Optimization Models

Ms Swapnil Shrivastava
CDAC Bangalore

Content

- Optimization Overview
- Linear Optimization Model
- Linear Optimization Application
- Integer Linear Optimization
- Non Linear Optimization
- Non Smooth Optimization

Optimization

- Optimization is the process of selecting values of decision variables that minimize and maximize some quantity of interest.
- It is the fundamental tool in prescriptive analytics.
- Been used extensively in operations and supply chains, finance, marketing and other disciplines to help managers allocate resources more efficiently and make lower-cost or more-profitable decisions.

Steps for Building Linear Optimization Model

- Identify the decision variables
  - They are the unknown values that the model seeks to determine.
  - Quantities of product to be produced, amount of money spent on R&D projects
- Identify the objective function
  - The quantity we seek to minimize or maximize
  - Minimize risk, maximize profit

Steps for Building Linear Optimization Model

- Identify all appropriate constraints
  - Limitations, requirements or restrictions imposed on any solution.
  - They are either from practical or technological considerations or by management policy.
- Write the objective function and constraints as mathematical expressions
  - The presence of constraints along with a large number of variables usually makes identifying an optimal solution considerably more difficult and necessitates the use of software tools.

Sklena Ski Company (SSC)

SSC is a small manufacturer of two types of popular all-terrain snow skis, The Jordanelle and the Deercrest models.

The manufacturing process consists of two principal departments: fabrication and finishing.

The fabrication department has 12 skilled workers, each of whom works 7 hours per day. The finishing department has 3 workers, who also work a 7-hour shift.

Each pair of Jordanelle skis requires 3.5 labor-hours in the fabricating department and 1 labor-hours in finishing. The Deercrest model requires 4 labor-hours in fabricating and 1.5 labor-hours in finishing.

The company operates 5 days per week.

SSC makes a net profit of $50 on the Jordanelle model and $65 on the Deercrest model.

Sklena Ski Company (SSC)

In anticipation of the next ski-sale season, SSC must plan its production of these two models. Because of the popularity of its products and limited production capacity, its products are in high demand, and SSC can sell all it can produce each session.

The company anticipates selling at least twice as many Deercrest models as Jordanelle models. The company wants to determine how many of each model should be produced on a daily basis to maximize net profit.

Sklena Ski Company (SSC)

- Step 1: Identify the decision variables

  Jordanelle = number of pairs of Jordanelle skis produced/day

  Deecrest = number of pairs of Deecrest skis produced/day
- Step 2: Identify the objective function

  maximize net profit
- Step 3: Identify the constraints

  Total labor hours used in fabrication cannot exceed the amount of labor hours available.

  Total labor hours used in finishing cannot exceed the amount of labor hours available.

  Number of pairs of Deecrest skis must be at least twice the number of pairs of Jordanelle skis.

  Negative value of decision variable cannot happen.

Sklena Ski Company (SSC)

- Step 4 : Modeling the Objective Function

Maximize total profit = $50 Jordanelle + $65 Deercrest

- Modeling the Constraints

Fabrication: 3.5 Jordanelle + 4 Deercrest <= 84

Finishing: 1 Jordanelle + 1.5 Deercrest <= 21

Deercrest – 2Jordanelle >= 0

Deercrest >= 0

Jordanelle >= 0

Optimization Model for the SSC problem

Maximize Total Profit = 50 Jordanelle + 65 Deercrest

3.5 Jordanelle + 4 Deercrest <= 84

1 Jordanelle + 1.5 Deercrest <= 21

Deercrest – 2Jordanelle >= 0

Deercrest >= 0

Jordanelle >= 0

More about Constraints

- Constraints may take on many different forms
  - The amount of money spent on R&D projects cannot exceed the assigned budget of $300,000
    - Amount spent on R&D <= 300,000
  - Contractual requirements specify that at least 500 units of product must be produced.
    - Number of units of product produced >= 500
  - A mixture of fertilizer must contain exactly 30% nitrogen
    - Amount of nitrogen in mixture/total amount in mixture = 0.30

