Third Week Report

Abhishek Daniel

Scattering Theory

On Week 3, a lecture on Scattering theory was given by Dr.Andrew Jackura. When we want to find the probability of for a certain interaction to occur this probability is called **Scattering Amplitude,** $\mathcal{M}$. It is defined by

$$i\mathcal{M} \propto \langle \text{final} \, |S-1| \, \text{initial}\rangle \tag{1}$$

For simplicity let's assume that the in number of particles in initial and final state like proton - proton scattering or neutron - neutron scattering. The transition, or interaction is mediated by an operator $S-1$ where S is a matrix known as $S-$ matrix which encodes the all the interactions possible for a given system from far past to far future. The S - matrix is subtracted by 1 which is an identity matrix which represents the no interaction scenario. By subtracting the no interaction from S - matrix, we are certain to expect an interaction. The operator $S-1$ has to be specified by a given theory we are looking into. This is proportional to the scattering amplitude $\mathcal{M}$ which is multiplied by i for convention. The proportionality is removed by introducing the kinematic factors.

The scattering amplitude is a function of some kinematic variables. But we are going to focus on the reduced form of scattering amplitude which is a function of center of momentum energy, E^* i.e. $\mathcal{M} = \mathcal{M}(E^*)$

We already know that the scattering amplitude is a complex function of energy in center of momentum frame. On previous week (also given in fig(1)), we saw the graph between E^*/m and $|\rho\mathcal{M}|$ and we know that the Probability $\propto |\mathcal{M}|^2$ and ρ is the kinematic phase space factor.

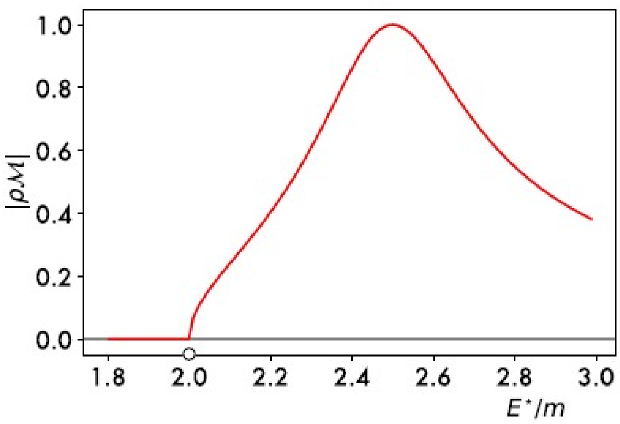

Figure 1: Graph between E^*/m and $|\rho\mathcal{M}|$

Our aim is to understand the scattering theory for the nuclear particles or generally hadronic particles. Hadrons are a class of composite particles which are made up of quarks and gluons. They come in 2 different kinds **Mesons**, particles with integer spin and **Baryons**, particles with half integer spin.

We want to understand the interactions from the theory of quarks and gluons, quantum chromodynamics(QCD). But QCD is not a simple theory, the interactions are so strong and approximation methods like perturbation theory fails. So we have to use other methods one of the popular method is lattice QCD to get the interactions and thus to form an S - matrix. But it is also tedious to extract the elements we require to find the scattering amplitude. Since they are not be extracted directly we are suppose to determine them indirectly.

Features

In order to use this technique to try to understanding the scattering from a very complicated theory. We need to understand some model independent features of the scattering theory. By word "model" we mean that we are not specifying the underlying interaction. So these are the general features of the scattering theory. So we have to impose these features regardless of the interaction from electron-electron scattering to hadronic scattering. These features can be seen even in atomic scattering to same extent. There are four model independent features in scattering theory. They are

1. **Symmetry**
   To keep the scattering process consistent with other theories and laws. we impose symmetries on the scattering process. This includes spacetime symmetry and some internal symmetries which is different for different particles.
2. **Unitarity**
   Unitarity directly implies to probability conservation. Unitary has a wide range of applications.
3. **Analyticity**
   Causality, for every action there is a cause which results in an output. This causality helps us to write our scattering amplitude as a complex function which has some very specific properties.
4. **Crossing**
   These are properties that relates particles and anti-particles like CPT symmetries.

We will discuss Symmetry and Unitarity in detail.

