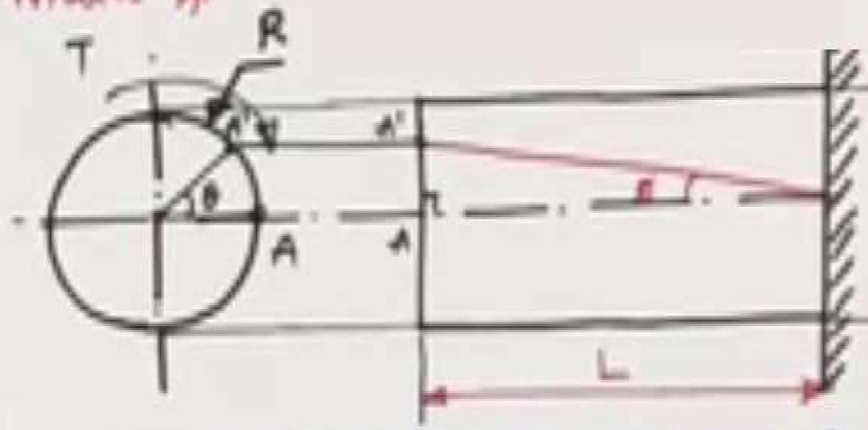
Torsion

A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft. The torque is equal to the product of the force applied (tangentially to the ends of a shaft) and radius of the shaft. Due to the application of the torques at the two ends, the shaft is subjected to a twisting moment. This causes the shear stresses and shear strains in the material of the shaft.

* Derivation of Torsional Formula *



∵ Length of arc AA' :-

$$\boxed{AA' = R\theta} \longrightarrow (1)$$

∴ $\tan\phi = \dfrac{AA'}{L}$

∵ ϕ is a very very small angle.

∴ $\tan\phi \simeq \phi$

∴ $\phi = \dfrac{AA'}{L}$

∴ $\phi = \dfrac{R\theta}{L}$

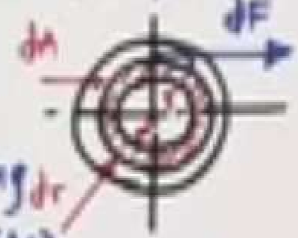
∵ $\phi = \dfrac{\text{Shear stress } (\tau)}{\text{Shear Modulus } (G)}$ $\simeq e = \dfrac{\sigma}{E}$

∴ $\left[\phi = \dfrac{\tau}{G}\right]$

∴ $\dfrac{\tau}{G} = \dfrac{R\theta}{L}$

∴ $$\boxed{\frac{\tau}{R} = \frac{G\theta}{L}} \longrightarrow (A)$$

→ Considering an elemental ring at radius (r) from the centre of shaft having small area (dA)



Small amount of shear stress acting on the ring.

$$\left[\tau = \frac{G\theta}{L}\, r\right]$$

→ Small amount of force acting on the ring :-

$$dF = \tau \cdot dA$$

$\therefore \quad dF = \frac{G\theta}{L} \cdot r \cdot dA$

$\therefore$ Small amount of Torque acting on the ring:-

$$dT = dF \cdot r$$

$$\therefore \quad dT = \frac{G\theta}{L} \cdot r\, dA \cdot r \Rightarrow \frac{G\theta}{L} r^2 dA$$

$\therefore$ Integrating to get total torque acting on the shaft of radius (R)

$$\therefore \quad \int dT = \frac{G\theta}{L} \int r^2 dA$$

$$\therefore \quad T = \frac{G\theta}{L} \cdot J \quad \left[\because J = \text{Polar Moment of inertia} \Rightarrow \int r^2 dA \right]$$

$$\therefore \quad \boxed{\frac{T}{J} = \frac{G\theta}{L}} \longrightarrow (B)$$

Equating (A) & (B)

$$\therefore \quad \frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

16.2.1. Assumptions Made in the Derivation of Shear Stress Produced in a Circular Shaft Subjected to Torsion. The derivation of shear stress produced in a circular shaft subjected to torsion, is based on the following assumptions :

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. The shaft is of uniform circular section throughout.
4. Cross-sections of the shaft, which are plane before twist remain plain after twist.
5. All radii which are straight before twist remain straight after twist.

