CENTRE OF GRAVITY

CENTROID

CENTRE OF GRAVITY

Consider the suspended body shown in Fig. 4.10(*a*). The self weight of various parts of this body are acting vertically downward. The only upward force is the force *T* in the string. To satisfy the equilibrium condition the resultant weight of the body. *W* must act along the line of string 1–1. Now, if the position is changed and the body is suspended again (Fig. 4.10(*b*)), it will reach equilibrium condition in a particular position. Let the line of action of the resultant weight be 2–2 intersecting 1–1 at *G*. It is obvious that if the body is suspended in any other position, the line of action of resultant weight *W* passes through *G*. This point is called the centre of gravity of the body. Thus *centre of gravity can be defined as the point through which the resultant of force of gravity of the body acts.*

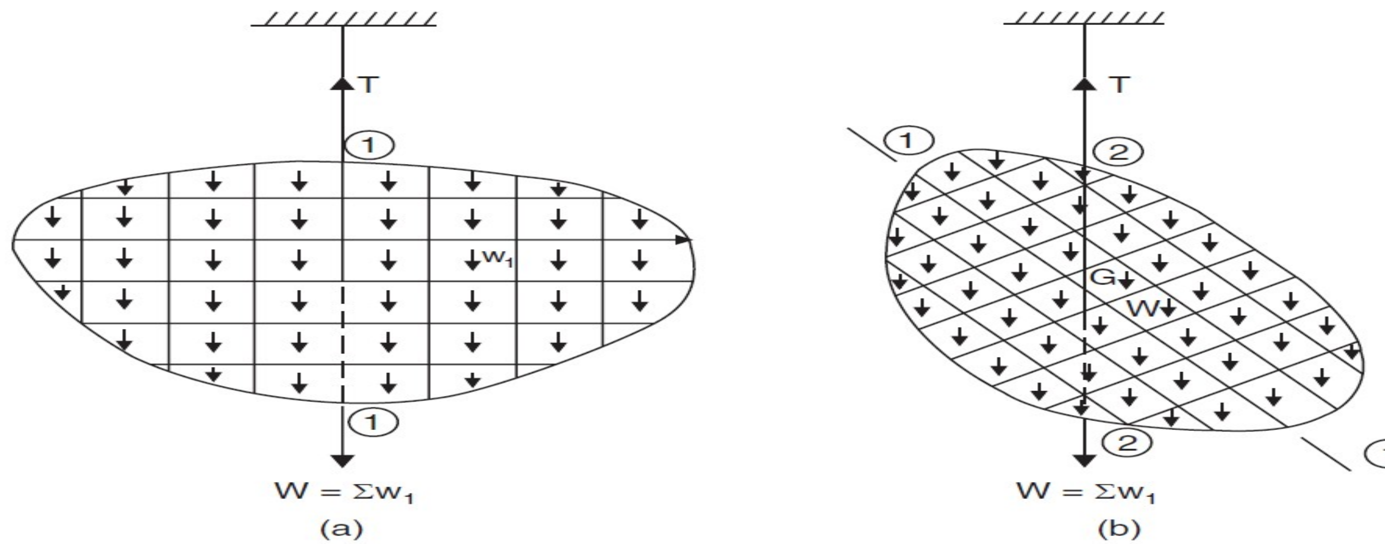


Fig. 4.10

The above method of locating centre of gravity is the practical method. If one desires to locating centre of gravity of a body analytically, it is to be noted that the resultant of weight of various portions of the body is to be determined. For this Varignon's theorem, which states the moment of resultant force is equal to the sum of moments of component forces, can be used.

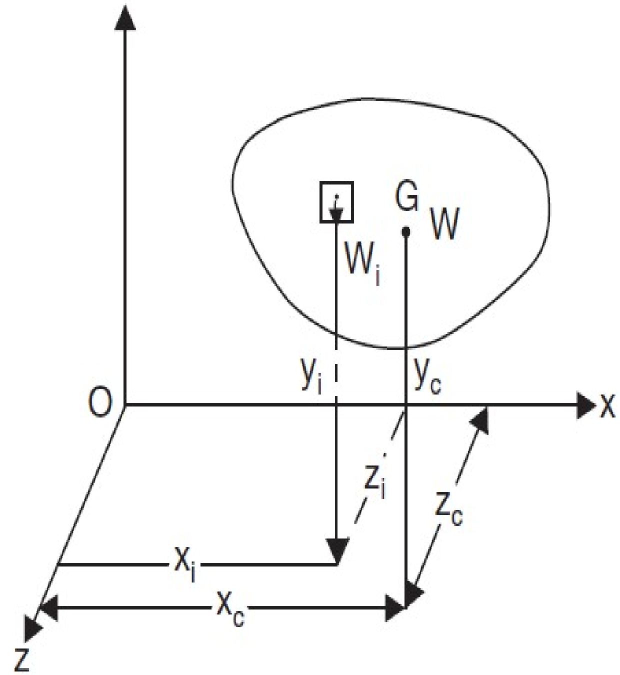


Fig. 4.11

Referring to Fig. 4.11, let W_i be the weight of an element in the given body. W be the total weight of the body. Let the coordinates of the element be x_i, y_i, z_i and that of centroid G be x_c, y_c, z_c. Since W is the resultant of W_i forces,

$$W = W_1 + W_2 + W_3 + \cdots$$
$$= \Sigma W_i$$

and
$$Wx_c = W_1x_1 + W_2x_2 + W_3x_3 + \cdots$$

$$\left.\begin{aligned} \therefore \quad & Wx_c = \Sigma W_i x_i = \oint x\,dw \\ \text{Similarly,} \quad & Wy_c = \Sigma W_i y_i = \oint y\,dw \\ \text{and} \quad & Wz_c = \Sigma W_i z_i = \oint z\,dw \end{aligned}\right\} \qquad \ldots(4.1)$$

If M is the mass of the body and m_i that of the element, then

$M = \dfrac{W}{g}$ and $m_i = \dfrac{W_i}{g}$, hence we get

$$\left.\begin{aligned} Mx_c &= \Sigma m_i x_i = \oint x_i dm \\ My_c &= \Sigma m_i y_i = \oint y_i dm \\ \text{and} \qquad Mz_c &= \Sigma m_i z_i = \oint z_i dm \end{aligned}\right\} \qquad \text{...(4.2)}$$

