Video 1:


TWO POPULATIONS:
HYPOTHESIS ABOUT THE MEAN DIFFERENCE
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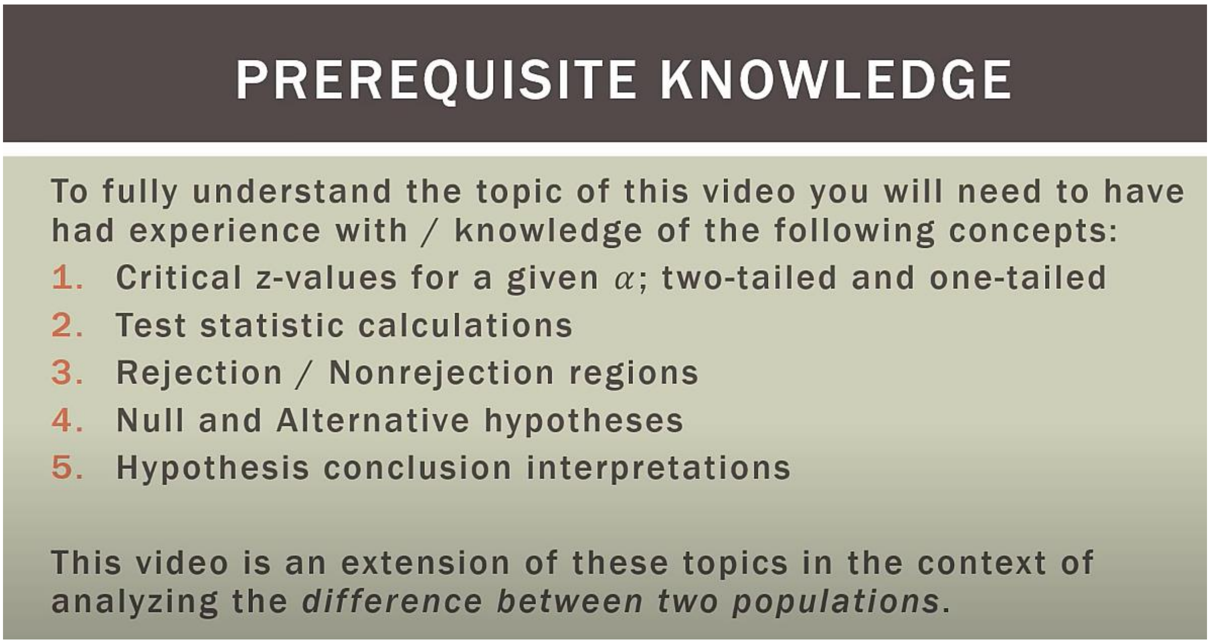

PREREQUISITE KNOWLEDGE
To fully understand the topic of this video you will need to have had experience with / knowledge of the following concepts:
1. Critical z-values for a given α; two-tailed and one-tailed
2. Test statistic calculations
3. Rejection / Nonrejection regions
4. Null and Alternative hypotheses
5. Hypothesis conclusion interpretations
This video is an extension of these topics in the context of analyzing the *difference between two populations*.

EXAMPLE 1: CALL CENTER COMPARISON

Let's say you are the manager of two call centers near Bangalore, India. These centers focus on in-depth problem solving for clients, therefore the calls last approximately 12 minutes. Each location has its own employees, management, and work culture.

As part of your performance analysis, you wish to see if there is a difference in the average call-length between both locations. The assumption is that the mean difference between the two locations is zero. That is, there is no difference in mean call length between the two call centers.

Call Center 1

μ_1

$\sigma_1 = 1.2$ minutes

Assumption?

Mean difference is ZERO.

$$\mu_1 - \mu_2 = 0$$

Independence?

These two centers are independent of each other.

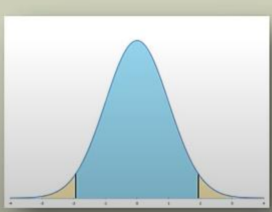
Call Center 2

μ_2

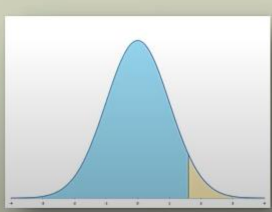
$\sigma_2 = 1.5$ minutes

HYPOTHESIS FORMATS AND REGIONS

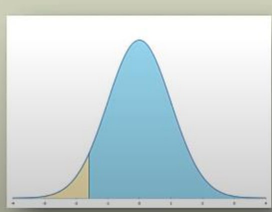
$H_0: \mu_1 - \mu_2 = D_0$	$H_0: \mu_1 - \mu_2 \leq D_0$	$H_0: \mu_1 - \mu_2 \geq D_0$
$H_a: \mu_1 - \mu_2 \neq D_0$	$H_a: \mu_1 - \mu_2 > D_0$	$H_a: \mu_1 - \mu_2 < D_0$



