Final Report

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Quantum Chromodynamics

Fundamental forces of nature are Gravitational force, Electromagnetic force, Strong nuclear force and weak nuclear force. The strong nuclear force is responsible for holding the protons and neutrons inside the nucleus of the atom and also responsible for the interaction between them. But protons and neutrons are not the fundamental particles. In fact they are composite particles made up of **quarks and gluons**. They also interact via strong nuclear force which is governed by theory of Quantum Chromodynamics.

The theory QCD has a conserved quantity like electric charge called the colour charge. There are three fundamental types of colour charge, **Red, Blue and Green**. All these charges have their own counterparts.

The quarks have another characteristic called the **'flavour'**. They have six different flavours. They are up, down, charm, strange, top and bottom. Every flavours of quark have different masses. For example, the top quark is the heaviest quark of mass approximately
173 GeV(in natural units). The gluons, the force carrying particle for strong nuclear force have no mass.

Quantum chromodynamics has confinement property. Confinement says that quarks are always bounded in a bound state in other words it is impossible to detect an individual quark. The theory of QCD is non perturbative at low energies this allows gluons to interactive with themselves also.

The excited states of QCD bound states decays rapidly at the order of 10^{-23}s. This makes their spectrum difficult to study.

Complex Analysis: Review

Complex numbers are useful tools in solving many physical problem. Its use can be seen extensively in wave physics and in quantum mechanics. For example, A classical wave can be defined as

$$A\cos(\omega t - kx) = Re\left[Ae^{i(\omega t - kx)}\right]$$

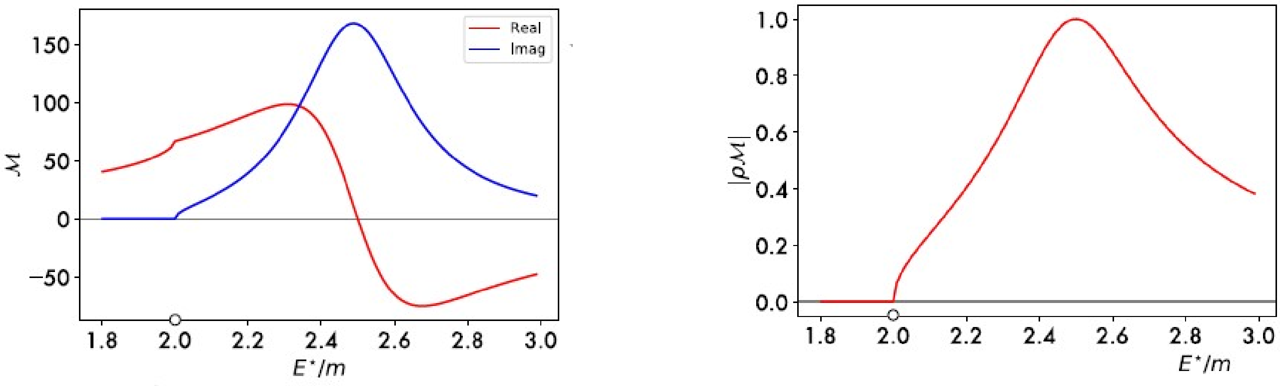
Where we neglect the imaginary part but in quantum mechanics, a wave function is defined as

$$\Psi(x,t) = Ae^{i(\omega t - kx)}$$

And the probability of finding a particle is,

$$\text{Probability} = \int_x |\Psi(x,t)|^2 dx = \int_x \Psi^*(x,t)\Psi(x,t)dx$$

The Probability of an interaction to occur Probability $\propto |\mathcal{M}|^2$ where $\mathcal{M}$ is the scattering amplitude which is a function of E^*, center of momentum energy i.e. $\mathcal{M} = \mathcal{M}(E^*)$ and $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$



(a) A graph between E^*/m and $\mathcal{M}$

(b) A graph between E^*/m and $|\rho\mathcal{M}|$

Figure 1: Graphing Scattering Amplitude $\mathcal{M}$

In fig(1a), a graph is drawn between E^*/m and $\mathcal{M}$. We can observe that there is a disturbance at $E^*/m = 2$ is the threshold energy $E^*_{th} = 2m$. As we can see the real part is zero when $E^*/m \leq 2$. This graph is a result theoretical calculation. But in laboratories, a kinematic factor known as phase space, ρ. A graph between E^*/m and $|\rho\mathcal{M}|$ is given in the fig(1b). Here also we can observe the threshold energy E^*_{th} is happen to be at $2m$.