If the body is made up of uniform material of unit weight γ, then we know $W_i = V_i \gamma$, where V represents volume, then equation 4.1 reduces to

$$\left.\begin{aligned} Vx_c &= \Sigma V_i x_i = \oint x dV \\ Vy_c &= \Sigma V_i y_i = \oint y dV \\ Vz_c &= \Sigma V_i z_i = \oint z dV \end{aligned}\right\} \qquad \text{...(4.3)}$$

If the body is a flat plate of uniform thickness, in x-y plane, $W_i = \gamma A_i t$ (Ref. Fig. 4.12). Hence equation 4.1 reduces to

$$\left.\begin{aligned} Ax_c &= \Sigma A_i x_i = \oint x\, dA \\ Ay_c &= \Sigma A_i y_i = \oint y\, dA \end{aligned}\right\} \qquad \text{...(4.4)}$$

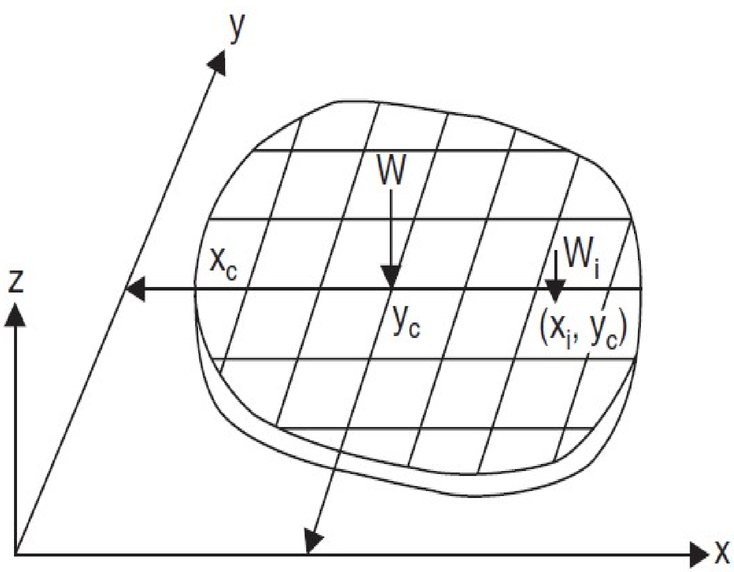


Fig. 4.12

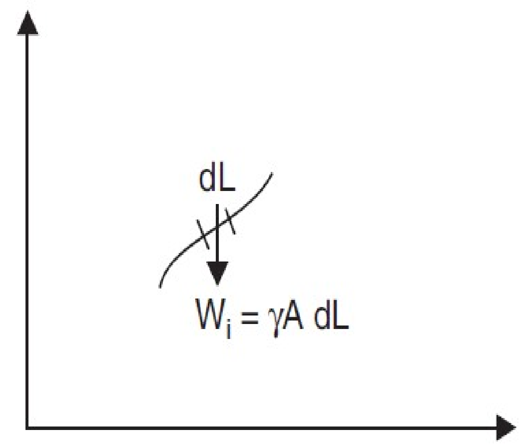


Fig. 4.13

If the body is a wire of uniform cross-section in plane x, y (Ref. Fig. 4.13) the equation 4.1 reduces to

$$\left.\begin{aligned} Lx_c &= \Sigma L_i x_i = \oint x\, dL \\ Ly_c &= \Sigma L_i y_i = \oint y\, dL \end{aligned}\right\} \qquad ...(4.5)$$

The term centre of gravity is used only when the gravitational forces (weights) are considered. This term is applicable to solids. Equations 4.2 in which only masses are used the point obtained is termed as *centre of mass*. The central points obtained for volumes, surfaces and line segments (obtained by eqn. 4.3, 4.4 and 4.5) are termed as *centroids*.

Difference between Centre of Gravity and Centroid

From the above discussion we can draw the following differences between centre of gravity and centroid:

(1) The term centre of gravity applies to bodies with weight, and centroid applies to lines, plane areas and volumes.

(2) Centre of gravity of a body is a point through which the resultant gravitational force (weight) acts for any orientation of the body whereas centroid is a point in a line plane area volume such that the moment of area about any axis through that point is zero.

4.3 CENTROID OF A LINE

Centroid of a line can be determined using equation 4.5. Method of finding the centroid of a line for some standard cases is illustrated below:

(*i*) ***Centroid of a straight line***

Selecting the x-coordinate along the line (Fig. 4.14)

$$Lx_c = \int_0^L x\,dx = \left[\frac{x^2}{2}\right]_0^L = \frac{L^2}{2}$$

$$\therefore \qquad x_c = \frac{L}{2}$$

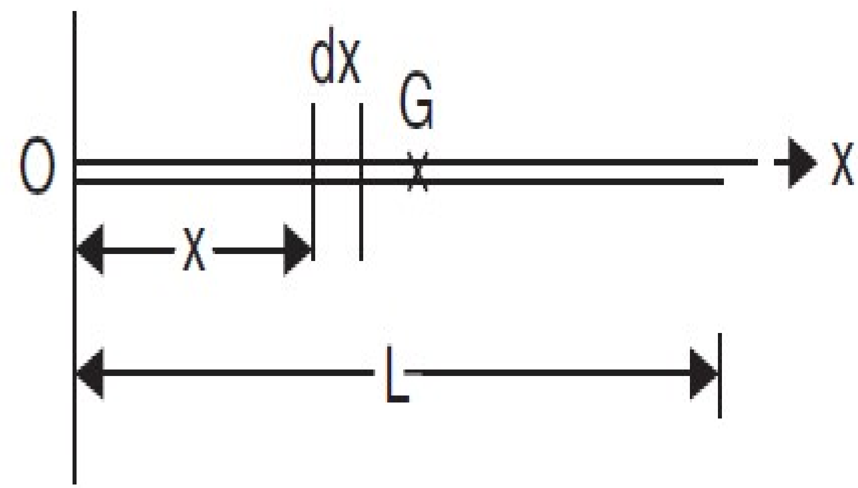


Fig. 4.14

Thus the centroid lies at midpoint of a straight line, whatever be the orientation of line (Ref. Fig. 4.15).

