MOMENT OF INERTIA

SECOND MOMENTS OF PLANE AREA

Consider the area shown in Fig. 4.37(*a*). dA is an elemental area with coordinates as x and y. The term $\Sigma y_i^2\, dA_i$ is called *moment of inertia* of the area about x axis and is denoted as I_{xx}. Similarly, the moment of inertia about y axis is

$$I_{yy} = \Sigma x_i^2\, dA_i$$

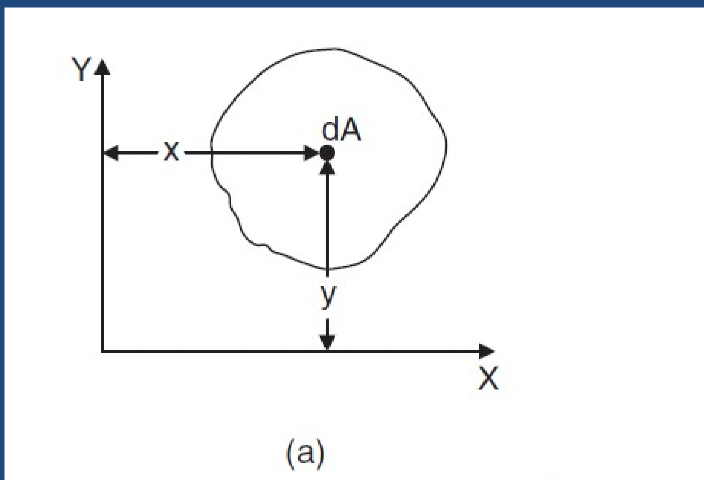


Fig. 4.37

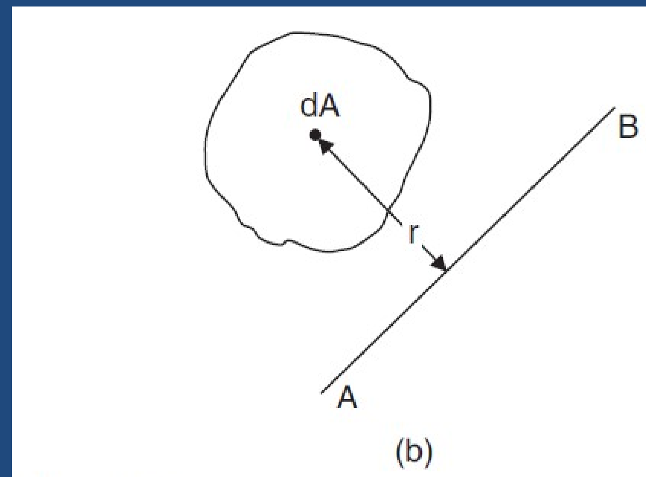


Fig. 4.37

In general, if r is the distance of elemental area dA from the axis AB [Fig. 4.37(*b*)], the sum of the terms $\Sigma r^2\, dA$ to cover the entire area is called moment of inertia of the area about the axis AB. If r and dA can be expressed in general term, for any element, then the sum becomes an integral. Thus,

$$I_{AB} = \Sigma r_i^2 dA_i = \int r^2\, dA \qquad \text{...(4.6)}$$

The term rdA may be called as moment of area, similar to moment of a force, and hence $r^2\, dA$ may be called as moment of area or the second moment of area. Thus, the moment of inertia of area is nothing but second moment of area. In fact, the term *'second moment of area'* appears to correctly signify the meaning of the expression $\Sigma r^2\, dA$. The term 'moment of inertia' is rather a misnomer.

Though moment of inertia of plane area is a purely mathematical term, it is one of the important properties of areas. The strength of members subject to bending depends on the moment of inertia of its cross-sectional area. Students will find this property of area very useful when they study subjects like strength of materials, structural design and machine design.

The moment of inertia is a fourth dimensional term since it is a term obtained by multiplying area by the square of the distance. Hence, in SI units, if metre (m) is the unit for linear measurements

used then m^4 is the unit of moment of inertia. If millimetre (mm) is the unit used for linear measurements, then mm^4 is the unit of moment of inertia. In MKS system m^4 or cm^4 and in FPS system ft^4 or in^4 are commonly used as units for moment of inertia.