Two-tailed



One-tailed upper



One-tailed lower

CALL CENTER HYPOTHESIS

The tentative assumption (null hypothesis) is that no difference exists between the mean call lengths at the two call centers; D_0.

$$H_0: \mu_1 - \mu_2 = 0 \qquad H_a: \mu_1 - \mu_2 \neq 0$$

Significance level: $\alpha = .05$

Since σ is known, we will use the z-distribution in our test

Decision Rule:

Since $\alpha = .05$ and we are using the z-distribution, H_0 will be rejected if the test statistic is > 1.96 or < -1.96

TEST STATISTIC FOR $\mu_1 - \mu_2$: σ'S KNOWN

Test Statistic for a Single Mean

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

μ_0 = hypothesized population mean

Test Statistic for the Difference of Two Independent Random Samples

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\left(\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}\right)}$$

D_0 = hypothesized **difference** between two population means

INTERVAL ESTIMATE FOR $\bar{d}$

Interval Estimate for a Single Mean

$\bar{x} \pm$ margin of error

Margin of error $= z_{a/2}\frac{\sigma}{\sqrt{n}}$

$\bar{x} \pm z_{a/2}\frac{\sigma}{\sqrt{n}}$

Interval Estimate of the Difference for Two Independent Random Samples

$\bar{d} \pm$ margin of error

Margin of error $= z_{a/2}\left(\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}\right)$

$\bar{d} \pm z_{a/2}\left(\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}\right)$

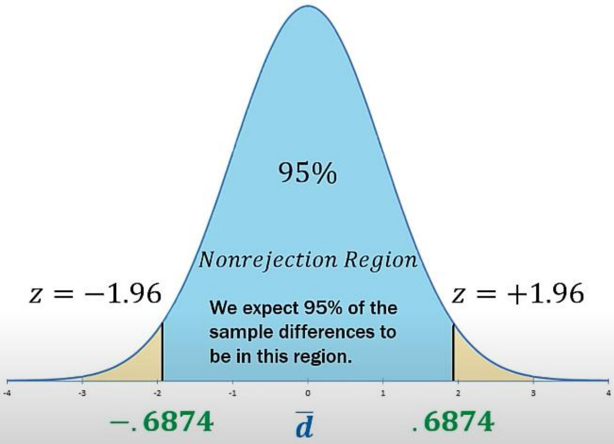
Nonrejection Region for the Difference

	Call Center 1	Call Center 2
Sample Size, n	30	30
σ (given)	1.20	1.50

$$\bar{d} \pm z_{a/2}\left(\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}\right)$$

$$\bar{d} \pm 1.96\left(\sqrt{\frac{1.20^2}{30} + \frac{1.50^2}{30}}\right)$$

$$\bar{d} \pm 0.6874$$

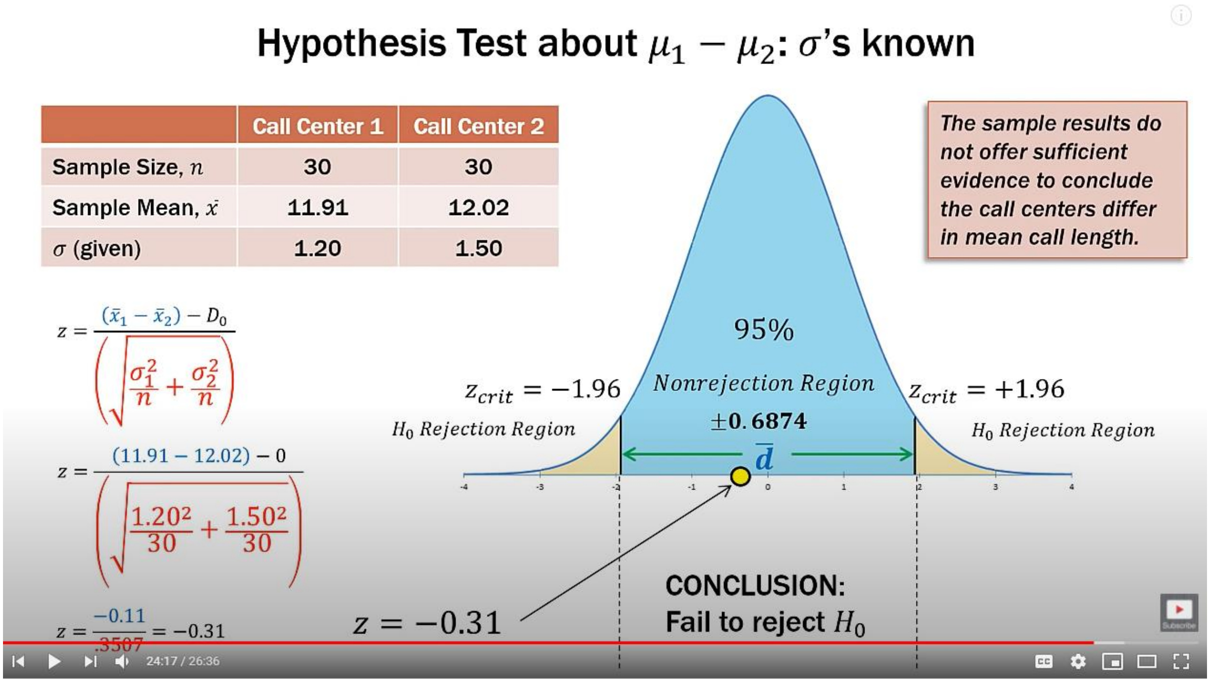


Distribution of the difference(s)