If a unique real valued complex function is differentiable on a domain D then the function is known as Analytic function. Singularity are the isolated points where the function is not differentiable.

Scattering Amplitude

The scattering Amplitude, $\mathcal{M}$ for an interaction is defined as

$$i\mathcal{M} \propto \langle \text{final} \, |S - 1| \text{initial} \rangle$$

QCD is a difficult theory to grasp.Therefore some model independent features of the scattering theory are:

1. **Symmetry**
   To keep the scattering process consistent with other theories and laws. we impose symmetries on the scattering process. Since we particles are moving with high velocity we want to ensure that the special relativity is obeyed in our scattering process, so impose Lorentz invariance on the system. This is the spacetime symmetry.
   We will also impose certain *internal symmetries*. These are all the conserved quantities preserved in a scattering theory. For example, If we consider baryonic scattering, we impose that certain quantities like baryon number, flavor, electric charge, color charge etc are conserved before and after the scattering process.

2. **Unitarity**
   Unitarity directly implies to probability conservation.By Probability conservation, we mean that the sum of probabilities of all the interactions possible for an initial state i to react the final state f is unit. Mathematically, $\sum_f \text{Prob}(i \to f) = 1$ It has the property when we

multiple S - matrix with its complex conjugate we get identity matrix. Mathematically, $S^{\dagger}S = 1$

From Unitarity we can express Im $\mathcal{M} = \rho|\mathcal{M}|^2 \quad$ for $\; E^* \geq 2m$
From unitarity, phase space function ρ which is a kinematic function is defined as

$$\rho = \frac{\xi q^*}{8\pi E^*}$$

where ξ parameter is characterizes weather the system under consideration has identical or distinguishable particles, $\xi = \begin{cases} \frac{1}{2} & \text{identical} \\ 1 & \text{otherwise} \end{cases}$

and q^* is the relative momentum between particles in center of momentum frame,

$$q^* = \frac{1}{2}\sqrt{E^*2 - 4m^2}$$

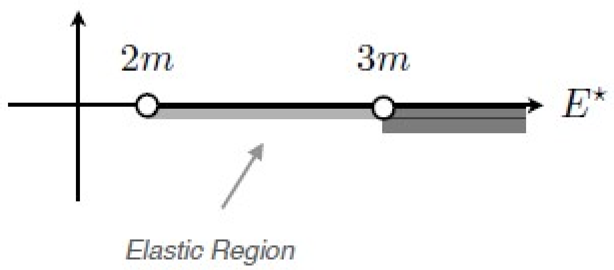


Figure 2: Elastic Region in a 2 particle scattering system

Since $E^*2 > 2m$ we get an elastic region where one can produce 2 stable particles. The Unitarity gives rise to 2 useful representation of scattering amplitude. They are,

- Phase shift representation: $\mathcal{M} = \frac{1}{\rho}e^{i\delta}\sin\delta$
- $\mathcal{K}$ matrix representation: $\mathcal{M} = \mathcal{K}\frac{1}{1-i\rho\mathcal{K}}$ where $\mathcal{K}^{-1} = \rho\cot\delta$

3. **Analyticity**
   Causality, for every action there is a cause which results in an output. This causality helps us to write our scattering amplitude as a complex function which has some very specific properties.

4. **Crossing**
   These are properties that relates particles and anti-particles like CPT symmetries.

QCD Spectroscopy

As we know the excited state of a QCD state is very unstable and decays rapidly. For example, the Roper, the first excited state of a nucleon, can decay either into a nucleon and a pion (Nπ)

for about 50 to 70 % of the time or a nucleon and 2 pions (N$\pi\pi$) for about 25 to 50 % of the time. As shown earlier the decay takes place in the order of 10^{-23}s.

Since the theory of QCD allows lot of possible bound states, we have many particles. The **Particle Data Group** has listed them all in their website https://pdg.lbl.gov/ where they provide all the information regarding the particle of interest can be found.