Polar Moment of Inertia

Moment of inertia about an axis perpendicular to the plane of an area is known as *polar moment of inertia*. It may be denoted as J or I_{zz}. Thus, the moment of inertia about an axis perpendicular to the plane of the area at O in Fig. 4.38 is called polar moment of inertia at point O, and is given by

$$I_{zz} = \Sigma r^2 dA \qquad \text{...(4.7)}$$

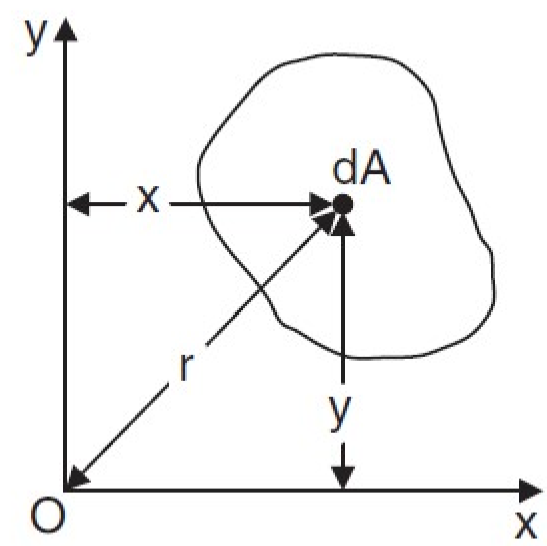


Fig. 4.38

Radius of Gyration

Radius of gyration is a mathematical term defined by the relation

$$k = \sqrt{\frac{I}{A}} \qquad \text{...(4.8)}$$

where k = radius of gyration,

I = moment of inertia,

and A = the cross-sectional area

Suffixes with moment of inertia I also accompany the term radius of gyration k. Thus, we can have,

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$

$$k_{AB} = \sqrt{\frac{I_{AB}}{A}}$$

and so on.

The relation between radius of gyration and moment of inertia can be put in the form:

$$I = Ak^2 \qquad \text{...(4.9)}$$

From the above relation a geometric meaning can be assigned to the term 'radius of gyration.' We can consider k as the distance at which the complete area is squeezed and kept as a strip of negligible width (Fig. 4.39) such that there is no change in the moment of inertia.

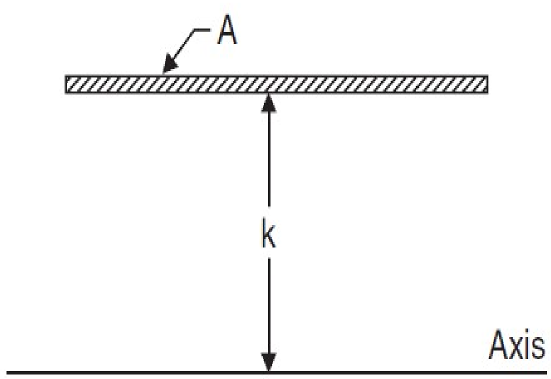


Fig. 4.39

Theorems of Moments of Inertia

There are two theorems of moment of inertia:
(1) Perpendicular axis theorem, and
(2) Parallel axis theorem.
These are explained and proved below.

Perpendicular Axis Theorem

The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point O and lying in the plane of the area.

Referring to Fig. 4.40, if z-z is the axis normal to the plane of paper passing through point O, as per this theorem,

$$I_{zz} = I_{xx} + I_{yy} \quad \text{... (4.10)}$$

The above theorem can be easily proved. Let us consider an elemental area dA at a distance r from O. Let the coordinates of dA be x and y. Then from definition:

$$I_{zz} = \Sigma r^2 dA$$
$$= \Sigma(x^2 + y^2)dA$$
$$= \Sigma x^2 dA + \Sigma y^2 dA$$
$$I_{zz} = I_{xx} + I_{yy}$$

