10

Stresses in Beams

As seen in the last chapter beams are subjected to bending moment and shear forces which vary from section to section. To resist them stresses will develop in the materials of the beam. For the simplicity in analysis, we consider the stresses due to bending and stresses due to shear separately.

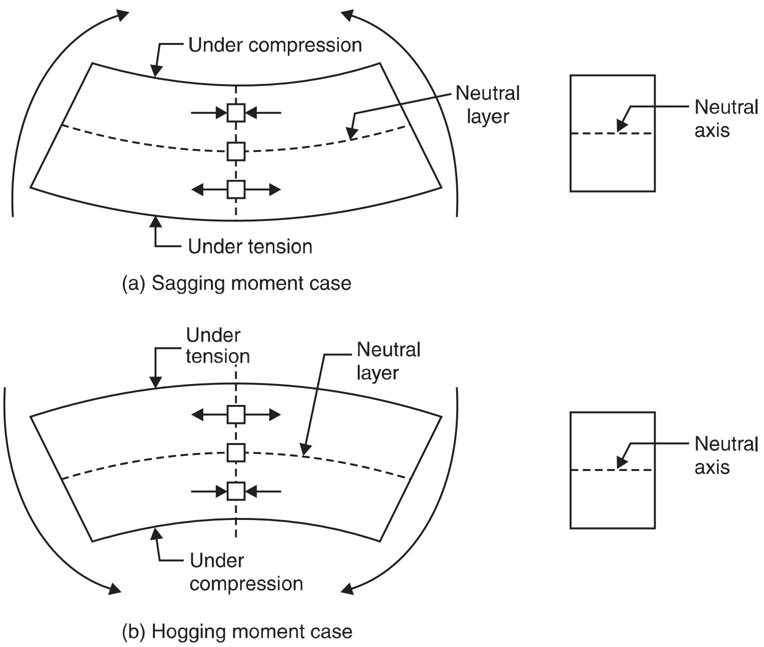

Fig. 10.1. Nature of Stresses in Beams

Due to pure bending, beams sag or hog depending upon the nature of bending moment as shown in Fig. 10.1. It can be easily observed that when beams sag, fibres in the bottom side get stretched while fibres on the top side are compressed. In other words, the material of the beam is subjected to tensile stresses in the bottom side and to compressive stresses in the upper side. In case of hogging the nature of bending stress is exactly opposite, *i.e.*, tension at top and compression at bottom. Thus bending stress varies from compression at one edge to tension at the other edge. Hence somewhere in between the two edges the bending stress should be zero. The layer of zero stress due to bending is called **neutral layer** and the trace of neutral layer in the cross-section is called **neutral axis** [Refer Fig. 10.1].

10.1 ASSUMPTIONS

Theory of simple bending is developed with the following assumptions which are reasonably acceptable:

(*i*) The material is homogeneous and isotropic.

(*ii*) Modulus of elasticity is the same in tension and in compression.

(*iii*) Stresses are within the elastic limit.

(*iv*) Plane section remains plane even after deformations.

(*v*) The beam is initially straight and every layer of it is free to expand or contract.

(*vi*) The radius of curvature of bent beam is very large compared to depth of the beam.

10.2 BENDING EQUATION

There exists a define relationship among applied moment, bending stresses and bending deformation (radius of curvature). This relationship can be derived in two steps:

(*i*) Relationship between bending stresses and radius of curvature.

(*ii*) Relationship between applied bending moment and radius of curvature.

(*i*) *Relationship between bending stresses and radius of curvature:* Consider an elemental length AB of the beam as shown in Fig. 10.2(*a*). Let EF be the neutral layer and CD the bottom most layer. If GH is a layer at distance y from neutral layer EF, initially $AB = EF = GH = CD$.

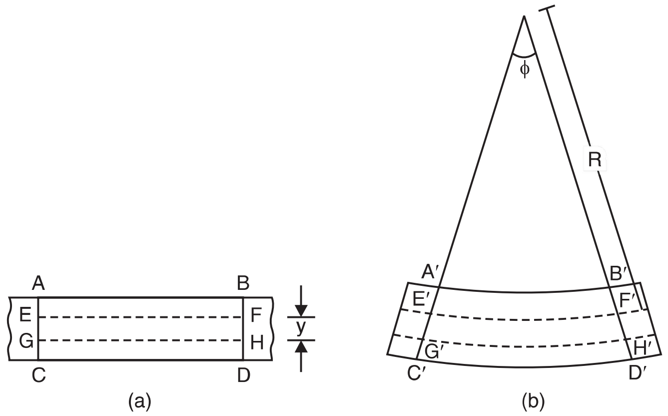

Fig. 10.2

Let after bending A, B, C, D, E, F, G and H take positions A', B', C', D', E', F', G' and H' respectively as shown in Fig. 10.2(*b*). Let R be the radius of curvature and ϕ be the angle subtended by $C'A'$ and $D'B'$ at centre of radius of curvature. Then,

$$EF = E'F', \quad \text{since } EF \text{ is neutral axis}$$

$$= R\phi \qquad \ldots(i)$$

$$\text{Strain in } GH = \frac{\text{Final length} - \text{Initial length}}{\text{Initial length}}$$

$$= \frac{G'H' - GH}{GH}$$

But $\quad GH = EF \quad$ (The initial length)

$$= R\phi$$

and $\quad G'H' = (R + y)\,\phi$

$\therefore$ Strain in layer GH

$$= \frac{(R + y)\,\phi - R\phi}{R\phi}$$

$$= \frac{y}{R} \qquad \text{...}(ii)$$

Since strain in GH is due to tensile forces, strain in $GH = f/E$ ...(*iii*)
where f is tensile stress and E is modulus of elasticity.