16.3. MAXIMUM TORQUE TRANSMITTED BY A CIRCULAR SOLID SHAFT

The maximum torque transmitted by a circular solid shaft, is obtained from the maximum shear stress induced at the outer surface of the solid shaft. Consider a shaft subjected to a torque T as shown in Fig. 16.3.

Let τ = Maximum shear stress induced at the outer surface

R = Radius of the shaft

q = Shear stress at a radius 'r' from the centre.

Consider an elementary circular ring of thickness 'dr' at a distance 'r' from the centre as shown in Fig. 16.3. Then the area of the ring,

$$dA = 2\pi r dr$$

From equation (16.2), we have

$$\frac{\tau}{R} = \frac{q}{r}$$

$\therefore$ Shear stress at the radius r,

$$q = \frac{\tau}{R} r = \tau \frac{r}{R}$$

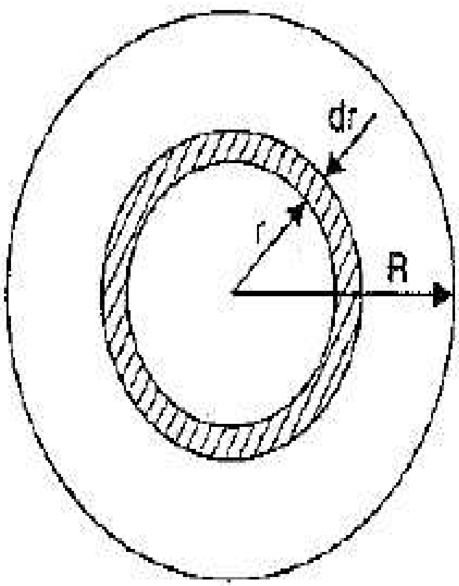


Fig. 16.3

$\therefore$ Turning force on the elementary circular ring

$$= \text{Shear stress acting on the ring} \times \text{Area of ring}$$

$$= q \times dA$$

$$= \tau \times \frac{r}{R} \times 2\pi r dr \qquad \left(\because \quad q = \tau \times \frac{r}{R}\right)$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr$$

Now turning moment due to the turning force on the elementary ring,

$$dT = \text{Turning force on the ring} \times \text{Distance of the ring from the axis}$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr \times r$$

$$= \frac{\tau}{R} \times 2\pi r^3 dr \qquad \text{...[16.3 (A)]}$$

$\therefore$ The total turning moment (or total torque) is obtained by integrating the above equation between the limits O and R

$$\therefore \qquad T = \int_0^R dT = \int_0^R \frac{\tau}{R} \times 2\pi r^3 dr$$

$$= \frac{\tau}{R} \times 2\pi \int_0^R r^3 dr = \frac{\tau}{R} \times 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\tau}{R} \times 2\pi \times \frac{R^4}{4} = \tau \times \frac{\pi}{2} \times R^3$$

$$= \tau \times \frac{\pi}{2} \times \left(\frac{D}{2}\right)^3 \qquad \left(\because \quad R = \frac{D}{2}\right)$$

$$= \tau \times \frac{\pi}{2} \times \frac{D^3}{8} = \tau \times \frac{\pi D^3}{16} = \frac{\pi}{16} \tau D^3 \qquad \text{...(16.4)}$$

16.4. TORQUE TRANSMITTED BY A HOLLOW CIRCULAR SHAFTS

Torque transmitted by a hollow circular shaft is obtained in the same way as for a solid shaft. Consider a hollow shaft. Let it is subjected to a torque T as shown in Fig. 16.4. Take an elementary circular ring of thickness 'dr' at a distance r from the centre as shown in Fig. 16.4.