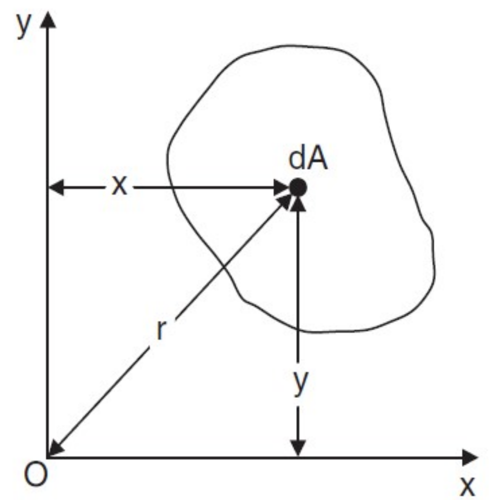


Fig. 4.40

Parallel Axis Theorem

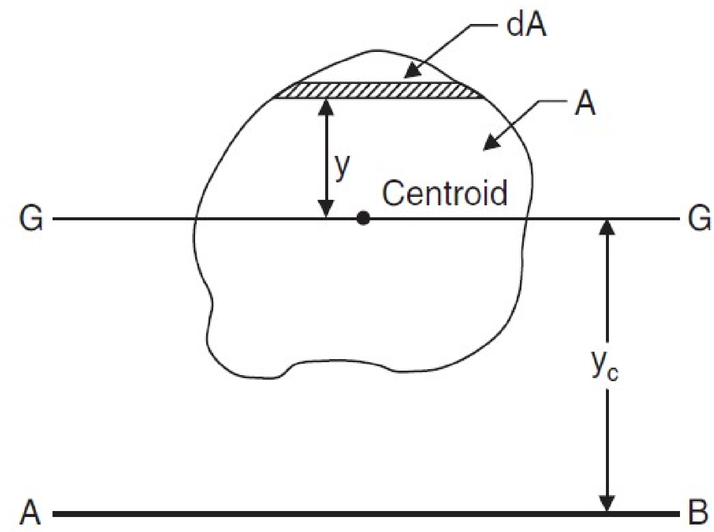


Fig. 4.41

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axis. Referring to Fig. 4.41 the above theorem means:

$$I_{AB} = I_{GG} + A\, y_c^2 \qquad \text{...(4.11)}$$

where

I_{AB} = moment of inertia about axis AB

I_{GG} = moment of inertia about centroidal axis GG parallel to AB.

A = the area of the plane figure given and

y_c = the distance between the axis AB and the parallel centroidal axis GG.

Proof: Consider an elemental parallel strip dA at a distance y from the centroidal axis (Fig. 4.41).

Then,

$$I_{AB} = \Sigma(y + y_c)^2 dA$$
$$= \Sigma(y^2 + 2y\, y_c + y_c^2)dA$$
$$= \Sigma y^2 dA + \Sigma 2y\, y_c\, dA + \Sigma\, y_c^2 dA$$

Now, $\Sigma y^2 dA$ = Moment of inertia about the axis GG $= I_{GG}$

$$\Sigma 2yy_c\, dA = 2y_c\, \Sigma y\, dA$$

$$= 2y_c A\, \frac{\Sigma ydA}{A}$$

In the above term $2y_c A$ is constant and $\frac{\Sigma ydA}{A}$ is the distance of centroid from the reference axis GG. Since GG is passing through the centroid itself $\frac{ydA}{A}$ is zero and hence the term $\Sigma 2yy_c dA$ is zero.

Now, the third term,

$$\Sigma y_c^2 dA = y_c^2 \Sigma dA$$

$$= Ay_c^2$$

$$\therefore \qquad I_{AB} = I_{GG} + Ay_c^2$$

Note: The above equation cannot be applied to any two parallel axis. One of the axis (GG) must be centroidal axis only.

Moment of inertia from first principles

For simple figures, moment of inertia can be obtained by writing the general expression for an element and then carrying out integration so as to cover the entire area. This procedure is illustrated with the following three cases:

(1) Moment of inertia of a rectangle about the centroidal axis

(2) Moment of inertia of a triangle about the base

(3) Moment of inertia of a circle about a diametral axis

(1) *Moment of Inertia of a Rectangle about the Centroidal Axis*: Consider a rectangle of width b and depth d (Fig. 4.42). Moment of inertia about the centroidal axis x-x parallel to the short side is required.

Consider an elemental strip of width dy at a distance y from the axis. Moment of inertia of the elemental strip about the centroidal axis xx is:

$$= y^2 dA$$

$$= y^2 b\, dy$$

$$\therefore \quad I_{xx} = \int_{-d/2}^{d/2} y^2 b\, dy$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$I_{xx} = \frac{bd^3}{12}$$

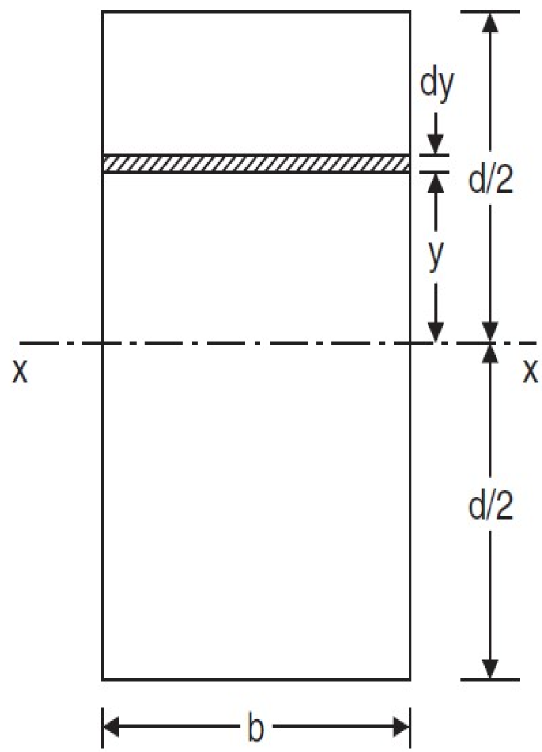


Fig. 4.42

(2) *Moment of Inertia of a Triangle about its Base*: Moment of inertia of a triangle with base width b and height h is to be determined about the base AB (Fig. 4.43).