From eqns. (*ii*) and (*iii*), we get

$$\frac{f}{E} = \frac{y}{R}$$

or

$$\boldsymbol{\frac{f}{y} = \frac{E}{R}} \qquad \text{...(10.1)}$$

(*ii*) *Relationship between bending moment and radius of curvature:* Consider an elemental area δa at distance y from neutral axis as shown in Fig. 10.3.

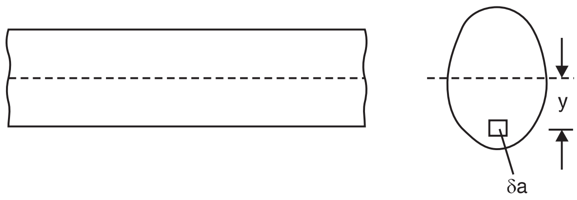

Fig. 10.3

From eqn. 10.1, stress on this element is

$$f = \frac{E}{R}\,y \qquad \text{...}(i)$$

$\therefore$ Force on this element

$$= \frac{E}{R}\,y\,\delta a$$

Moment of resistance of this elemental force about neutral axis

$$= \frac{E}{R}\,y\,\delta a\,y$$

$$= \frac{E}{R}\,y^2\,\delta a$$

$\therefore$ Total moment resisted by the section M' is given by

$$M' = \sum \frac{E}{R} y^2 \, \delta a$$

$$= \frac{E}{R} \sum y^2 \, \delta a$$

From the definition of moment of inertia (second moment of area) about centroidal axis, we know

$$I = \Sigma y^2 \, \delta a$$

$$\therefore \qquad M' = \frac{E}{R} I$$

From equilibrium condition, $M = M'$ where M is applied moment.

$$\therefore \qquad M = \frac{E}{R} I$$

or
$$\frac{M}{I} = \frac{E}{R} \qquad \text{...(10.2)}$$

From eqns. (10.1) and (10.2), we get

$$\boldsymbol{\frac{M}{I} = \frac{f}{y} = \frac{E}{R}} \qquad \text{...(10.3)}$$

where M = bending moment at the section
I = moment of inertia about centroid axis
f = bending stress
y = distance of the fibre from neutral axis
E = modulus of elasticity and
R = radius of curvature of bent section.

Equation (10.3) is known as bending equation.

10.3 LOCATING NEUTRAL AXIS

Consider an elemental area δa at a distance y from neutral axis [Ref. Fig. 10.3].

If 'f' is the stress on it, force on it = $f \, \delta a$

But $f = \frac{E}{R} y$, from eqn. (10.1).

$\therefore$ Force on the element $= \frac{E}{R} y \, \delta a$

Hence total horizontal force on the beam

$$= \sum \frac{E}{R} y \, \delta a$$

$$= \frac{E}{R} \Sigma y \, \delta a$$

Since there is no other horizontal force, equilibrium condition of horizontal forces gives

$$\frac{E}{R} \Sigma y \, \delta a = 0$$

As $\frac{E}{R}$ is not zero,

$$\Sigma y \, \delta a = 0 \qquad ...(i)$$

If A is total area of cross-section, from eqn. (i), we get

$$\sum \frac{y \, \delta a}{A} = 0 \qquad ...(ii)$$

Noting that $\Sigma y \delta a$ is the moment of area about neutral axis, $\frac{\Sigma y \delta a}{A}$ should be the distance of centroid of the area from the neutral axis. Hence $\frac{\Sigma y \delta a}{A} = 0$ means the *neutral axis coincides with the centroid of the cross-section.*

10.4 MOMENT CARRYING CAPACITY OF A SECTION

From bending equation, we have

$$\frac{M}{I} = \frac{f}{y}$$

i.e.,

$$f = \frac{M}{I} y \qquad ...(i)$$

Hence bending stress is maximum, when y is maximum. In other words, maximum stress occurs in the extreme fibres. Denoting extreme fibre distance from neutral fibre as y_{max} equation (i) will be

$$f_{max} = \frac{M}{I} y_{max} \qquad ...(ii)$$

In a design f_{max} is restricted to the permissible stress in the material. If f_{per} is the permissible stress, then from equation (ii),

$$f_{per} = \frac{M}{I} y_{max}$$

$$\therefore \qquad M = \frac{I}{y_{max}} f_{per}$$

The moment of inertia I and extreme fibre distance from neutral axis y_{max} are the properties of section. Hence $\frac{I}{y_{max}}$ is the property of the section of the beam. This term is known as **modulus of section** and is denoted by Z. Thus

$$Z = \frac{I}{y_{max}} \qquad ...(10.4)$$

and $$M = f_{per}\, Z \qquad ...(10.5)$$

Note : If moment of inertia has unit mm^4 and y_{max} has mm, Z has the unit mm^3.

