - 9a + 2b$

$\Rightarrow 11a + b = 0$

$\Rightarrow \frac{b}{a} = -11$

Now, we are also given that,

$g(x) = x^2$

Therefore,

$$g\left(\frac{b}{a}\right) = (-11)^2 = 121$$

30. **(c)**

First, let's find the function $g(2x - 1)$:

$g(2x - 1) = 7(2x - 1)^3 + 4(2x - 1)^2 - (2x - 1) + 8$

Now let's substitute $g(2x - 1)$ into the function $f(x)$:

$h(x) = f(g(2x - 1)) = 3[g(2x - 1)]^2 - 2[g(2x - 1)] + 5$

To find the nature of the function $h(x)$, we'll evaluate $h(-x)$ and compare it to $h(x)$:

$h(-x) = f(g(-2x - 1))$

Now let's find $g(-2x - 1)$:

$g(-2x - 1) = 7(-2x - 1)^3 + 4(-2x - 1)^2 - (-2x - 1) + 8$

$= -7(2x+1)^3 + 4(2x+1)^2 + (2x+1) + 8$

So, neither $g(2x-1) = g(-2x-1)$, nor $g(-2x - 1) = -g(2x-1)$

Therefore, since, $g(2x- 1)$ is neither even, nor an odd function, so, $h(x)$ is neither even, nor an odd function.