Normal Distribution $E(Y), Var(Y)$

We find the expected value of a function by finding the sum of the products for each possible outcome and its chance of occurring.

Step 1: For continuous variables, this means using an integral going from negative infinity to infinity. The chance of each outcome occurring is given by the PDF, $f(y)$, so $\mathrm{E(Y)} = \int_{-\infty}^{\infty} y f(y)\, dy$.

Step 2: The PDF for a Normal Distribution is the following expression: $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$

Step 3: Thus, the expected value equals: $\mathrm{E(Y)} = \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$

Step 4: Since sigma and pi are constant numbers, we can take them out of the integral:

$$\mathrm{E(Y)} = \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$

Normal Distribution $E(Y), Var(Y)$

Step 5: We will substitute **t** in for $\frac{\boldsymbol{y-\mu}}{\sqrt{\boldsymbol{2}}\boldsymbol{\sigma}}$ to make the integral more manageable. To do so, we need to transform **y** and **dy**. If t = $\frac{\boldsymbol{y-\mu}}{\sqrt{\boldsymbol{2}}\boldsymbol{\sigma}}$, then clearly $y = \mu + \sqrt{2}\sigma t$. Knowing this, $\frac{dy}{dt} = \sqrt{2}\sigma$, so $dy = \sqrt{2}\sigma\, dt$. Therefore, we can substitute and take the constant out of the integral, before simplifying:

$$\frac{\mathbf{1}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} y\, \boldsymbol{e}^{\frac{-(\boldsymbol{y}-\boldsymbol{\mu})^{2}}{2\sigma^{2}}}\, dy = \frac{\mathbf{1}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mu + \sqrt{2}\sigma t)\ \boldsymbol{e}^{-\boldsymbol{t}^{2}}\sqrt{2}\sigma\, dt = \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mu + \sqrt{2}\sigma t)\ \boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\mu + \sqrt{2}\sigma t)\ \boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt$$

Step 6: We expand the expression within parenthesis and split the integral:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\mu + \sqrt{2}\sigma t)\ \boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt = \frac{1}{\sqrt{\pi}}\left[\mu\int_{-\infty}^{\infty}\boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt + \sqrt{2}\sigma\int_{-\infty}^{\infty}\boldsymbol{t}\,\boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt\right]$$

Step 7: We solve the two simpler integrals:

$$\frac{1}{\sqrt{\pi}}\left[\mu\int_{-\infty}^{\infty}\boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt + \sqrt{2}\sigma\int_{-\infty}^{\infty}\boldsymbol{t}\,\boldsymbol{e}^{-\boldsymbol{t}^{2}}\, dt\right] = \frac{1}{\sqrt{\pi}}\left[\mu\sqrt{\boldsymbol{\pi}} + \sqrt{2}\sigma\left(-\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{e}^{-\boldsymbol{t}^{2}}\right)_{-\infty}^{\infty}\right]$$

Step 8: Since the exponential tends to 0, we get the following:

$$\frac{1}{\sqrt{\pi}}\left[\mu\sqrt{\pi} + \sqrt{2}\sigma\left(-\frac{1}{2}e^{-t^2}\right)_{-\infty}^{\infty}\right] = \frac{1}{\sqrt{\pi}}[\mu\sqrt{\pi} + \mathbf{0}] = \frac{\mu\sqrt{\pi}}{\sqrt{\pi}} = \mu$$

Step 9: Using Calculus we just showed that for a variable y which follows a Normal Distribution and has a PDF of $f(y) = \frac{\mathbf{1}}{\sigma\sqrt{2\pi}}\boldsymbol{e}^{\frac{-(y-\mu)^2}{2\sigma^2}}$, the expected value equals μ.