Z-STATISTIC FOR CALL LENGTH

	Call Center 1	Call Center 2
Sample Size, n	30	30
Sample Mean, $\bar{x}$	11.91	12.02
σ (given)	1.20	1.50

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\left(\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}\right)} \quad z = \frac{(11.91 - 12.02) - 0}{\left(\sqrt{\frac{1.20^2}{30} + \frac{1.50^2}{30}}\right)} \quad z = \frac{-0.11}{.3507} = -0.31$$



Hypothesis Test about $\mu_1 - \mu_2$: σ's known
Call Center 1
Call Center 2
Sample Size, n
30
30
Sample Mean, $\bar{x}$
11.91
12.02
σ (given)
1.20
1.50
$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\left(\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}\right)}$
$z = \frac{(11.91 - 12.02) - 0}{\left(\sqrt{\frac{1.20^2}{30} + \frac{1.50^2}{30}}\right)}$
$z = \frac{-0.11}{.3507} = -0.31$
95%
$z_{crit} = -1.96$
Nonrejection Region
$z_{crit} = +1.96$
H_0 Rejection Region
± 0.6874
$\bar{d}$
H_0 Rejection Region
-4
-3
-2
-1
0
1
2
3
4
The sample results do not offer sufficient evidence to conclude the call centers differ in mean call length.
$z = -0.31$
CONCLUSION:
Fail to reject H_0
24:17 / 26:36

Video 2: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INTERVAL ESTIMATE OF THE MEAN DIFFERENCE
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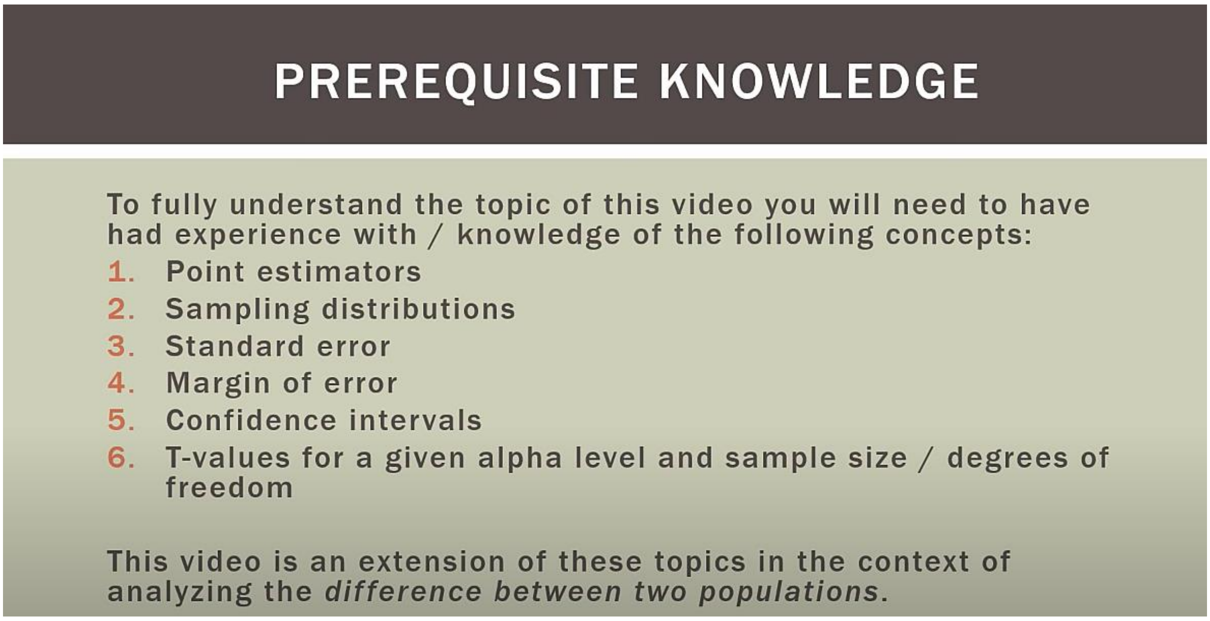

PREREQUISITE KNOWLEDGE
To fully understand the topic of this video you will need to have had experience with / knowledge of the following concepts:
1. Point estimators
2. Sampling distributions
3. Standard error
4. Margin of error
5. Confidence intervals
6. T-values for a given alpha level and sample size / degrees of freedom
This video is an extension of these topics in the context of analyzing the difference between two populations.