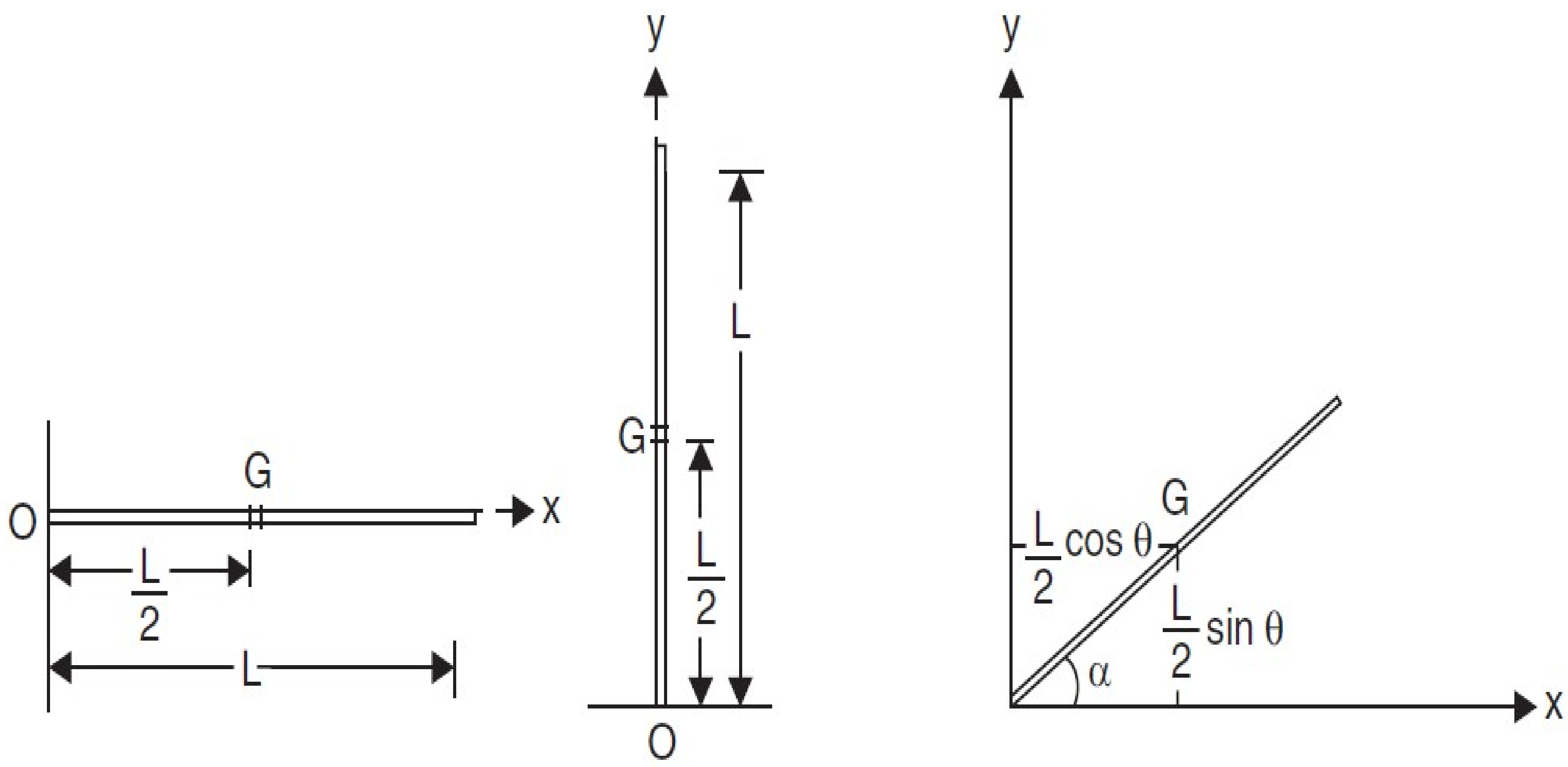


Fig. 4.15

(*ii*) ***Centroid of an arc of a circle***

Referring to Fig. 4.16,

$$L = \text{Length of arc} = R\,2\alpha$$

$$dL = Rd\theta$$

Hence from eqn. 4.5

$$x_c L = \int_{-\alpha}^{\alpha} x\,dL$$

i.e.,

$$x_c\,R\,2\alpha = \int_{-\alpha}^{\alpha} R\cos\theta \cdot Rd\theta$$

$$= R^2\Big[\sin\theta\Big]_{-\alpha}^{\alpha}$$

$\therefore$

$$x_c = \frac{R^2 \times 2\sin\alpha}{2R\alpha} = \frac{R\sin\alpha}{\alpha}$$

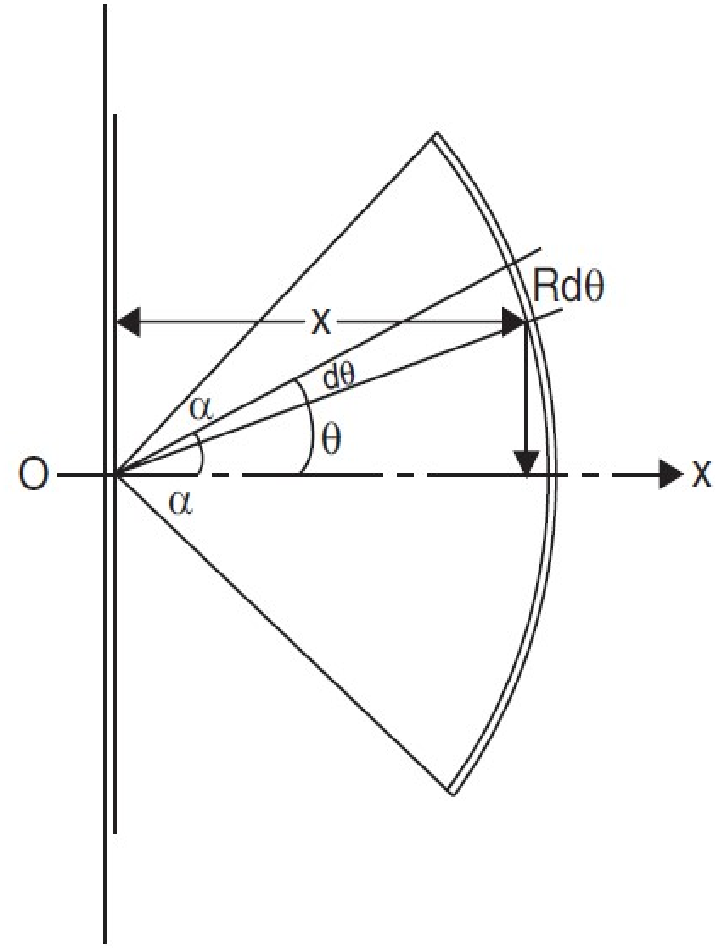


Fig. 4.16

and $$y_c L \int_{-\alpha}^{\alpha} y\, dL = \int_{-\alpha}^{\alpha} R \sin\theta \cdot R d\theta$$

$$= R^2 \Big[\cos\theta \Big]_{-\alpha}^{\alpha} \qquad \ldots(ii)$$

$$= 0$$

$\therefore$ $$y_c = 0$$

From equation (i) and (ii) we can get the centroid of semicircle shown in Fig. 4.17 by putting $\alpha = \pi/2$ and for quarter of a circle shown in Fig. 4.18 by putting α varying from zero to $\pi/2$.