To find the Variance of the distribution, we need to use the relationship between Expected Value and Variance we already know, namely:

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

Step 1: We already know the expected value, so we can plug in μ^2 for $[E(Y)]^2$, hence:

$$Var(Y) = E(Y^2) - \mu^2$$

Step 2: To compute the expected value for Y^2, we need to go over the same process we did when calculating the expected value for Y, so let's quickly go over the obvious simplifications.

$$\mathrm{E}(Y^2) - \mu^2 = \int_{-\infty}^{\infty} y^2 \frac{\mathbf{1}}{\sigma\sqrt{2\pi}} \boldsymbol{e}^{\frac{-(y-\mu)^2}{2\sigma^2}} dy - \mu^2 =$$

$$= \frac{\mathbf{1}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \, \boldsymbol{e}^{\frac{-(y-\mu)^2}{2\sigma^2}} dy \; - \mu^2 =$$

$$= \frac{\mathbf{1}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma t + \mu\right)^{\mathbf{2}} \boldsymbol{e}^{-t^2} \sqrt{2}\sigma \, dt - \mu^2 =$$

$$= \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma t + \mu\right)^{\mathbf{2}} e^{-t^2} dt - \mu^2 =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma t + \mu\right)^{\mathbf{2}} e^{-t^2} dt - \mu^2 =$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} (\boldsymbol{2\sigma^2 t^2} + \boldsymbol{2\sqrt{2}\sigma\mu t} \; + \; \boldsymbol{\mu^2}) \; e^{-t^2} dt \right] - \mu^2 =$$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{\infty} t^2 \, e^{-t^2} dt + 2\sqrt{2}\sigma\mu \int_{-\infty}^{\infty} t \, e^{-t^2} dt + \mu^2 \int_{-\infty}^{\infty} e^{-t^2} dt \right] - \mu^2$$

Step 3: We already evaluated two of the integrals when finding the expected value, so let's just use the results and simplify.

$$\frac{1}{\sqrt{\pi}}\left[2\sigma^2\int_{-\infty}^{\infty} t^2\, e^{-t^2}dt + 2\sqrt{2}\sigma\mu\int_{-\infty}^{\infty} t\, e^{-t^2}\,dt + \mu^2\int_{-\infty}^{\infty} e^{-t^2}dt\right] - \mu^2 =$$

$$= \frac{1}{\sqrt{\pi}}\left[2\sigma^2\int_{-\infty}^{\infty} t^2\, e^{-t^2}dt + 2\sqrt{2}\sigma\mu\ \times \mathbf{0} + \mu^2\sqrt{\boldsymbol{\pi}}\right] - \mu^2 =$$

$$= \frac{1}{\sqrt{\pi}}\left[2\sigma^2\int_{-\infty}^{\infty} t^2\, e^{-t^2}dt\right] + \frac{1}{\sqrt{\pi}}\mu^2\sqrt{\pi} - \mu^2 =$$

$$= \frac{1}{\sqrt{\pi}}\left[2\sigma^2\int_{-\infty}^{\infty} t^2\, e^{-t^2}dt\right] + \mu^2 - \mu^2 =$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2\, e^{-t^2}dt$$

Step 4: We need to integrate by parts next:

$$\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2\, e^{-t^2}dt = \frac{2\sigma^2}{\sqrt{\pi}}\left(\left[-\frac{t}{2}e^{-t^2}\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-t^2}dt\right)$$

Normal Distribution $E(Y), Var(Y)$

Step 5: The exponential tends to 0 once again, so we get the following:

$$\frac{2\sigma^2}{\sqrt{\pi}}\left(\left[-\frac{t}{2}e^{-t^2}\right]_{-\infty}^{\infty}+\frac{1}{2}\int_{-\infty}^{\infty}e^{-t^2}dt\right)=$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\left(\mathbf{0}+\frac{1}{2}\int_{-\infty}^{\infty}e^{-t^2}dt\right)=$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\frac{1}{2}\int_{-\infty}^{\infty}e^{-t^2}dt=$$

$$=\frac{\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}dt$$

Step 6: As we computed earlier, $\int_{-\infty}^{\infty}e^{-t^2}dt=\sqrt{\pi}$, which means:

$$\frac{\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}dt=\frac{\sigma^2}{\sqrt{\pi}}\sqrt{\boldsymbol{\pi}}=\boldsymbol{\sigma^2}$$

Thus, the variance for a Variable, whose PDF looks like: $f(y)=\frac{\mathbf{1}}{\sigma\sqrt{2\pi}}\boldsymbol{e}^{\frac{-(y-\mu)^2}{2\sigma^2}}$, equals $\boldsymbol{\sigma^2}$.