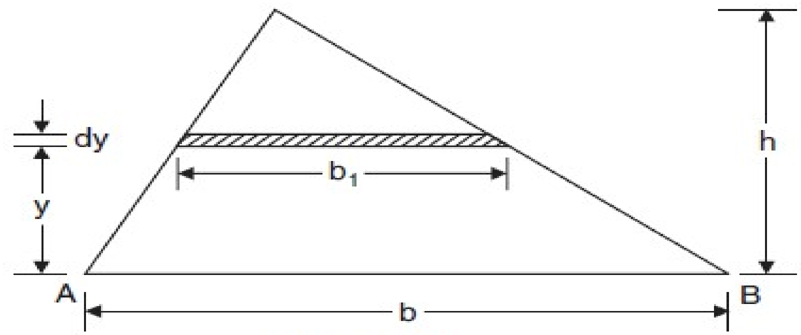


Fig. 4.43

Consider an elemental strip at a distance y from the base AB. Let dy be the thickness of the strip and dA its area. Width of this strip is given by:

$$b_1 = \frac{(h - y)}{h} \times b$$

Moment of inertia of this strip about AB

$$= y^2 dA$$

$$= y^2\, b_1\, dy$$

$$= y^2\, \frac{(h - y)}{h} \times b \times dy$$

$\therefore$ Moment of inertia of the triangle about AB,

$$I_{AB} = \int_0^h \frac{y^2\,(h - y) b dy}{h}$$

$$= \int_0^h \left(y^2 - \frac{y^3}{h} \right) b dy$$

$$= b \left[\frac{y^3}{3} - \frac{y^4}{4h} \right]_0^h$$

$$= b \left[\frac{h^3}{3} - \frac{h^4}{4h} \right]$$

$$I_{AB} = \frac{bh^3}{12}$$

(3) *Moment of Inertia of a Circle about its Diametral Axis*: Moment of inertia of a circle of radius R is required about it's diametral axis as shown in Fig. 4.44

Consider an element of sides $rd\theta$ and dr as shown in the figure. It's moment of inertia about the diametral axis x-x:

$$= y^2 dA$$
$$= (r \sin\theta)^2 \, r \, d\theta \, dr$$
$$= r^3 \sin^2\theta \, d\theta \, dr$$

$\therefore$ Moment of inertia of the circle about x-x is given by

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2\theta \, d\theta \, dr$$

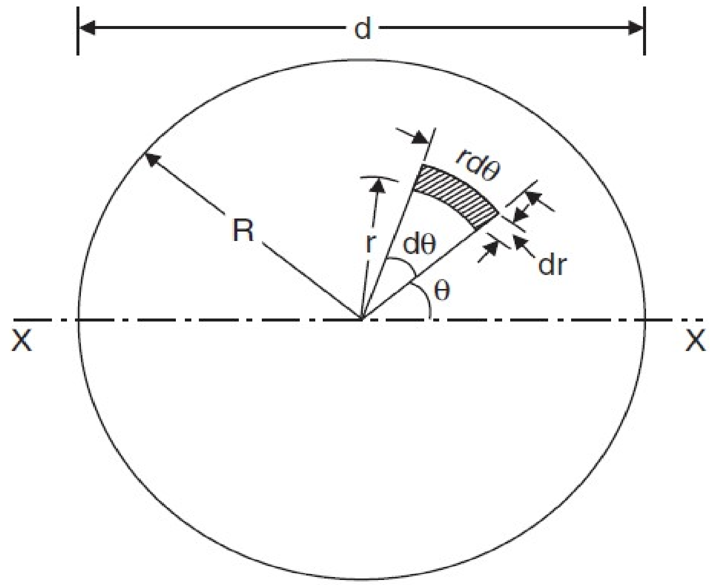


Fig. 4.44

$$= \int_0^R \int_0^{2\pi} \frac{(1-\cos 2\theta)}{2} d\theta \, dr$$

$$= \int_0^R \frac{r^3}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} dr$$

$$= \left[\frac{r^4}{8}\right]_0^R [2\pi - 0 + 0 - 0] = \frac{2\pi}{8}R^4$$

$$I_{xx} = \frac{\pi R^4}{4}$$

If d is the diameter of the circle, then

$$R = \frac{d}{2}$$

$$\therefore \qquad I_{xx} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4$$

$$I_{xx} = \frac{\pi d^4}{64}$$

Moment of Inertia of a Semicircle: (*a*) *About Diametral Axis*:

If the limit of integration is put as 0 to π instead of 0 to 2π in the derivation for the moment of inertia of a circle about diametral axis the moment of inertia of a semicircle is obtained. It can be observed that the moment of inertia of a semicircle (Fig. 4.49) about the diametral axis *AB*:

$$= \frac{1}{2} \times \frac{\pi d^4}{64} = \frac{\pi d^4}{128}$$

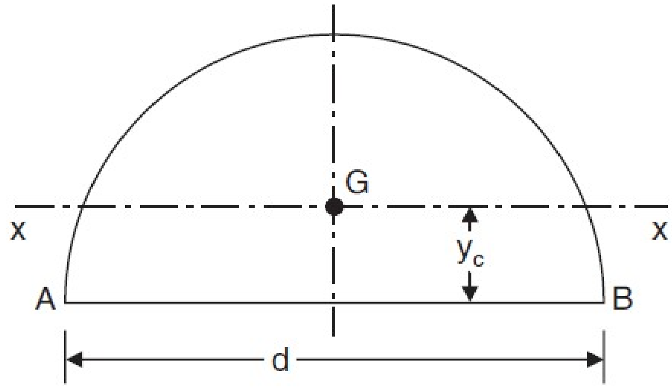


Fig. 4.49

(*b*) *About Centroidal Axis x-x*:

Now, the distance of centroidal axis y_c from the diametral axis is given by:

$$y_c = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

and, $$\text{Area } A = \frac{1}{2} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{8}$$

From parallel axis theorem,

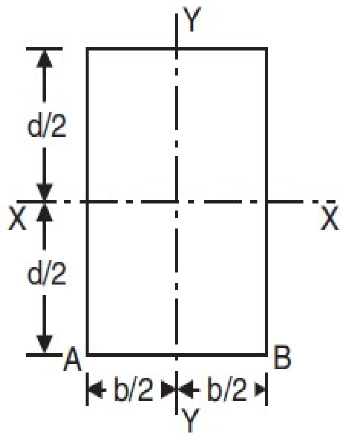
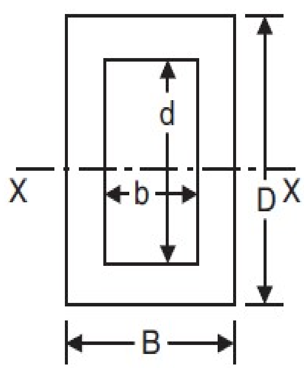
$$I_{AB} = I_{xx} + Ay_c^2$$

$$\frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \left(\frac{2d}{3\pi}\right)^2$$

$$I_{xx} = \frac{\pi d^4}{128} - \frac{d^4}{18\pi}$$

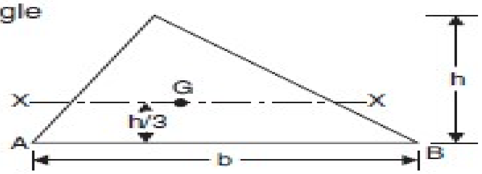
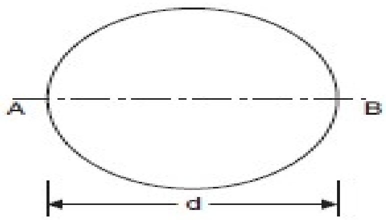
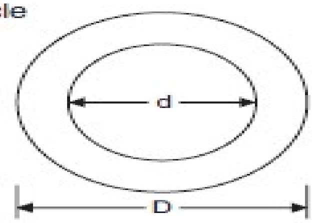
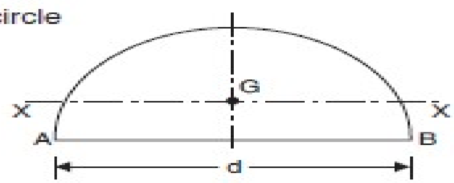
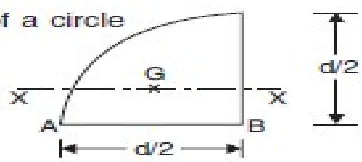
$$= 0.0068598\, d^4$$

Table 4.4 Moment of Inertia of Standard Sections

Shape	*Axis*	*Moment of Inertia*
Rectangle 	(a) Centroidal axis *x-x* (b) Centroidal axis *y-y* (c) *A – B*	$I_{xx} = \dfrac{bd^3}{12}$ $I_{yy} = \dfrac{db^3}{12}$ $I_{AB} = \dfrac{bd^3}{3}$
Hollow Rectangle 	Centroidal axis *x-x*	$I_{xx} = \dfrac{BD^3 - bd^3}{12}$

(Contd.)