The eqn. (10.5) gives permissible maximum moment on the section and is known as **moment carrying capacity of the section**. Since there is definite relation between bending moment and the loading given for a beam it is possible to find the load carrying capacity of the beam by equating maximum moment in the beam to moment carrying capacity of the section. Thus

$$M_{max} = f_{per}\, Z \qquad ...(10.6)$$

If permissible stresses in tension and compressions are different for a material, moment carrying capacity in tension and compression should be found separately and equated to maximum values of moment creating tension and compression separately to find the load carrying capacity. The lower of the two values obtained should be reported as the load carrying capacity.

10.5 SECTION MODULI OF STANDARD SECTIONS

Section modulus expressions for some of the standard sections are presented below:

(*i*) **Rectangular section:** Let width be '*b*' and depth be '*d*' as shown in Fig. 10.4.
Since *N-A* is in the mid depth

$$y_{max} = d/2$$

$$I = \frac{1}{12} bd^3$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{1/12\, bd^3}{d/2}$$

i.e., $$\mathbf{Z = 1/6\ bd^2} \qquad ...(10.7)$$

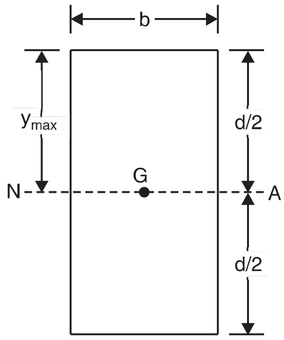

Fig. 10.4

(*ii*) **Hollow rectangular section.** Figure 10.5 shows a typical hollow rectangular section with symmetric opening. For this,

$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12}(BD^3 - bd^3)$$

$$y_{max} = D/2$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{1}{12}\frac{(BD^3 - bd^3)}{D/2}$$

i.e. $$\mathbf{Z = \frac{1}{6}\frac{BD^3 - bd^3}{D}} \qquad ...(10.8)$$

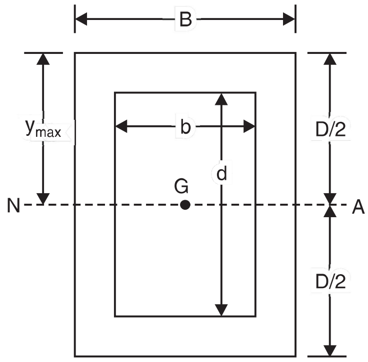

Fig. 10.5

(*iii*) **Circular section of diameter 'd'**. Typical section is shown in Fig. 10.6. For this,

$$I = \frac{\pi}{64} d^4$$

$$y_{max} = d/2$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\pi/64\, d^4}{d/2}$$

i.e.,
$$\mathbf{Z = \frac{\pi}{32} d^3}$$

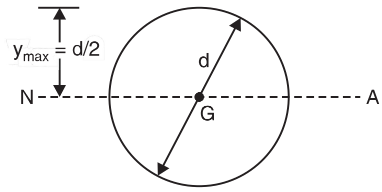

Fig. 10.6

(*iv*) **Hollow circular tube of uniform section.** Referring to Fig. 10.7,

$$I = \frac{\pi}{64} D^4 - \frac{\pi}{64} d^4$$

$$= \frac{\pi}{64} (D^4 - d^4)$$

$$y_{max} = D/2$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\pi}{64} \frac{(D^4 - d^4)}{D/2}$$

i.e.,
$$\mathbf{Z = \frac{\pi}{32} \frac{D^4 - d^4}{D}} \qquad \text{...(10.9)}$$

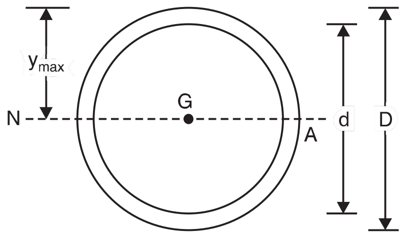

Fig. 10.7

(*v*) **Triangular section of base width *b* and height '*h*'.** Referring to Fig. 10.8,

$$I = \frac{bh^3}{36}$$

$$y_{max} = \frac{2}{3} h$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{bh^3/36}{2/3\, h}$$

i.e.,
$$\mathbf{Z = \frac{bh^2}{24}} \qquad \text{...(10.10)}$$

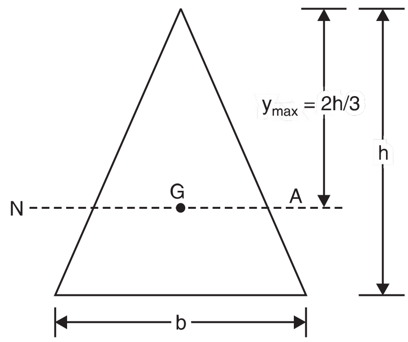

Fig. 10.8

Example 10.1. *A simply supported beam of span 3.0 m has a cross-section 120 mm × 180 mm. If the permissible stress in the material of the beam is 10 N/mm^2, determine*

(*i*) *maximum udl it can carry*

(*ii*) *maximum concentrated load at a point 1 m from support it can carry.*

Neglect moment due to self weight.

