Q. For a given data in the Table, (a) fit the curve with the equation $y = a_0 + a_1 e^{2x} + a_2 e^{-2x}$

x_i	1	2	3	4
y_i	2	4	4	2

(b) Find the constants (a_1 and a_2) using the least square method if $a_0 = 0$

(c) Suppose the objective is to fit $y = b_1 x e^{-b_2 x}$, find b_1 and b_2 using least square method.

$\Rightarrow$ (a) The equation for error can be written as

$$E = \sum_{i=1}^{4} (a_0 + a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i)^2$$

$$\frac{\partial E}{\partial a_0} = 2 \sum_{i=1}^{4} (a_0 + a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i) = 0$$

$$\frac{\partial E}{\partial a_1} = 2 \sum_{i=1}^{4} e^{2x_i} (a_0 + a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i) = 0$$

$$\frac{\partial E}{\partial a_2} = 2 \sum_{i=1}^{4} e^{-2x_i} (a_0 + a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i) = 0$$

$$\begin{bmatrix} n & \sum e^{2x_i} & \sum e^{-2x_i} \\ \sum e^{2x_i} & \sum e^{4x_i} & n \\ \sum e^{-2x_i} & n & \sum e^{-4x_i} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum e^{2x_i} y_i \\ \sum e^{-2x_i} y_i \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3446.37 & 0.1564 \\ 3446.37 & 9051900.86 & 4 \\ 0.1564 & 4 & 0.0186 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7808.080 \\ 0.3545 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4.3656 \\ -0.0008 \\ -17.4790 \end{bmatrix}$$

(b) $$E = \sum_{i=1}^{4} (a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i)^2$$

$$\frac{\partial E}{\partial a_1} = 2 \sum_{i=1}^{4} e^{2x_i} (a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i) = 0$$

$$\frac{\partial E}{\partial a_2} = 2 \sum_{i=1}^{4} e^{-2x_i} (a_1 e^{2x_i} + a_2 e^{-2x_i} - y_i) = 0$$

$$\begin{bmatrix} \sum e^{4x_i} & n \\ n & \sum e^{-4x_i} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum e^{2x_i} y_i \\ \sum e^{-2x_i} y_i \end{bmatrix}$$

$$\begin{bmatrix} 9051900.86 & 4 \\ 4 & 0.0186 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7808.80 \\ 0.3545 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.0009 \\ 18.8754 \end{bmatrix}$$

(c) If $y = b_1 x e^{-b_2 x}$ then $\ln y = \ln b_1 + \ln x - b_2 x$

$$E = \sum_{i=1}^{4} (\ln b_1 + \ln x_i - b_2 x_i - \ln y_i)^2$$

$$\frac{\partial E}{\partial \ln b_1} = 2 \sum_{i=1}^{4} (\ln b_1 + \ln x_i - b_2 x_i - \ln y_i) = 0$$

$$\frac{\partial E}{\partial b_2} = -2 \sum_{i=1}^{4} x_i (\ln b_1 + \ln x_i - b_2 x_i - \ln y_i) = 0$$

$$\begin{bmatrix} n & -\sum x_i \\ -\sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \ln b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \sum \ln y_i - \sum \ln x_i \\ -\sum x_i \ln y_i + \sum x_i \ln x_i \end{bmatrix}$$

$$= \begin{bmatrix} \ln \frac{\prod y_i}{\prod x_i} \\ \ln \frac{\prod x_i^{x_i}}{\prod y_i^{x_i}} \end{bmatrix}$$

$$\begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} \ln b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.9808 \\ -0.1698 \end{bmatrix} \Rightarrow \begin{bmatrix} \ln b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1.3863 \\ 0.4564 \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0.4564 \end{bmatrix}$$