Table 4.4 *(Contd.)*

Shape	*Axis*	*Moment of Inertia*
Triangle 	(a) Centroidal axis *x-x* (b) Base *AB*	$I_{xx} = \frac{bh^3}{36}$ $I_{AB} = \frac{bh^3}{12}$
Circle 	Diametral axis	$I = \frac{\pi d^4}{64}$
Hollow circle 	Diametral axis	$I = \frac{\pi}{64}\ (D^4 - d^4)$
Semicircle 	(a) *A – B* (b) Centroidal axis	$I_{AB} = \frac{\pi d^4}{128}$ $I_{xx} = 0.0068598\ d^4$
Quarter of a circle 	(a) *A – B* (b) Centroidal axis x-x	$I_{AB} = \frac{\pi d^4}{256}$ $I_{xx} = 0.00343\ d^4$

Example 4.12. *Determine the moment of inertia of the section shown in Fig. 4.53 about an axis passing through the centroid and parallel to the top most fibre of the section. Also determine moment of inertia about the axis of symmetry. Hence find radii of gyration.*

Solution: The given composite section can be divided into two rectangles as follows:

Area $A_1 = 150 \times 10 = 1500 \text{ mm}^2$

Area $A_2 = 140 \times 10 = 1400 \text{ mm}^2$

Total Area $A = A_1 + A_2 = 2900 \text{ mm}^2$

Due to symmetry, centroid lies on the symmetric axis y-y.

The distance of the centroid from the top most fibre is given by:

$$y_c = \frac{\text{Sum of moment of the areas about the top most fibre}}{\text{Total area}}$$

$$= \frac{1500 \times 5 + 1400(10 + 70)}{2900}$$

$$= 41.21 \text{ mm}$$

Referring to the centroidal axis x-x and y-y, the centroid of A_1 is g_1 (0.0, 36.21) and that of A_2 is g_2 (0.0, 38.79).

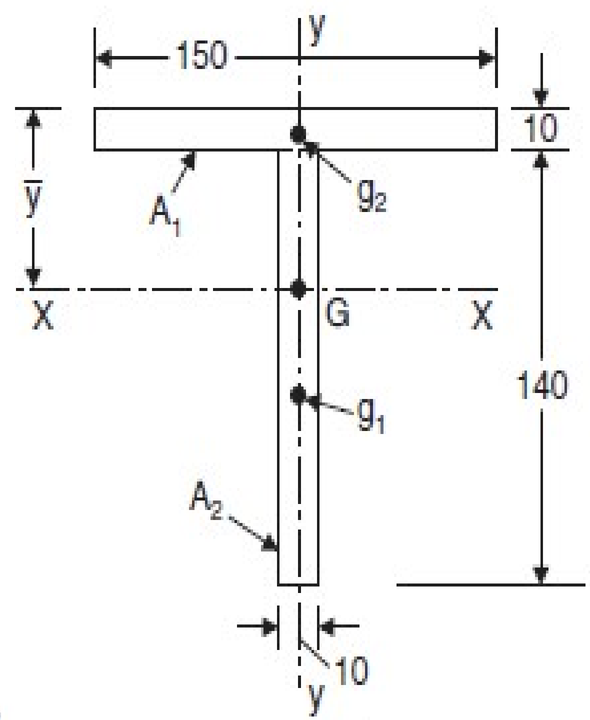


Fig. 4.53

Moment of inertia of the section about x-x axis

I_{xx} = moment of inertia of A_1 about x-x axis + moment of inertia of A_2 about x-x axis.

$$\therefore \quad I_{xx} = \frac{150 \times 10^3}{12} + 1500\,(36.21)^2 + \frac{10 \times 140^3}{12} + 1400\,(38.79)^2$$

i.e., $\quad \boldsymbol{I_{xx} = 63\,72442.5\ \mathrm{mm}^4}$

Similarly,

$$I_{yy} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = \mathbf{2824{,}166.7\ mm^4}$$

Hence, the moment of inertia of the section about an axis passing through the centroid and parallel to the top most fibre is 6372442.5 mm^4 and moment of inertia of the section about the axis of symmetry is 2824166.66 mm^4.

The radius of gyration is given by:

$$k = \sqrt{\frac{I}{A}}$$

$$\therefore \quad k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$= \sqrt{\frac{6372442.5}{2900}}$$

$$\boldsymbol{k_{xx} = 46.88\ \mathrm{mm}}$$

Similarly, $$k_{yy} = \sqrt{\frac{2824166.66}{2900}}$$

$$\boldsymbol{k_{yy} = 31.21\ \mathrm{mm}}$$

Example 4.13. Determine the moment of inertia of the L-section shown in the Fig. 4.54 about its centroidal axis parallel to the legs. Also find out the polar moment of inertia.