Solution:

Here $b = 120$ mm, $d = 180$ mm, $I = \frac{1}{12} bd^3$, $y_{max} = \frac{d}{2}$

$$\therefore \quad Z = \frac{1}{6} bd^2$$

$$= \frac{1}{6} \times 120 \times 180^2 = 648000 \text{ mm}^3$$

$$f_{per} = 10 \text{ N/mm}^2$$

$\therefore$ Moment carrying capacity of the section

$$= f_{per} \times Z$$

$$= 10 \times 648000 \text{ N-mm}$$

(*i*) Let maximum *udl* beam can carry be *w*/metre length as shown in Fig. 10.9.

In this case, we know that maximum moment occurs at mid span and is equal to $M_{max} = \frac{wL^2}{8}$.

Equating it to moment carrying capacity, we get,

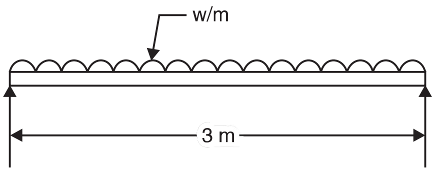

Fig. 10.9

$$\frac{w \times 3^2}{8} \times 10^6 = 10 \times 648000$$

$$\therefore \quad \boldsymbol{w = 5.76 \text{ kN/m.}}$$

(*ii*) Concentrated load at distance 1 m from the support be *P* kN. Referring to Fig. 10.10.

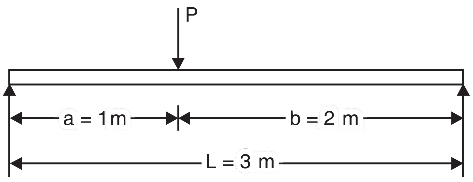

Fig. 10.10

$$M_{max} = \frac{P \times a \times b}{L} = \frac{P \times 1 \times 2}{3}$$

$$= \frac{2P}{3} \text{ kN-m}$$

$$= \frac{2P}{3} \times 10^6 \text{ N-mm}$$

Equating it to moment carrying capacity, we get

$$\frac{2P}{3} \times 10^6 = 10 \times 648000$$

$$\therefore \quad \boldsymbol{P = 9.72 \text{ kN-m.}}$$

Example 10.2. *A circular steel pipe of external diameter 60 mm and thickness 8 mm is used as a simply supported beam over an effective span of 2 m. If permissible stress in steel is 150 N/mm², determine the maximum concentrated load that can be carried by it at mid span.*

Solution:

External diameter $D = 60$ mm

Thickness $= 8$ mm

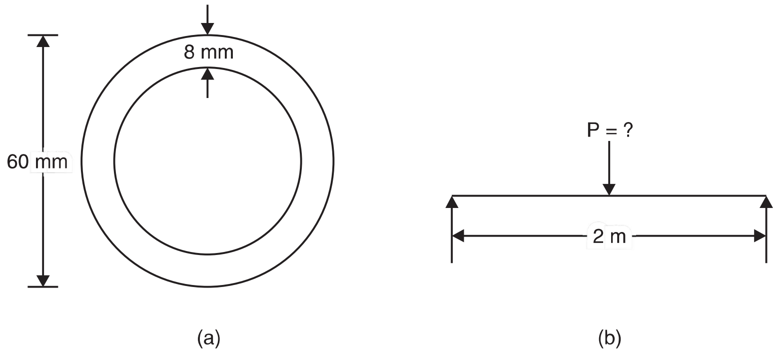

Fig. 10.11

$\therefore$ Internal diameter $= 60 - 2 \times 8 = 44$ mm.

$$I = \frac{\pi}{64}(60^4 - 44^4) = 452188 \text{ mm}^4$$

$$y_{max} = 30 \text{ mm.}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{452188}{30} = 15073 \text{ mm}^3.$$

Moment carrying capacity

$$M = f_{per}\, Z = 150 \times 15073 \text{ N-mm.}$$

Let maximum load it can carry be P kN.

Then maximum moment $= \dfrac{PL}{4}$

$$= \frac{P \times 2}{4} \text{ kN-m}$$

$$= 0.5\, P \times 10^6 \text{ N-mm.}$$

Equating maximum bending moment to moment carrying capacity, we get

$$0.5P \times 10^6 = 150 \times 15073$$

$$\therefore \quad \mathbf{P = 4.52 \text{ kN.}}$$

Example 10.3: *Figure 10.12 (a) shows the cross-section of a cantilever beam of 2.5 m span. Material used is steel for which maximum permissible stress is 150 N/mm^2. What is the maximum uniformly distributed load this beam can carry?*

Solution: Since it is a symmetric section, centroid is at mid depth.