For semicircle $$x_c = \frac{2R}{\pi}$$

$$y_c = 0$$

For quarter of a circle,

$$x_c = \frac{2R}{\pi}$$

$$y_c = \frac{2R}{\pi}$$

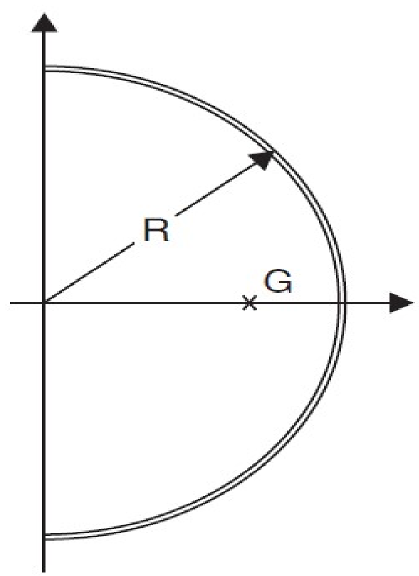


Fig. 4.17

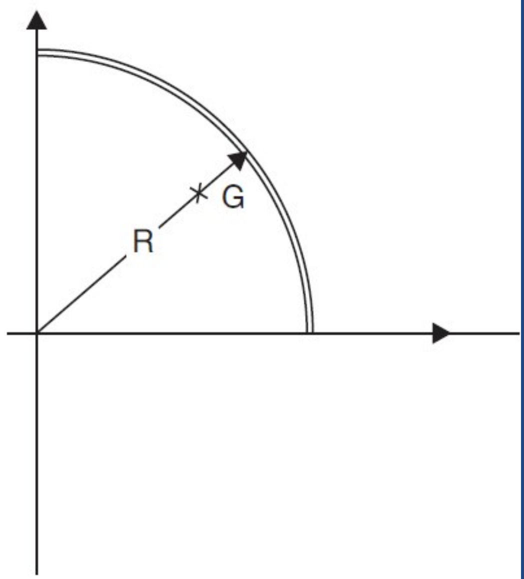


Fig. 4.18

(iii) ***Centroid of composite line segments***

The results obtained for standard cases may be used for various segments and then the equations 4.5 in the form

$$x_c L = \Sigma L_i x_i$$

$$y_c L = \Sigma L_i y_i$$

may be used to get centroid x_c and y_c. If the line segments is in space the expression $z_c L = \Sigma L_i z_i$ may also be used. The method is illustrated with few examples below:

Example 4.1. *Determine the centroid of the wire shown in Fig. 4.19.*

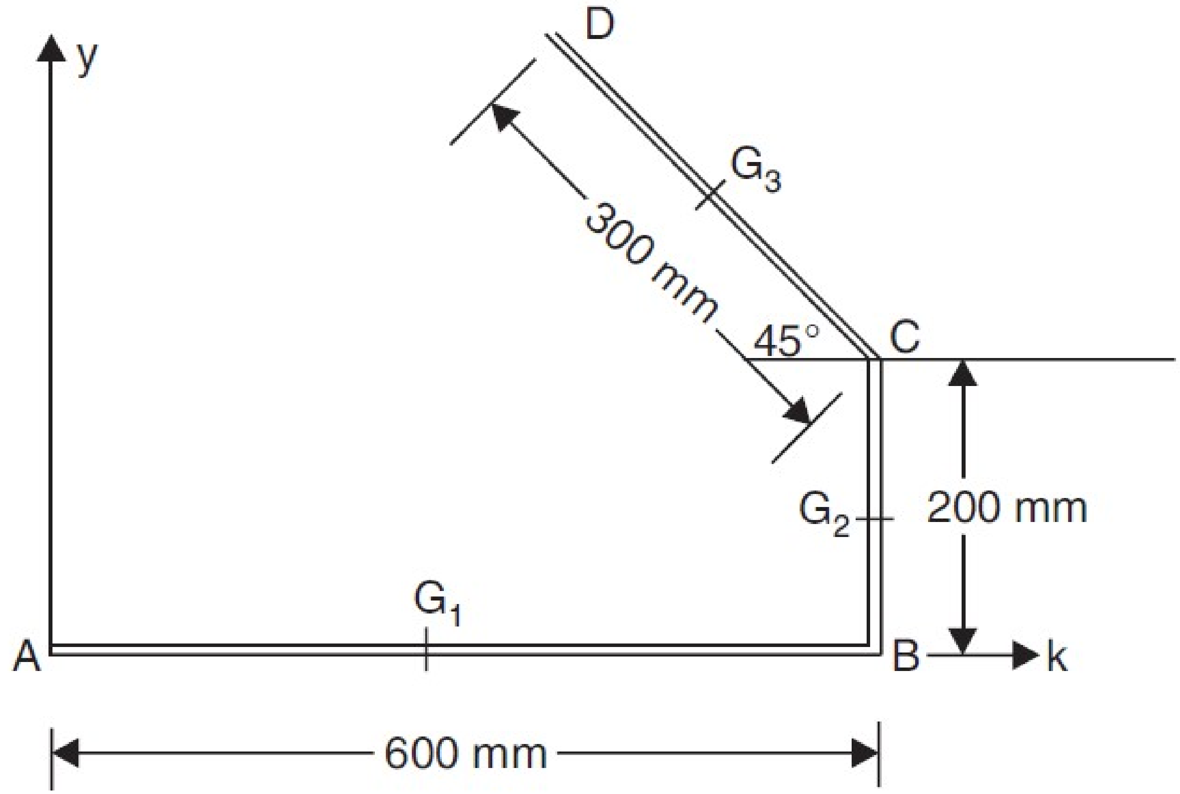


Fig. 4.19

Solution: The wire is divided into three segments *AB, BC* and *CD*. Taking *A* as origin the coordinates of the centroids of *AB, BC* and *CD* are