σ UNKNOWN, s AS ESTIMATE

- When the population standard deviations, σ, are unknown we must estimate them using the sample standard deviation, s
- Since s is by definition an estimate, it will not be perfect
- Therefore we switch from using the standard normal distribution (z-distribution) to utilizing the t-distribution with a certain degrees of freedom
- In this case, degrees of freedom are computed using an ugly formula we will learn later (its not that bad actually)
- There are other ways to compute the degrees of freedom, however we are going to learn the one that requires the fewest assumptions and is universally applicable

CALL CENTER COMPARISON

Let’s say you are the manager of two call centers near Bangalore, India. These centers focus on in-depth problem solving for clients, therefore the calls last approximately 12 minutes. Each location has its own employees, management, and work culture.

As part of your performance analysis, you wish to see if there is a difference in the average call-length between both locations. The assumption is that the mean difference between the two locations is zero. That is, there is no difference in mean call length between the two call centers.

A random sample of $n = 30$ will be taken from both call centers.

Call Center 1

μ_1

$s_1 = 1.2$ minutes

Assumption?

Mean difference is ZERO.

$\mu_1 - \mu_2 = 0$

Or $\mu_1 = \mu_2$

Independence?

These two centers are independent of each other.

Call Center 2

μ_2

$s_2 = 1.5$ minutes

DISTRIBUTION OF DIFFERENCES: $\mu_1 - \mu_2$

μ_1 = mean call length for Call Center 1

μ_2 = mean call length for Call Center 2

What we are really talking about:

1. Take an independent sample from μ_1 which will be $\bar{x}_1$
2. Take an independent sample from μ_2 which will be $\bar{x}_2$
3. Find $\bar{x}_1 - \bar{x}_2 = d_1$ (difference)
4. (Theoretically) return to step #1 and repeat process
5. Then create a distribution of the differences, d_i.

$\bar{x}_1 - \bar{x}_2$ or d **is a point estimator of** $\mu_1 - \mu_2$

DISTRIBUTION OF DIFFERENCES

Sample ($n = 30$)	$\bar{x}_1$	$\bar{x}_2$	d_i
1	11.91	12.02	-0.11
2	11.36	12.02	-0.66
3	11.75	12.05	-0.3
4	12.14	12.18	-0.04
5	11.72	12.11	-0.39
6	11.61	12.07	-0.46
7	11.85	12.05	-0.2
8	12.16	11.64	0.52
9	11.91	12.39	-0.48
10	12.12	11.65	0.47

Mean call length in minutes

Each d_i is a point estimate of $\mu_1 - \mu_2$

The d_i values form a distribution of differences in the same way single sample means have a sampling distribution

$\boxed{\bar{d} = \text{mean of the differences}}$

STANDARD ERROR OF $\bar{d}$, σ UNNKOWN

Standard Error of the Mean

$$s_{\bar{x}} = \frac{s}{\sqrt{n}}$$

Standard Error of the Difference for Two Independent Random Samples

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$s_{\bar{d}} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

INTERVAL ESTIMATE FOR $\bar{d}$, σ UNNKOWN

Interval Estimate for a Single Mean		Interval Estimate of the **Difference** for Two Independent Random Samples
$\bar{x} \pm \text{margin of error}$	→	$\bar{d} \pm \text{margin of error}$
$\text{Margin of error} = t_{a/2}\dfrac{s}{\sqrt{n}}$	→	$\text{Margin of error} = t_{a/2}\left(\sqrt{\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}}\right)$
$\bar{x} \pm t_{a/2}\dfrac{s}{\sqrt{n}}$	→	$\bar{d} \pm t_{a/2}\left(\sqrt{\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}}\right)$

DEGREES OF FREEDOM CALCULATION

- Finding the degrees of freedom when using the t-distribution with two independent random samples uses a more involved formula

$$df = \frac{\left(\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}\right)}{\dfrac{1}{n_1-1}\left(\dfrac{s_1^2}{n_1}\right)+\dfrac{1}{n_2-1}\left(\dfrac{s_2^2}{n_2}\right)}$$

s_1 = standard deviation of sample 1

n_1 = sample size of sample 1

s_2 = standard deviation of sample 2

n_2 = sample size of sample 2

DEGREES OF FREEDOM CALCULATION

Finding the degrees of freedom when using the t-distribution with two independent random samples requires a more involved formula

$s_1 = 1.2$ minutes $\quad n_1 = 30$

$s_2 = 1.5$ minutes $\quad n_2 = 30$

$$df = \frac{\left(\frac{1.2^2}{30} + \frac{1.5^2}{30}\right)}{\frac{1}{30-1}\left(\frac{1.2^2}{30}\right) + \frac{1}{30-1}\left(\frac{1.5^2}{30}\right)}$$

$$df = \frac{(.048 + .075)}{\frac{1}{29}(.048) + \frac{1}{29}(.075)} \qquad df = \frac{(.123)}{.00167 + .00259} \qquad df = 28.873 \cong 28$$