$$I = MI \text{ of 3 rectangles about centroid}$$

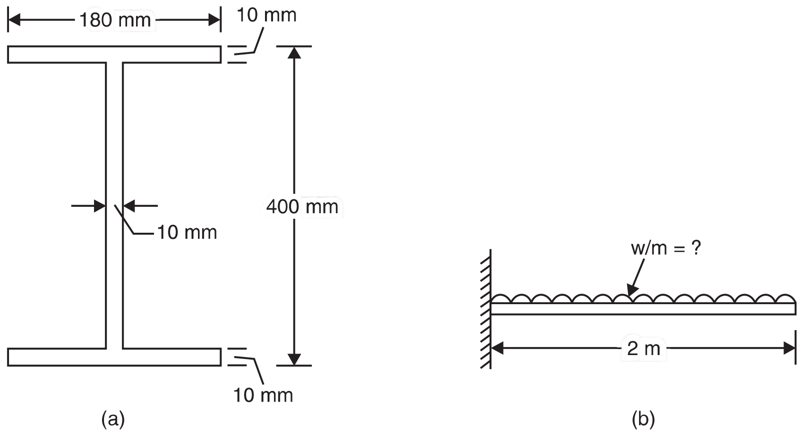

Fig. 10.12

$$= \frac{1}{12} \times 180 \times 10^3 + 180 \times 10\ (200 - 5)^2$$
$$+ \frac{1}{12} \times 10 \times (400 - 20)^3 + 10 \times (400 - 20) \times 0^2$$
$$+ \frac{1}{12} \times 180 \times 10^3 + 180 \times 10\ (200 - 5)^2$$
$$= 182.6467 \times 10^6 \text{ mm}^4$$

[**Note:** Moment of above section may be calculated as difference between *MI* of rectangle of size 180 × 400 and 170 × 380. *i.e.*,

$$I = \frac{1}{12} \times 180 \times 400^3 - \frac{1}{12} \times 170 \times 380^3$$

$$y_{max} = 200 \text{ mm.}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{182.6467 \times 10^6}{200} = 913233 \text{ mm}^3.$$

∴ Moment carrying capacity

$$= f_{per} \times Z$$
$$= 180 \times 913233$$
$$= 136985000 \text{ N-mm.}$$

If *udl* is *w* kN/m, maximum moment in cantilever

$$= wL = 2w \text{ kN-mm}$$
$$= 2w \times 10^6 \text{ N-mm}$$

Equating maximum moment to movement carrying capacity of the section, we get

$$2w \times 10^6 = 136985000$$

$$\therefore \quad \mathbf{w = 68.49 \text{ kN/m}}$$

Example 10.4. *Compare the moment carrying capacity of the section given in example 10.3 with equivalent section of the same area but*

(*i*) *square section*

(*ii*) *rectangular section with depth twice the width and*

(*iii*) *a circular section.*

Solution:

Area of the section $= 180 \times 10 + 380 \times 10 + 180 \times 10$

$= 7400 \text{ mm}^2$

(*i*) Square section

If '*a*' is the size of the equivalent square section,

$$a^2 = 7400 \qquad \therefore \quad a = 86.023 \text{ mm.}$$

Moment of inertia of this section

$$= \frac{1}{12} \times 86.023 \times 86.023^3$$

$$= 4563333 \text{ mm}^4$$

$$Z = \frac{I}{y_{\max}} = \frac{4563333}{86.023/2} = 106095.6 \text{ mm}^3$$

Moment carrying capacity $= fZ = 150 \times 106095.6$

$= 15.914 \times 10^6$ N-mm

$$\therefore \quad \frac{\textbf{Moment carrying capacity of I section}}{\textbf{Moment carrying capacity of equivalent square section}} = \frac{136985000}{15.914 \times 10^6}$$

$$= \mathbf{8.607.}$$

(*ii*) Equivalent rectangular section of depth twice the width.

Let b be the width

$\therefore$ Depth $d = 2b$.

Equating its area to area of *I*-section, we get

$$b \times 2b = 7400$$

$$b = 60.8276 \text{ mm}$$

$$y_{\max} = d/2 = b = 60.8276$$

$$M = f\,\frac{I}{y_{\max}} = 150 \times \frac{1}{12} \times \frac{b \times (2b)^3}{b}$$

$$= 150 \times \frac{8}{12}\, b^3 = 150 \times \frac{8}{12} \times 60.8276^3$$

$$= 22506193 \text{ N-mm.}$$

$$\therefore \quad \frac{\textbf{Moment carrying capacity of I section}}{\textbf{Moment carrying capacity of this section}} = \frac{136985000}{22506193} = \mathbf{6.086.}$$

(*iii*) Equivalent circular section.

Let diameter be d.

Then,
$$\frac{\pi d^2}{4} = 7400$$
$$d = 97.067$$
$$I = \frac{\pi}{64} d^4$$
$$y_{max} = d/2$$
$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\pi}{32} d^3.$$
$$M = f_{per}\, Z = 150 \times \frac{\pi}{32} \times 97.067^3 = 13468024$$
$$\therefore \quad \frac{\textbf{Moment carrying capacity of I section}}{\textbf{Moment carrying capacity of circular section}} = \frac{136985000}{13468024} = \mathbf{10.17.}$$

[**Note.** *I* section of same area resists more bending moment compared to an equivalent square, rectangular or circular section. Reason is obvious because in *I*-section most of the area of material is in heavily stressed zone.]