$$G_1(300, 0);\ G_2(600, 100) \text{ and } G_3\ (600 - 150 \cos 45°,\ 200 + 150 \sin 45°)$$

i.e., $G_3\ (493.93,\ 306.07)$

$$L_1 = 600 \text{ mm},\ L_2 = 200 \text{ mm},\ L_3 = 300 \text{ mm}$$

$\therefore$ Total length $L = 600 + 200 + 300 = 1100$ mm

$\therefore$ From the eqn. $Lx_c = \Sigma L_i x_i$, we get

$$1100\ x_c = L_1 x_1 + L_2 x_2 + L_3 x_3$$

$$= 600 \times 300 + 200 \times 600 + 300 \times 493.93$$

$\therefore$ $\mathbf{x_c = 407.44}$ **mm**

Now, $Ly_c = \Sigma L_i y_i$

$$1100\ y_c = 600 \times 0 + 200 \times 100 + 300 \times 306.07$$

$\therefore$ $\mathbf{y_c = 101.66}$ **mm**

Centroid of a Triangle

Consider the triangle ABC of base width b and height h as shown in Fig. 4.25. Let us locate the distance of centroid from the base. Let b_1 be the width of elemental strip of thickness dy at a distance y from the base. Since ΔAEF and ΔABC are similar triangles, we can write:

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left(\frac{h-y}{h}\right)b = \left(1-\frac{y}{h}\right)b$$

$\therefore$ Area of the element

$$= dA = b_1 dy$$

$$= \left(1-\frac{y}{h}\right)b\ dy$$

Area of the triangle $A = \frac{1}{2}bh$

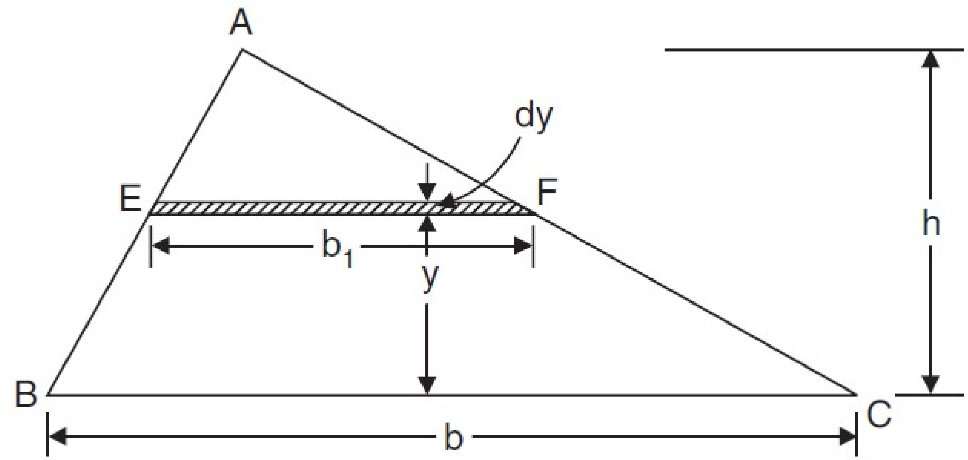


Fig. 4.25

$\therefore$ From eqn. 4.4

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\int y dA}{A}$$

Now,

$$\int y dA = \int_0^h y\left(1-\frac{y}{h}\right)b\ dy$$

$$= \int_0^h \left(y - \frac{y^2}{h} \right) b \, dy$$

$$= b \left[\frac{y^2}{2} - \frac{y^3}{3h} \right]_0^h$$

$$= \frac{bh^2}{6}$$

$$\therefore \qquad \bar{y} = \frac{\int y dA}{A} = \frac{bh^2}{6} \times \frac{1}{\frac{1}{2} bh}$$

$$\therefore \qquad \bar{y} = \frac{h}{3}$$

Thus the centroid of a triangle is at a distance $\frac{h}{3}$ from the base (or $\frac{2h}{3}$ from the apex) of the triangle, where h is the height of the triangle.

Centroid of a Semicircle

Consider the semicircle of radius R as shown in Fig. 4.26. Due to symmetry centroid must lie on y axis. Let its distance from diametral axis be $\bar{y}$. To find $\bar{y}$, consider an element at a distance r from the centre O of the semicircle, radial width being dr and bound by radii at θ and $\theta + d\theta$.

Area of element $= r\, d\theta\, dr$.

Its moment about diametral axis x is given by:

$$r d\theta \times dr \times r \sin\theta = r^2 \sin\theta\, dr\, d\theta$$

$\therefore$ Total moment of area about diametral axis,

$$\int_0^{\pi}\int_0^{R} r^2 \sin\theta\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^R \sin\theta\, d\theta$$

$$= \frac{R^3}{3}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{R^3}{3}[1+1] = \frac{2R^3}{3}$$

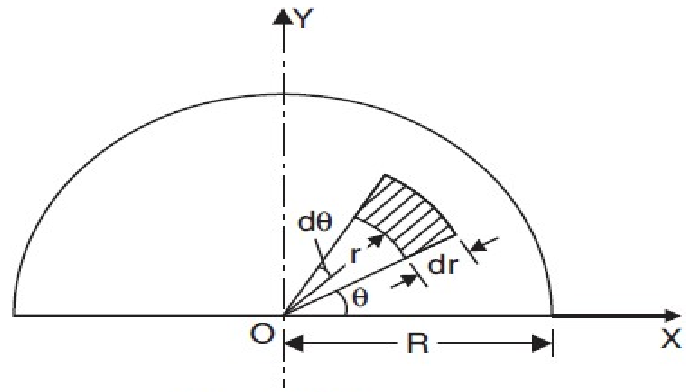


Fig. 4.26

Area of semicircle $\quad A = \frac{1}{2}\pi R^2$

$$\therefore \qquad \bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2}$$

$$= \frac{4R}{3\pi}$$

Thus, the centroid of the circle is at a distance $\frac{4R}{3\pi}$ from the diametral axis.

Centroid of Sector of a Circle

Consider the sector of a circle of angle 2α as shown in Fig. 4.27. Due to symmetry, centroid lies on x axis. To find its distance from the centre O, consider the elemental area shown.