Solution: The given section is divided into two rectangles A_1 and A_2.

Area $\quad A_1 = 125 \times 10 = 1250 \text{ mm}^2$

Area $\quad A_2 = 75 \times 10 = 750 \text{ mm}^2$

Total Area $= 2000 \text{ mm}^2$

First, the centroid of the given section is to be located.

Two reference axis (1)–(1) and (2)–(2) are chosen as shown in Fig. 4.54.

The distance of centroid from the axis (1)–(1)

$$= \frac{\text{sum of moment of areas } A_1 \text{ and } A_2 \text{ about(1)} - (1)}{\text{Total area}}$$

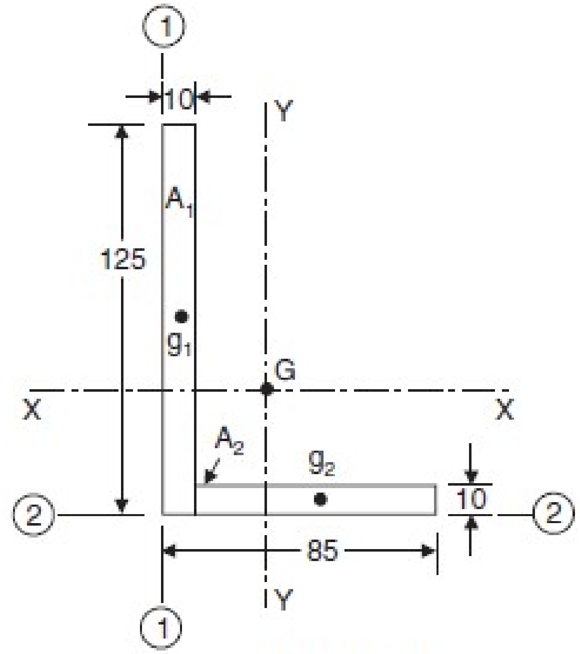


Fig. 4.54

i.e., $$\bar{x} = \frac{120 \times 5 + 750\left(10 + \frac{75}{2}\right)}{2000}$$

$$= 20.94 \text{ mm}$$

Similarly,

the distance of the centroid from the axis (2)–(2)

$$= \bar{y} = \frac{1250 \times \frac{125}{2} + 750 \times 5}{2000} = 40.94 \text{ mm}$$

With respect to the centroidal axis *x*-*x* and *y*-*y*, the centroid of A_1 is g_1 (15.94, 21.56) and that of A_2 is g_2 (26.56, 35.94).

$\therefore$ I_{xx} = Moment of inertia of A_1 about *x*-*x* axis + Moment of inertia of A_2 about *x*-*x* axis

$$\therefore \quad I_{xx} = \frac{10 \times 125^3}{12} + 1250 \times 21.56^2 + \frac{75 \times 10^3}{12} + 750 \times 39.94^2$$

i.e., $\mathbf{I_{xx} = 3411298.9 \text{ mm}^4}$

Similarly,

$$I_{yy} = \frac{125 \times 10^3}{12} + 1250 \times 15.94^2 + \frac{10 \times 75^3}{12} + 750 \times 26.56^2$$

i.e., $\mathbf{I_{yy} = 1208658.9 \text{ mm}^4}$

Polar moment of inertia $= I_{xx} + I_{yy}$

$= 3411298.9 + 12{,}08658.9$

$\mathbf{I_{zz} = 4619957.8 \text{ mm}^4}$

Example 14. Determine the moment of inertia of the symmertic I-section shown in Fig. 4.55 about its centroidal axis x-x and y-y.

Also, determine moment of inertia of the section about a centroidal axis perpendicular to x-x axis and y-y axis.

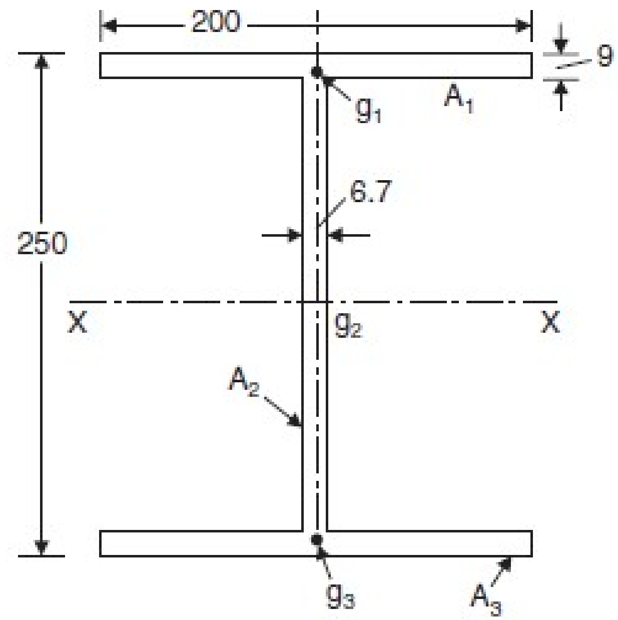


Fig. 4.55

Solution: The section is divided into three rectangles A_1, A_2 and A_3.