Example 10.5. *A symmetric I-section of size 180 mm × 40 mm, 8 mm thick is strengthened with 240 mm × 10 mm rectangular plate on top flange as shown is Fig. 10.13. If permissible stress in the material is 150 N/mm², determine how much concentrated load the beam of this section can carry at centre of 4 m span. Given ends of beam are simply supported.*

Solution: Area of section A

$$= 240 \times 10 + 180 \times 8 + 384 \times 8 + 180 \times 8 = 8352 \text{ mm}^2$$

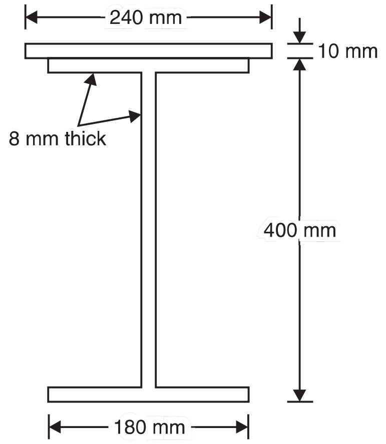

Fig. 10.13

Let centroid of the section be at a distance y from the bottom most fibre. Then

$$A\ \bar{y} = 240 \times 10 \times 405 + 180 \times 8 \times (400 - 4) + 384 \times 8 \times 200 + 180 \times 8 \times 4$$

i.e., $8352\,\bar{y} = 2162400$

$\therefore$ $\bar{y} = 258.9$ mm

$$I = \frac{1}{12} \times 240 \times 10^3 + 240 \times 10\,(405 - 258.9)^2$$

$$+ \frac{1}{12} \times 180 \times 8^3 + 180 \times 8\,(396 - 258.9)^2$$

$$+ \frac{1}{12} \times 8 \times 384^3 + 8 \times 384\,(200 - 258.9)^2$$

$$+ \frac{1}{12} \times 180 \times 8^3 + 180 \times 8\,(4 - 258.9)^2$$

$$= 220.994 \times 10^6 \text{ mm}^4$$

$\therefore$ $y_{top} = 405 - 258.9 = 146.1$ mm

$y_{bottom} = 258.9$ mm.

$\therefore$ $y_{max} = 258.9$ mm

$\therefore$ $$Z = \frac{I}{y_{max}} = \frac{220.994 \times 10^6}{258.9} = 853588.3$$

$\therefore$ Moment carrying capacity of the section

$$= f_{per}\,Z = 150 \times 853588.3$$

$$= 128038238.7 \text{ N-mm}$$

$$= 128.038 \text{ kN-m.}$$

Let P kN be the central concentrated load the simply supported beam can carry. Then max bending movement in the beam

$$= \frac{P \times 4}{4} = P \text{ kN-m}$$

Equating maximum moment to moment carrying capacity, we get

$$\boldsymbol{P = 128.038 \text{ kN.}}$$

Example 10.6. *The cross-section of a cast iron beam is as shown in Fig. 10.14(a). The top flange is in compression and bottom flange is in tension. Permissible stress in tension is 30 N/mm^2 and its value in compression is 90 N/mm^2. What is the maximum uniformly distributed load the beam can carry over a simply supported span of 5 m?*

Solution:

Cross-section area $A = 75 \times 50 + 25 \times 100 + 150 \times 50$

$$= 13750 \text{ mm}^2$$

Let neutral axis lie at a distance $\bar{y}$ from bottom most fibre. Then

$$A\bar{y} = 75 \times 50 \times 175 + 25 \times 100 \times 100 + 150 \times 50 \times 25$$

$$13750 \times \bar{y} = 1093750$$

$\therefore$ $\bar{y} = 79.54$ mm

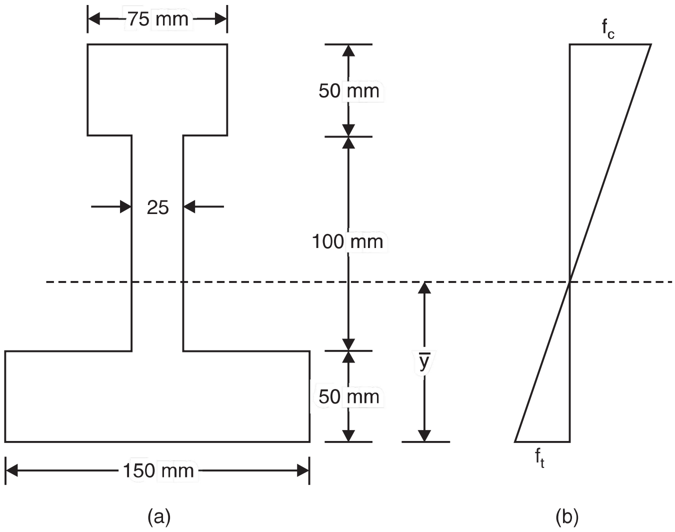

Fig. 10.14

$$\therefore \qquad I = \frac{1}{12} \times 75 \times 50^3 + 75 \times 50\,(175 - 79.54)^2$$