Area of the element $= rd\theta\, dr$

Its moment about y axis

$$= rd\theta \times dr \times r \cos\theta$$

$$= r^2 \cos\theta\, drd\theta$$

$\therefore$ Total moment of area about y axis

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r^2 \cos\theta\, drd\theta$$

$$= \left[\frac{r^3}{3}\right]_0^R \left[\sin\theta\right]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3}\, 2\sin\alpha$$

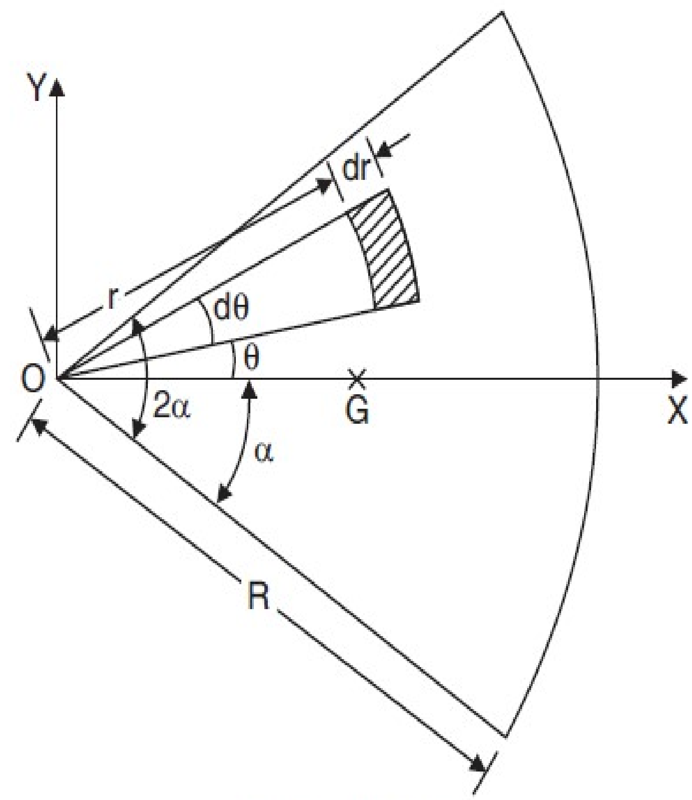


Fig. 4.27

Total area of the sector

$$= \int_{-\alpha}^{\alpha} \int_{0}^{R} r dr d\theta$$

$$= \int_{-\alpha}^{\alpha} \left[\frac{r^2}{2} \right]_0^R d\theta$$

$$= \frac{R^2}{2} \left[\theta \right]_{-\alpha}^{\alpha}$$

$$= R^2 \alpha$$

$\therefore$ The distance of centroid from centre O

$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$= \frac{\frac{2R^3}{3} \sin \alpha}{R^2 \alpha} = \frac{2R}{3\alpha} \sin \alpha$$

Centroid of Parabolic Spandrel

Consider the parabolic spandrel shown in Fig. 4.28. Height of the element at a distance x from O is $y = kx^2$

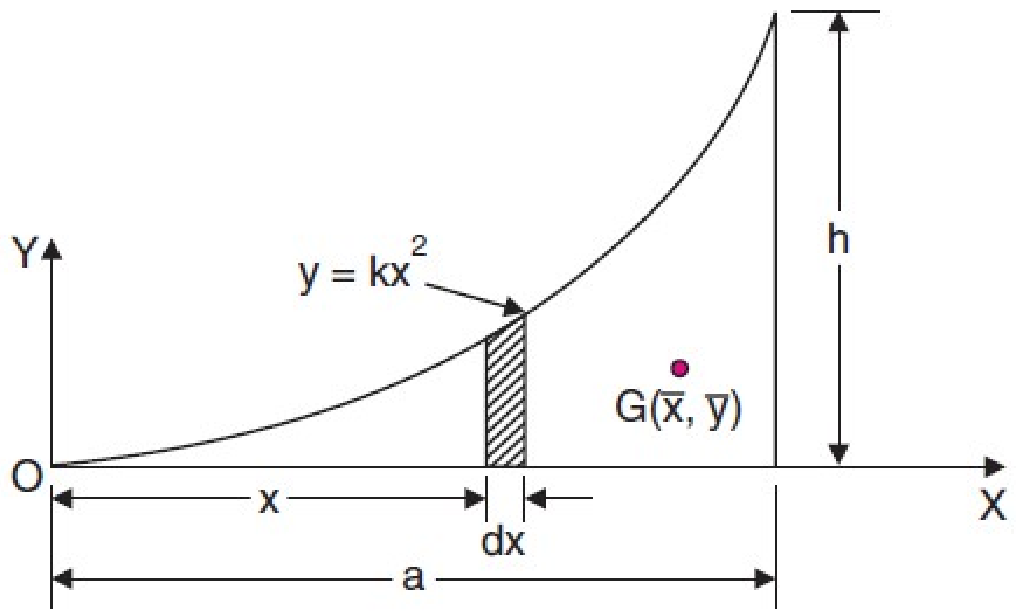


Fig. 4.28

Width of element $= dx$

$\therefore$ Area of the element $= kx^2dx$

$\therefore$ Total area of spandrel $= \int_0^a kx^2dx = \left[\frac{kx^3}{3}\right]_0^a$

$$= \frac{ka^3}{3}$$

Moment of area about y axis

$$= \int_0^a kx^2dx \times x$$

$$= \int_0^a kx^3\, dx$$

$$= \left[\frac{k\,x^4}{4}\right]_0^a$$

$$= \frac{ka^4}{4}$$

Moment of area about x axis $= \int_0^a dA \cdot \frac{y}{2}$

$$= \int_0^a kx^2 dx \ \frac{kx^2}{2} = \int_0^a \frac{k^2x^4}{2}\, dx$$

$$= \frac{k^2a^5}{10}$$

$$\therefore \qquad \bar{x} = \frac{ka^4}{4} \div \frac{ka^3}{3} = \frac{3a}{4}$$

$$\bar{y} = \frac{k^2a^5}{10} \div \frac{ka^3}{3} = \frac{3}{10}\ ka^2$$

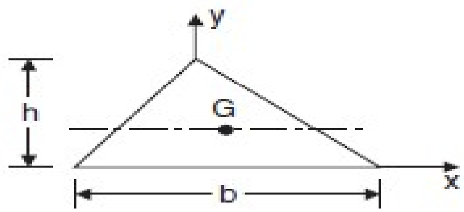
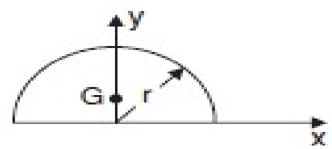
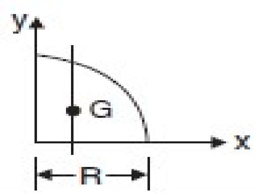
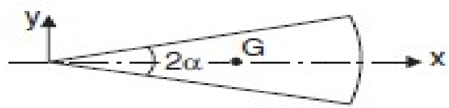
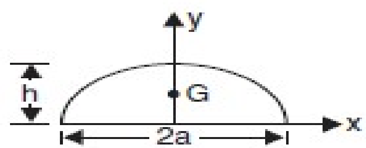
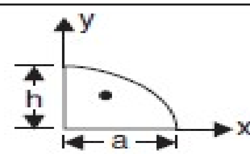
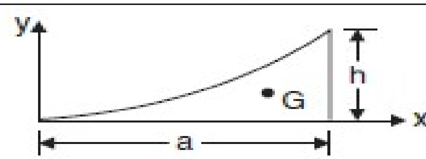
From the Fig. 4.28, at $x = a$, $y = h$