Some of the QCD bound states and their properties like mass, mean life and prominent decay modes are tabulated below

Particle	Mass	Mean life	Decay channels
π^+	140 MeV	$\sim 3 \times 10^{-8}s$	$\mu^+\nu_\mu$ ($\sim$ 100%)
π^0	135 MeV	$\sim 9 \times 10^{-17}s$	2γ ($\sim$ 99%)
η	550 MeV	$\sim 5 \times 10^{-17}s$ ($\Gamma \sim$ 1.3 keV)	2γ($\sim$ 39%) $3\pi^0$($\sim$ 32%) $\pi^+\pi^-\pi^0$($\sim$ 23%)
$f_0(500)/\sigma$	400 - 550 MeV	$\sim 10^{-24}$s ($\Gamma \sim 400 - 700$ MeV)	$\pi\pi$($\sim$ 100%)
ρ	770 MeV	$\sim 10^{-23}$s ($\Gamma \sim$ 147 MeV)	$\pi\pi$($\sim$ 100%)

Table 1: Some unflavored mesons and their properties are tabulated above.

In Table(1), Γ is the reciprocal of the mean life(τ) and the colored texts represent what type of mechanism through which it decays, the violet represents Weak mechanism, the blue represents quantum electrodynamics (QED) and the red represents quantum chromodynamics (QCD).

Since we are interested in the dynamics of QCD, we usually tend to consider a simplified version of standard model where we can remove the weak and electromagnetic force and just considering QCD on its own. Under this assumptions, we can get lot of information about the dynamics of a QCD state. Under this consideration, we can see that many particles which are unstable in original theory is now stable under this simplified version of standard model. These states are known as "QCD stable" states.

Here we can understand that the heavier the particle of the QCD bound state the more likely to decay into lighter QCD bound states. These states are called "QCD unstable" states.

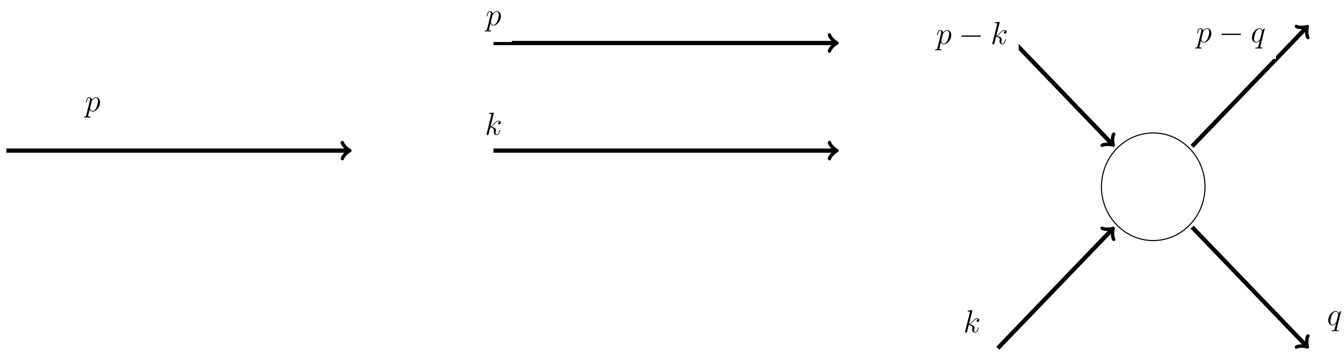
Feynman Diagram

Feynman diagrams are useful tools to visualize the trajectories of the particles interacting and useful in visualizing the mathematical description for the given interaction.

In order to undergo scattering process, the Energy of the system during the collision has to greater than $2m$(for 2 particle system). It is well known that when a system of particles undergo interaction there is a non zero probability that they will create some more particles which is the result of excess energy converts into mass as per the Einstein's mass energy equivalence principle. This phenomenon was not well explained in the quantum mechanics. The full explanation comes from quantum field theory(QFT)

Now consider the simplest QFT model, a 4 scalar field $\lambda\varphi^4$ here either of the following can happen:

1. It propagates freely i.e. no interaction takes place as shown in figure(3a) & figure(3b). Note figure(3b) has a non zero probability so this cannot be discarded.
2. Some interactions takes place as shown in figure(3c). Note this is not the only interaction possible. There are infinitely many interactions can be taken place, like interactions can be happen twice or thrice and so on.