Characteristics of Linear Optimization Models

- A linear optimization model has two basic properties
  - The objective function and all constraints are linear functions of the decision variables.
  - All variables are continuous, meaning that they may assume any real value (typically, nonnegative)

$$\frac{0.2x + 0.33\,y}{x + y} = 0.3$$

Implementing Linear Optimization Models on Spreadsheets

- To solve optimization models using an Excel tool called Solver.
  - Put the objective function coefficients, constraint coefficients, and right-hand values in a logical format in the spreadsheet.
  - Define a set of cells (either rows or columns) for the values of the decision variables.
  - Define separate cells for the objective function and each constraint function (the left-hand side of a constraint).

Solving Linear Optimization Model

- Seek values of the decision variables that maximize or minimize the objective function and also satisfy all constraints.
- Any solution that satisfies all constraints of a problem is called a feasible solution.
- Finding optimal solution among the infinite number of possible feasible solutions is a difficult task.
- To guarantee finding an optimal solution, systematic mathematical solution procedure is necessary.
  – MS Excel Solver tool, Risk Solver Platform – Premium Solver

Using Standard Solver

- Standard Solver is found in the Analysis group under the Data tab in Excel.
- We define the objective, decision variables and constraints within Solver.
- It provides three options for the solving method:
  - GRG Nonlinear – used for solving nonlinear optimization problems.
  - Simplex LP – used for solving linear and linear integer optimization model.
  - Evolutionary – used for solving complex nonlinear and nonlinear integer problems.

Solver Report

- Solver generates three reports:
  - Answer
  - Sensitivity
  - Limits

Solver Outcomes and Solution Messages

- Unique optimal solution: there is exactly one solution that will result in the maximum (or minimum) objective.
- Alternate optimal solution: the objective is maximized (or minimized) by more than one combination of decision variables, all of which have the same objective function value.
  - 50 Jordanelle + 75 Deecrest
- Unbounded solution: the objective can be increased or decreased without bound while the solution remains feasible.
  - Remove finishing and fabrication constraint
- Infeasible solution: there is no solution that satisfies all constraints together. E.g. demand requirement is higher than available capacity.
  - Reverse inequality sign

Crebo Manufacturing

Crebo Manufacturing produces four types of structural support fittings- plugs, rails, rivets, and clips. They are machined on two CNC machining centers.

The machining centers have a capacity of 280,000 minutes per year. The gross margin per unit and machining requirements are:

Product	**Plugs**	**Rails**	**Rivets**	**Clips**
Gross margin / unit	$0.30	$1.30	$0.75	$1.20
Minutes / unit	1	2.5	1.5	2

How many of each product should be made to maximize gross profit margin?

Optimization Model

Let X_1, X_2, X_3 and X_4 be the number of plugs, rails, rivets, and clips to produce.

The problem is maximize gross margin

$$= 0.3X_1 + 1.3X_2 + 0.75X_3 + 1.2X_4$$

Subject to the constraint that limits the machining capacity and non negativity of the variables:

$$1X_1 + 2.5X_2 + 1.5X_3 + 2X_4 <= 280{,}000$$

$$X_1, X_2, X_3, X_4 >= 0$$

Types of Constraints

- Simple bounds: constrain the value of a single variable.
  - Y >= 300
- Limitations: involve the allocation of scarce resources.
  - Amount of material in production cannot exceed the amount available in inventory.
- Requirements: specification of minimum levels of performance.
  - Production must be sufficient to meet promised customer orders.
- Proportional Relationships: found in problems involving mixtures or blends of materials or strategies.
  - The amount invested in aggressive growth stocks cannot be more than twice the amount invested.
- Balance Constraints: essentially state that input=output and ensure that the flow of material or money is accounted for at locations or between time periods.
  - Production in June plus any available inventory must equal June's demand plus inventory held to July.

