Second Week Report

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On week 2, a review on complex numbers was given by Dr. Jackura. Complex numbers are useful tools in solving many physical problem. Its use can be seen extensively in wave physics and in quantum mechanics. For example, A classical wave can be defined as

$$\Psi(x,t) = A\cos(\omega t - kx) = Re\left[Ae^{i(\omega t - kx)}\right]$$

But in quantum mechanics, complex numbers are more fundamental than a useful math trick. A wave function $\Psi(x,t)$ is defined as

$$\Psi(x,t) = Ae^{i(\omega t - kx)}$$

And the probability of finding a particle is,

$$\text{Probability} = \int_x |\Psi(x,t)|^2 dx$$

$$= \int_x \Psi^*(x,t)\Psi(x,t)dx$$

Where $\Psi^*(x,t)$ is the complex conjugate of $\Psi(x,t)$ defined as $\Psi^*(x,t) = Ae^{-i(\omega t - kx)}$

Scattering Amplitude

The Probability of an interaction to occur Probability $\propto |\mathcal{M}|^2$ where $\mathcal{M}$ is the scattering amplitude which is a function of E^*, center of momentum energy i.e. $\mathcal{M} = \mathcal{M}(E^*)$ and $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$

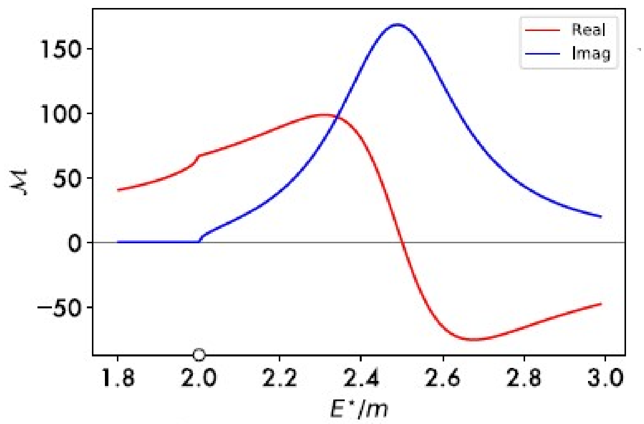

Figure 1: A graph between E^*/m and $\mathcal{M}$

In fig(1), a graph is drawn between E^*/m and $\mathcal{M}$. We can observe that there is a spike or a disturbance at $E^*/m = 2$ is known as the threshold energy $E^*_{th} = 2m$. As we can see the real part is zero when $E^*/m \leq 2$. This graph is a result theoretical calculation.

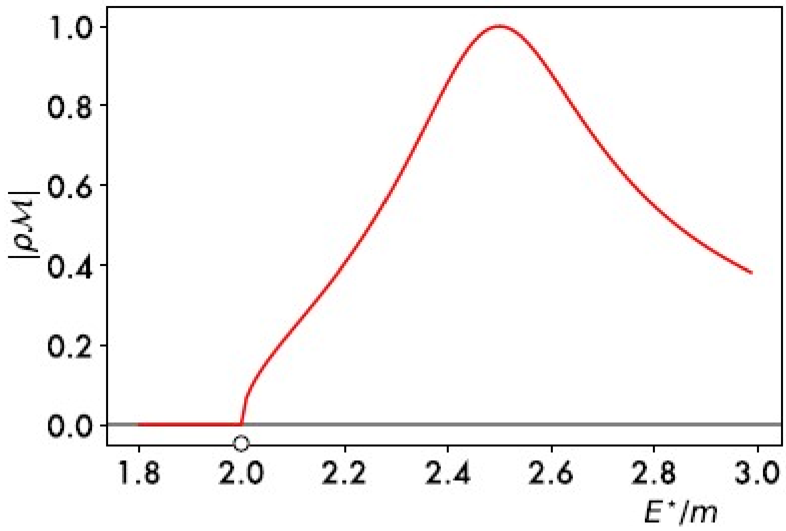

Figure 2: A graph between E^*/m and $|\rho\mathcal{M}|$

In laboratories, a kinematic factor known as phase space, ρ. A graph between E^*/m and $|\rho\mathcal{M}|$ is given in the fig(2). Here also we can observe the threshold energy E^*_{th} is happen to be at $2m$.

Basics of Complex Numbers

Now lets focus on the basic concepts of Complex numbers. A complex number $z \in \mathbb{C}$ is given by $z = x + iy$ where $x, y \in \mathbb{R}$ and $i \equiv \sqrt{-1}$ such that the real part of z is $Re\ z = x$ and the imaginary part of z is $Im\ z = y$. It's complex conjugate is defined as $z^* = x - iy$. The magnitude of a complex number z is $|z| = \sqrt{z.z^*} = \sqrt{x^2 + y^2}$. Some properties of Complex numbers are,

1. $(z^*)^* = z$
2. $Re\ z = \frac{z+z^*}{2}$
3. $Im\ z = \frac{z-z^*}{2i}$
4. $Re\ z^* = Re\ z$
5. $Im\ z^* = -Im\ z$
6. $|z^*| = |z|$
7. $\frac{1}{z} = \frac{z^*}{|z|}$

One can represent a complex number in polar form as $z = re^{i\varphi}$ where r is the magnitude of the complex number i.e.

$$r = |z| = \sqrt{x^2 + y^2}$$

and φ is the phase or argument of the complex number,

$$\varphi = \begin{cases} 2\arctan\left(\frac{y}{\sqrt{x^2+y^2}+x}\right) & \text{if } x > 0 \text{ or when } y \neq 0 \\ \pi & \text{if } x < 0 \text{ or when } y = 0 \\ undefined & \text{if } x = 0 \text{ or when } y = 0 \end{cases}$$

whose ranges are $0 \leq r\infty$ and $\varphi \in (-\pi, \pi]$. In polar form, $x = rcos\varphi$ and $y = rsin\varphi$ and from Euler's formula, $e^{i\varphi} = cos\varphi + isin\varphi$

Analytic function

Analytic functions are defined as complex functions in domain D such that the function is uniquely defined i.e. single valued at every point in domain D and differentiable in the domain D. For example, a power series,

$$f(z) = \sum_{n=0}^{N} a_n z^n$$

is an analytic function in the entire complex plane $\mathbb{C}$

Pole singularity

Singularity are the isolated points at which the function is not differentiable. Pole singularity is the simplest non analyticity.

In complex analysis, a powerful way to express a given function is in form of Laurent series. A Laurent series is the extension of power series and defined around a disc of radius r_0 as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

If there exist $a_n \neq 0$ when $n = -1$, Then the function $f(z)$ has a pole of order 1 at the point z_0. On the simplest way the pole singularity of order 1 will be like

$$f(z) = \frac{a_{-1}}{(z - z_0)}$$

The residue of $f(z)$ at the point z_0 is a_{-1} i.e. Res $(f, z_0) = a_{-1}$ finding the residue of a function can be useful when we want to perform complex integration. From residue theorem,

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{ Res } (f, z_k)$$

Cauchy's Theorem

Cauchy's theorem states that if a function $f(z)$ with $z \in \mathbb{C}$ is analytic everywhere within and on a closed contour C, then

$$\oint_C f(z)dz = 0$$

Cauchy's theorem can be used to prove the Cauchy integral formula, which states that if a function $f(z)$ with $z \in \mathbb{C}$ is analytic everywhere within and on a closed contour C, then

$$\oint_C \frac{f(z)}{z - z_0} \mathrm{d}z \ = 2\pi i f(z_0)\,,$$

provided that z_0 lies within the contour. We define the residue of a function as $f(z_0)$, which is the value of the function at the pole.