Round down (more conservative)

INTERVAL ESTIMATE FOR CALL LENGTH

	Call Center 1	Call Center 2
Sample Size, n	30	30
σ (given)	1.20	1.50
$df = 28$	$\alpha = .05$	$t_{crit} = \pm 2.048$

95% confidence

$$\bar{d} \pm t_{a/2}\left(\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right)$$

$$\bar{d} \pm 2.048\left(\sqrt{\frac{1.20^2}{30} + \frac{1.50^2}{30}}\right) \qquad \bar{d} \pm 2.048(.3507) \qquad \bar{d} \pm \mathbf{0.7183}$$

95% confidence interval estimate for $\bar{d}$

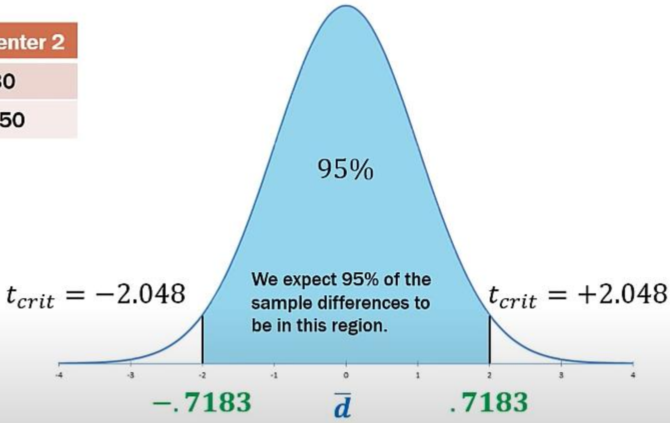
Sampling Distribution of $\bar{d}$

	Call Center 1	Call Center 2
Sample Size, n	30	30
s (given)	1.20	1.50

$$\bar{d} \pm 2.048\left(\sqrt{\frac{1.20^2}{30}+\frac{1.50^2}{30}}\right)$$

$$\bar{d} \pm 2.048(.3507)$$

$$\bar{d} \pm \mathbf{0.7183}$$



INTERVAL ESTIMATE FOR CALL LENGTH

	Call Center 1	Call Center 2
Sample Size, n	30	30
Sample Mean, $\bar{x}$	11.91	12.02
s (given)	1.20	1.50
$df = 28$	$\alpha = .05$	$t_{crit} = \pm 2.048$

95% confidence

$$-.11 \pm t_{\alpha/2}\left(\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}\right)$$

$$-.11 \pm 2.048\left(\sqrt{\frac{1.20^2}{30}+\frac{1.50^2}{30}}\right)$$

$$-.11 \pm 2.048(.3507)$$

$$-.11 \pm \mathbf{0.7183}$$

$$-0.8283 \text{ to } 0.6083$$

minutes

95% confidence interval estimate of the difference between μ_1 and μ_2

Sampling Distribution of $\bar{d}$

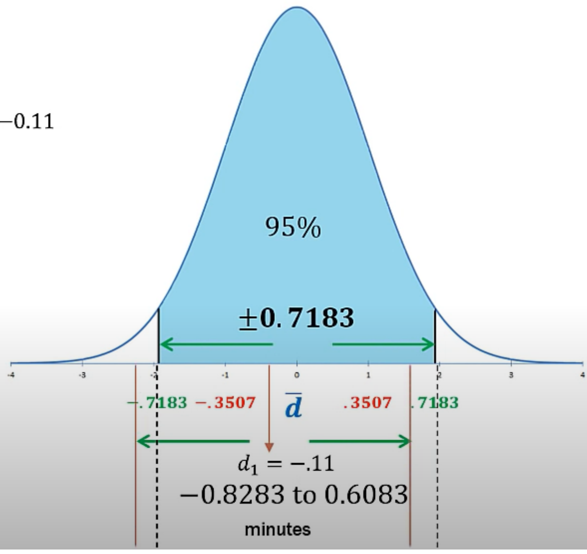
	Call Center 1	Call Center 2
Sample Size, n	30	30
Sample Mean, $\dot{x}$	11.91	12.02
s (given)	1.20	1.50
$df = 28$	$\alpha = .05$	$t_{crit} = \pm 2.048$

$d_1 = -0.11$

$$-.11 \pm 2.048\left(\sqrt{\frac{1.20^2}{30} + \frac{1.50^2}{30}}\right)$$

$$-.11 \pm 2.048(.3507)$$

$$-.11 \pm \mathbf{0.7183}$$



CALL CENTER CONCLUSION

Based on our samples:

- the margin of error is ± 0.7183 minutes
- the 95% confidence interval estimate of the difference between the two call center call population means is -0.11 ± 0.7183

-0.8283 minutes to 0.6083 minutes.

There is a 95% probability that this interval contains the true mean difference $\bar{d}$ between μ_1 and μ_2.

Is ZERO in this 95% interval? YES.