Transportation Models

- Many practical models in supply chain optimization stem from a very simple model called the transportation problem.
- This involves determining how much to ship from a set of sources of supply (factories, warehouses etc) to a set of demand locations (warehouse, customers etc) at minimum cost.
- Decision Variables: Amount to ship between sources of supply and destinations
- Objective Function: Minimize total transportation cost
- Typical Constraints: Limited availability at sources, required demands met at destinations

General Appliance Corporation (GAC)

GAC produces refrigerators at two plants: Marietta, Georgia and Minneapolis, Minnesota.

They ship them to major distribution centres in Cleveland, Baltimore, Chicago and Phoenix.

The GAC cost, capacity and demand data are:

	Distribution Center				
Plant	Cleveland	Baltimore	Chicago	Phoenix	Capacity
Marietta	\$12.60	\$14.35	\$11.52	\$17.58	1200
Minneapolis	\$9.75	\$16.26	\$8.11	\$17.92	800
Demand	150	350	500	1000	

GAC's supply chain manager faces the problem of determining how much to ship between each plant and distribution center to minimize the total transportation cost, not exceed available capacity, and meet customer demand.

GAC Optimization Model

Define X_{ij} – amount shipped from plant I to distribution center j.

Minimize $12.60X_{11} + 14.35X_{12} + 11.52X_{13} + 17.58X_{14} + 9.75X_{21} + 12.63X_{22} + 8.11\,X_{23} + 17.92\,X_{24}$

$X_{11} + X_{12} + X_{13} + X_{14} <= 1200$

$X_{21} + X_{22} + X_{23} + X_{24} <= 800$

$X_{11} + X_{21} = 150$

$X_{12} + X_{22} = 350$

$X_{13} + X_{23} = 500$

$X_{14} + X_{24} = 1000$

$X_{ij} >= 0$, for all i and j

Integer Linear Optimization

- The some of or all the variables are restricted to being integer.
- If only a subset variable is restricted to being integer while others are continuous, we call this a mixed-integer linear optimization model.
- A special type of integer problem is one in which variable can be only 0 or 1: these are used to model logical yes-or-no decisions.
- Revisit: Sklena Ski Company (SSC)

Nonlinear Optimization Model

- The objective functions and or constraint functions are nonlinear functions of the decision variables.
- The terms cannot be written as a constant multiplied by a variable.
- Some examples: 3x2, 4/y and 6xy
- They require more creativity and analytical expertise.

Pricing Decision Models

Consider a simple nonlinear optimization model for finding the optimal price to maximize revenue. In this example, a market research study has collected data that estimate the expected annual sales for different levels of pricing.

Analysts determined that sales can be expressed by the following:

sales= -2.9485*price + 3240.9

Using the fact that revenue equals price x sales,

total revenue= -2.9485xprice2 + 3240.9xprice

Solve the Pricing Decision Model

Multiperiod Production Planning Models

- The basic decisions are how much to produce in each time period to meet anticipated demand over each period.
- It may be advantageous to produce more than needed in earlier time periods when production costs may be lower and store excess production as inventory for use in later time periods, thereby letting lower production costs offset the costs of holding the inventory.
- So best decision is often not obvious.
- Decision Variables: Quantities of product to produce in each of several time periods, amount of inventory to hold between periods.
- Objective Function: Minimize total production and inventory cost
- Typical Constraints: Limited production rates, material balance equations