Let R_0 = Outer radius of the shaft

R_i = Inner radius of the shaft

r = Radius of elementary circular ring

dr = Thickness of the ring

τ = Maximum shear stress induced at outer surface of the shaft

q = Shear stress induced on the elementary ring

dA = Area of the elementary circular ring

$= 2\pi r \times dr$

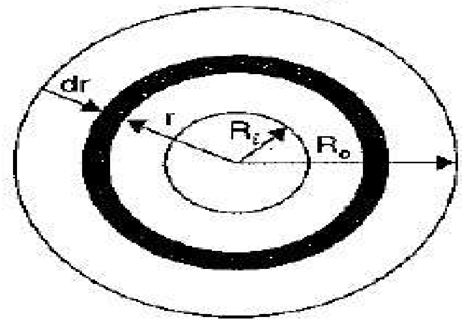


Fig. 16.4. Hollow shaft.

Shear stress at the elementary ring is obtained from equation (16.2) as

$$\frac{\tau}{R_0} = \frac{q}{r} \qquad (\because \text{ Here outer radius } R = R_0)$$

$$\therefore \quad q = \frac{\tau}{R_0} \times r$$

$\therefore$ Turning force on the ring = Stress × Area = $q \times dA$

$$= \frac{\tau}{R_0} r \times 2\pi r dr \qquad \left(\because \ q = \frac{\tau}{R_0} r\right)$$

$$= 2\pi \frac{\tau}{R_0} r^2 \, dr$$

Turning moment (dT) on the ring,

dT = Turning force × Distance of the ring from centre

$$= 2\pi \frac{\tau}{R_0} r^2 dr \times r = 2\pi \frac{\tau}{R_0} r^3 \, dr$$

The total turning moment (or total torque T) is obtained by integrating the above equation between the limits R_i and R_0.

$$\therefore \quad T = \int_{R_i}^{R_0} dT = \int_{R_i}^{R_0} 2\pi \frac{\tau}{R_0} r^3 \, dr$$

$$= 2\pi \frac{\tau}{R_0} \int_{R_i}^{R_0} r^3 \, dr$$

($\because$ τ and R_0 are constant and can be taken outside the integral)

$$= 2\pi \frac{\tau}{R_0}\left[\frac{r^4}{4}\right]_{R_i}^{R_0} = 2\pi \frac{\tau}{R_0}\left[\frac{R_0^4 - R_i^4}{4}\right]$$

$$= \frac{\pi}{2}\tau\left[\frac{R_0^4 - R_i^4}{R_0}\right] \qquad \text{...(16.5)}$$

Let D_0 = Outer diameter of the shaft

D_i = Inner diameter of the shaft.

Then $$R_0 = \frac{D_0}{2} \quad \text{and} \quad R_i = \frac{D_i}{2}.$$

Substituting the values of R_0 and R_i in equation (16.5),

$$T = \frac{\pi}{2}\tau\left[\frac{\left(\frac{D_0}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4}{\left(\frac{D_0}{2}\right)}\right] = \frac{\pi}{2}\tau\left[\frac{\frac{D_0^4}{16} - \frac{D_i^4}{16}}{\frac{D_0}{2}}\right]$$

$$= \frac{\pi}{2}\tau\left[\frac{D_0^4 - D_i^4}{16} \times \frac{2}{D_0}\right]$$

$$= \frac{\pi}{16}\tau\left[\frac{D_0^4 - D_i^4}{D_0}\right] \qquad \text{...(16.6)}$$

Problems

Problem 16.2. *The shearing stress is a solid shaft is not to exceed 40 N/mm² when the torque transmitted is 20000 N-m. Determine the minimum diameter of the shaft.*

Sol. Given :

Maximum shear stress, $\tau = 40 \text{ N/mm}^2$

Torque transmitted, $T = 20000 \text{ N-m} = 20000 \times 10^3 \text{ N-mm}$

Let D = Minimum diameter of the shaft in mm.

Using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$D = \left(\frac{16T}{\pi\tau}\right)^{1/3} = \left(\frac{16 \times 20000 \times 10^3}{\pi \times 40}\right)^{1/3} = \mathbf{136.2 \text{ mm.}} \quad \textbf{Ans.}$$

or

Problems

Problem 16.3. *In a hollow circular shaft of outer and inner diameters of 20 cm and 10 cm respectively, the shear stress is not to exceed 40 N / mm². Find the maximum torque which the shaft can safely transmit.*

Sol. Given :

Outer diameter, $D_0 = 20 \text{ cm} = 200 \text{ mm}$

Inner diameter, $D_i = 10 \text{ cm} = 100 \text{ mm}$

Maximum shear stress, $\tau = 40 \text{ N/mm}^2$

Let T = Maximum torque transmitted by the shaft.