(a) A particle propagates freely with a momentum p

(b) Two particles of momenta p and k propagates without interacting

(c) Two particles undergoes into some interaction

Figure 3: Some Possible interactions in $\lambda\varphi^4$ field. Remember there are infinite possible interactions for a given system. Note that the time flows from left to right.

The Feynman Rules for $\lambda\varphi^4$ field are:

1. **Rule 1:** Every vertex contributes to $-i\lambda$ where λ is some parameter which we don't know but we can tune in such a way to match with the experimental results.
2. **Rule 2:** For every external legs, a factor of $(+1)$ is multiplied.
3. **Rule 3:** For internal lines also known as propagators the term $\left(\frac{i}{p^2-m^2+i\epsilon}\right)$ where p is the 4 momentum of the propagators and for most of the case we ultimately set $\epsilon = 0$
4. **Rule 4:** For each intermediate undetermined momentum then we should integrate over it i.e. $\int \frac{d^4k_1}{(2\pi)^4}$, $\int \frac{d^4k_2}{(2\pi)^4}$ and so on.
5. **Rule 5:** Divide the expression by the symmetry factors. These symmetry factor are equal to number of permutations of internal lines which leave the diagram unchanged.

QCD of deuteron

Deuteron is the bound state of proton and neutron. It is one of the simplest interacting case of proton and neutron. The interaction of deuteron is seen inside the Sun where proton proton cycle takes place.

So our focus is now on the nucleon nucleon (NN) scattering of deuteron. Assume that the proton and neutron have same mass i.e. $m_p = m_n = m \approx 940$ MeV

The scattering amplitude is

$$\mathcal{M} = \frac{1}{\mathcal{K}^{-1} - i\rho} = \frac{8\pi E^*/\xi}{q^* cot\delta - iq^*}$$

Here $q^* = \frac{1}{2}\sqrt{E^*2 - 4m^2}$ and $\xi = 1/2$. It is useful to represent $\mathcal{K} = q^* cot\delta$ as in the fprm called Effective range expansion form which is defined as

$$q^* cot\delta = -\frac{1}{a} + \frac{1}{2} r q^* + \mathcal{O}(q^{*4})$$

where a is the scattering length and r is the effective range.

Since our goal is deuteron, we need to have proton neutron are be a bound state. Thus our scattering process is an attractive interaction. Therefore energy for the bound state has to be lesser than the threshold energy i.e. $E^*_{bs} < E^*_{th}$

Let $q^* = i\kappa$ where κ is the binding momentum.
So the mass of the bound state is $E^*_{bs} = 2\sqrt{m^2 - \kappa^2} = m_{bs}$ Thus clearly, $m_{bs} < E^*_{th}$

The binding energy is $E^*_{binding} = E^*_{th} - E^*_{bs}$

From experiments, we know that $m_N = 940$ MeV, $a = 5.425$ fm and $r = 1.749$ fm and 1 fm $\approx \frac{1}{200 \text{ MeV}}$

Now we have to find the pole position for the above measurement and by approximating $q^* cot\delta \approx -\frac{1}{a}$ we get $q^* = \frac{i}{a}$ therefore, $\kappa \approx 37$ MeV

Therefore, $m_{bs} = 2\sqrt{m^2 - \kappa^2} \approx 1878$ MeV

Thus Binding Energy is $E^*_{binding} = E^*_{th} - E^*_{bs} \approx 1880 \text{ MeV} - 1878 \text{ MeV} \approx 2$ MeV

Conclusion

I would like to thank Dr. Raul Briceno, Dr. Andrew Jackura, and the Old Dominion University for creating this opportunity in spite of the global pandemic and making this mentorship program possible. I have learned fundamental concepts deeper than ever and learned a lot of new concepts during the program. The program increased my confidence level. I enjoyed the program a lot. I am very much passionate about continuing my journey in this field. I am very excited to do more research in this field.