$$+ \frac{1}{12} \times 25 \times 100^3 + 25 \times 100\,(100 - 79.54)^2$$

$$+ \frac{1}{12} \times 150 \times 50^3 + 150 \times 50\,(25 - 79.54)^2$$

$$= 61.955493 \times 10^6 \text{ mm}^4.$$

Extreme fibre distances are

$$y_{\text{bottom}} = \bar{y} = 79.54 \text{ mm.}$$

$$y_{\text{top}} = 200 - \bar{y} = 200 - 79.54 = 120.46 \text{ mm.}$$

Top fibres are in compression. Hence from consideration of compression strength, moment carrying capacity of the beam is given by

$$M_1 = f_{\text{per}} \text{ in compression} \times \frac{I}{y_{\text{top}}}$$

$$= 90 \times \frac{61.955493 \times 10^6}{120.46}$$

$$= 46.289178 \times 10^6 \text{ N-mm}$$

$$= 46.289178 \text{ kN-m.}$$

Bottom fibres are in tension. Hence from consideration of tension, moment carrying capacity of the section is given by

$$M_2 = f_{\text{per}} \text{ in tension} \times \frac{I}{y_{\text{bottom}}}$$

$$= \frac{30 \times 61.955493 \times 10^6}{79.54}$$

$$= 21.367674 \times 10^6 \text{ N-mm}$$

$$= 21.367674 \text{ kN-m.}$$

Actual moment carrying capacity is the lower value of the above two values. Hence moment carrying capacity of the section is

$$= 21.367674 \text{ kN-m.}$$

Maximum moment in a simply supported beam subjected to *udl* of *w*/unit length and span *L* is

$$= \frac{wL^2}{8}$$

Equating maximum moment to moment carrying capacity of the section, we get maximum load carrying capacity of the beam as

$$w \times \frac{5^2}{8} = 21.367674$$

$$\therefore \quad \mathbf{w = 6.838 \text{ kN/m.}}$$

Example 10.7. *The diameter of a concrete flag post varies from 240 mm at base to 120 mm at top as shown in Fig. 10.15. The height of the post is 10 m. If the post is subjected to a horizontal force of 600 N at top, find the section at which stress is maximum. Find its value also.*

Solution: Consider a section y metres from top. Diameter at this section is

$$d = 120 + \frac{y}{10}(240 - 120)$$

$$= 120 + 12y \text{ mm}$$

$$\therefore \quad I = \frac{\pi}{64} d^4$$

$$Z = \frac{I}{d/2} = \frac{\pi}{32} d^3$$

$$= \frac{\pi}{32} [120 + 12y]^3$$

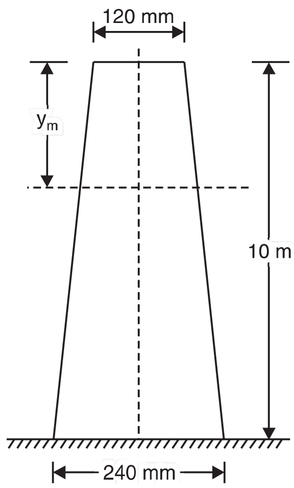

Fig. 10.15

At this section, moment is given by

$$M = 600\, y \text{ N-m}$$

$$= 600000\, y \text{ N-mm.}$$

Equating moment of resistance to moment at the section, we get

$$f\, Z = M$$

where f is extreme fibre stress

$$\therefore \quad f \cdot \frac{\pi}{32} [120 + 12y]^3 = 600000\, y$$

$$\therefore \quad f = 600000 \times 32 \frac{y}{\pi [120 + 12y]^3}$$

For 'f' to be maximum, $\dfrac{df}{dy} = 0$

$$600000 \times 32\,[(120 + 12y)^{-3} + y(-3)\,(120 + 12y)^{-4} \times 12] = 0$$

i.e., $$(120 + 12y)^{-3} = 36\,(120 + 12y)^{-4}\,y$$

i.e., $$1 = 36\,(120 + 12y)^{-1}\,y$$

i.e., $$120 + 12y = 36y$$

$\therefore$ $$\mathbf{y = 5\ m.}$$

Stress at this section f is given by

$$f = 600000 \times 32 \times \frac{5}{\pi\,(120 + 12 \times 5)^3}$$

$$\mathbf{f = 5.24\ N/mm^2.}$$

Example 10.8. *A circular log of timber has diameter D. Find the dimensions of the strongest rectangular section one can cut from this.*

Solution: Let the width and depth of strongest section that can be cut from the log be 'b' and 'd' respectively. Then,

$$D^2 = b^2 + d^2$$

or $$d^2 = D^2 - b^2$$

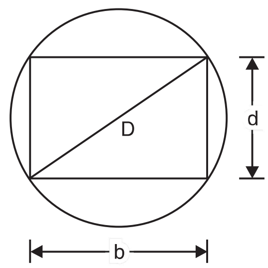

Fig. 10.16

For rectangular section

$$I = \frac{1}{12}\,bd^3$$

$$y_{max} = d/2.$$

$\therefore$ $$z = \frac{I}{y_{max}} = \frac{1}{6}\,bd^2$$

$$= \frac{1}{6}\,b\,(D^2 - b^2) = \frac{1}{6}\,(bD^2 - b^3)$$

The beam is strongest if section modulus is maximum. Hence the condition is

$$\frac{dz}{db} = 0$$

$$\frac{1}{6}\,[D^2 - 3b^2] = 0$$

i.e., $$D^2 = 3b^2$$

or $$b = \frac{D}{\sqrt{3}}.$$

$\therefore$ $$d = \sqrt{(D^2 - b^2)} = \sqrt{D^2 - \frac{D^2}{3}} = D\,\sqrt{2/3}$$