Using equation (16.6),

$$T = \frac{\pi}{16}\tau\left[\frac{D_0^{\ 4} - D_i^{\ 4}}{D_0}\right] = \frac{\pi}{16} \times 40\left[\frac{200^4 - 100^4}{200}\right]$$

$$= \frac{\pi}{16} \times 40\left[\frac{16 \times 10^8 - 1 \times 10^8}{200}\right] = 58904860 \text{ Nmm}$$

$$= \mathbf{58904.86 \text{ Nm.}} \quad \textbf{Ans.}$$

Problems

Problem 16.4. *Two shafts of the same material and of same lengths are subjected to the same torque, if the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal diameter is 2/3 of the outside diameter and the maximum shear stress developed in each shaft is the same, compare the weights of the shafts.*

Problems

Sol. Given :

Two shafts of the same material and same lengths (one is solid and other is hollow) transmit the same torque and develops the same maximum stress.

Let T = Torque transmitted by each shaft

τ = Max. shear stress developed in each shaft

D = Outer diameter of the solid shaft

D_0 = Outer diameter of the hollow shaft

D_i = Inner diameter of the hollow shaft = $\frac{2}{3} D_0$

W_s = Weight of the solid shaft

W_h = Weight of the hollow shaft

L = Length of each shaft

w = Weight density of the material of each shaft.

Torque transmitted by the solid shaft is given by equation (16.4)

$$T = \frac{\pi}{16} \tau D^3 \quad \text{...}(i)$$

Torque transmitted by the hollow shaft is given by equation (16.6),

$$T = \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0}\right] = \frac{\pi}{16} \tau \left[\frac{D_0^4 - (2/3\, D_0)^4}{D_0}\right]$$

$$= \frac{\pi}{16} \tau \left[\frac{D_0^4 - \frac{16}{81} D_0^4}{D_0}\right] = \frac{\pi}{16} \tau \times \frac{65\, D_0^4}{81 \times D_0}$$

$$= \frac{\pi}{16} \tau \times \frac{65\, D_0^3}{81} \quad \text{...}(ii)$$

Problems

As torque transmitted by solid and hollow shafts are equal, hence equating equations (*i*) and (*ii*),

$$\frac{\pi}{16}\tau D^3 = \frac{\pi}{16}\tau \times \frac{65}{81} D_0{}^3$$

Cancelling $\frac{\pi}{16}\tau$ to both sides

or $$D^3 = \frac{65}{81} D_0{}^3$$

Problems

$$\therefore \quad D = \left[\frac{65}{81} D_0^{\,3}\right]^{1/3} = \left(\frac{65}{81}\right)^{1/3} D_0 = 0.929\, D_0 \qquad \text{...}(iii)$$

Now weight of solid shaft, W_s = Weight density × Volume of solid shaft

= w × Area of cross-section × Length

$$= w \times \frac{\pi}{4} D^2 \times L \qquad \text{...}(iv)$$

Weight of hollow shaft,

W_h = w × Area of cross-section of hollow shaft × Length

$$= w \times \frac{\pi}{4} [D_0^{\,2} - D_i^{\,2}] \times L = w \times \frac{\pi}{4} [D_0^{\,2} - (2/3\, D_0)^2] \times L$$

$$= w \times \frac{\pi}{4}\left[D_0^{\,2} - \frac{4}{9} D_0^{\,2}\right] \times L = w \times \frac{\pi}{4} \times \frac{5}{9} D_0^{\,2} \times L \qquad \text{...}(v)$$