Area $A_1 = 200 \times 9 = 1800 \text{ mm}^2$

Area $A_2 = (250 - 9 \times 2) \times 6.7 = 1554.4 \text{ mm}^2$

Area $A_3 = 200 \times 9 = 1800 \text{ mm}^2$

Total Area $A = 5154.4 \text{ mm}^2$

The section is symmetrical about both x-x and y-y axis. Therefore, its centroid will coincide with the centroid of rectangle A_2.

With respect to the centroidal axis x-x and y-y, the centroid of rectangle A_1 is g_1 (0.0, 120.5), that of A_2 is g_2 (0.0, 0.0) and that of A_3 is g_3 (0.0, 120.5).

I_{xx} = Moment of inertia of A_1 + Moment of inertia of A_2 + Moment of inertia of A_3 about x-x axis

$$I_{xx} = \frac{200 \times 9^3}{12} + 1800 \times 120.5^2 + \frac{6.7 \times 232^3}{12} + 0 + \frac{200 \times 9^3}{12} + 1800(120.5)^2$$

$$\mathbf{I_{xx} = 5,92,69,202 \text{ mm}^4}$$

Similarly,

$$I_{xy} = \frac{9 \times 200^3}{12} + \frac{232 \times 6.7^3}{12} + \frac{9 \times 200^3}{12}$$

$$\boldsymbol{I_{yy} = 1{,}20{,}05{,}815 \text{ mm}^4}$$

Moment of inertia of the section about a centroidal axis perpendicular to *x-x* and *y-y* axis is nothing but polar moment of inertia, and is given by:

$$I_{xx} = I_{xx} + I_{yy}$$

$$= 59269202 + 12005815$$

$$\boldsymbol{I_{yy} = 7{,}12{,}75{,}017 \text{ mm}^4}$$

Example 4.22. Find moment of inertia of the shaded area shown in the Fig. 4.63 about the axis AB.

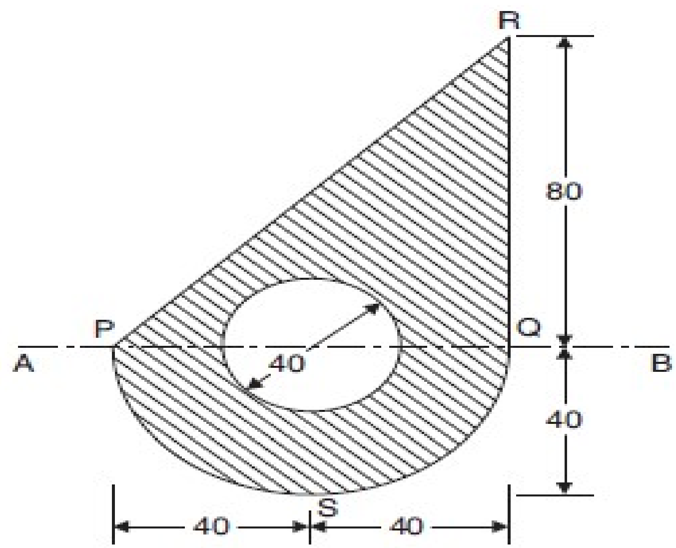


Fig. 4.63

Solution: The section is divided into a triangle *PQR*, a semicircle *PSQ* having base on axis *AB* and a circle having its centre on axis *AB*.

Now,

$$\left.\begin{array}{l}\text{Moment of inertia of the}\\ \text{section about axis } AB\end{array}\right\} = \left\{\begin{array}{l}\text{Moment of inertia of triangle } PQR \text{ about}\\ AB + \text{Moment of inertia of semicircle}\\ PSQ \text{ about } AB - \text{moment of inertia of}\\ \text{circle about } AB\end{array}\right.$$

$$= \frac{80 \times 80^3}{12} + \frac{\pi}{128} \times 80^4 - \frac{\pi}{64} \times 40^4$$

$$\mathbf{I_{AB} = 42{,}92{,}979\ mm^4.}$$