Thus the dimensions of strongest beam

$$= \frac{\boldsymbol{D}}{\sqrt{\mathbf{3}}}\ \textbf{wide} \times \sqrt{\mathbf{2/3}}\ \boldsymbol{D}\ \textbf{deep.}$$

10.6 PROPORTIONING SECTIONS

In designing beams, span of the beam is known and load expected on the beam can be estimated. Hence bending moment to be resisted by the beam can be calculated. A designer has to select suitable section of the beam of desirable materials. Theoretically speaking, the section required changes along the span. Usually uniform sections are used. Hence the section selected should be capable of resisting the maximum moment. In case of circular sections we may find the diameter required, since section modulus required depends only on diameter. In case of rectangular sections, section modulus depends upon width and depth. Hence usually width is assumed and depth is calculated or else ratio of width to depth is assumed and section is selected. For steel sections, Indian Standard Hand Book may be used to identify standard section that satisfies the required section modulus value. This process of proportioning sections is known as **Design**. The design process is illustrated with the following examples:

Example 10.9. *Design a timber beam is to carry a load of 5 kN/m over a simply supported span of 6 m. Permissible stress in timber is 10 N/mm*^{2}*. Keep depth twice the width.*

Solution:

$$w = 5 \text{ kN/m}, \quad L = 6 \text{ m}.$$

$\therefore$ Maximum bending moment

$$= \frac{wL^2}{8} = \frac{5 \times 6^2}{8} = 22.5 \text{ kN-m}$$

$$= 22.5 \times 10^6 \text{ N-mm}$$

Let b be the width and d the depth. Then in this problem $d = 2b$

$$\therefore \qquad Z = \frac{1}{6} bd^2 = \frac{1}{6} \times b\ (2b)^2$$

$$= \frac{2}{3} b^3$$

$$f = 10 \text{ N/mm}^2 \quad \text{(given)}$$

Hence design requirement is

$$fZ = M$$

i.e.,
$$10 \times \frac{2}{3} b^3 = 22.5 \times 10^6$$

$$\therefore \qquad b = 150 \text{ mm.}$$

$$\therefore \qquad d = 2b = 300 \text{ mm.}$$

Use 150 mm × 300 mm section.

Example 10.10. *A cantilever of 3 m span, carrying uniformly distributed load of 3 kN/m is to be designed using cast iron rectangular section. Permissible stresses in cast iron are f = 30 N/mm*^{2} *in tension and* f_c *= 90 N/mm*^{2} *in compression. Proportion the section suitably.*

Solution:

Span of cantilever = 3 m $\quad w = 3$ kN/m

$$\therefore \text{ Maximum moment} \quad = \frac{wL^2}{2} = \frac{3 \times 3^2}{2} = 13.5 \text{ kN-m}$$

$$= 13.5 \times 10^6 \text{ N-mm}$$

Let b be the width and d the depth.

$$\therefore \qquad Z = \frac{1}{6}\, bd^2$$

Since it is rectangular section, N-A lies at mid-depth, and stresses at top and bottom are same. Hence, permissible tensile stress value is reached earlier and it governs the design.

$$f_{per} = 30 \text{ N/mm}^2$$

$\therefore$ Design condition is

$$fZ = M$$

$$30 \times \frac{1}{6}\, bd^2 = 13.5 \times 10^6$$

$$bd^2 = 2700000$$

Using $b = 100$ mm, we get

$$d^2 = \frac{2700000}{100}$$

$$\therefore \qquad d = 164.3 \text{ mm}$$

Use 100 mm wide and 165 mm deep section.

Example 10.11. *A circular bar of simply supported span 1 m has to carry a central concentrated load of 800 N. Find the diameter of the bar required, if permissible stress is 150 N/mm^2.*

Solution: Let the diameter of the bar be 'd'. Now, $W = 800$ N $L = 1$ m $= 1000$ mm.

$$\therefore \quad \text{Maximum moment} = \frac{WL}{4}$$

$$= \frac{800 \times 1000}{4} = 200000 \text{ N-mm}$$

$$f = 150 \text{ N/mm}^2$$

$$Z = \frac{\pi}{32}\, d^3$$

$\therefore$ Design condition is,

$$150 \times \frac{\pi}{32}\, d^3 = 200000$$

$$\therefore \qquad d = 23.8 \text{ mm}$$

Select 25 mm bar (which is available in market)

10.7 SHEAR STRESS DISTRIBUTION

In the 9th chapter we have seen that in a beam bending moment as well as shearing forces act. Shear force gives rise to shearing stresses in the beam. In this article expression for shearing stress is derived and its variation across the section is discussed. A designer has to see that the beam is safe not only in bending but in shear also.