INFERENCES ABOUT μ_1 AND μ_2; σ UNKOWN

- μ_1 is the mean of population 1 and μ_2 is the mean of population 2.
- We are interested in the differences between the means; $\mu_1 - \mu_2$.
- To conduct this analysis we will select two **independent random samples**, n_1 and n_2 from μ_1 and μ_2 respectively. They do not have to be the same size but ideally are close in size.
- When σ is not given for the populations we use s and the t-distribution as a model of the sampling distribution of differences
- We must also calculate the appropriate degrees of freedom
- From there, we proceed in finding the confidence interval for $\bar{d}$ in the same manner as we would find the C.I. for $\bar{x}$

$$\bar{x} \pm t_{a/2} \frac{s}{\sqrt{n}} \quad \longrightarrow \quad \bar{d} \pm t_{a/2} \left(\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

Video 3:

TWO POPULATIONS:

MATCHED SAMPLES T-TEST

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1. Point estimators
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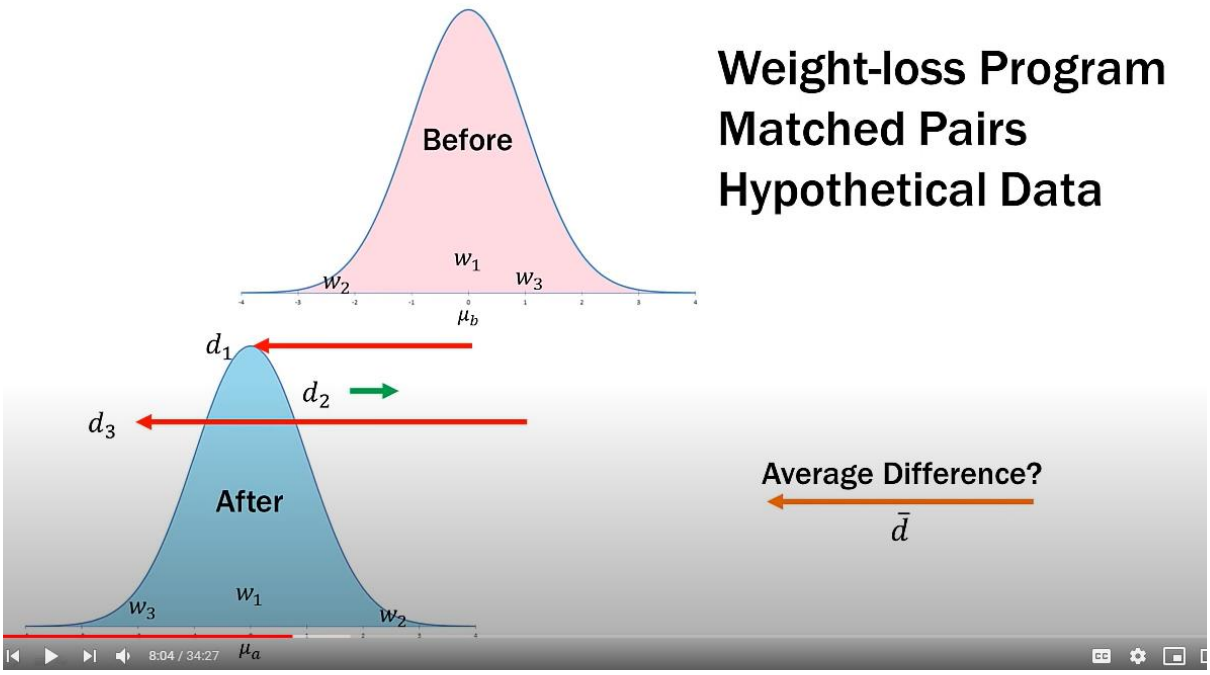
4. Margin of error
5. Confidence intervals
6. T-values for a given alpha level and sample size / degrees of freedom; both one-tailed and two-tailed

This video is an extension of these topics in the context of analyzing the *difference between two populations*.



THE FIGHT THE FLAB PROGRAM

A new diet and exercise craze called "Fight the Flab" has hit popular culture. While on this diet you can eat anything you want every two hours but the item can be no larger than the size of your closed hand. You must also exercise for 30 minutes 3 times a week (minimum).

To test how well this program works, 10 men are randomly selected to participate in a study. They are to follow the program for one year. Body weight measurements will be taken before the diet begins and again after one year.

At the end of the study, we will test whether or not there has been a significant reduction in mean weight for the 10 men by comparing before diet weight with after diet weight.

Before Diet Program

Assumption?

Mean of the matched differences is zero.

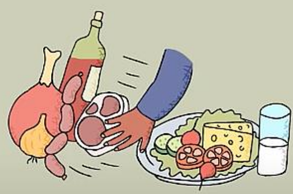
$\mu_d = 0$

Independence?

These two measures are NOT independent of each other.

They are MATCHED.