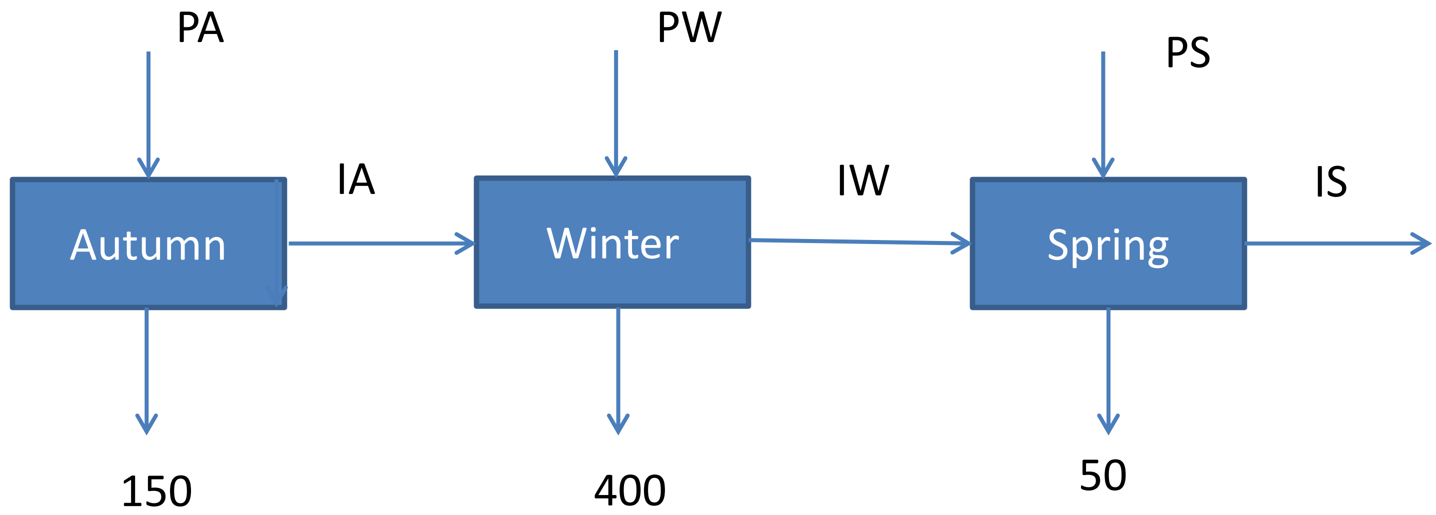
K&L Designs

K&L designs is a home-based company that makes hand-painted jewelry boxes for teenage girls. Forecasts for sales for the next year are 150 in the autumn, 400 in the winter, and 50 in the spring.

Plain jewelry boxes are purchased from a supplier for $20. The cost of capital is estimated to be 24% per year (or 6% per quarter), thus the holding cost per item is 0.06($20) = $1.20 per quarter.

The company hires art students part-time to craft designs during the autumn, and they earn $5.50 per hour. Because of the high demand for part-time help during the winter holiday seasons, labor rates are higher in the winter, and workers earn $7.00 per hour. In the spring labor is more difficult to keep, and the owner must pay $6.25 per hour to retain qualified help. Each jewelry box takes 2 hours to complete. How should production be planned over the three quarters to minimize the combined production and inventory holding costs?

Material Balance Constraint Structure



PA = amount to produce in autumn
PW = amount to produce in winter
PS = amount to produce in spring
IA = inventory held at the end of autumn
IW = inventory held at the end of winter
IS = inventory held at the end of spring

K&L designs Optimization Model

Minimize $11PA + 14PW + 12.50PS + 1.20IA + 1.20\,IW + 1.20IS$

Subject to

$PA - IA = 150$

$PW + IA - IW = 400$

$PS + IW - IS = 50$

$Pi >= 0$, for all i

$Ij >= 0$, for all j

Non-smooth Optimization

- Excel functions such as IF, ABS, MIN and MAX lead to non-smooth models.
- Solver’s Standard Evolutionary algorithm uses metaheuristics approach.
  - Intelligent rules for systematically searching among solutions
  - That remember the best solutions they find and then modify or combine them in attempting to find better solutions.

K&L designs Optimization Model

Minimize 11PA + 14PW + 12.50PS + 1.20IA + 1.20 IW + 1.20IS + IF(PA > 0,65,0) + IF(PW >0,65,0) + IF(PS > 0,65,0)

Subject to

PA – IA = 150

PW + IA – IW = 400

PS + IW – IS = 50

Pi >= 0, for all i

Ij >= 0, for all j