Symmetry

Since we particles are moving with high velocity we want to ensure that the special relativity is obeyed in our scattering process, so impose Lorentz invariance on the system. If we are considering a very low energy system we can neglect Lorentz invariance. However we are considering relativistic scattering so we will impose Lorentz invariance on the system. This is a *spacetime symmetry.*

We will also impose certain *internal symmetries.* These are all the conserved quantities preserved in a scattering theory. For example, If we consider baryonic scattering, we impose that certain quantities like baryon number, flavor, electric charge, color charge etc are conserved before and after the scattering process.

Unitarity

It describes the probability conservation. Probability conservation plays an important role in physics. When we impose Probability conservation on the S - matrix, it becomes an unitary operator.

By Probability conservation, we mean that the sum of probabilities of all the interactions possible for an initial state i to react the final state f is unit. Mathematically,

$$\sum_f \text{Prob}(i \to f) = 1$$

This probability conservation imposes that S - matrix has to be an unitary operator. It has the property when we multiple S - matrix with its complex conjugate we get identity matrix. Mathematically,

$$S^\dagger S = 1 \tag{2}$$

Where $S^\dagger$ is an Hermitian conjugate of S and 1 is the identity matrix. After some work with the help of this property, for a scattering amplitude in a limited energy i.e. for must obey the following relation,

$$\text{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2 \qquad \text{for} \;\; E^* \geq 2m \tag{3}$$

In fig(2) we can see that for E^* less than $2m$ the Im($\mathcal{M}$) is zero. We also know that this energy $2m$ is the threshold energy where one can create 2 stable particles and the next threshold energy is at $3m$ where one can create a third particle. In that case the equation(3) will get an additional terms. The region above $3m$ is known as first inelastic region. The range between $2m$ and $3m$ is know as the elastic region.

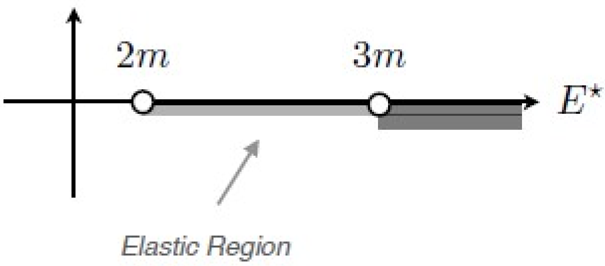

Figure 2: E^* vs Im $\mathcal{M}$

Now focusing on phase space function ρ which is a kinematic function and it characterizes on-shell scattering of two particles. It is defined by

$$\rho = \frac{\xi q^*}{8\pi E^*} \tag{4}$$

where ξ parameter is characterizes weather the system under consideration has identical or distinguishable particles,

$$\xi = \begin{cases} \frac{1}{2} & \text{identical} \\ 1 & \text{otherwise} \end{cases}$$

and q^* is the relative momentum between particles in center of momentum frame,

$$q^* = \frac{1}{2}\sqrt{E^*2 - 4m^2}$$

Different representations of Scattering amplitude

Unitarity enforces some useful properties of scattering amplitude which allows us to represent the scattering amplitude in different representation. We will focus on Phase shift representation and $\mathcal{K}$-matrix representation.

Phase shift representation

For a fixed energy, the scattering amplitude can written using Euler's theorem,

$$\mathcal{M} = |\mathcal{M}|e^{i\delta} \tag{5}$$

Thus the scattering amplitude can be determined by just 2 real parameters magnitude of $\mathcal{M}$ and the phase δ. Imposing the unitarity result, equation (3) we can show that

$$|\mathcal{M}| = \frac{1}{\rho}\sin\delta \tag{6}$$

Thus substituting equation (6) in equation (5), the scattering amplitude can be reduced to just one real parameter δ at a fixed energy as

$$\mathcal{M} = \frac{1}{\rho}e^{i\delta}\sin\delta \tag{7}$$

$\mathcal{K}$ - matrix representation

One can rewrite equation (3) as

$$\mathrm{Im}\,\mathcal{M}^{-1} = -\rho$$

One can show that,

$$\mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho \tag{8}$$

Where $\mathcal{K}$ is real function which characterizes short range forces between two particles. The $\mathcal{K}$ matrix cannot be known generally but we need to specify the interaction to get appropriate $\mathcal{K}$ matrix.

Thus the scattering amplitude can be represented as a function in $\mathcal{K}$ matrix as

$$\mathcal{M} = \mathcal{K}\frac{1}{1 - i\rho\mathcal{K}} \tag{9}$$

Also we can relate the $\mathcal{K}$ matrix as a function in ρ as

$$\mathcal{K}^{-1} = \rho\cot\delta \tag{10}$$