After Diet Program



Subject 1	weight_{before}	weight_{after}
Subject 2	weight_{before}	weight_{after}
⋮	⋮	⋮

DIFFERENCES OF MATCHED PAIRS

$weight_{before}$ = subject weight before diet

$weight_{after}$ = subject weight after 12 months

What we are really talking about:

1. Take an measurement of subject's weight before diet
2. Take an measurement of subject's weight after 12 months
3. Find $w_{before} - w_{after} = d_1$ (difference)
4. Return to step #1 and repeat process for all subjects
5. Then conduct analysis on new variable; the mean of the differences, $\bar{d}$.

FIGHT THE FLAB HYPOTHESIS

The research hypothesis (alternative hypothesis) is that FTF **reduced** subject weight. The mean of the differences is < 0.

$$H_0: \mu_d \geq 0 \qquad H_a: \mu_d < 0$$

Significance level: $\alpha = .05$

Since this is a matched pairs test with $n = 10$, we will use the t-distribution with $df = 9$.

Decision Rule:

Since $\alpha = .05$ and we are using lower-tail t-distribution, H_0 will be rejected if the test statistic is < -1.833

MEAN OF THE DIFFERENCES

Subject	w_b	w_a	$d_i = w_a - w_b$
1	220	195	-25
2	260	202	-58
3	253	220	-33
4	241	241	0
5	230	205	-25
6	295	220	-75
7	224	208	-16
8	305	245	-60
9	235	230	-5
10	250	270	+20

The d_i values form a distribution of differences in the same way single sample means have a sampling distribution

$\boxed{\bar{d} = \text{mean of the differences}}$

$\bar{d} = -27.7$ pounds

$s_d = 29.7$

STANDARD ERROR OF $\bar{d}$, MATCHED PAIRS

Standard Error of the Mean

$$s_{\bar{x}} = \frac{s}{\sqrt{n}}$$

Standard Error of the Difference for Two Matched Measurements

$$s_{\bar{d}} = \frac{s_d}{\sqrt{n}}$$

The only difference is that we are using…DIFFERENCES! ☺

T-TEST STATISTIC, MATCHED PAIRS

Single Sample t-Test

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

$$\text{df} = n - 1$$

Matched Pair t-Test

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$$

$$\text{df} = n - 1$$

The only difference is that we are using…DIFFERENCES! ☺

INTERVAL ESTIMATE FOR $\bar{d}$, MATCHED PAIRS

Interval Estimate for a Single Mean		Standard Error of the Difference for Two Matched Measurements
$\bar{x} \pm$ margin of error	→	$\bar{d} \pm$ margin of error
Margin of error $= t_{a/2}\dfrac{s}{\sqrt{n}}$	→	Margin of error $= t_{a/2}\dfrac{s_d}{\sqrt{n}}$ (two tailed)
$\bar{x} \pm t_{a/2}\dfrac{s}{\sqrt{n}}; with\ df = n-1$	→	$\bar{d} \pm t_{a/2}\dfrac{s_d}{\sqrt{n}}; with\ df = n-1$

CRITICAL WEIGHT VALUE FOR FLAB

Sample Size, n (pairs)	10	
s_d	29.7 pounds	
$df = 9$	$\alpha = .05$	$t_{crit} = -1.833$

95% confidence

$$\bar{d} - t_a\frac{s_d}{\sqrt{n}}$$

$$with\ df = n-1$$

$$\bar{d} - 1.833\frac{29.7}{\sqrt{10}}$$

$$\bar{d} - 1.833(9.39)$$

$$\bar{d} - 17.22$$

A mean of differences representing a greater than 17.22 pound weight loss will lead to a rejection of H_0.

T-Test: Paired Two Sample for Means

Sample Size, n (pairs)	10	
s_d	29.7 pounds	
$df = 9$	$\alpha = .05$	$t_{crit} = -1.833$

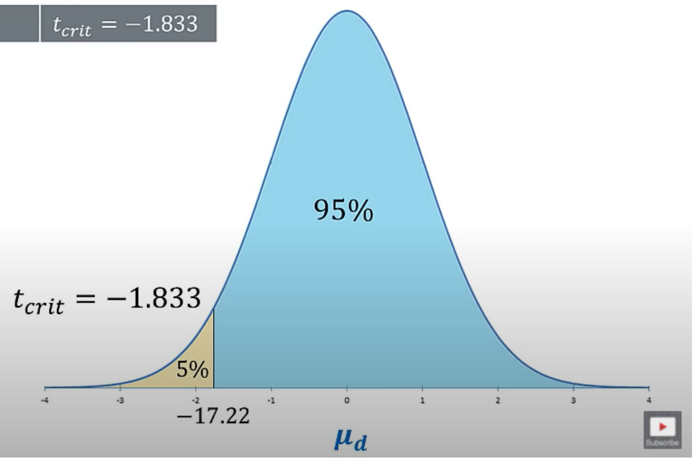
$$\bar{d} - t_{\alpha}\frac{s_d}{\sqrt{n}}$$