$$\therefore \qquad h = ka^2 \text{ or } k = \frac{h}{a^2}$$

$$\therefore \qquad \bar{y} = \frac{3}{10} \times \frac{h}{a^2} a^2 = \frac{3h}{10}$$

Thus, centroid of spandrel is $\left(\frac{3a}{4}, \frac{3h}{10}\right)$

Table 4.2 Centroid of Some Common Figures

Shape	*Figure*	$\bar{x}$	$\bar{y}$	*Area*
Triangle		—	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha}$ sin a	0	αR^2
Parabola		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Semi parabola		$\frac{3}{8}a$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

Example 4.4. Locate the centroid of the T-section shown in the Fig. 4.29.

Solution: Selecting the axis as shown in Fig. 4.29, we can say due to symmetry centroid lies on y axis, *i.e.* $\bar{x} = 0$. Now the given *T*-section may be divided into two rectangles A_1 and A_2 each of size 100×20 and 20×100. The centroid of A_1 and A_2 are $g_1(0, 10)$ and $g_2(0, 70)$ respectively.

$\therefore$ The distance of centroid from top is given by:

$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

$= 40$ mm

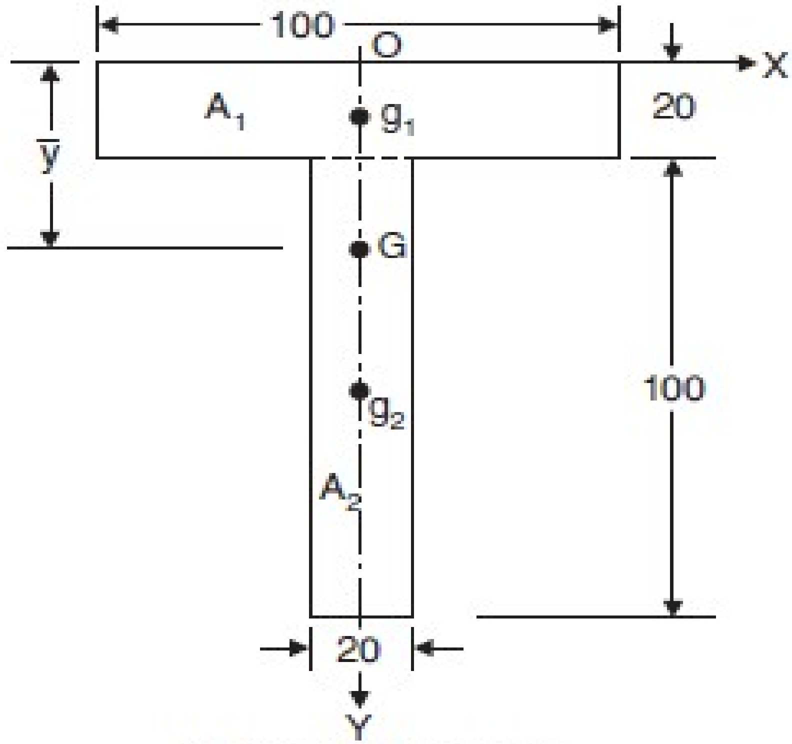


Fig. 4.29

Hence, centroid of T-section is on the symmetric axis at a distance 40 mm from the top.

Example 4.5. *Find the centroid of the unequal angle 200 × 150 × 12 mm, shown in Fig. 4.30.*

Solution: The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

Total area $\quad A = A_1 + A_2 = 4056 \text{ mm}^2$

Selecting the reference axis x and y as shown in Fig. 4.30. The centroid of A_1 is g_1 (75, 6) and that of A_2 is:

$$g_2\left[6, 12 + \frac{1}{2}(200 - 12)\right]$$

i.e., $\quad g_2$ (6, 106)

$$\therefore \quad \bar{x} = \frac{\text{Moment about } y \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1x_1 + A_2x_2}{A}$$

$$= \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm}$$

$$\bar{y} = \frac{\text{Moment about } x \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1y_1 + A_2y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm}$$

Thus, the centroid is at $\bar{x}$ = 36.62 mm and $\bar{y}$ = 61.62 mm as shown in the figure.

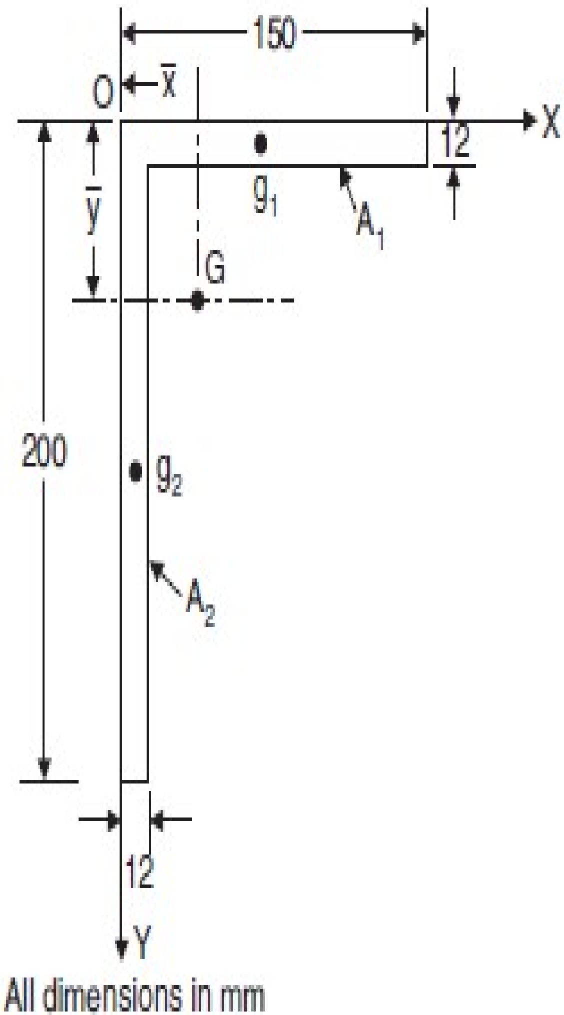


Fig. 4.30

Example 4.6. Locate the centroid of the I-section shown in Fig. 4.31.

