Third Week Exercise

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1 Exercise

1.1 Plot ρ(phase space) for identical particles in the range $1.8 \leq E^* \leq 3.2$

The phase space, ρ is a kinematic function. It characterizes on-shell scattering of two particles.

It is mathematically defined as,
$$\rho = \frac{\xi q^*}{8\pi E^*}$$

for $E^* \geq 2m$ where q^* is the relative momentum between particles in center of momentum frame and defined as $q^* = \frac{1}{2}\sqrt{E^{*2} - 4m^2}$ and ξ parameter defined as

$$\xi = \begin{cases} \frac{1}{2}, & \text{identical} \\ 1, & \text{otherwise} \end{cases}$$

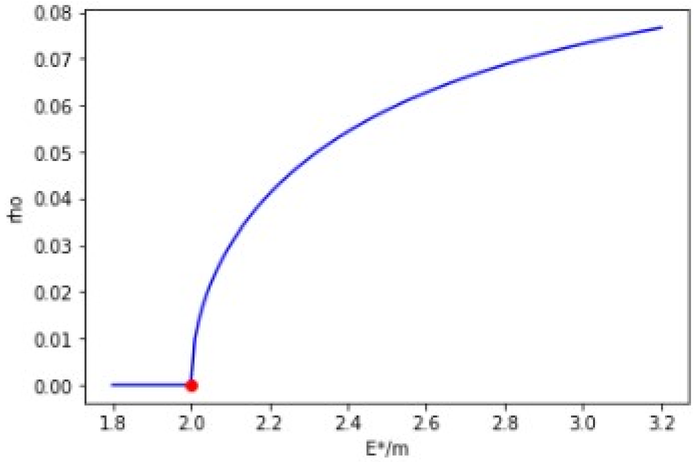
Here, we are considering the case for identical particles. Thus, $\xi = \frac{1}{2}$. Substituting all this values will give

$$\rho = \begin{cases} 0, & E^* \leq 2m \\ \frac{1}{32\pi}\frac{\sqrt{E^{*2}-4m^2}}{E^*}, & E^* > 2m \end{cases}$$

We are assigned to plot in the range $1.8 \leq E^*/m \leq 3.2$. The python code was shown below

```python
import numpy as np
import matplotlib.pyplot as plt

def func(x):
    return np.piecewise(x, [x < 2.0, x > 2.0], [lambda x: x * 0, lambda x: (1/32*np.pi)*(np.sqrt(x**2-4)/x)])


E = np.arange(1.8, 3.201, 0.01)

vfunc = np.vectorize(func)
rho = vfunc(E)

print('E*/m = ', E)
print('rho = ', rho)

plt.plot(E, rho,'b')
plt.plot(2, 0, marker ='o', color = 'r')
plt.xlabel('E*/m')
plt.ylabel('rho')
plt.show()

```

The output for the python program is given below



1.2 Derivation of the phase shift representation for the scattering amplitude

The scattering amplitude can be written as

$$\mathcal{M} = |\mathcal{M}|e^{i\delta}$$

But from Euler's formula,

$$|\mathcal{M}|e^{i\delta} = |\mathcal{M}|\left(\cos\delta + i\sin\delta\right)$$
$$\mathcal{M} = |\mathcal{M}|e^{i\delta} = |\mathcal{M}|\cos\delta + i|\mathcal{M}|\sin\delta$$

Since $\mathcal{M}$ is complex valued it can be expressed as,

$$\text{Re } \mathcal{M} = |\mathcal{M}|\cos\delta$$
$$\text{Im } \mathcal{M} = |\mathcal{M}|\sin\delta$$

From unitarity we know that, Im $\mathcal{M} = \rho|\mathcal{M}|^2$

Therefore,

$$\rho|\mathcal{M}|^2 = |\mathcal{M}|\sin\delta$$
$$|\mathcal{M}| = \frac{1}{\rho}\sin\delta$$
$$\therefore \mathcal{M} = \frac{1}{\rho}e^{i\delta}\sin\delta$$

1.3 Show that Im $\mathcal{M}^{-1} = -\rho$

Let $\mathcal{M} = a + ib$. Thus Im $\mathcal{M} = b = \rho|\mathcal{M}|^2$. We know that $|\mathcal{M}|^2 = \mathcal{M}^\dagger\mathcal{M} = a^2 + b^2$. Therefore, $b = \rho(a^2 + b^2)$

Now,

$$
\begin{aligned}
\frac{1}{\mathcal{M}} &= \frac{1}{a+ib}\frac{a-ib}{a-ib} \\
&= \frac{a-ib}{a^2+b^2} \\
\mathcal{M}^{-1} &= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\text{Im } \mathcal{M}^{-1} &= -\frac{b}{a^2+b^2} \\
&= -\frac{\rho(a^2+b^2)}{a^2+b^2} \\
\text{Im } \mathcal{M}^{-1} &= -\rho
\end{aligned}
$$

1.4 Show that $\mathcal{K}^{-1} = \rho\cot\delta$

Since $\mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho$ and also $\mathcal{M}^{-1} = \frac{1}{|\mathcal{M}|}e^{-i\delta}$ such that

$$
\begin{aligned}
\text{Re } \mathcal{M}^{-1} = \mathcal{K}^{-1} &= \frac{1}{|\mathcal{M}|}\cos\delta \\
\text{Im } \mathcal{M}^{-1} = -\rho &= -\frac{1}{|\mathcal{M}|}\sin\delta
\end{aligned}
$$

Therefore, $\rho = \frac{1}{|\mathcal{M}|}\sin\delta$ or $|\mathcal{M}| = \frac{1}{\rho}\sin\delta$ Thus,

$$
\begin{aligned}
\mathcal{K}^{-1} &= \rho\left(\frac{\cos\delta}{\sin\delta}\right) \\
\therefore \mathcal{K}^{-1} &= \rho\cot\delta
\end{aligned}
$$