$with\ df = n - 1$

$$\bar{d} - 1.833\frac{29.7}{\sqrt{10}}$$

$$\bar{d} - 1.833(9.39)$$

$$\bar{d} - 17.22$$



TEST STATISTIC FOR FIGHT THE FLAB

Sample Size, n (pairs)	10	
$\bar{d}$	-27.7 pounds	
s_d	29.7 pounds	
$df = 9$	$\alpha = .05$	$t_{crit} = -1.833$

$$t = \frac{\bar{d} - D_0}{\frac{s_d}{\sqrt{n}}}$$

$\mathrm{df} = n - 1$

$$t = \frac{-27.7 - 0}{\frac{29.7}{\sqrt{10}}} \qquad t = \frac{-27.7}{9.39} \qquad t = -2.95$$

Our test statistic is well beyond (below) the t_{crit}. It is in the rejection region.

T-Test: Paired Two Sample for Means

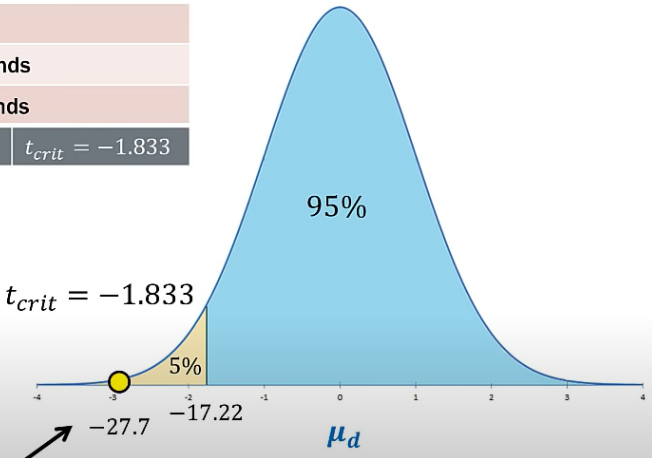
Sample Size, n (pairs)	10	
$\bar{d}$	-27.7 pounds	
s_d	29.7 pounds	
$df = 9$	$\alpha = .05$	$t_{crit} = -1.833$

$$t = \frac{\bar{d} - D_0}{\frac{s_d}{\sqrt{n}}}$$

$$t = \frac{-27.7 - 0}{\frac{29.7}{\sqrt{10}}}$$

$$t = \frac{-27.7}{9.39}$$

$$t = -2.95$$



INTERVAL ESTIMATE FOR FIGHT THE FLAB

Sample Size, n (pairs)	10	
$\bar{d}$	-27.7 pounds	
s_d	29.7 pounds	
$df = 9$	$\alpha = .05$	$t_{crit} = -1.833$

$$\bar{d} \pm t_{a/2}\frac{s_d}{\sqrt{n}}; with\ df = n - 1$$

$$-27.7 \pm 1.833\frac{29.7}{\sqrt{10}}$$

$$-27.7 \pm 1.833(9.39)$$

$$-27.7 \pm 17.21$$

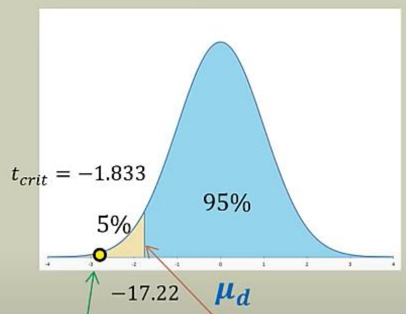
$$-44.91 \text{ to } -10.48 \text{ pounds}$$

Does this interval contain ZERO? No!

FIGHT THE FLAB RESULTS IN EXCEL

t-Test: Paired Two Sample for Means

	After	*Before*
Mean	223.6	251.3
Variance	539.3777778	824.9
Observations	10	10
Pearson Correlation	0.36166805	
Hypothesized Mean Difference	0	
df	9	
t Stat	-2.949833192	
P(T<=t) one-tail	0.008112664	
t Critical one-tail	1.833112933	
~~P(T<=t) two-tail~~	~~0.016225328~~	
~~t Critical two-tail~~	~~2.262157163~~	



$t = -2.95$

$t_{crit} = -1.833$

Also notice the p −value of 0.008 is well below $\alpha = 0.05$.

FIGHT THE FLAB CONCLUSION

Based on our matched sample the 95% confidence interval estimate of the difference between the matched pair means is

$$-44.91 \text{ to } -10.48 \text{ pounds}$$

We are 95% confident this interval contains the true difference between before and after weight.

Does this interval estimate include the hypothesizes mean difference of ZERO? No. Zero is outside of this confidence interval and the true mean of the differences appears much lower than zero pounds.

CONCLUSION:

We reject the null hypothesis H_0 that the mean of the before/after weight differences is ZERO. The test statistic and $\bar{d}$ are both beyond t_{crit} for $\mathrm{df} = 9$. Therefore based on this data it appears the diet works.