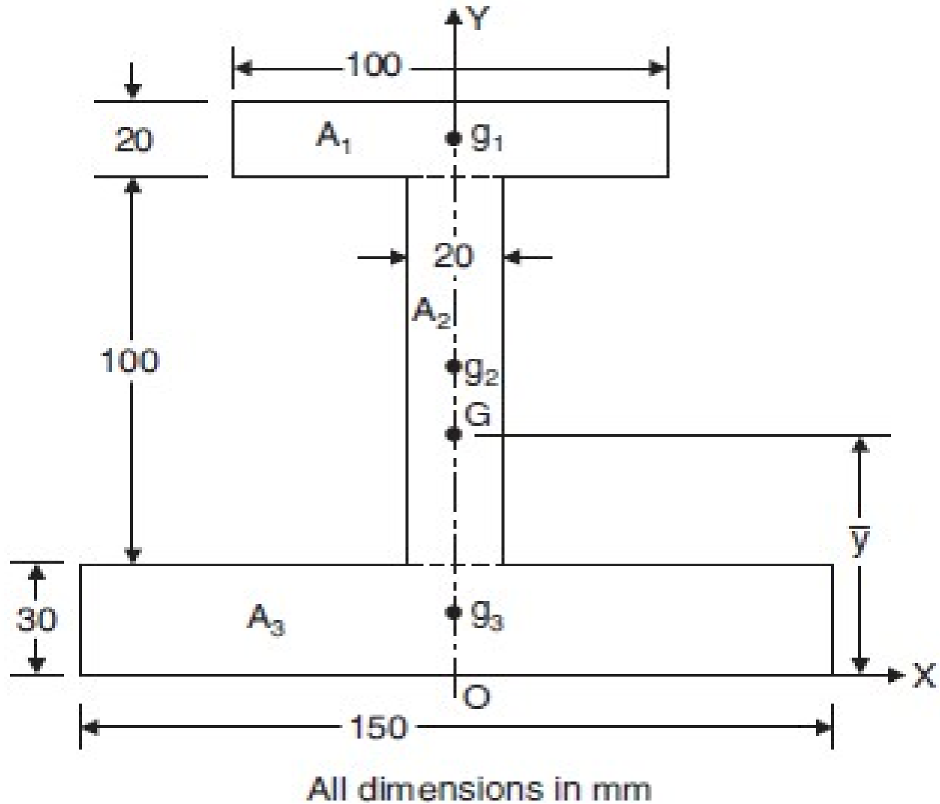


Fig. 4.31

Solution: Selecting the coordinate system as shown in Fig. 4.31, due to symmetry centroid must lie on y axis,

i.e., $$\bar{x} = 0$$

Now, the composite section may be split into three rectangles

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2$$

Centroid of A_1 from the origin is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly $$A_2 = 100 \times 20 = 2000 \text{ mm}^2$$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2,$$

and $$y_3 = \frac{30}{2} = 15 \text{ mm}$$

$\therefore$ $$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A}$$

$$= \frac{2000 \times 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500}$$

$$= \mathbf{59.71 \text{ mm}}$$

Thus, the centroid is on the symmetric axis at a distance 59.71 mm from the bottom as shown in Fig. 4.31.

Example 4.10. *Determine the coordinates x_c and y_c of the centre of a 100 mm diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 4.35 (All dimensions are in mm).*

Solution: If x_c and y_c are the coordinates of the centre of the circle, centroid also must have the coordinates x_c and y_c as per the condition laid down in the problem. The shaded area may be considered as a rectangle of size 200 mm × 150 mm *minus* a triangle of sides 100 mm × 75 mm and a circle of diameter 100 mm.

$\therefore$ Total area

$$= 200 \times 150 - \frac{1}{2} \times 100 \times 75 - \left(\frac{\pi}{4}\right)100^2$$

$$= 18396 \text{ mm}^2$$

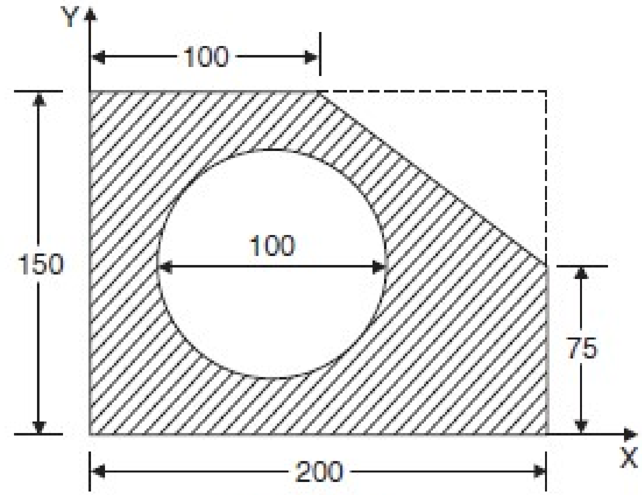


Fig. 4.35

$$x_c = \bar{x} = \frac{200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times \left[200 - \left(\frac{100}{3}\right)\right] - \frac{\pi}{4} \times 100^2 \times x_c}{18396}$$

$$\therefore \qquad x_c(18396) = 200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times 166.67 - \frac{\pi}{4} \times 100^2\, x_c$$

$$\mathbf{x_c = \frac{2375000}{26250} = 90.48\ mm}$$

Similarly,

$$18396\, y_c = 200 \times 150 \times 75 - \frac{1}{2} \times 100 \times 75 \times (150 - 25) - \frac{\pi}{4} \times 100^2\, y_c$$

$$\therefore \qquad \mathbf{y_c = \frac{1781250.0}{26250} = 67.86\ mm}$$

Centre of the circle should be located at (90.48, 67.86) so that this point will be the centroid of the remaining shaded area as shown in Fig. 4.35.

Note: The centroid of the given figure will coincide with the centroid of the figure without circular hole. Hence, the centroid of the given figure may be obtained by determining the centroid of the figure without the circular hole also.