Dividing equation (iv) by equation (v),

$$\frac{W_s}{W_h} = \frac{w \times \frac{\pi}{4} D^2 \times L}{w \times \frac{\pi}{4} \times \frac{5}{9} D_0^{\,2} \times L} = \frac{9D^2}{5D_0^{\,2}}$$

$$= \frac{9}{5} \times \frac{(0.929 D_0)^2}{D_0^{\,2}} \qquad [\because \quad D = 0.929\, D_0 \text{ from equation } (iii)]$$

$$= \frac{9}{5} \times 0.929^2 \times \frac{D_0^{\,2}}{D_0^{\,2}} = \frac{1.55}{1}$$

$$\therefore \quad \frac{\text{Weight of solid shaft}}{\text{Weight of hollow shaft}} = \frac{\mathbf{1.55}}{\mathbf{1}}. \quad \textbf{Ans.}$$

Problems

Problem 16.7. *A hollow shaft of external diameter 120 mm transmits 300 kW power at 200 r.p.m. Determine the maximum internal diameter if the maximum stress in the shaft is not to exceed 60 N/mm².* (AMIE, Summer 1990)

Sol. Given :

External dia., $D_0 = 120$ mm

Power, $P = 300$ kW $= 300{,}000$ W

Speed, $N = 200$ r.p.m.

Max. shear stress, $\tau = 60$ N/mm²

Let D_i = Internal dia. of shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60} \quad \text{or} \quad 300{,}000 = \frac{2\pi \times 200 \times T}{60}$$

$$\therefore \quad T = \frac{300{,}000 \times 60}{2\pi \times 200} = 14323.9 \text{ Nm}$$

$$= 14323.9 \times 1000 \text{ Nmm} = 14323900 \text{ Nmm}$$

Now using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$\text{or} \quad 14323900 = \frac{\pi}{16} \times 60 \times \frac{(120^4 - D_i^4)}{120}$$

$$\text{or} \quad \frac{14323900 \times 16 \times 120}{\pi \times 60} = 120^4 - D_i^4$$

$$145902000 = 207360000 - D_i^4$$

$$\text{or} \quad D_i^4 = 207360000 - 145902000 = 61458000$$

$$\therefore \quad D_i = (61458000)^{1/4} = \mathbf{88.5 \text{ mm.}} \quad \textbf{Ans.}$$

Problems

Problem 16.13. *Determine the diameter of a solid steel shaft which will transmit 90 kW at 160 r.p.m. Also determine the length of the shaft if the twist must not exceed 1° over the entire length. The maximum shear stress is limited to 60 N/mm². Take the value of modulus of rigidity = 8 × 10⁴ N/mm².*

Sol. Given :

Power, $P = 90 \text{ kW} = 90 \times 10^3 \text{ W}$

Speed, $N = 160 \text{ r.p.m.}$

Angle of twist, $\theta = 1^\circ$ or $\frac{\theta}{180}$ radian $\left(\because \; 1^\circ = \frac{\pi}{180} \text{ radian}\right)$

Max. shear stress, $\tau = 60 \text{ N/mm}^2$

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let D = Diameter of the shaft and

L = Length of the shaft.

Problems

(i) Diameter of the shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60}$$

or

$$90 \times 10^3 = \frac{2\pi \times 160 \times T}{60}$$

$\therefore$

$$T = \frac{90 \times 10^3 \times 60}{2\pi \times 160} = 5371.48 \text{ N-m} = 5371.48 \times 10^3 \text{ N-mm}$$

Now using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$5371.48 \times 10^3 = \frac{\pi}{16} \times 60 \times D^3$$

$\therefore$

$$D^3 = \frac{5371.48 \times 10^3 \times 16}{\pi \times 60} = 455945$$

$\therefore$

$$D = (455945)^{1/3} = \mathbf{76.8\ mm.} \quad \textbf{Ans.}$$

Problems

(ii) Length of the shaft

Using equation (16.7),

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

or $$\frac{60}{\left(\frac{76.8}{2}\right)} = \frac{8 \times 10^4 \times \pi}{L \times 180} \qquad \left(\because \; R = \frac{D}{2} = \frac{76.8}{2} \text{ mm}, \theta = \frac{\pi}{180} \text{ radian}\right)$$

or $$L = \frac{8 \times 10^4 \times \pi \times 76.8}{60 \times 180 \times 2} = \mathbf{893.6 \text{ mm.}} \quad \textbf{Ans.}$$